A geometric view of Ralescu's many-valued cardinality

Eduard Bartl, Radim Belohlavek
Department of Computer Science, Palacký University Olomouc, Czech Republic

***Abstract*—We develop a geometric view of Ralescu's concept of a many-valued cardinality of fuzzy sets. The view facilitates an easy understanding of this concept and helps elucidate its nature. We demonstrate that the view enables to obtain properties of this concept as consequences of theorems regarding the geometric view that are derived by a straightforward geometric reasoning. We observe that the developed view applies to a natural generalization of Ralescu's concept for which it reveals why this generalization is more difficult to analyze.**

***Index Terms*—Many-valued set, fuzzy set, cardinality, fuzzy cardinality.**

I. Our aim

THE notion of cardinality entered considerations on many-valued sets and, in particular, fuzzy sets in the early stage of development; see, e.g., [7] and [4] for the first contributions and [11] for a comprehensive account of the developments. In his well-known paper [9], Ralescu proposed interesting notions of both, the so-called fuzzy and scalar cardinality of fuzzy sets. Our aim is to provide a geometric interpretation of Ralescu's cardinality concepts. For one, the view we develop reveals a new understanding of Ralescu's notions of cardinality. Secondly, the view enables a straightforward geometric analysis of Ralescu's notions. We demonstrate this by providing simple proofs of some properties of Ralescu's fuzzy and scalar cardinality obtained in the original paper [9] as well as in the recent paper [2], and by presenting new considerations on the cardinality concepts that are naturally offered by the geometric view.

II. Ralescu's concept of fuzzy and scalar cardinality

CONSIDER a fuzzy set $A : U \to [0,1]$ in a finite universe $U = \{u_1, \dots, u_n\}$. That is, A represents a many-valued characteristic function of a collection with an unsharp, gradual boundary to which each element u in U is a member to the degree $A(u) \in [0,1]$. Denote by $A_{(1)}, \dots, A_{(n)}$ the membership degrees $A(u_1), \dots, A(u_n)$ ordered in a non-increasing manner, i.e.,

$$A_{(1)} \geq \cdots \geq A_{(n)} \quad \text{and put} \quad A_{(0)} = 1 \text{ and } A_{(n+1)} = 0. \quad (1)$$

The non-increasing sequence $A_{(1)}, \dots, A_{(n)}$ thus results by a permutation of $A(u_1), \dots, A(u_n)$. Ralescu [9] defined the so-called fuzzy cardinality of A as the fuzzy set $|A|_{\mathrm{Rf}}$ assigning to a non-negative integer $k = 0, 1, \dots, n$ the degree

$$|A|_{\mathrm{Rf}}(k) = \min\{A_{(k)}, 1 - A_{(k+1)}\}, \quad (2)$$

which is interpreted as the truth degree of the statement "A has k elements." Furthermore, he defined the scalar cardinality $|A|_{\mathrm{Rs}}$ of A by

$$|A|_{\mathrm{Rs}} = \begin{cases} 0 & \text{if } A = \emptyset, \\ j & \text{if } A \neq \emptyset \text{ and } A_{(j)} \geq 0.5, \\ j-1 & \text{if } A \neq \emptyset \text{ and } A_{(j)} < 0.5, \end{cases} \quad (3)$$

where

$$j = \max\{k\,;\, 1 \leq k \leq n \text{ and } A_{(k-1)} + A_{(k)} > 1\}. \quad (4)$$

We present examples which shall be used in the rest of this paper.

Example 1. For the fuzzy set

$$A = \{{}^{0.85}\!/u_1, {}^{0.8}\!/u_2, {}^{0.6}\!/u_3, {}^{0.6}\!/u_4, {}^{0.3}\!/u_5, {}^{0.2}\!/u_6\}, \quad (5)$$

one has $A_{(0)} = 1$, $A_{(1)} = 0.85$, $A_{(2)} = 0.8$, $A_{(3)} = 0.6$, $A_{(4)} = 0.6$, $A_{(5)} = 0.3$, $A_{(6)} = 0.2$, and $A_{(7)} = 0$. One easily verifies that

$$|A|_{\mathrm{Rf}} = \{{}^{0.15}\!/0, {}^{0.2}\!/1, {}^{0.4}\!/2, {}^{0.4}\!/3, {}^{0.6}\!/4, {}^{0.3}\!/5, {}^{0.2}\!/6\}$$

and

$$|A|_{\mathrm{Rs}} = 4. \qquad \square$$

Example 2. For the fuzzy set

$$B = \{{}^{0.3}\!/u_1, {}^{0.8}\!/u_2, {}^{0.2}\!/u_3, {}^{0.5}\!/u_4, {}^{0.85}\!/u_5, {}^{0.6}\!/u_6\},$$

we obtain $B_{(0)} = 1$, $B_{(1)} = 0.85$, $B_{(2)} = 0.8$, $B_{(3)} = 0.6$, $B_{(4)} = 0.5$, $B_{(5)} = 0.3$, $B_{(6)} = 0.2$, $B_{(7)} = 0$, and one has

$$|B|_{\mathrm{Rf}} = \{{}^{0.15}\!/0, {}^{0.2}\!/1, {}^{0.4}\!/2, {}^{0.5}\!/3, {}^{0.5}\!/4, {}^{0.3}\!/5, {}^{0.2}\!/6\}$$

$$|B|_{\mathrm{Rs}} = 4. \qquad \square$$

Example 3. For the fuzzy set

$$C = \{{}^{0.85}\!/u_1, {}^{0.8}\!/u_2, {}^{0.6}\!/u_3, {}^{0.5}\!/u_4, {}^{0.5}\!/u_5, {}^{0.5}\!/u_6, {}^{0.5}\!/u_7, {}^{0.3}\!/u_8, {}^{0.2}\!/u_9\},$$

the rearrangement yields $C_{(0)} = 1$, $C_{(1)} = 0.85$, $C_{(2)} = 0.8$, $C_{(3)} = 0.6$, $C_{(4)} = 0.5$, $C_{(5)} = 0.5$, $C_{(6)} = 0.5$, $C_{(7)} = 0.5$, $C_{(8)} = 0.3$, $C_{(9)} = 0.2$, $C_{(10)} = 0$, and one gets

$$|C|_{\mathrm{Rf}} = \{{}^{0.15}\!/0, {}^{0.2}\!/1, {}^{0.4}\!/2, {}^{0.5}\!/3, {}^{0.5}\!/4, {}^{0.5}\!/5, {}^{0.5}\!/6, {}^{0.5}\!/7, {}^{0.3}\!/8, {}^{0.2}\!/9\}$$

and

$$|C|_{\mathrm{Rs}} = 7. \qquad \square$$

Remark 1. *The purpose of both cardnality concepts is to express the size of a given fuzzy set A. While $|A|_{\mathrm{Rs}}$ is a non-negative integer, $|A|_{\mathrm{Rf}}$ is a fuzzy set of non-negative integers for which $|A|_{\mathrm{Rf}}(k)$ may be regarded as a degree to which it is* *plausible to consider k as the number of elements in A. One easily checks that both $|A|_{\mathrm{Rs}}$ and $|A|_{\mathrm{Rf}}$ generalize the notion of a cardinality of a classical finite set: If A is a characteristic function of a classical set with k elements, then $|A|_{\mathrm{Rs}} = k$ and, moreover, $|A|_{\mathrm{Rf}}(k) = 1$ and $|A|_{\mathrm{Rf}}(i) = 0$ for $i \neq k$.*

Ralescu's notions of scalar and fuzzy cardinality are interesting in that they both take into account the relationships among the membership degrees $A(u)$ rather than just the individual degrees like most alternative approaches. Yet, since the original paper [9] does not primarily focus on the properties of the cardinality concepts, it does not answer some natural questions. For instance, since the definition (2) of $|A|_{\mathrm{Rf}}$ expresses—in a many-valued setting—a natural idea that the cardinality of A is k if A contains k but not $k+1$ elements, one naturally asks for the maximum of $|A|_{\mathrm{Rf}}$ and for the integers at which the maximum is attained, i.e., for the most plausible cardinalities. Moreover, since the definition of $|A|_{\mathrm{Rs}}$ is—due to (4)—basically iterative in nature, one asks for a direct formula for $|A|_{\mathrm{Rs}}$.

The above questions, along with those regarding the relationship between the two cardinality concepts, are studied in a recent paper [2]. Our aim is to show that considerations on both of Ralescu's cardinality concepts may be conducted by using a natural geometric view which clarifies both concepts. In addition, we show that the properties obtained in [2] as well as other properties can be obtained within this geometric view. For this purpose, the following notions shall be used. First, let

$$\max |A|_{\mathrm{Rf}} = \max\{|A|_{\mathrm{Rf}}(0), |A|_{\mathrm{Rf}}(1), |A|_{\mathrm{Rf}}(2), \dots\},$$

i.e., $\max |A|_{\mathrm{Rf}}$ is the largest membership degree attained by $|A|_{\mathrm{Rf}}$. Furthermore, let

$$\arg\max |A|_{\mathrm{Rf}} = \{i\,;\, |A|_{\mathrm{Rf}}(i) = \max |A|_{\mathrm{Rf}}\},$$

i.e., $\arg\max |A|_{\mathrm{Rf}}$ denotes the set of non-negative integers for which $\max |A|_{\mathrm{Rf}}$ is attained.

III. Geometric view behind the concepts of cardinality

OUR geometric view may be explained as follows. The degrees $A_{(0)}, A_{(1)}, \dots, A_{(n+1)}$ ordered as in (1) can be represented by points $P_0, P_1, \dots, P_n$ in the xy-plane defined as

$$P_k = \langle A_{(k)}, A_{(k+1)}\rangle \quad \text{for } k = 0, 1, \dots, n. \tag{6}$$

In what follows, we shall write $P_k = \langle x_k, y_k\rangle$. Since $0 \leq x_k \leq 1$ and $x_k = A_{(k)} \geq A_{(k+1)} = y_k$ for all k, every point P_k is located below or on the main diagonal of the unit square. A point P_k located below the main diagonal, i.e., with $x_k > y_k$, shall be called a *subdiagonal point*, while a point P_k positioned on the main diagonal, i.e., with $x_k = y_k$, shall be called a *diagonal point*. Since $A_{(0)} = 1$ and $A_{(n+1)} = 0$ by definition, the sequence $P_0, P_1, \dots, P_n$ contains at least one subdiagonal point for each fuzzy set A. It is also immediate that a fuzzy set A is crisp, i.e., $A(u)$ is either 0 or 1 for each $u \in U$, iff $P_0, P_1, \dots, P_n$ contains just one subdiagonal point.

Because the y-coordinate of P_k coincides with the x-coordinate of P_{k+1}, the points $P_0, P_1, \dots, P_n$ form a step-like geometric pattern. Formally, for $P_k = \langle x_k, y_k\rangle$ being a subdiagonal point, a *step $S(P_k)$ in P_k* is the union

$$S(P_k) = [\langle y_k, y_k\rangle, \langle x_k, y_k\rangle] \cup [\langle x_k, y_k\rangle, \langle x_k, x_k\rangle]$$

of the horizontal line

$$[\langle y_k, y_k\rangle, \langle x_k, y_k\rangle] = \{\langle x, y_k\rangle \mid y_k \leq x \leq x_k\}$$

connecting the main diagonal with P_k and the vertical line

$$[\langle x_k, y_k\rangle, \langle x_k, x_k\rangle] = \{\langle x_k, y\rangle \mid y_k \leq y \leq x_k\}$$

connecting P_k with the main diagonal.

The notion of a step in P_k is illustrated in figure 1.

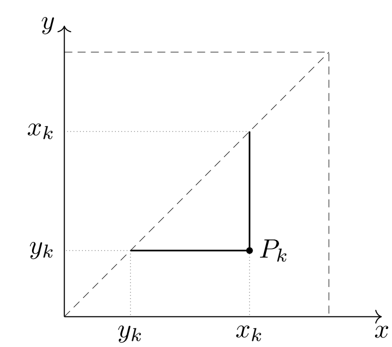

Fig. 1. Step in P_k.

A *step-pattern* corresponding to a fuzzy set A is defined as a union of steps in all subdiagonal points

$$\bigcup\{S(P_k) \mid P_k \text{ is subdiagonal point}\}.$$

For instance, for the fuzzy set (5) defined in example 1 the corresponding step-pattern is shown in figure 2. Note that as a result of $A_{(3)} = A_{(4)}$, P_2 and P_3 have the same y-coordinates, and P_3 and P_4 share their x-coordinates.

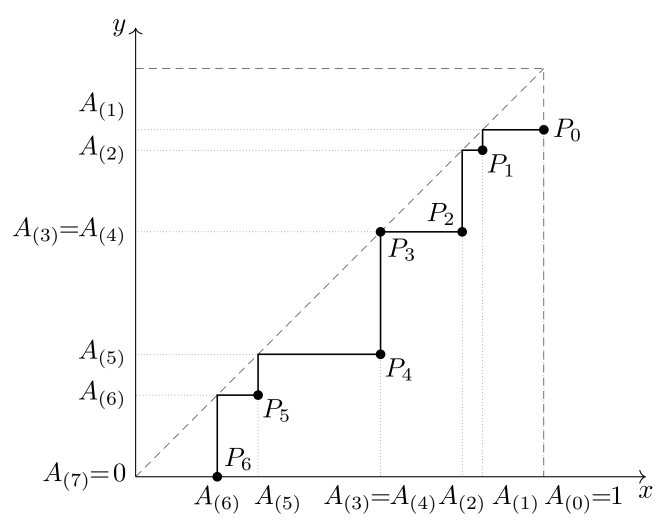

Fig. 2. Step-pattern corresponding to the fuzzy set A in example 1.

Formula (2) for the fuzzy cardinality $|A|_{\mathrm{Rf}}$ is based on the function $f : [0,1]^2 \to [0,1]$ defined as

$$f(x, y) = \min\{x, 1 - y\}$$

in that

$$|A|_{\mathrm{Rf}}(k) = f(x_k, y_k). \tag{7}$$

The graph of f is depicted in figure 3. In figure 4, we display the projections of several contour lines of f on the xy-plane. For brevity, we shall write $f(P_k)$ instead of $f(x_k, y_k)$, and speak of "contours" instead of "projections of the contour lines." For given $a \in [0,1]$, the set of points

$$f^{-1}(a) = \{\langle x, y\rangle \mid f(x, y) = a\}$$

shall be referred to as the *a-contour*. The 0.5-contours of the three fuzzy sets used in our examples are depicted as the red line segments in figures 5 and 6.

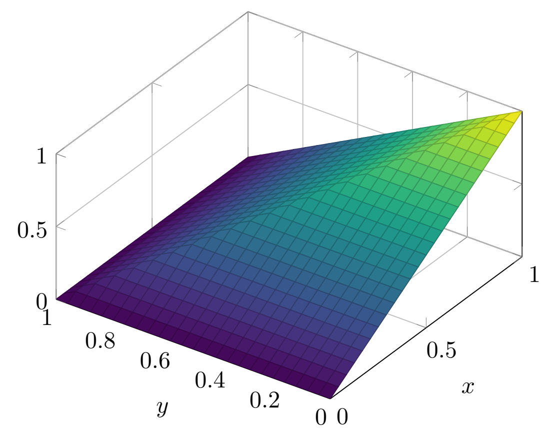

Fig. 3. Graph of $f(x, y) = \min\{x, 1 - y\}$.

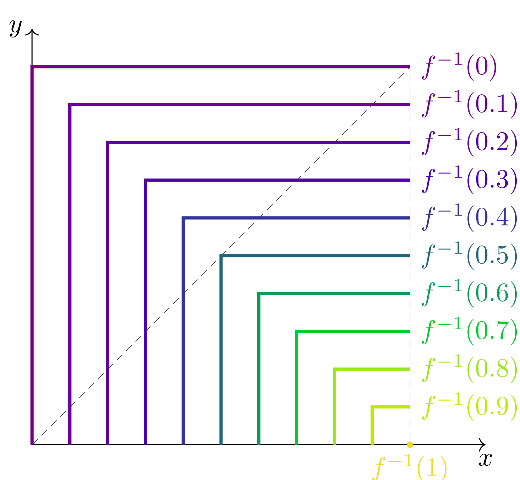

Fig. 4. Projections of the contour lines of $f(x, y) = \min\{x, 1 - y\}$ on the xy-plane.

We now consider the role of 0.5-contours in the geometric considerations of Ralescu's cardinality. We shall say that P_k is *on* the 0.5-contour if $f(P_k) = 0.5$; *above* the 0.5-contour if $f(P_k) > 0.5$; and *below* the 0.5-contour if $f(P_k) < 0.5$. For instance, in the left part of figure 6, P_2 and P_3 are below and on the 0.5-contour, respectively; in figure 5, P_4 is above the 0.5-contour.

The following assertions provide a basic insight needed for our analysis. They all concern arbitrary fuzzy sets A with the corresponding points $P_0, P_1, \ldots, P_n$ given by (6).

Lemma 1. *There exists at most one point above the 0.5-contour.*

Proof. *Suppose $P_k = \langle x_k, y_k\rangle$ is above the 0.5-contour, i.e., $x_k > 0.5$ and $y_k < 0.5$. Recall that due to the ordering of the membership degrees of A, $x_k \geq y_k$. If $k > 0$ then for every $l = 0, \ldots, k-1$, P_l is below the 0.5-contour. Indeed, P_{k-1} is below the 0.5-contour since $y_{k-1} = x_k > 0.5$ and $x_{k-1} \geq y_{k-1} > 0.5$. By a similar reasoning, $P_{k-2}, \ldots, P_0$ are all below the 0.5-contour. If $k < n$ then P_{k+1} is below the 0.5-contour since $x_{k+1} = y_k < 0.5$ and $y_{k+1} \leq x_{k+1} < 0.5$, and similarly for $P_{k+2}, \ldots, P_n$. Both cases are demonstrated in figure 5.* □

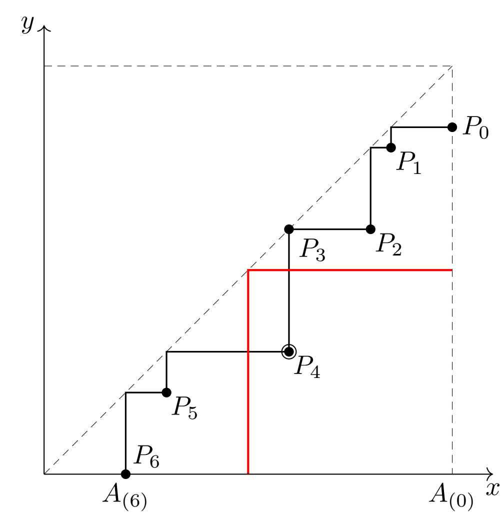

Fig. 5. Step-pattern corresponding to the fuzzy set A in example 1. Point P_4 is the only point located above 0.5-contour.

The step-patterns in figure 6 corresponding to the fuzzy sets B and C of examples 2 and 3, respectively, demonstrate that it may happen that none of P_is is above the 0.5-contour.

We now turn to the question of how many points may actually lie on the 0.5-contour. Figure 5 demonstrates that there may actually be no such point. On the other hand, figure 6 makes it clear that there may be two or three such points; an easy modification would provide fuzzy sets for which four, five, etc., points are on the 0.5-contour. The following lemma shows that no other option exists.

Lemma 2. *One of the following cases occurs:*

(i) *There is no point on the 0.5-contour.*
(ii) *There are at least two points on the 0.5-contour.*

In the latter case, if $P_k, P_{k+1}, \ldots, P_{k+r-1}$ are all the $r \geq 2$ points on the 0.5-contour, P_k and P_{k+r-1} are subdiagonal $P_{k+1}, \ldots, P_{k+r-2}$ are diagonal. Moreover, all the other points, $P_0, \ldots, P_{k-1}, P_{k+r}, \ldots, P_n$, are below the 0.5-contour.

Proof. We show that if there exists a point on the 0.5-contour, there must be at least two such points. Consider thus a fuzzy set having at least one point on the 0.5-contour and suppose P_k is the first of them.

If P_k is diagonal, i.e., $P_k = \langle 0.5, 0.5 \rangle$, then (1) implies $k > 0$. Therefore $x_{k-1} \geq 0.5$ and $y_{k-1} = 0.5$, i.e., P_{k-1} is on the 0.5-contour as well, contradicting the assumption that P_k is the first point on the 0.5-contour.

Let thus P_k be subdiagonal. If P_k is on the vertical part of the 0.5-contour, i.e., $x_k = 0.5$ and $y_k < 0.5$, then again (1) implies $k > 0$. Now, we can easily see that $y_{k-1} = 0.5$ so P_{k-1} is on the 0.5-contour as well which again contradicts the fact that P_k is the first point on the 0.5-contour.

To sum up, the first point on 0.5-contour is subdiagonal and lies on the horizontal part of the 0.5-contour. In other words, $P_k = \langle x_k, 0.5 \rangle$ with $x_k > 0.5$. Obviously, $x_{k+1} = 0.5$ and $y_{k+1} \geq 0.5$ so the immediately following point P_{k+1} is on the vertical part of the 0.5-contour. Therefore, there are at least two points on this contour.

We can now consider the previous arguments in a dual manner, in which case we obtain that the last point P_l on the 0.5-contour is subdiagonal and lies on the vertical part of this contour, i.e., $x_l = 0.5$ and $y_l > 0.5$ (and, moreover, $l < n$), and the immediately previous point P_{l-1} is on the horizontal part of the 0.5-contour.

Putting together: There are at least two points on the 0.5-contour—namely, $P_k = \langle x_k, 0.5 \rangle$ with $x_k > 0.5$ and $P_l = \langle 0.5, x_l \rangle$ with $y_l < 0.5$—and any other point on the 0.5-contour must be diagonal with coordinates $\langle 0.5, 0.5 \rangle$. □

The following lemma provides a further insight into the possible configurations of the points.

Lemma 3. *There is no point on the* 0.5*-contour if and only if there exists just one point above the* 0.5*-contour.*

Proof.

"⇒": Suppose that there exists no point on the 0.5-contour. In addition, assume that P_0 is above the 0.5-contour; this case happens whenever $y_0 = A_{(1)} < 0.5$. By applying lemma 1 we immediately obtain that P_0 is the unique point above the 0.5-contour.
Conversely, let us suppose P_0 is below the 0.5-contour, i.e., $y_0 = A_{(1)} > 0.5$. Consider the set

$$M = \{P_i \,|\, x_i > 0.5 \text{ and } y_i > 0.5\}.$$

Since M is nonempty (indeed, $P_0 \in M$) and finite (because U is a finite universe), we may consider the greatest k such that $P_k \in M$. Note also that $P_n \notin M$ because $y_n = A_{(n+1)} = 0 \not> 0.5$, so $P_k \neq P_n$ and we can thus consider the point P_{k+1} with $x_{k+1} = y_k > 0.5$.t As $P_{k+1} \notin M$ and there is no point on the 0.5-contour, we conclude $y_{k+1} < 0.5$. Therefore, P_{k+1} is above the 0.5-contour. By employing lemma 1 we obtain that P_{k+1} is the only point above the 0.5-contour.

"⇐": Let P_k, $0 < k < n$, be the only point above the 0.5-contour. We thus have $x_k > 0.5$ and $y_k < 0.5$ and, therefore, $x_{k-1} \geq x_k > 0.5$ and $y_{k-1} = x_k > 0.5$, which implies that P_{k-1} is below the 0.5-contour. Analogously, $x_{k+1} = y_k < 0.5$ and $y_{k+1} \leq x_{k+1} < 0.5$, implying that P_{k+1} is below the 0.5-contour.
Now, let P_i and P_j ($i \neq 0$, $j \neq n$) be points such that $x_i > 0.5$, $y_i > 0.5$ and $x_j < 0.5$, $y_j < 0.5$. That is, both points are below the 0.5-contour. Then P_{i-1} and P_{j+1} are below the 0.5-contour as well; indeed, $x_{i-1} \geq x_i > 0.5$, $y_{i-1} = x_i > 0.5$ and $x_{j+1} = y_j < 0.5$, $y_{j+1} \leq x_{j+1} < 0.5$. By induction we then obtain that the points $P_0, \ldots, P_{k-1}$ and $P_{k+1}, \ldots, P_n$ are bellow the 0.5-contour, so there is no point on this contour.
One proceeds in a similar way if either P_0 or P_n is the only point above the 0.5-contour. □

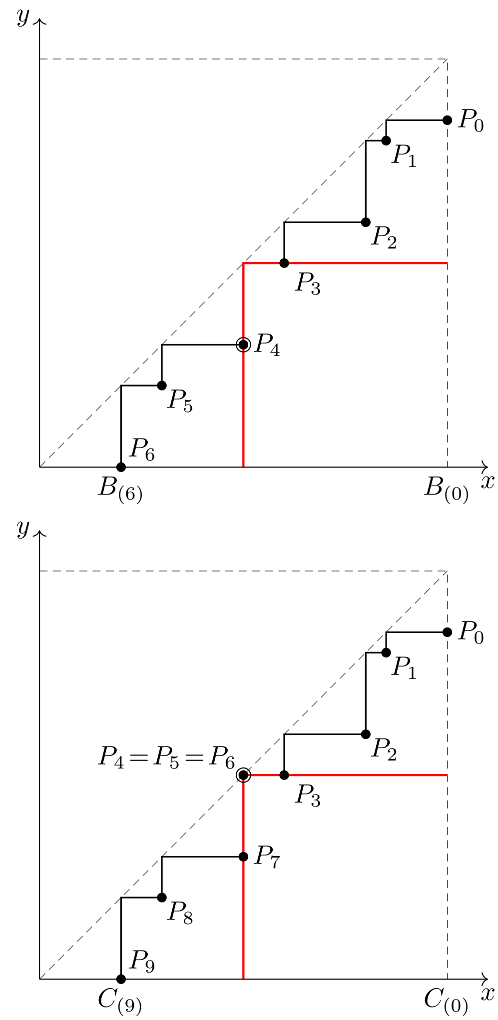

Fig. 6. Left: Step-pattern corresponding to the fuzzy set B in example 2. Points P_3 and P_4 are on 0.5-contour. Right: Step-pattern corresponding to the fuzzy set C in example 3. Points $P_3, \ldots, P_7$ are on 0.5-contour.

As a consequence of theorem 2 and 3, two situations may occur for an arbitrary fuzzy set A and the corresponding points P_0, P_1, …, P_n given by (6). The first one is demonstrated in figure 5: Exactly one of the points is above the 0.5-contour and all other points are below this contour. The second one is depicted in figure 6: Two or more points are on the 0.5-contour and all other points are below this contour.

IV. Properties of Ralescu's cardinality via the geometric view

In this section, we derive basic properties of and relationships between Ralescu's two cardinality concepts. Some of these results appeared in somewhat modified or different forms in [2]. We also show how the results in [2] can be derived from the assertions presented in this section.

Theorem 1.

(a) *Let there be no point on the* 0.5*-contour and let P_k be the only point above the* 0.5*-contour (according to lemma 3). Then*
$$\max |A|_{\mathrm{Rf}} = f(P_k) > 0.5,$$
and
$$\arg\max |A|_{\mathrm{Rf}} = \{k\}.$$

(b) *Let there be points on the* 0.5*-contour and let P_k be the first of them. Then there exists $r \geq 2$ such that*
$$\max |A|_{\mathrm{Rf}} = f(P_k) = \cdots = f(P_{k+r-1}) = 0.5,$$
and
$$\arg\max |A|_{\mathrm{Rf}} = \{k, \ldots, k+r-1\}.$$

Proof.

(a) If P_k and P_j are above and below the 0.5-contour, respectively, it obviously holds that $f(P_k) > f(P_j)$. So we have
$$\max\{f(P_0), \ldots, f(P_n)\} = f(P_k) > 0.5,$$
and
$$\arg\max\{f(P_0), \ldots, f(P_n)\} = \{k\}.$$
Since $|A|_{\mathrm{Rf}}(l) = f(P_l)$, $l = 0, \ldots, n$, see (7), we immediately obtain the claim.

(b) From lemma 2 (b) we have that there is $r \geq 2$ such that the points $P_k, \ldots, P_{k+r-1}$ are on the 0.5-contour and the rest of the points are below this contour. If P_i and P_j are on and below the 0.5-contour, respectively, it holds that $f(P_i) > f(P_j)$. Therefore we have
$$\max\{f(P_0), \ldots, f(P_n)\} = f(P_k) = \cdots = f(P_{k+r-1}) = 0.5,$$
and
$$\arg\max\{f(P_0), \ldots, f(P_n)\} = \{k, \ldots, k+r-1\}.$$
Again, to obtain the claim we use (7). □

Remark 2. The first part of theorem 3 in [2] claiming that
$$\#\arg\max |A|_{\mathrm{Rf}} = p + 1,$$
where $p = \#\{u \in U; A(u) = 0.5\}$, can be viewed as a consequence of lemma 2 and theorem 1 (the second part shall be discussed in remark 3). Indeed, if there is no point on the 0.5-contour (i.e., there is no $u \in U$ such that $A(u) = 0.5$) then we have $\#\arg\max |A|_{\mathrm{Rf}} = 1 = p + 1$. On the other hand, if there is a point on 0.5-contour, then there are $r \geq 2$ points $P_k, \ldots, P_{k-p+1}$ with
$$x_k > 0.5, x_{k+1} = 0.5, \ldots, x_{k-r+1} = 0.5,$$
i.e., $p = r - 1$. From theorem 1 we then get
$$\#\arg\max |A|_{\mathrm{Rf}} = r = p + 1.$$

Theorem 2.

(a) *Let there be no point on the* 0.5*-contour and let P_k be the only point above the* 0.5*-contour. Then $|A|_{\mathrm{Rs}} = k$.*

(b) *Let there be points on the* 0.5*-contour and let P_k be the first of them. Then $|A|_{\mathrm{Rs}} = k + 1$.*

Proof. The definition of index j (see equation 4) can equivalently be stated as:
$$j = \min\{k\,;\, 0 \leq k \leq n-1 \text{ and } A_{(k)} + A_{(k+1)} \leq 1\}.$$
The condition $A_{(k)} + A_{(k+1)} \leq 1$ can be easily rewritten as $y_k \leq 1 - x_k$, so $P_j = \langle x_j, y_j \rangle$ is the first point lying below or on the secondary diagonal.

(a) Suppose P_k is the only point above the 0.5-contour, i.e., $x_k > 0.5$ and $y_k < 0.5$. Then we have $y_{k-1} > 0.5$ and $x_{k-1} > 0.5$, since
$$y_{k-1} = x_k > 0.5 \quad \text{and} \quad x_{k-1} \geq y_{k-1} > 0.5.$$
Therefore,
$$1 - x_{k-1} < 0.5 \quad \text{and} \quad y_{k-1} > 1 - x_{k-1}.$$
That is, P_{k-1} is above the secondary diagonal. Obviously, all the preceding points $P_0, \ldots, P_{k-2}$ lie above the secondary diagonal too.
Now, we consider two possibilities: (i) P_k is the point lying below or on the secondary diagonal, i.e., $j = k$. Since $A_{(j)} = A_{(k)} = x_k > 0.5$, (3) yields
$$|A|_{\mathrm{Rs}} = j = k.$$
(ii) Let P_k be above the secondary diagonal. Then
$$y_{k+1} \leq x_{k+1} = y_k < 0.5,$$
$$1 - x_{k+1} > 0.5.$$
So
$$y_{k+1} \leq 1 - x_{k+1},$$
which means that P_{k+1} is the first point lying below the secondary diagonal, i.e., $j = k + 1$. Because $A_{(j)} = A_{(k+1)} = x_{k+1} < 0.5$, (3) implies
$$|A|_{\mathrm{Rs}} = j - 1 = k + 1 - 1 = k.$$

(b) Now, suppose P_k is the first point on the 0.5-contour. Then $x_k > 0.5$ and $y_k = 0.5$ as we have already shown in the proof of lemma 2. That is,
$$y_k = 0.5 > 1 - x_k,$$
so P_k is above the secondary diagonal. Obviously, all the previous points $P_0, \ldots, P_{k-1}$ lie above the secondary diagonal too.
Now, lemma 2 implies that P_{k+1} is on the 0.5-contour, and $x_{k+1} = 0.5$ and $y_{k+1} \leq 0.5$. Then $y_{k+1} \leq 0.5 = 1 - x_{k+1}$, so P_{k+1} is the first point below the secondary diagonal. We thus obtain $j = k + 1$ and since $A_{(j)} = A_{(k+1)} = x_{k+1} = 0.5$, we finally have
$$|A|_{\mathrm{Rs}} = j = k + 1.$$
□

Remark 3. The second part of theorem 3 in [2] claiming that

$$\arg\max |A|_{\mathrm{Rf}} = \{k_1, \ldots, k_{p+1}\},$$

where $k_1 = |A|_{\mathrm{Rs}}$, $k_2 = |A|_{\mathrm{Rs}} - 1$, $k_3 = |A|_{\mathrm{Rs}} + 1$, $k_4 = |A|_{\mathrm{Rs}} + 2$, $\ldots$, $k_{p+1} = |A|_{\mathrm{Rs}} + p - 1$, and $p = \#\{u \in U; A(u) = 0.5\}$, can be viewed as a consequence of theorems 1 and 2. Namely, consider two cases.

First, there is no point on the 0.5-contour and let P_k be the only point above the 0.5-contour. Then from theorem 1 (a) we have $\arg\max |A|_{\mathrm{Rf}} = \{k\}$, and from theorem 2 (a) we have $k = |A|_{\mathrm{Rs}}$. So

$$\arg\max |A|_{\mathrm{Rf}} = \{|A|_{\mathrm{Rs}}\} = \{k_1\}. \tag{8}$$

Second, let there be points on the 0.5-contour and let P_k be the first of them. From theorem 1 (b) we have that there exists $r \geq 2$ such that $\arg\max |A|_{\mathrm{Rf}} = \{k, \ldots, k+r-1\}$ and from theorem 2 (b) we have $k = |A|_{\mathrm{Rs}} - 1$. Since $p = r - 1$ (see also remark 2), we get

$$\begin{aligned} \arg\max |A|_{\mathrm{Rf}} &= \{|A|_{\mathrm{Rs}} - 1, |A|_{\mathrm{Rs}}, \ldots, |A|_{\mathrm{Rs}} + r - 2\} \\ &= \{|A|_{\mathrm{Rs}} - 1, |A|_{\mathrm{Rs}}, \ldots, |A|_{\mathrm{Rs}} + p - 1\} \\ &= \{k_1, \ldots, k_{p+1}\}. \end{aligned} \tag{9}$$

Remark 4. Theorem 4 in [2] restoring $|A|_{\mathrm{Rs}}$ from $|A|_{\mathrm{Rf}}$ is a direct consequence of remark 3. Indeed, from (8) we immediately obtain that $|A|_{\mathrm{Rs}}$ is the unique element in $\arg\max |A|_{\mathrm{Rf}}$ if $\arg\max |A|_{\mathrm{Rf}}$ is a singleton, and from (9) we get that $|A|_{\mathrm{Rs}}$ is the second smallest element in $\arg\max |A|_{\mathrm{Rf}}$ if $\arg\max |A|_{\mathrm{Rf}}$ is not a singleton.

Remark 5. Theorem 5 in [2] stating that

$$\max |A|_{\mathrm{Rf}} = \begin{cases} \min\{A_{(|A|_{\mathrm{Rs}})}, 1 - A_{(|A|_{\mathrm{Rs}}+1)}\} > 0.5 & \text{if } p = 0, \\ 0.5 & \text{if } p > 0, \end{cases}$$

where $p = \#\{u \in U; A(u) = 0.5\}$, can be easily derived from theorems 1 and 2.

Indeed, let there be no point on the 0.5-contour, i.e., $p = 0$, and let P_k be the only point above the 0.5-contour, then from theorem 2 (a) we have $|A|_{\mathrm{Rs}} = k$, so

$$\begin{aligned} f(P_k) &= \min\{x_k, 1 - y_k\} = \min\{A_{(k)}, 1 - A_{(k+1)}\} \\ &= \min\{A_{(|A|_{\mathrm{Rs}})}, 1 - A_{(|A|_{\mathrm{Rs}}+1)}\}. \end{aligned}$$

From theorem 1 (a) we now obtain

$$\max |A|_{\mathrm{Rf}} = f(P_k) = \min\{A_{(|A|_{\mathrm{Rs}})}, 1 - A_{(|A|_{\mathrm{Rs}}+1)}\} > 0.5.$$

If there are points on the 0.5-contour, i.e., $p > 0$, then theorem 1 (b) yileds

$$\max |A|_{\mathrm{Rf}} = 0.5.$$

We now present two other important properties on Ralescu's cardinality concepts which are derived from the geometric view. The first one is convexity, for which the geometric view provides a straightforward argument (cf. [9]).

Theorem 3. *The fuzzy cardinality $|A|_{\mathrm{Rf}}$ of any fuzzy set A is a convex fuzzy set.*

Proof. We consider the common notion of convexity of a fuzzy set: Its membership function is bell-shaped, i.e., is nondecreasing until it reaches its maximum and then becomes nonincreasing.

First, we assume that P_k is the only point above the 0.5-contour. From theorem 1 (a) we have that $|A|_{\mathrm{Rf}}$ has the maximal value in k. Using the geometric view it is now easy to see that $j_1 \leq j_2 \leq k$ implies

$$\begin{aligned} |A|_{\mathrm{Rf}}(j_1) = f(P_{j_1}) = \min\{x_{j_1}, 1 - y_{j_1}\} = 1 - y_{j_1} \leq \\ 1 - y_{j_2} = \min\{x_{j_2}, 1 - y_{j_2}\} = f(P_{j_2}) = |A|_{\mathrm{Rf}}(j_2), \end{aligned}$$

i.e., $|A|_{\mathrm{Rf}}$ is nondecreasing for $j \in \{0, \ldots, k\}$. Similarly, it is simple to check that $k \leq l_1 \leq l_2$ implies

$$\begin{aligned} |A|_{\mathrm{Rf}}(l_1) = f(P_{l_1}) = \min\{x_{l_1}, 1 - y_{l_1}\} = x_{l_1} \geq \\ x_{l_2} = \min\{x_{l_2}, 1 - y_{l_2}\} = f(P_{l_2}) = |A|_{\mathrm{Rf}}(l_2), \end{aligned}$$

i.e., $|A|_{\mathrm{Rf}}$ is noninceasing for $l \in \{k, \ldots, n\}$.

For the second case, i.e., if points on the 0.5-contour exist, one proceeds in a similar way by taking an arbitrary point on the 0.5-contour as P_k. □

The following statement presents an alternative closed-form expression of scalar cardinality which is directly based on the developed geometric view, and is more compact compared to theorem 2.

Theorem 4. *Let $P_0, P_1, \ldots, P_n$ be points given by* (6). *Then*

$$|A|_{\mathrm{Rs}} = \#\{P_i\,;\, x_i > 0.5, y_i \geq 0.5\}.$$

Proof. Due to lemma 1–lemma 3, it is sufficient to distinguish the following two cases. First, let P_k be the only point above the 0.5-contour. Then, as we have pointed out above,

$$\begin{aligned} &x_i > 0.5, y_i > 0.5 \text{ for } i < k, \\ &x_k > 0.5, y_k < 0.5, \\ &x_j < 0.5, y_j < 0.5, \text{ for } j > k. \end{aligned}$$

Hence, theorem 2 implies

$$\#\{P_i\,;\, x_i > 0.5, y_i \geq 0.5\} = \#\{P_0, \ldots, P_{k-1}\} = k = |A|_{\mathrm{Rs}}.$$

Second, let P_k be the first point on the 0.5-contour. Then

$$\begin{aligned} &x_i > 0.5, y_i > 0.5 \text{ for } i < k, \\ &x_k > 0.5, y_k = 0.5, \\ &x_j \leq 0.5, y_j \leq 0.5, \text{ for } j > k. \end{aligned}$$

Therefore, theorem 2 yields

$$\begin{aligned} \#\{P_i\,;\, x_i > 0.5, y_i \geq 0.5\} &= \#\{P_0, \ldots, P_k\} \\ &= k + 1 = |A|_{\mathrm{Rs}}. \end{aligned}$$ □

V. Geometric view of a more general interpretation of many-valued cardinality

In the concluding remarks of [2], a natural generalization of Ralescu's fuzzy cardinality has been suggested which is based on the formula

$$|A|_{\mathrm{Rf}}(k) = A_{(k)} \otimes \neg A_{(k+1)}, \tag{10}$$

with $\otimes$ being a truth function of a conjunction, such as a t-norm, and $\neg$ being a truth function of a negation. Ralescu's fuzzy cardinality (2) then becomes a particular case for $a \otimes b =$

$\min(a, b)$ and $\neg a = 1 - a$ in (10). In the rest of this section we provide some properties of the generalized concept for the two other fundamental continuous t-norms $\otimes$, namely the Łukasiewicz and the product t-norms, which can easily be derived using our geometric view.

For the well-known Łukasiewicz connectives, i.e., $a \otimes b = \max\{0, a + b - 1\}$ and $\neg a = 1 - a$, we obtain

$$\begin{aligned} |A|_{\mathrm{Rf}}(k) &= \max\{0, A_{(k)} + (1 - A_{(k+1)}) - 1\} \\ &= \max\{0, A_{(k)} - A_{(k+1)})\} = A_{(k)} - A_{(k+1)}, \end{aligned} \tag{11}$$

since $A_{(k)} - A_{(k+1)} \geq 0$ for all $k = 0, \ldots, n-1$. Consider the corresponding function $f : [0,1]^2 \to [0,1]$ defined as

$$f(x, y) = x - y.$$

The contours of f, whose graph is depicted in figure 7, form the lines parallel with the main diagonal as illustrated in figure 8. In this case, we clearly have

$$\arg\max |A|_{\mathrm{Rf}} = \{k\,; \text{the drop from } A_{(k)} \text{ to } A_{(k+1)} \text{ is maximal}\}$$

and $\max |A|_{\mathrm{Rf}}$ is the extent of the maximal drop, as demonstrated in the following example.

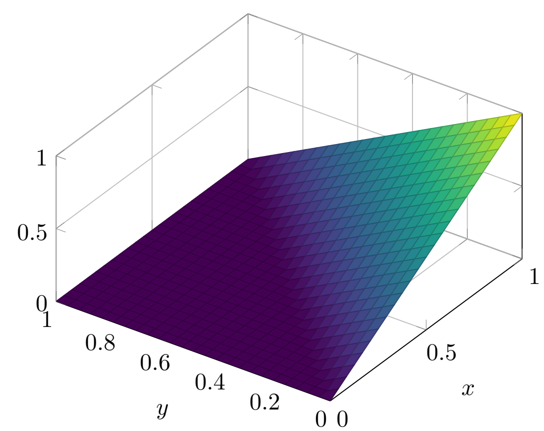

Fig. 7. Graph of $f(x, y) = x - y$.

Example 4. For the fuzzy set

$$D = \{{}^{0.9}/u_1, {}^{0.6}/u_2, {}^{0.5}/u_3, {}^{0.5}/u_4, {}^{0.2}/u_5, {}^{0.1}/u_6\}$$

(11) implies

$$|D|_{\mathrm{Rf}} = \{{}^{0.1}/0, {}^{0.3}/1, {}^{0.1}/2, {}^{0.3}/4, {}^{0.1}/5, {}^{0.1}/6\}.$$

Figure 9 shows the step-pattern corresponding to the fuzzy set D. The maximal drop from $x_k = A_{(k)}$ to $y_{k+1} = A_{(k+1)}$ occurs for $k = 1$ and $k = 4$ (the corresponding points P_1 and P_4 are denoted by a double circle). Therefore,

$$\arg\max |D|_{\mathrm{Rf}} = \{1, 4\}. \qquad \square$$

The geometric view gives us the following simple interpretation of fuzzy cardinality based on formula (11).

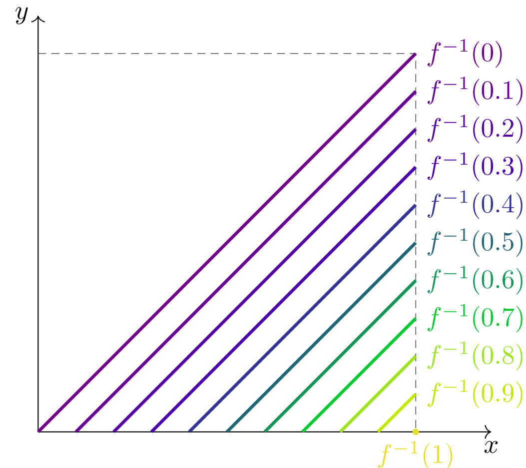

Fig. 8. Projections of the contour lines of $f(x, y) = x - y$ on the xy-plane.

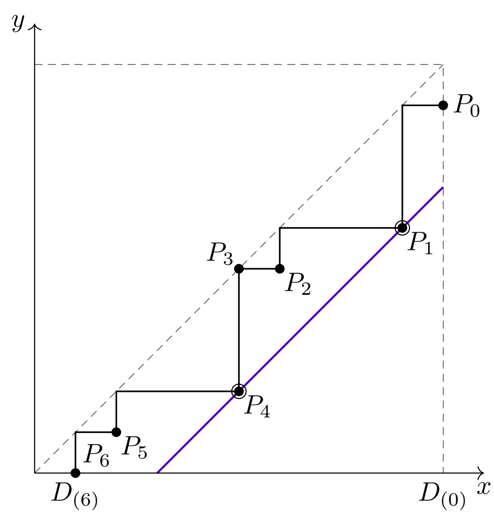

Fig. 9. Step-pattern corresponding to the fuzzy set D in example 4. Points P_1 and P_4 are on the 0.3-contour of $f(x, y) = x - y$.

Theorem 5. *For the fuzzy cardinality based on Łukasiewicz t-norm, the following holds:*

$$|A|_{\mathrm{Rf}}(k) = \sqrt{2} \cdot d_k,$$

where d_k is the Euclidean distance of P_k from the main diagonal.

Proof. From figure 10 we can easily see that $d_k = |P_k S_k|$, where $S_k = [s_k, s_k]$ with

$$s_k = \frac{x_k + y_k}{2}.$$

Now we have

$$\begin{aligned} d_k &= \sqrt{(x_k - s_k)^2 + (y_k - s_k)^2} \\ &= \sqrt{\left(\frac{x_k - y_k}{2}\right)^2 + \left(\frac{y_k - x_k}{2}\right)^2} \\ &= \sqrt{2 \cdot \frac{(x_k - y_k)^2}{4}} \\ &= \frac{\sqrt{2}}{2} \cdot (x_k - y_k) \\ &= \frac{\sqrt{2}}{2} \cdot f(x_k, y_k), \end{aligned}$$

so

$$|A|_{\mathrm{Rf}}(k) = f(x_k, y_k) = \frac{2}{\sqrt{2}} \cdot d_k = \sqrt{2} \cdot d_k. \qquad \square$$

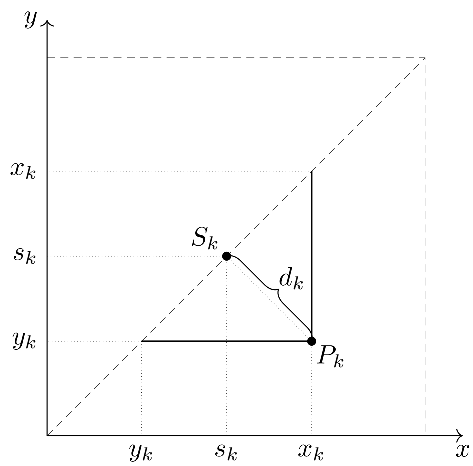

Fig. 10. The distance d_k of point P_k from the main diagonal.

Unlike the original case, i.e., with $\otimes = \min$, it may happen in the Lukasiewicz case that the corresponding step-pattern contains points that do not immediately follow each other but still lie on the same contour, i.e., they have the same distance from the main diagonal. In fact, the fuzzy set of example 4 has this property as is apparent from figure 9. Since the value $|A|_{\mathrm{Rf}}(k)$ is simply $\sqrt{2}$ times the distance of P_k from the main diagonal, the fuzzy cardinality $|A|_{\mathrm{Rf}}$ need not be a convex fuzzy set; see again figure 9. In particular, $\arg\max |A|_{\mathrm{Rf}}$ is not a set of consecutive natural numbers in general, as in the case of Ralescu's original fuzzy cardinality (theorem 1). This fact makes it impossible to define the scalar cardinality based on the Łukasiewicz t-norm in such a way that it is easily describable, as we obtained for Ralescu's scalar cardinality in theorem 2.

For the product t-norm, i.e., $a \otimes b = a \cdot b$, and with $\neg$ set again to $\neg a = 1 - a$, (10) yields

$$|A|_{\mathrm{Rf}}(k) = A_{(k)} \cdot (1 - A_{(k+1)}). \qquad (12)$$

As above, consider the corresponding function $f : [0,1]^2 \to [0,1]$, i.e.,

$$f(x, y) = x \cdot (1 - y).$$

The graph of f is depicted in figure 11 and some of the contours of f are shown in figure 12. As in the Łukasiewicz

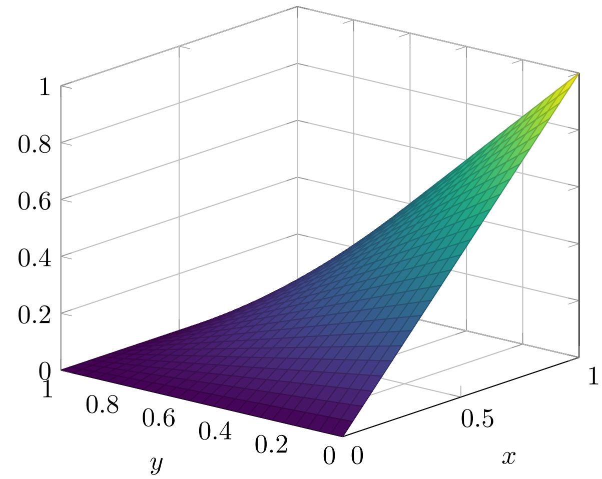

Fig. 11. Graph of $f(x, y) = x \cdot (1 - y)$.

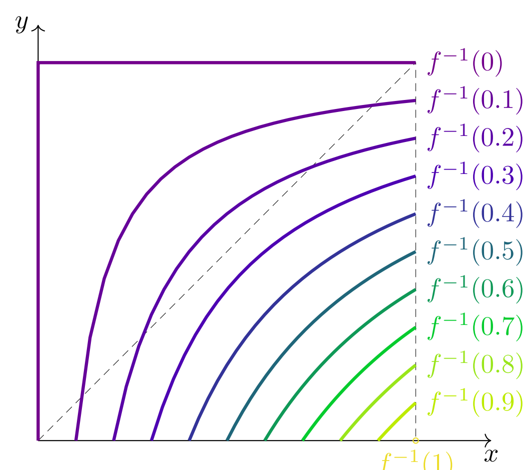

Fig. 12. Projections of the contour lines of $f(x, y) = x \cdot (1 - y)$ on the xy-plane.

t-norm, $|A|_{\mathrm{Rf}}$ is not convex fuzzy set in general, as we can see in example 5. A more detailed description of the fuzzy cardinality (12) is even more involved than for Ralescu's original fuzzy cardinality (2) or the fuzzy cardinality based on Łukasiewicz t-norm (11), because the contours of f are mostly non-linear.

Example 5. For the fuzzy set

$$E = \{{}^{0.7}/u_1, {}^{0.6}/u_2, {}^{0.5}/u_3, {}^{0.4}/u_4, {}^{0.35}/u_5, {}^{0.3}/u_6\}$$

(12) yields

$$|E|_{\mathrm{Rf}} = \{{}^{0.3}/0, {}^{0.28}/1, {}^{0.3}/2, {}^{0.3}/3, {}^{0.26}/4, {}^{0.245}/5, {}^{0.3}/6\}.$$

One immediately observes that

$$\arg\max |E|_{\mathrm{Rf}} = \{0, 2, 3, 6\}.$$

Figure 13 shows the step-pattern corresponding to E. $\square$

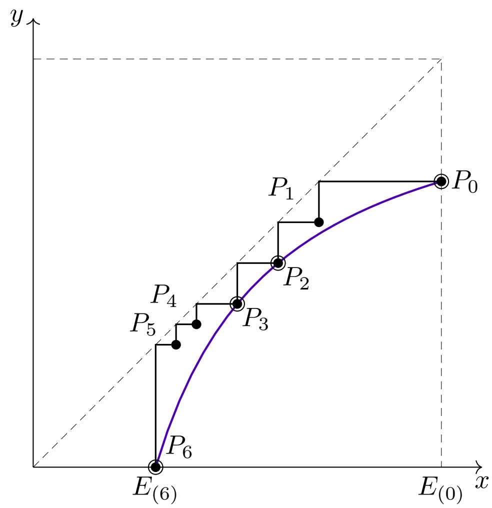

Fig. 13. Step-pattern corresponding to the fuzzy set E in example 5. Points P_0, P_2, P_3 and P_6 are on the 0.3-contour of $f(x, y) = x \cdot (1 - y)$.

VI. Conslusion

In this paper, we present a geometric view of both fuzzy and scalar cardinality of a fuzzy set introduced by Ralescu [9]. This view is visually appealing, easy to understand, and provides an alternative justification for the results concerning these cardinalities presented in [2]. Moreover, the proposed view reveals a new description of scalar cardinality that was previously unknown.

We also discuss a geometric view of a natural generalization of Ralescu's concept, i.e., the fuzzy cardinalities based on the Łukasiewicz and the product t-norms. The view reveals two important properties that are satisfied by Ralescu's concept, make it considerably easier to analyze, and enable a simple closed-form definition of the scalar cardinality. These two properties are convexity of fuzzy cardinality (which is not satisfied by either the Łukasiewicz- and the product-based fuzzy cardinality) and piecewise linearity of contours (which is not satisfied by the fuzzy cardinality based on the product t-norm).

To sum up, the proposed geometric view not only enables alternative proofs of existing results, but reveals a new, significant insight into Ralescu's concepts of fuzzy and scalar cardinality.

References

[1] Bartl, E., Belohlavek, R.: "Cardinality of fuzzy sets and accumulation of small membership." *IEEE Transactions on Fuzzy Systems* **32**(6), 3779–3789 (2024).

[2] Bartl, E., Belohlavek, R.: "On Ralescu's cardinality of fuzzy sets." *Fuzzy Sets and Systems* **498**, 109118 (2025).

[3] Blanchard, N.: "Cardinal and ordinal theories about fuzzy sets." In: Gupta M. M. and Sanchez E. (eds.), *Fuzzy Information and Decision Processes*, 149–157, North-Holland, Amsterdam (1982).

[4] De Luca, A., Termini, S.: "A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory." *Information and Control* **20**, 301–312 (1972).

[5] Dubois, D., Prade, H.: *Fuzzy Sets and Systems, Theory and Applications.* Academic Press, New York (1980).

[6] Dubois, D., Prade, H.: "Fuzzy cardinality and the modeling of imprecise quantification," *Fuzzy Sets and Systems* **16**, 199–230 (1985).

[7] Klaua, D.: "Stetige Gleichmächtigkeiten kontinuierlich-wertiger Mengen [Continuous equicardinalities of continuum-valued sets]," *Monatsberichte der Deutschen Akademie der Wissenschaften zu Berlin* **12**, 749–758 (1970).

[8] Klir, G. J., Yuan, B.: *Fuzzy Sets and Fuzzy Logic. Theory and Applications.* Prentice-Hall (1995).

[9] Ralescu, D.: "Cardinality, quantifiers, and the aggregation of fuzzy criteria." *Fuzzy Sets and Systems* **69** (3), 355–365 (1995).

[10] Wygralak, M.: "An axiomatic approach to scalar cardinalities of fuzzy sets." *Fuzzy Sets and Systems* **110** (2), 175–179 (2000).

[11] Wygralak, M.: *Cardinalities of Fuzzy Sets.* Springer (2003).