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# On Ralescu's cardinality of fuzzy sets

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ABSTRACT

We provide a direct formula for Ralescu's scalar cardinality. Unlike the original, iterative definition, the formula reveals intuitive shortcomings of this concept of cardinality. These are apparent from examples and reflected formally in that, as we show, the concept violates one of the axioms of cardinality of fuzzy sets. In addition, we provide a relationship of this concept to Ralescu's concept of fuzzy cardinality which unveils a tight link between the two concepts and points out another counterintuitive property of the concept of scalar cardinality. We argue that the discussed concept of fuzzy cardinality represents an interesting proposition, suggest its geometric interpretation, and provide preliminary observations as a basis for future considerations.

## 1. Introduction

Since the early years of fuzzy logic, the notion of cardinality of a fuzzy set has played an important role. The reader is referred, e.g., to [5] for more information about this notion and its development. Arguably the most important among the various concepts of cardinality of fuzzy sets is the so-called scalar cardinality, i.e., a function assigning to a given fuzzy set a non-negative number representing the size of the fuzzy set. In particular, the so-called sigma-count of a fuzzy set, introduced by De Luca and Termini as the "power of a fuzzy set" [2, p. 304] and defined for a fuzzy set $A : U \to [0,1]$ in a finite universe $U$ by $|A| = \sum_{u\in U} A(u)$, is the most widespread due to its intuitive appeal and practical relevance. In addition, approaches to the so-called fuzzy cardinality, i.e., cardinality being a fuzzy set of non-negative numbers rather than a single number, have been explored.

In his interesting paper [3], Ralescu proposed and examined a novel approach to cardinality of fuzzy sets, both scalar and fuzzy. The originality of this approach consists in that it takes mutual relationships between the degrees of membership in a fuzzy set into account. Moreover, Ralescu mentions [3, pp. 361 and 363] as one of the advantages of his new concept that—unlike the standard sigma-count—it alleviates the problem of accumulation of a large number of small degrees of membership.[1]

In our paper, we argue that while Ralescu's concept is based on a natural idea, his concept of scalar cardinality does not yield intuitive results as regards the problem of accumulation of small degrees of membership and, moreover, yields counterintuitive results on a more substantial ground, which is formally manifested by its violation of one of the Wygralak's axioms [4] of scalar cardinality. For this purpose, we present a formula for Ralescu's concept of scalar cardinality. While the original paper [3] provides an indirect definition in terms of an iterative procedure, our formula clarifies the essence of this concept. The formula, along with related observations, help us see shortcomings of Ralescu's concept which we discuss from several viewpoints. In the subsequent part of our paper, we examine Ralescu's concept of fuzzy cardinality also introduced in [3]. We provide an important link between

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[1] The problem consists in that the sigma-count of a fuzzy set with very small degrees of membership may be a rather large number, which is counterintuitive in many situations. See, e.g., the recent paper [1] on this topic, which also proposes a solution to this problem.

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Ralescu's scalar and fuzzy cardinality which reveals yet another aspect of the peculiar nature of the scalar cardinality. We emphasize the natural idea behind Ralescu's fuzzy cardinality, suggest a geometric view of it, and propose a perspective of a future development of this concept.

## 2. Ralescu's scalar cardinality and its problem

Recall first Ralescu's concept of scalar cardinality.[2] Let $A : U \to [0,1]$ be a fuzzy set in a finite universe, i.e.,

$$A : U \to [0,1] \quad \text{for} \quad U = \{u_1, \dots, u_n\}.$$

Denote by $A_{(1)}, \dots, A_{(n)}$ the membership degrees $A(u_1), \dots, A(u_n)$ ordered in a non-increasing manner, i.e.,

$$A_{(1)} \geq \cdots \geq A_{(n)} \text{ and put, furthermore, } A_{(0)} = 1 \text{ and } A_{(n+1)} = 0. \tag{1}$$

That is, (1) holds for some permutation $\pi$ of the set $\{1, \dots, n\}$ with $A_{(1)} = A(u_{\pi(1)}), \dots, A_{(n)} = A(u_{\pi(n)})$. The (scalar) cardinality $|A|_{\mathrm{Rs}}$ of $A$ is defined [3] by

$$|A|_{\mathrm{Rs}} = \begin{cases} 0 & \text{if } A = \emptyset, \\ j & \text{if } A \neq \emptyset \text{ and } A_{(j)} \geq 0.5, \\ j-1 & \text{if } A \neq \emptyset \text{ and } A_{(j)} < 0.5, \end{cases} \tag{2}$$

where

$$j = \max\{k\,;\, 1 \leq k \leq n \text{ and } A_{(k-1)} + A_{(k)} > 1\}. \tag{3}$$

**Remark 1.** In the classical, non-fuzzy case, condition (3) is clearly equivalent to $A$ having $j$ elements. Condition (3) thus generalizes the classical case. As regards an interpretation of (3) in the fuzzy setting, rewriting the condition $A_{(k-1)} + A_{(k)} > 1$ as $A_{(k)} > 1 - A_{(k-1)}$ offers a natural interpretation of $j$, namely, as the largest number such that the degree to which the $j$th element, according to (1), belongs to $A$ is larger than degree to which the $(j-1)$th element of $U$ does not belong to $A$. This condition obviously characterizes the size of a finite classical set, and may hence be considered as a basis for a corresponding concept for fuzzy sets. □

We start our analysis by the following observation:

**Lemma 1.** *Let $j$ be defined by (3). Then*

*(a)* $A_{(j-1)} > 0.5$.
*(b)* *Either* $A_{(j)} = 0.5$ *or* $A_{(j)} > 0.5$*, and both options may occur.*
*(c)* *Either* $A_{(j+1)} = 0.5$ *or* $A_{(j+1)} < 0.5$*, and both options may occur.*
*(d)* *If* $A_{(k)} = 0.5$ *then* $j \leq k$ *and* $A_{(j)} = \cdots = A_{(k-1)} = A_{(k)} = 0.5$.

**Proof.** (a) $A_{(j-1)} \leq 0.5$ cannot be the case because then, $0.5 \geq A_{(j-1)} \geq A_{(j)}$ would imply $A_{(j-1)} + A_{(j)} \leq 1$, a contradiction to (3).

(b) For both fuzzy sets $A = \{{}^{1}/u_1, {}^{0.6}/u_2, {}^{0.1}/u_3\}$ and $A = \{{}^{1}/u_1, {}^{0.5}/u_2, {}^{0.1}/u_3\}$, definition (3) yields $j = 2$.

(c) $A_{(j+1)} > 0.5$ cannot be the case because from $A_{(j)} \geq A_{(j+1)}$ one would obtain $A_{(j)} + A_{(j+1)} > 1$, which contradicts (3); therefore, $A_{(j+1)} \leq 0.5$. Moreover, $A_{(j+1)} = 0.5$ occurs, for instance, for the fuzzy set $A = \{{}^{1}/u_1, {}^{0.5}/u_2, {}^{0.5}/u_3\}$, since (3) yields $j = 2$ for this $A$, whence $A_{(j+1)} = A(u_3) = 0.5$.

(d) First, $k < j$ cannot be the case, because then (1) implies $0.5 = A_{(k)} \geq A_{(j-1)} \geq A_{(j)}$, whence $A_{(j-1)} + A_{(j)} \leq 0.5 + 0.5 = 1$, a contradiction to (3). Hence $j \leq k$.

In view of (1) and $A_{(k)} = 0.5$, to check $A_{(j)} = \cdots = A_{(k-1)} = A_{(k)} = 0.5$, it suffices to verify $A_{(j)} = 0.5$. If $k = j$, we are done. If $k > j$, then since $A_{(j)} \geq A_{(k)} = 0.5$ in view of (1), it suffices to observe that $A_{(j)} > 0.5$ cannot be the case. Indeed, along with $A_{(j+1)} \geq A_{(k)} = 0.5$, assuming $A_{(j)} > 0.5$ would imply $A_{(j)} + A_{(j+1)} > 1$, which contradicts (3). □

**Remark 2.** Part (d) of Lemma 1 already alludes to the problem with Ralescu's cardinality discussed below. Namely, it asserts that when 0.5 occurs among the membership degrees of $A$ then $A_{(j)} = 0.5$. In this case, $j$ equals $|A|_{\mathrm{Rs}}$ and hence, due to (a), $|A|_{\mathrm{Rs}}$ is the index of the first occurrence of 0.5 in (1).

[2] In his paper [3], Ralescu starts with the definition of fuzzy cardinality (p. 357) and only then provides his definition of scalar cardinality (definition 1 on p. 359). It is apparent from the text that he considers his concept of scalar cardinality as derived from fuzzy cardinality. This, however, is not so because his definition of scalar cardinality does not refer to his concept of fuzzy cardinality. Yet, it is clear that Ralescu's motivation for scalar cardinality came from his considerations on fuzzy cardinality. We explore the relationships between these two notions of cardinality in section 3.

Ralescu's definition (2) does not render a straightforward meaning of the value $|A|_{\mathrm{Rs}}$. Rather, it represents an iterative procedure to determine $|A|_{\mathrm{Rs}}$, which is explicitly described in [3, p. 359]. A clear meaning is provided by the following theorem via a direct formula for $|A|_{\mathrm{Rs}}$.[3]

**Theorem 1** *(clarification of Ralescu's cardinality).*

$$|A|_{\mathrm{Rs}} = \begin{cases} \#\{u \in U\,;\, A(u) > 0.5\} & \text{if } A(u) \neq 0.5 \text{ for each } u \in U, \\ \#\{u \in U\,;\, A(u) > 0.5\} + 1 & \text{if } A(u) = 0.5 \text{ for some } u \in U. \end{cases}$$

**Proof.** First, let $A(u) \neq 0.5$ for each $u \in U$. Distinguish two cases in the definition (2):

If $A_{(j)} \geq 0.5$ then $|A|_{\mathrm{Rs}} = j$. Due to the assumption $A_{(j)} \neq 0.5$, we have $A_{(j)} > 0.5$, whence $A_{(j+1)} < 0.5$ since otherwise one would obtain $A_{(j)} + A_{(j+1)} > 1$ contradicting (3). Therefore, $j$ is the number of elements $u \in U$ for which $A(u) > 0.5$, i.e., $|A|_{\mathrm{Rs}} = \#\{u \in U\,;\, A(u) > 0.5\}$.

If $A_{(j)} < 0.5$ then $|A|_{\mathrm{Rs}} = j - 1$. In this case, $A_{(j-1)} > 0.5$, because otherwise $A_{(j-1)} + A_{(j)} \leq 1$ which contradicts (3). Since $A_{(1)} \geq \cdots \geq A_{(j-1)} > 0.5$, it follows that $j - 1$ is the number of elements $u \in U$ for which $A(u) > 0.5$, i.e., $|A|_{\mathrm{Rs}} = \#\{u \in U\,;\, A(u) > 0.5\}$.

Second, let $A(u) = 0.5$ for some $u \in U$. According to Lemma 1 (d), $A_{(j)} = 0.5$, and according to (2) we have $|A|_{\mathrm{Rs}} = j$. Condition (3) implies $A_{(j-1)} > 0.5$, and since $A_{(1)} \geq \cdots \geq A_{(j-1)} > 0.5$, there are just $j - 1$ elements $u \in U$ for which $A(u) > 0.5$. It hence follows that $|A|_{\mathrm{Rs}} = j = j - 1 + 1 = \#\{u \in U\,;\, A(u) > 0.5\} + 1$. □

Theorem 1 reveals two problems of Ralescu's cardinality which we address below. The first one regards sensitivity of cardinality to small changes in membership degrees. The second one concerns an unnatural behavior w.r.t. fuzzy sets containing multiple elements with membership equal to 0.5. The second one seems more serious because it implies violation of finite additivity, i.e., a violation of a commonly accepted intuitive condition that has been proposed as an axiom of scalar cardinality [4,5].

Consider first the fuzzy sets

$$\begin{aligned} A &= \{\ {}^{1}/u_1,\ {}^{1}/u_2, \cdots, {}^{1}/u_n\}, \\ B &= \{\ {}^{0.5+\varepsilon}/u_1, {}^{0.5+\varepsilon}/u_2, \cdots, {}^{0.5+\varepsilon}/u_n\}, \text{ and} \\ C &= \{\ {}^{0.5-\varepsilon}/u_1, {}^{0.5-\varepsilon}/u_2, \cdots, {}^{0.5-\varepsilon}/u_n\}, \end{aligned}$$

with an arbitrary $0 < \varepsilon \leq 0.5$. Using Theorem 1 it is immediate that

$$|A|_{\mathrm{Rs}} = n, \quad |B|_{\mathrm{Rs}} = n, \quad \text{and} \quad |C|_{\mathrm{Rs}} = 0.$$

These values are counterintuitive as a result of the threshold-based definition of Ralescu's cardinality revealed above. First, no matter how small $\varepsilon$ is, hence how close the membership degrees in $B$ are to 0.5, the cardinality of $B$ remains the same as the cardinality of $A$, i.e., the cardinality of the whole set $U$. Second, an arbitrarily small change from $0.5 + \varepsilon$ to $0.5 - \varepsilon$, makes the cardinality drop from the largest possible value to the smallest possible value (i.e., from $n$ of $B$ to 0 of $C$). That is, the considered scalar cardinality is undesirably sensitive to small changes of membership degrees.[4] In addition, the threshold-like nature results in an inappropriate behavior, in our view, from the perspective of the problem of accumulation of small membership degrees (see Note 1 and [1]): While small membership degrees—and, in fact, all degrees smaller than 0.5—do not accumulate, they are completely ignored by the scalar cardinality $|\cdot|_{\mathrm{Rs}}$. That is, all the membership degrees smaller than 0.5 are treated the same way as the elements with null membership. Moreover, of all the elements with membership 0.5, only one is counted (and contributes to the cardinality by the value of 1), whereas the others are dismissed.

A more serious problem becomes apparent by observing that

$$\left|\{{}^{1}/u_1, {}^{0.8}/u_2, {}^{0.5}/u_3\}\right|_{\mathrm{Rs}} = \left|\{{}^{1}/u_1, {}^{0.8}/u_2, {}^{0.5}/u_3, {}^{0.5}/u_4\}\right|_{\mathrm{Rs}}$$

and, to make the point stronger, that

$$\left|\{{}^{0.5}/u_1\}\right|_{\mathrm{Rs}} = \left|\{{}^{0.5}/u_1, {}^{0.5}/u_2\}\right|_{\mathrm{Rs}} = \cdots = \left|\{{}^{0.5}/u_1, {}^{0.5}/u_2, \ldots, {}^{0.5}/u_n\}\right|_{\mathrm{Rs}} = 1$$

for arbitrary $n$. That is, of all the elements $u$ with membership 0.5, only one gets counted by Ralescu's cardinality, which is easily seen from the formula for $|A|_{\mathrm{Rs}}$ in Theorem 1. We contend that this property contradicts intuition regarding the cardinality of fuzzy sets.

The latter clash with intuition manifests itself formally by violation of one of the commonly accepted axioms of scalar cardinality, namely, the axiom of finite additivity [4,5]. According to this axiom, a scalar cardinality $\mathrm{card}(\cdot)$ needs to satisfy

$$\mathrm{card}\left(A_1 \cup \cdots \cup A_k\right) = \mathrm{card}(A_1) + \cdots + \mathrm{card}(A_k)$$

for arbitrary disjoint finite fuzzy sets $A_1, \ldots, A_k$; i.e., fuzzy sets satisfying $\min(A_i(u), A_j(u)) = 0$ for each $u \in U$ and $i \neq j$.

[3] In this theorem and the remainder of this paper, $\#V$ stands for the number of elements (i.e., ordinary cardinality) of an ordinary set $V$.

[4] A reviewer of this paper does not agree with our view and maintains that Ralescu's scalar cardinality treats $B$ and $C$ properly.

**Theorem 2.** *Ralescu's cardinality violates the axiom of finite additivity.*

**Proof.** Theorem 1 yields that while

$$\left|\{{}^{0.5}/u_1\} \cup \{{}^{0.5}/u_2\}\right|_{\mathrm{Rs}} = \left|\{{}^{0.5}/u_1, {}^{0.5}/u_2\}\right|_{\mathrm{Rs}} = 1,$$

one has

$$\left|\{{}^{0.5}/u_1\}\right|_{\mathrm{Rs}} + \left|\{{}^{0.5}/u_2\}\right|_{\mathrm{Rs}} = 1 + 1 = 2. \quad \square$$

One easily checks that Theorem 2 in fact follows from a stronger claim, which describes all the instances on which finite additivity fails: For disjoint fuzzy sets $A_1, \ldots, A_k$, one has $\mathrm{card}\left(A_1 \cup \cdots \cup A_k\right) \neq \mathrm{card}(A_1) + \cdots + \mathrm{card}(A_k)$ if and only if at least two of these fuzzy sets contain elements with a membership degree equal to 0.5.

**Remark 3.** (a) Theorem 1 enables easier considerations on $|\cdot|_{\mathrm{Rs}}$. For instance, while Proposition 5 in [3], according to which $|A|_{\mathrm{Rs}} \leq |B|_{\mathrm{Rs}}$ for $A \leq B$, is an immediate consequence of Theorem 1, the proof in [3] occupies almost a half a page.

(b) Theorem 1 and the examples above show that Proposition 9 in [3], which claims that $|A|_{\mathrm{Rs}}$ is the number of elements for which $A(u) \geq 0.5$, is not true.

(c) The modification

$$|A|_{\mathrm{Rs1}} = \#\{u \in U\,;\, A(u) \geq 0.5\}$$

of $|A|_{\mathrm{Rs}}$ is known to satisfy finite additivity, as well as the other axioms of scalar cardinality [4,5]. Still, it does not properly address the problem of accumulation of small membership degrees as the above argument with the fuzzy sets with $B(u_i) = 0.5 + \varepsilon$ and $C(u_i) = 0.5 - \varepsilon$ still remains. The same is true of

$$|A|_{\mathrm{Rs2}} = \#\{u \in U\,;\, A(u) > 0.5\} + \frac{\#\{u \in U\,;\, A(u) = 0.5\}}{2},$$

which may be regarded as representing a compromise between $|A|_{\mathrm{Rs}}$ and $|A|_{\mathrm{Rs1}}$.

## 3. Relationship to Ralescu's fuzzy cardinality

We now briefly consider Ralescu's notion of fuzzy cardinality. First, it comes as a natural question to clarify the relationship of the two notions of cardinality, and we provide such a clarification in the present section. Interestingly, the clarification makes the somewhat peculiar character of the notion of scalar cardinality apparent from a different perspective. Second, since we regard Ralescu's notion of fuzzy cardinality as a compelling concept, we present further properties of this concept, its geometrical interpretation, and remarks on a possible future exploration which are provided in the last section.

Let $A_{(k)}$ be as in (1). For a fuzzy set $A : U \to [0,1]$ and a non-negative integer $k = 0, 1, \ldots, n$, Ralescu defines

$$|A|_{\mathrm{Rf}}(k) = \min\{A_{(k)}, 1 - A_{(k+1)}\}. \tag{4}$$

Formula (4) determines a fuzzy set $|A|_{\mathrm{Rf}}$ for which the membership degree $|A|_{\mathrm{Rf}}(k)$ may be interpreted as the truth degree of the statement "$A$ has $k$ elements."

Note first that (4) appears to be a rather natural definition. Namely, interpreting $1 - A_{(k+1)}$ as the negation of $A_{(k+1)}$, $|A|_{\mathrm{Rf}}(k)$ may be regarded as the truth degree of the statement "the $k$th element belongs to $A$ but the $(k+1)$th element does not," provided the elements are ordered as in (1). It is obvious that this definition naturally generalizes the ordinary case: If $A$ is an ordinary set then $|A|_{\mathrm{Rf}}(k) = 1$ if $\#A = k$, i.e., if $A$ has exactly $k$ elements, and $|A|_{\mathrm{Rf}}(k) = 0$ if $\#A \neq k$.[5] In addition, as has been pointed out by a reviewer of this paper, Ralescu's fuzzy cardinality satisfies finite additivity generalized in that the addition of scalar cardinalities (non-negative integers) is replaced by the addition of fuzzy cardinalities (fuzzy sets of non-negative integers), and hence behaves naturally from this viewpoint.

To reveal a link between the two notions of cardinality, $|\cdot|_{\mathrm{Rs}}$ and $|\cdot|_{\mathrm{Rf}}$, we denote by $\max |A|_{\mathrm{Rf}}$ the largest membership degree of $|A|_{\mathrm{Rf}}$, i.e.,

$$\max |A|_{\mathrm{Rf}} = \max\{|A|_{\mathrm{Rf}}(0), |A|_{\mathrm{Rf}}(1), |A|_{\mathrm{Rf}}(2), \ldots\}.$$

Clearly, $\max |A|_{\mathrm{Rf}}$ is well defined as $|A|_{\mathrm{Rf}}(n+1)$, $|A|_{\mathrm{Rf}}(n+2), \ldots$ are all equal to 0. Furthermore, $\arg\max |A|_{\mathrm{Rf}}$ shall denote the set of non-negative integers for which $\max |A|_{\mathrm{Rf}}$ is attained, i.e.,

$$\arg\max |A|_{\mathrm{Rf}} = \{i\,;\, |A|_{\mathrm{Rf}}(i) = \max |A|_{\mathrm{Rf}}\}.$$

For the remainder of this section, let

[5] Following common usage, we identify an ordinary set with the corresponding crisp fuzzy set, i.e., with its characteristic function.

$$p = \#\{u \in U\,;\ A(u) = 0.5\},$$

i.e., $p$ be the number of elements $u \in U$ for which $A(u) = 0.5$. The following theorems describe connections between $|\cdot|_{\mathrm{Rs}}$ and $|\cdot|_{\mathrm{Rf}}$.[6]

**Theorem 3.** *We have*

$$\#\arg\max |A|_{\mathrm{Rf}} = p + 1.$$

*In particular, putting*

$$k_1 = |A|_{\mathrm{Rs}}, k_2 = |A|_{\mathrm{Rs}} - 1, k_3 = |A|_{\mathrm{Rs}} + 1, k_4 = |A|_{\mathrm{Rs}} + 2, k_5 = |A|_{\mathrm{Rs}} + 3, \dots,$$

*we have*

$$\arg\max |A|_{\mathrm{Rf}} = \{k_1, \dots, k_{p+1}\}.$$

**Proof.** Put

$$\kappa = \max\{k\,;\ A_{(k)} > 0.5\}.$$

First, suppose $p = 0$, i.e., $A(u) \neq 0.5$ for each $u \in U$. Theorem 1 implies that in this case, $|A|_{\mathrm{Rs}} = \kappa$. We verify that

$$\arg\max |A|_{\mathrm{Rf}} = \{\kappa\} \tag{5}$$

by checking that $|A|_{\mathrm{Rf}}(k) < |A|_{\mathrm{Rf}}(\kappa)$ for every $k = 0, \dots, n$ with $k < \kappa$ or $k > \kappa$. Since $A_{(\kappa)} > 0.5$ and $A_{(\kappa+1)} < 0.5$, and thus $1 - A_{(\kappa+1)} > 0.5$, we obtain

$$|A|_{\mathrm{Rf}}(\kappa) = \min(A_{(\kappa)}, 1 - A_{(\kappa+1)}) > 0.5.$$

For $k < \kappa$, one has $A_{(k)} > 0.5$ and $A_{(k+1)} > 0.5$, hence $1 - A_{(k+1)} < 0.5$, from which we obtain

$$|A|_{\mathrm{Rf}}(k) = \min(A_{(k)}, 1 - A_{(k+1)}) < 0.5 < |A|_{\mathrm{Rf}}(\kappa).$$

For $k > \kappa$, a similar reasoning applied to $A_{(k)} < 0.5$ and $A_{(k+1)} < 0.5$ yields $|A|_{\mathrm{Rf}}(k) < |A|_{\mathrm{Rf}}(\kappa)$ as well.

Second, suppose $p \neq 0$, i.e., $A(u) = 0.5$ for some $u \in U$. In this case, we have

$$A_{(\kappa)} > 0.5, A_{(\kappa+1)} = \cdots = A_{(\kappa+p)} = 0.5, \text{ and } A_{(\kappa+p+1)} < 0.5. \tag{6}$$

Using (1), a similar reasoning as used for $p = 0$ yields that $|A|_{\mathrm{Rf}}(k) < |A|_{\mathrm{Rf}}(\kappa)$ both for $k < \kappa$ and $k > \kappa + p$. Now, since (6) implies

$$|A|_{\mathrm{Rf}}(\kappa) = |A|_{\mathrm{Rf}}(\kappa + 1) = \cdots = |A|_{\mathrm{Rf}}(\kappa + p) = 0.5, \tag{7}$$

we obtain

$$\arg\max |A|_{\mathrm{Rf}} = \{\kappa, \kappa + 1, \dots, \kappa + p\}. \tag{8}$$

Since by virtue of Theorem 1, $|A|_{\mathrm{Rs}} = \kappa + 1$ in this case, the proof is finished by observing how the sequence $k_1, \dots, k_{p+1}$ is constructed. □

Put another way, Theorem 3 claims that

$$\arg\max |A|_{\mathrm{Rf}} = \begin{cases} \{|A|_{\mathrm{Rs}}\} & \text{if } p = 0, \\ \{|A|_{\mathrm{Rs}} - 1, |A|_{\mathrm{Rs}}, \dots, |A|_{\mathrm{Rs}} + p - 1\} & \text{if } p > 0. \end{cases}$$

Furthermore, it has the following consequence.

**Theorem 4** *(restoring $|A|_{\mathrm{Rs}}$ from $|A|_{\mathrm{Rf}}$).*

$$|A|_{\mathrm{Rs}} = \begin{cases} \textit{the unique element in } \arg\max |A|_{\mathrm{Rf}} \\ \qquad\qquad \textit{if } \arg\max |A|_{\mathrm{Rf}} \textit{ is a singleton;} \\ \textit{the second smallest element in } \arg\max |A|_{\mathrm{Rf}} \\ \qquad\qquad \textit{if } \arg\max |A|_{\mathrm{Rf}} \textit{ is not a singleton.} \end{cases}$$

**Proof.** Directly from Theorem 3. □

[6] Note that Ralescu only mentions in his paper, without giving a proof, that his scalar cardinality is a value at which the fuzzy cardinality attains its maximum [3, p. 359]. Clearly, this does not characterize the scalar cardinality as there may be several such values. Our results below are stronger and more informative, and answer the question of connections between $|\cdot|_{\mathrm{Rs}}$ and $|\cdot|_{\mathrm{Rf}}$ completely.

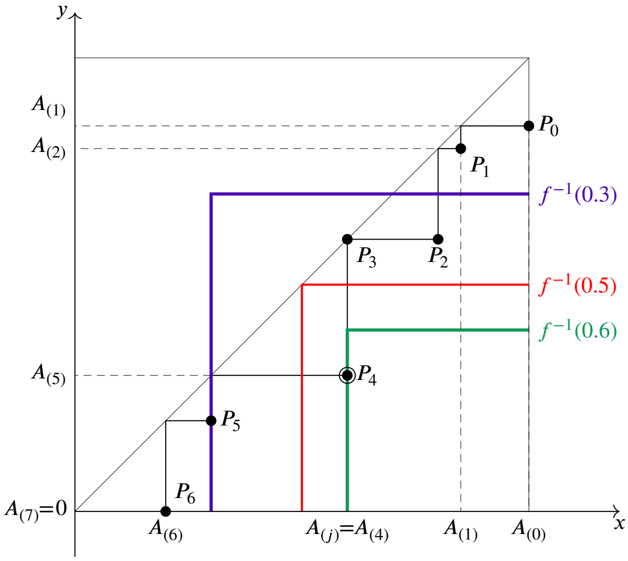


**Fig. 1.** $\arg\max |A|_{\mathrm{Rf}}$ is a singleton. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

As regards the largest value of $|A|_{\mathrm{Rf}}$, it is described by the following theorem.

**Theorem 5.**

$$\max |A|_{\mathrm{Rf}} = \begin{cases} \min(A_{(|A|_{\mathrm{Rs}})}, 1 - A_{(|A|_{\mathrm{Rs}}+1)}) > 0.5 & \textit{if } p = 0, \\ 0.5 & \textit{if } p > 0. \end{cases}$$

**Proof.** Consider the proof of Theorem 3. For $p = 0$, the claim follows from $\kappa = |A|_{\mathrm{Rs}}$, (5), and $|A|_{\mathrm{Rf}}(\kappa) = \min(A_{(\kappa)}, 1 - A_{(\kappa+1)})$. For $p > 0$, it is a direct consequence of (7) and (8). □

**Remark 4.** The explored notion of fuzzy cardinality lends itself to a simple geometric interpretation. This interpretation appears useful as it lets one observe some of the claims above in a geometric manner, which we now illustrate.

The degrees $A_{(0)}, A_{(1)}, \ldots, A_{(n+1)}$ ordered as in (1) can be visualized as points $P_0, P_1, \ldots, P_n$ in the $xy$-plane, where $P_k = \langle A_{(k)}, A_{(k+1)} \rangle$, $k = 0, 1, \ldots, n$. Since $0 \le A_{(k)} \le 1$ and $A_{(k)} \ge A_{(k+1)}$ for all $k$, every point $P_k$ is located below or on the main diagonal of the unit square. Moreover, the $y$-coordinate of $P_k$ coincides with the $x$-coordinate of $P_{k+1}$, so the points $P_0, P_1, \ldots, P_n$ form a step-like geometric pattern. For instance, for a fuzzy set $A$ with $A_{(0)} = 1$, $A_{(1)} = 0.85$, $A_{(2)} = 0.8$, $A_{(3)} = 0.6$, $A_{(4)} = 0.6$, $A_{(5)} = 0.3$, $A_{(6)} = 0.2$, and $A_{(7)} = 0$, the corresponding step-like pattern is shown in Fig. 1. Furthermore, in Fig. 1 we show projections of several contour lines[7] of the function $f(x, y) = \min\{x, 1 - y\}$ onto the $xy$-plane, i.e., of the function representing the evaluation of fuzzy cardinality (4). In particular, $f^{-1}(0.5)$ is depicted in Fig. 1 as the red line segment connecting all points $\langle x, y \rangle$ for which $f(x, y) = 0.5$.

The step-like pattern together with the contours lets us observe geometrically some of the above claims regarding $|A|_{\mathrm{Rf}}$ and $|A|_{\mathrm{Rs}}$. For instance, for any fuzzy set $A$ there is $k \in \{1, 2, \ldots, n+1\}$ such that the line $P_{k-1}P_k$ intersects the contour $f^{-1}(0.5)$. Obviously, the fuzzy cardinality reaches its maximum for the smallest such $k$, i.e., $k \in \arg\max |A|_{\mathrm{Rf}}$. Two situations then may occur:

(a) Line $P_{k-1}P_k$ intersects the contour $f^{-1}(0.5)$ but $P_k$ does not lie on it, i.e., there is no $u \in U$ such that $A(u) = 0.5$. This situation occurs with the fuzzy set $A$ defined above, as we can see in Fig. 1. Then, as one easily observes, $|A|_{\mathrm{Rf}}(k) > |A|_{\mathrm{Rf}}(k+1)$, hence $\arg\max A = \{k\}$.

(b) Line $P_{k-1}P_k$ intersects the contour $f^{-1}(0.5)$ and $P_k$ lies on it, i.e., there is at least one $u \in U$ for which $A(u) = 0.5$. Suppose that there is exactly one such $u$, as in the case of $B = \{{}^{0.85}/u_1, {}^{0.8}/u_2, {}^{0.6}/u_3, {}^{0.5}/u_4, {}^{0.3}/u_5, {}^{0.2}/u_6\}$, which situation is illustrated in Fig. 2. Then, clearly, the next point $P_{k+1}$ must lie on the contour $f^{-1}(0.5)$ as well, and, furthermore, $P_{k+2}$ must lie on a contour $f^{-1}(c)$ for some $c < 0.5$. Therefore, $\arg\max |B|_{\mathrm{Rf}} = \{k, k+1\}$.

Observe now that both of the conclusions in (a) and (b) coincide with the claim of Theorem 3.

[7] For brevity, we shall speak of "contours" instead of "projections of the contour lines" in what follows.

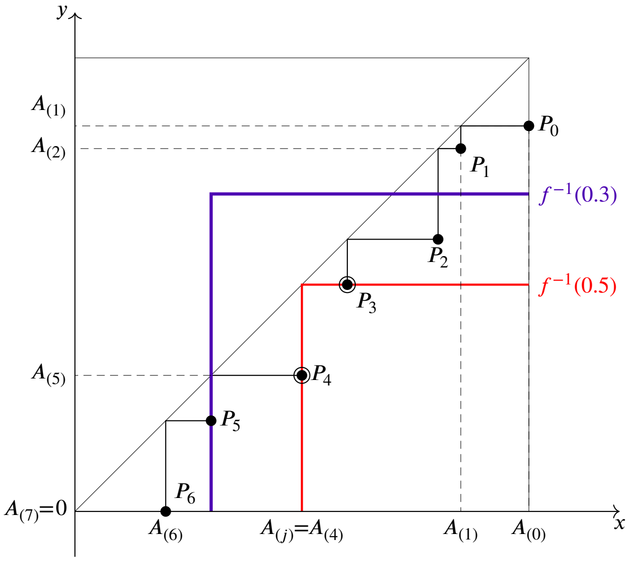


**Fig. 2.** $\arg\max |B|_{\mathrm{Rf}}$ has two elements.

## 4. Concluding considerations

In this paper, we present a direct formula for Ralescu's concept of scalar cardinality which is implied by the original, iterative definition. This formula reveals problems of this concept, both on intuitive and formal grounds, which are described in section 2. Moreover, we provide in section 3 a link of this concept to Ralescu's other concept of cardinality, namely, fuzzy cardinality, which exposes another perspective on the peculiarity of the scalar cardinality concept.

As mentioned at the beginning of section 3, Ralescu's fuzzy cardinality is based on a rather intuitive formula and, in our view, is hence worth further exploration. Our preliminary exploration shows that an interesting possibility is to consider a more general interpretation of Ralescu's idea of fuzzy cardinality, namely to consider a formula

$$|A|_{\mathrm{Rf}}(k) = A_{(k)} \otimes \neg A_{(k+1)} \tag{9}$$

with $\otimes$ being a truth function of a conjunction, such as a t-norm, and $\neg$ being a truth function of a negation. Clearly, Ralescu's fuzzy cardinality results from (9) by taking $a \otimes b = \min(a,b)$ and $\neg a = 1 - a$. It seems noteworthy that by varying the interpretation of the involved logical connectives, one obtains quantitatively rather different concepts of cardinality drawn together qualitatively by the clear logical meaning of the formula. In particular, while for the Gödel (minimum) t-norm, i.e., $a \otimes b = \min(a,b)$, the sets $\arg\max |A|_{\mathrm{Rf}}$ and the value $\max |A|_{\mathrm{Rf}}$ are described by Theorems 3 and 5, one may check that for $\otimes$ being the Łukasiewicz t-norm, i.e., $a \otimes b = \max(0, a+b-1)$, one obtains

$$|A|_{\mathrm{Rf}}(k) = A_{(k)} - A_{(k+1)},$$

and hence

$$\arg\max |A|_{\mathrm{Rf}} = \{k;\ \text{the drop from } A_{(k)} \text{ to } A_{(k+1)} \text{ is maximal}\}$$

and $\max |A|_{\mathrm{Rf}}$ is the extent of the maximal drop. For the product t-norm, the resulting cardinality is yet different but its description is more involved. In the same spirit in which different t-norms make a single fuzzy-logic-based model fit for different applications, the different concepts of fuzzy cardinality obtained from the different t-norms are likely to fit different situations in which the notion of a cardinality of a fuzzy sets occurs. It hence seems that both from the theoretical as well as application viewpoint, Ralescu's fuzzy cardinality calls for further examination.

### CRediT authorship contribution statement

**Eduard Bartl:** Investigation. **Radim Belohlavek:** Investigation.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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