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Bivalent and other solutions of fuzzy relational equations via linguistic hedges

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Abstract

We show that the well-known results regarding solutions of fuzzy relational equations and their systems can easily be generalized to obtain criteria regarding constrained solutions such as solutions which are crisp relations. When the constraint is empty, constrained solutions are ordinary solutions. The generalization is obtained by employing intensifying and relaxing linguistic hedges, conceived in this paper as certain unary functions on the scale of truth degrees. One aim of the paper is to highlight the problem of constrained solutions and to demonstrate that this problem naturally appears when identifying unknown relations. The other is to emphasize the role of linguistic hedges as constraints.

Keywords: Fuzzy logic; Fuzzy relation; Fuzzy relational equation; Linguistic hedge

1. Motivation and preliminaries

1.1. Motivation

Fuzzy relational equations play an important role in fuzzy set theory and its applications, see [18] for Sanchez's seminal paper, and e.g. [8–10,12,17,15,16]. Namely, it is often the case that the problem in a particular application of fuzzy logic may be transformed to the problem of identifying an unknown fuzzy relation. The problem to determine an unknown fuzzy relation R between universe sets X and Y such that $R \odot S = T$, where S and T are given (known) fuzzy relations between Y and Z, and X and Z, respectively, and $\odot$ is an operation of composition of fuzzy relations, is called the problem of fuzzy relational equations. Alternatively, given R and T, the problem is to determine S. The respective fuzzy relational equations with unknown U are denoted by

$$U \odot S = T \quad \text{and} \quad R \odot U = T, \tag{1}$$

and every fuzzy relation U satisfying the first or the second equality is called a solution of the respective fuzzy relational equation.

The nature of the unknown relationship represented by U may impose additional constraints on U. For example, one may require that U be a bivalent (crisp) relation (see Section 3 for an illustrative example). More generally,

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one may require that the truth degrees assigned by U are taken from a subset K of the set L of truth degrees, such as $K = \{0, 1\}$ in case one requires U to be crisp, $K = \{0, \frac{1}{2}, 1\}$ in case one requires U to assign degrees representing "bad", "acceptable", "good", etc. In this paper, we show that such constrained solutions of ordinary fuzzy relational equations naturally appear as solutions of a new type of fuzzy relational equations, namely equations that use modified compositions of fuzzy relations. The modification consists in inserting linguistic hedges [21] in the description of the compositions. As an example, instead of "there exists y such that x and y are R-related and y and z are S-related" we use "there exists y such that x and y are very much R-related and y and z are S-related" as a condition for x being related to z. Such a modified condition may be used to obtain constrained solutions with the unknown relation R. The particular constraint depends on how one interprets the hedge "very much", i.e. on the truth function used to interpret the hedge. We show that the well-known results regarding solutions of fuzzy relational equations may easily be generalized to the more general case which involves intensifying or relaxing hedges. This idea emphasizes the role of linguistic hedges as constraints (see [4–7] for similar utilizations of linguistic hedges).

1.2. Preliminaries

We assume that the reader is familiar with basic notions of fuzzy set theory and refer the reader to [11,13,17] for details. Our results below are developed for complete residuated lattices as the structures of truth degrees. Recall that a complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$. A reader, who is not familiar with residuated lattices may assume that the set of truth degrees L is the real unit interval [0, 1], that $\otimes$ and $\rightarrow$ are a left-continuous t-norm and its residuum, respectively, and that $\wedge$ and $\vee$ are infimum and supremum.

Linguistic hedges are natural language expressions, such as "very", "highly", "more or less", that modify the meaning of other expressions. The importance of hedges was recognized by Zadeh in [21] and hedges have played an important role in fuzzy logic ever since. In this paper, we use both the intensifying (or truth-stressing) hedges such as "very" or "highly", and the relaxing (or truth-depressing) hedges such as "more or less" or "roughly". We understand these hedges as unary functions $^* : L \rightarrow L$ on the set of truth degrees. If $\|\varphi\|$ denotes the truth degree of formula φ and * corresponds to "very", then the truth degree of formula very φ is $\|\varphi\|^*$, i.e. $\|\text{very } \varphi\| = \|\varphi\|^*$. Note that intensifying hedges have been investigated from the point of view of fuzzy logic in the narrow sense by Hájek [13,14].

In this paper, we assume that an *intensifying hedge* is a unary function $^* : L \rightarrow L$ satisfying

$$a^* \leq a, \tag{2}$$

$$a \leq b \text{ implies } a^* \leq b^*, \tag{3}$$

$$a^{**} = a^*. \tag{4}$$

Note that (2) and (3) are obvious requirements for an intensifying hedge. (2), subdiagonality, says that the truth degree of very φ is less than or equal to the truth degree of φ ("this apple is red" is at least as true as "this apple is very red"); (3), isotony, says that if "x is red" is less true than "y is red" then "x is very red" is less true than "y is very red"; (4), idempotency, distinguishes certain hedges, namely those for which "very very φ" has the same truth degree as "very φ", provided "very" is the term representing the hedge. Arguably, not all intensifying hedges satisfy (4) but we use this property for technical reasons. Note that Hájek in [14] does not use (4). Instead, he uses $(a \vee b)^* \leq a^* \vee b^*$ and a stronger condition than (3), namely $(a \rightarrow b)^* \leq a^* \rightarrow b^*$. In terms of interior structures, a mapping satisfying (2)–(4) is an interior operator on the lattice $\langle L, \leq\rangle$ of truth degrees. On every complete residuated lattice, there are two important intensifying hedges:

(i) identity, i.e. $a^* = a$ ($a \in L$);
(ii) globalization [19], i.e.

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Note that globalization agrees with Baaz's Δ-operation [1] in case of linearly ordered residuated lattices.

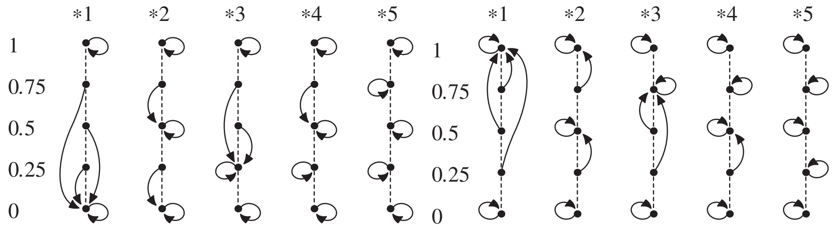

Fig. 1. Intensifying (left) and relaxing (right) hedges on a five-element chain.

In addition, we assume that a *relaxing hedge* is a unary function $^* : L \to L$ satisfying

$$a \le a^*, \tag{6}$$

(3) and (4). Requiring (6) instead of (2) for relaxing hedges corresponds to the fact that the truth degree of "this apple is more or less red" needs to be at least as high as the truth degree of "this apple is red". Therefore, while intensifying hedges are interior operators, relaxing hedges are closure operators on the lattice $\langle L, \le \rangle$ of truth degrees. On every complete residuated lattice, there are two important relaxing hedges: one is the identity described in (i) above, the other is dual to globalization [20] and is defined by

$$a^* = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{7}$$

Example 1. Fig. 1 (left) shows five intensifying hedges on a chain $L = \{0, 0.25, 0.5, 0.75, 1\}$ (the chain may be made a complete residuated lattice by adding, for example, Łukasiewicz operations). Hedges $*_1$ and $*_5$ are globalization and identity, respectively. Fig. 1 (right) shows relaxing hedges which are dual to the intensifying ones. Namely, $*_1$ on the left (globalization) corresponds to $*_1$ on the right defined by (7).

2. Constrained solutions of fuzzy relational equations

Let $\odot$ denote a composition of fuzzy relations, let K be a non-empty subset of the whole set L of truth degrees. A solution U of a fuzzy relational equation $U \odot S = T$, see (1), is called a *K-solution* (a solution constrained by K) if $U(x, y) \in K$ for every $x \in X$ and $y \in Y$. The concept of a K-solution of $R \odot U = T$ as well as the concept of a K-solution of a system of fuzzy relational equations are defined analogously. The set K presents a constraint on the truth degrees that are "allowed" to be assigned by U to elements x and y. For example, if $K = \{0, 1\}$, the constraint says that we are looking for solutions which are bivalent (crisp) relations. If $K = L$, the constraint is empty in the sense that any solution is a K-solution.

Let us recall that the $\circ$-composition, $\triangleleft$-composition, and $\triangleright$-composition (see, e.g. [2,3]) of fuzzy relations $R : X \times Y \to L$ (a fuzzy relation between sets X and Y) and $S : Y \times Z \to L$ (between Y and Z) are defined by

$$(R \circ S)(x, z) = \bigvee_{y \in Y} (R(x, y) \otimes S(y, z)), \tag{8}$$

$$(R \triangleleft S)(x, z) = \bigwedge_{y \in Y} (R(x, y) \to S(y, z)), \tag{9}$$

for every $x \in X, z \in Z$. For the particular case of $\odot$ being $\circ$ and $\triangleleft$, (1) become the well-known types of fuzzy relational equations, namely

$$U \circ S = T, \quad R \circ U = T, \quad U \triangleleft S = T, \quad \text{and} \quad R \triangleleft U = T.$$

We show that the problems regarding *K*-solutions of these equations may be reduced to the corresponding problems regarding solutions of other, natural fuzzy relational equations that involve hedges. In this reduction, the set *K* plays the role of the set of fixpoints of an (intensifying or relaxing) hedge $*$. Recall that the set of fixpoints of $*$ is defined as

$$\mathrm{fix}(*) = \{a \in L | a^* = a\}.$$

The following lemma follows directly from the well-known facts that the sets of fixpoints of interior (closure) operators are just the sets closed under arbitrary suprema (infima).

Lemma 1. (i) $K = \mathrm{fix}(*)$ *for an intensifying hedge* $* : L \to L$ *if and only if K is closed w.r.t. arbitrary suprema.*
(ii) $K = \mathrm{fix}(*)$ *for a relaxing hedge* $* : L \to L$ *if and only if K is closed w.r.t. arbitrary infima.*

For the practically important case when $\langle L, \leq \rangle$ is a chain and *K* is finite we get:

Corollary 2. *Let* $\langle L, \leq \rangle$ *be a chain with* 0 *and* 1 *being the least and the greatest element, respectively, and K be a finite subset of L. If* $0 \in K$ *then there exists an intensifying hedge* $*$ *such that* $K = \mathrm{fix}(*)$. *If* $1 \in K$ *then there exists a relaxing hedge* $*$ *such that* $K = \mathrm{fix}(*)$.

Therefore, using intensifying and relaxing hedges as defined above is convenient from the user's point of view since in practical situations, these conditions do not impose any serious constraints on *K*.

Note also that the intensifying hedge for which $K = \mathrm{fix}(*)$ for a given *K* according to Lemma 1 is given by: $a^* = \bigvee\{b \in K | b \leq a\}$, which in case $\langle L, \leq \rangle$ is a chain and *K* is finite means that a^* is the largest element from *K* which is smaller than or equal to *a*. A dual condition holds for relaxing hedges: $a^* = \bigwedge\{b \in K | a \leq b\}$; if $\langle L, \leq \rangle$ is a chain and *K* finite, a^* is the smallest element from *K* which is larger than or equal to *a*.

Definition 3. Let *R* and *S* be fuzzy relations between *X* and *Y* and between *Y* and *Z*. Fuzzy relations $(R \circ^* S)$, $(R \lhd^* S)$, and $(R \lhd_* S)$ between *X* and *Z* are defined by

$$(R \circ^* S)(x, z) = \bigvee_{y \in Y} (R(x, y)^* \otimes S(y, z)), \tag{10}$$

$$(R \lhd^* S)(x, z) = \bigwedge_{y \in Y} (R(x, y)^* \to S(y, z)), \tag{11}$$

$$(R \lhd_* S)(x, z) = \bigwedge_{y \in Y} (R(x, y) \to S(y, z)^*), \tag{12}$$

for all $x \in X, z \in Z$.

Clearly, $(R \circ^* S) = (R^*) \circ S$, $(R \lhd^* S) = (R^*) \lhd S$, and $(R \lhd_* S) = R \lhd (S^*)$. In addition, one need not define $R \circ_* S$ because $R \circ (S^*) = (S^{-1} \circ^* R^{-1})^{-1}$.

If $*$ is an intensifying hedge (representing "very"), then $(R \circ^* S)(x, z)$ is the truth degree of proposition "there is $y \in Y$ such that it is very true that $\langle x, y \rangle$ is in *R* and $\langle y, z \rangle$ is in *S*". Likewise, if $*$ is a relaxing hedge (representing "more or less"), $R \lhd_* S(x, z)$ is the truth degree of "for every $y \in Y$: if $\langle x, y \rangle$ is in *R* then it is more or less true that $\langle y, z \rangle$ is in *S*". Therefore, the new compositions may be seen as generalizations of the ordinary ones that result by inserting intensifying or relaxing hedges into the verbal descriptions of the compositions.

We now consider the following three types of equations with the unknown fuzzy relation *U*:

$$U \circ^* S = T, \tag{13}$$

$$U \lhd^* S = T, \tag{14}$$

$$R \lhd_* U = T. \tag{15}$$

Note that one need not consider $R \circ^* U = T$, $R \lhd^* U = T$, or $U \lhd_* S = T$, because they coincide with the ordinary equations $R^* \circ U = T$, $R^* \lhd U = T$, or $U \lhd S^* = T$.

For intensifying hedges $*_1$ and $*_2$, we say that $*_1$ is *stronger* than $*_2$ if and only if $a^{*_1} \le a^{*_2}$ for every $a \in L$. Dually, for relaxing hedges $*_1$ and $*_2$, we say that $*_1$ is *stronger* than $*_2$ if and only if $a^{*_1} \ge a^{*_2}$ for every $a \in L$.

Lemma 4. *Let $*_1$ and $*_2$ be both intensifying or both relaxing hedges.*

(i) *Then* $\mathrm{fix}(*_1) \subseteq \mathrm{fix}(*_2)$ *if and only if $*_1$ is stronger than $*_2$.*
(ii) *Let $*_1$ be stronger than $*_2$. If U is a* $\mathrm{fix}(*_1)$*-solution of $U \circ^{*_2} S = T$ then U is a solution of $U \circ^{*_1} S = T$. If U is a solution of $U \circ^{*_1} S = T$ then U^{*_1} is a* $\mathrm{fix}(*_1)$*-solution of $U \circ^{*_2} S = T$.*

Proof. The claim of (i) regarding intensifying hedges is proved in [5]; the claim regarding relaxing hedges is dual to the first claim. (ii): If U is a $\mathrm{fix}(*_1)$-solution of $U \circ^{*_2} S = T$ then $U(x, y) \in \mathrm{fix}(*_1)$, hence $U = U^{*_1}$. Moreover, since $*_1$ is stronger than $*_2$, (i) yields that $\mathrm{fix}(*_1) \subseteq \mathrm{fix}(*_2)$, hence $U^{*_2} = U$, whence $U^{*_1} = U^{*_2}$. Therefore, $U \circ^{*_1} S = U^{*_1} \circ S = U^{*_2} \circ S = U \circ^{*_2} S = T$, proving that U is a solution of $U \circ^{*_1} S = T$. If U is a solution of $U \circ^{*_1} S = T$ then since $U^{*_1}(x, y) \in \mathrm{fix}(*_1)$, we have $(U^{*_1})^{*_2} = U^{*_1}$, whence $U^{*_1} \circ^{*_2} S = (U^{*_1})^{*_2} \circ S = U^{*_1} \circ S = T$, proving that U^{*_1} is a $\mathrm{fix}(*_1)$-solution of $U \circ^{*_2} S = T$. □

The following assertion shows that the constrained solutions of ordinary fuzzy relational equations are just solutions of (13)–(15).

Theorem 5. *Let $K = \mathrm{fix}(*)$ for $*$ being an intensifying or relaxing hedge $*$.*

(i) *An ordinary equation $U \circ S = T$ has a K-solution if and only if $U \circ^* S = T$ has a solution. In particular, if U is a K-solution of $U \circ S = T$ then U is a solution of $U \circ^* S = T$. If U is a solution of $U \circ^* S = T$ then U^* is a K-solution of $U \circ S = T$.*
(ii) *The same is true for $U \lhd S = T$ and $U \lhd^* S = T$.*
(iii) *The same is true for $R \lhd U = T$ and $R \lhd_* U = T$.*

Proof. Immediately from Lemma 4, observing that $U \circ S = T$ coincides with $U \circ^* S = T$ for $*$ being identity. □

In view of Theorem 5, to obtain results regarding constrained solutions of ordinary fuzzy relational equations, it is sufficient to provide results regarding solutions of equations (13)–(15). This is the subject of the remainder of this section. We first consider solvability of single equations.

Theorem 6. *Let $*$ be an intensifying hedge. Equation $U \circ^* S = T$ has a solution iff $(S \lhd T^{-1})^{-1}$ is its solution.*

Proof. Since $U \circ^* S = U^* \circ S$, if U is a solution then $U^* \subseteq (S \lhd T^{-1})^{-1}$ (use adjointness and standard manipulation). Due to (3) and (4), $U^* \subseteq (S \lhd T^{-1})^{-1}$ implies $U^* \subseteq ((S \lhd T^{-1})^{-1})^*$. Therefore, using (2),

$$T = U \circ^* S = U^* \circ S \subseteq ((S \lhd T^{-1})^{-1})^* \circ S \subseteq (S \lhd T^{-1})^{-1} \circ S \subseteq T,$$

which shows that $(S \lhd T^{-1})^{-1} \circ^* S = T$. □

Theorems 5 and 6 yield the following corollary.

Corollary 7. *Let $U \circ S = T$ have a* $\mathrm{fix}(*)$*-solution for $*$ being an intensifying hedge. Then $((S \lhd T^{-1})^{-1})^*$ is the greatest* $\mathrm{fix}(*)$*-solution of $U \circ S = T$.*

Example 2. Let $*$ be globalization and consider the Łukasiewicz operations on $L = [0, 1]$. Let S and T be represented by matrices $\binom{0.4}{0.8}$ and $\binom{0.5}{0.3}$. One can check (e.g. using Theorem 6 for $*$ being identity) that $U \circ S = T$ has a solution. One such solution is $(S \lhd T^{-1})^{-1}$ whose matrix is $\left(\begin{smallmatrix} 1 & 0.7 \\ 0.9 & 0.5 \end{smallmatrix}\right)$, i.e.

$$\begin{pmatrix} 1 & 0.7 \\ 0.9 & 0.5 \end{pmatrix} \circ \begin{pmatrix} 0.4 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}.$$

On the other hand, there is no {0, 1}-solution (i.e. crisp solution) of $U \circ S = T$. This follows from Theorems 5 and 6 by checking that $(S \lhd T^{-1})^{-1}$ is not a solution of $U \circ^* S = T$. That is, there is no Boolean matrix B for which

$$B \circ \begin{pmatrix} 0.4 \\ 0.8 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}.$$

Theorem 8. *Let $*$ be an intensifying hedge. Equation $U \lhd^* S = T$ has a solution iff $T \lhd S^{-1}$ is its solution.*

Proof. Since $U \lhd^* S = U^* \lhd S$, one can see that if U is a solution then $U^* \subseteq T \lhd S^{-1}$ (use adjointness and standard manipulation). Due to (3) and (4), $U^* \subseteq T \lhd S^{-1}$ implies $U^* \subseteq (T \lhd S^{-1})^*$. Therefore, using (2),

$$T = U \lhd^* S = U^* \lhd S \supseteq (T \lhd S^{-1})^* \lhd S \supseteq (T \lhd S^{-1}) \lhd S = T,$$

which shows that $(T \lhd S^{-1}) \lhd^* S = T$. □

Corollary 9. *Let $U \lhd S = T$ have a* fix($*$)*-solution for $*$ being an intensifying hedge. Then $(T \lhd S^{-1})^*$ is the greatest* fix($*$)*-solution of $U \lhd S = T$.*

Example 3. Consider the Łukasiewicz operations on $L = \{0, 0.25, 0.5, 0.75, 1\}$. Let S and T be represented by matrices

$$\begin{pmatrix} 0.75 & 0.25 & 0 \\ 1 & 0.75 & 1 \\ 0 & 0.5 & 0.5 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.75 & 0.25 & 0 \end{pmatrix}.$$

Suppose we want to find a {0, 0.5, 1}-solution of $U \lhd S = T$. The intensifying hedge $*_2$ from Example 1 satisfies $\{0, 0.5, 1\} = \mathrm{fix}(*_2)$. One can check using Theorem 8 that $U \lhd^{*_2} S = T$ has a solution. Namely, one such solution is $T \lhd S^{-1}$. According to Theorem 5, $(T \lhd S^{-1})^{*_2}$, whose matrix is

$$\begin{pmatrix} 0.5 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

is a {0, 0.5, 1}-solution of $U \lhd S = T$.

Theorem 10. *Let $*$ be a relaxing hedge. Equation $R \lhd_* U = T$ has a solution iff $R^{-1} \circ T$ is its solution.*

Proof. Since $R \lhd_* U = R \lhd U^*$, we see that if U is a solution then $R^{-1} \circ T \subseteq U^*$ (again, use adjointness and standard manipulation). Due to (3) and (4), $R^{-1} \circ T \subseteq U^*$ implies $(R^{-1} \circ T)^* \subseteq U^*$. Therefore, using (6),

$$T \subseteq R \lhd (R^{-1} \circ T) \subseteq R \lhd (R^{-1} \circ T)^* \subseteq R \lhd U^* = R \lhd_* U = T,$$

which shows that $R \lhd_* (R^{-1} \circ T) = T$. □

Corollary 11. *Let $R \lhd U = T$ have a* fix($*$)*-solution for $*$ being an relaxing hedge. Then $(R^{-1} \circ T)^*$ is the least* fix($*$)*-solution of $R \lhd U = T$.*

Example 4. Consider the Łukasiewicz operations on $L = \{0, 0.25, 0.5, 0.75, 1\}$. Let R and T be represented by matrices

$$\begin{pmatrix} 0.5 & 1 \\ 0.5 & 0.25 \end{pmatrix}, \quad \begin{pmatrix} 0.5 & 0.75 \\ 0.5 & 1 \end{pmatrix}.$$

Suppose we want to find a {0, 0.75, 1}-solution of $R \lhd U = T$. Therefore, we need a relaxing hedge $*$ which maps both 0.25 and 0.5 to 0.75 and maps the other degrees to themselves. Such hedge is $*_3$ from Fig. 1 (right). One can check using Theorem 10 that $R \lhd_* U = T$ has a solution. Namely, one such solution is $R^{-1} \circ T$. According to Theorem 5, $(R^{-1} \circ T)^*$, whose matrix is

$$\begin{pmatrix} 0 & 0.75 \\ 0.75 & 1 \end{pmatrix},$$

is a $\{0, 0.75, 1\}$-solution of $R \lhd U = T$.

Next, we describe solutions of systems of fuzzy relational equations (13)–(15). To this end, we use the following two properties of hedges:

$$\left(\bigwedge_{i\in I} a_i\right)^* \leq \bigwedge_{i\in I} a_i^*, \tag{16}$$

$$\left(\bigvee_{i\in I} a_i\right)^* \geq \bigvee_{i\in I} a_i^*. \tag{17}$$

One can easily see that both of them are direct consequences of (3) and hence are satisfied by intensifying as well as relaxing hedges.

Theorem 12. *Let $*$ be an intensifying hedge. A system $\mathcal{E} = \{U \circ^* S_j = T_j \mid j \in J\}$ of equations has a solution iff $\bigcap_{j\in J}(S_j \lhd T_j^{-1})^{-1}$ is a solution of $\mathcal{E}$.*

Proof. If U is a solution of $\mathcal{E}$ then $U \circ^* S_j = T_j$ for every $j \in J$. By Theorem 6, $(S_j \lhd T_j^{-1})^{-1}$ is a solution of the j-th equation of system $\mathcal{E}$. Moreover, from the proof of Theorem 6 we get that $U^* \subseteq (S_j \lhd T_j^{-1})^{-1}$ for all $j \in J$, therefore $U^* \subseteq \bigcap_{j\in J}(S_j \lhd T_j^{-1})^{-1}$. Due to (3), (4) and using (16), $U^* \subseteq \bigcap_{j\in J}(S_j \lhd T_j^{-1})^{-1}$ implies $U^* \subseteq \bigcap_{j\in J}((S_j \lhd T_j^{-1})^{-1})^*$. Therefore,

$$T_j = U \circ^* S_j = U^* \circ S_j \subseteq \bigcap_{i\in J}((S_i \lhd T_i^{-1})^{-1})^* \circ S_j$$

$$\subseteq \bigcap_{i\in J}(S_i \lhd T_i^{-1})^{-1} \circ S_j \subseteq (S_j \lhd T_j^{-1})^{-1} \circ S_j = T_j.$$

So $\bigcap_{i\in J}(S_i \lhd T_i^{-1})^{-1} \circ^* S_j = T_j$ for every $j \in J$. □

Corollary 13. *Let $\mathcal{E} = \{U \circ S_j = T_j \mid j \in J\}$ have a* fix($*$)*-solution for $*$ being an intensifying hedge. Then $(\bigcap_{j\in J}(S_j \lhd T_j^{-1})^{-1})^*$ is the greatest* fix($*$)*-solution of $\mathcal{E}$.*

Theorem 14. *Let $*$ be an intensifying hedge. A system $\mathcal{E} = \{U \lhd^* S_j = T_j \mid j \in J\}$ of equations has a solution iff $\bigcap_{j\in J}(T_j \lhd S_j^{-1})$ is a solution of $\mathcal{E}$.*

Proof. If U is a solution of $\mathcal{E}$ then $U \lhd^* S_j = T_j$ for every $j \in J$. By Theorem 8, $T_j \lhd S_j^{-1}$ is a solution of the j-th equation of system $\mathcal{E}$. Moreover, from the proof of Theorem 8 we get that $U^* \subseteq T_j \lhd S_j^{-1}$ for all $j \in J$, therefore $U^* \subseteq \bigcap_{j\in J}(T_j \lhd S_j^{-1})$. Due to (3), (4) and using (16), $U^* \subseteq \bigcap_{j\in J}(T_j \lhd S_j^{-1})$ implies $U^* \subseteq \bigcap_{j\in J}(T_j \lhd S_j^{-1})^*$. Therefore,

$$T_j = U \lhd^* S_j = U^* \lhd S_j \supseteq \bigcap_{i\in J}(T_i \lhd S_i^{-1})^* \lhd S_j$$

$$\supseteq \bigcap_{i\in J}(T_i \lhd S_i^{-1}) \lhd S_j \supseteq (T_j \lhd S_j^{-1}) \lhd S_j = T_j,$$

showing that $\bigcap_{i\in J}(T_i \lhd S_i^{-1}) \lhd^* S_j = T_j$ for every $j \in J$.

Corollary 15. *Let* $\mathcal{E} = \{U \triangleleft S_j = T_j \mid j \in J\}$ *have a* fix($*$)*-solution for* $*$ *being an intensifying hedge. Then* $(\bigcap_{j \in J}(T_j \triangleleft S_j^{-1}))^*$ *is the greatest* fix($*$)*-solution of* $\mathcal{E}$.

Theorem 16. *Let* $*$ *be a relaxing hedge. A system* $\mathcal{E} = \{R_j \triangleleft_* U = T_j \mid j \in J\}$ *of equations has a solution iff* $\bigcup_{j \in J}(R_j^{-1} \circ T_j)$ *is a solution of* $\mathcal{E}$.

Proof. If U is a solution of $\mathcal{E}$ then $R_j \triangleleft_* U = T$ for every $j \in J$. By Theorem 10, $R_j^{-1} \circ T_j$ is a solution of the j-th equation of system $\mathcal{E}$. Moreover, from the proof of Theorem 10 we get that $R_j^{-1} \circ T_j \subseteq U^*$ for all $j \in J$, therefore $\bigcup_{j \in J}(R_j^{-1} \circ T_j) \subseteq U^*$. Due to (3), (4) and using (17), $\bigcup_{j \in J}(R_j^{-1} \circ T_j) \subseteq U^*$ implies $\bigcup_{j \in J}(R_j^{-1} \circ T_j)^* \subseteq U^*$. Therefore,

$$T_j = R_j \triangleleft_* U = R_j \triangleleft U^* \supseteq R_j \triangleleft \bigcup_{j \in J}(R_j^{-1} \circ T_j)^*$$

$$\supseteq R_j \triangleleft \bigcup_{j \in J}(R_j^{-1} \circ T_j) \supseteq R_j \triangleleft (R_j^{-1} \circ T_j) = T_j,$$

showing that $R_j \triangleleft_* \bigcup_{j \in J}(R_j^{-1} \circ T_j) = T_j$ for every $j \in J$. □

Corollary 17. *Let* $\mathcal{E} = \{R_j \triangleleft U = T_j \mid j \in J\}$ *have a* fix($*$)*-solution for* $*$ *being a relaxing hedge. Then* $(\bigcup_{j \in J}(R_j^{-1} \circ T_j))^*$ *is the least* fix($*$)*-solution of* $\mathcal{E}$.

3. Illustrative example

The example presented in this section demonstrates the results from the previous section as well as the usefulness of constrained solutions of fuzzy relational equations.

Consider a supply chain problem with m producers and k haulers. Each producer uses services of at least one hauler to deliver its goods to customers. Therefore, there is a relationship between the m producers and k haulers that can be represented by an $m \times k$ binary matrix U with $U_{ij} = 1$ (or $= 0$) meaning that producer i uses (or does not use) services of hauler j. By its nature, such a relationship is bivalent but for outsiders of the supply chain (e.g. a competing delivery company) it may be unknown. If a new delivery company wants to offer its services to producers and compete with established haulers, it may be useful to reveal information about U using market and customer quality surveys. This example shows an approach to this problem using fuzzy relational equations.

The quality of services of a producer is considerably affected by the quality of services of its haulers. Namely, if a hauler is a source of common issues (e.g. delay of articles) then the producer is perceived as unreliable (e.g. slow) by its customers. The information about quality of services of haulers and producers can be obtained by customer surveys and described by a $k \times n$ graded matrix S and an $m \times n$ graded matrix T, respectively, where n is the number of attributes with negative effect in a supply chain (e.g. delayed packaged or high price) which are monitored by the survey. Let us simplify the situation by excluding other factors and assuming that the quality of results of a producer depends solely on the quality of services provided by the haulers (such an assumption is natural for a new competing delivery company). The relationship between the matrices may then be described by $U \circ S = T$. If S and T are obtained by the survey, one may want to obtain U as a binary solution of this relational equation.

Imagine the following scenario. A new competing delivery company wants to address producers with an offer of business cooperation. The goal is to address producers to whom the company may bring benefit and thus increase income. The knowledge about a weak link of particular supply chain gives the new company a sound argument for convincing producers to accept its services. Consider producers $p_1, \dots, p_5$ and haulers $h_1, \dots, h_4$ and attributes $y_1, \dots, y_6$ that are being surveyed. The attributes have the following meaning:

y_1 … delayed packages,
y_2 … high price,

y_3 ... difficult claim process,
y_4 ... damaged packages,
y_5 ... mismatched packages,
y_6 ... impossibility of online tracking.

Attributes $y_1, \ldots, y_5$ are graded, whereas y_6 is binary. Suppose matrix S is the following one:

$$S = \begin{pmatrix} 0.50 & 0.00 & 0.25 & 0.00 & 0.25 & 0.00 \\ 0.25 & 0.75 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.75 & 1.00 & 0.00 & 0.25 & 1.00 \\ 0.75 & 0.25 & 0.00 & 0.75 & 0.00 & 0.00 \end{pmatrix}.$$

Notice that from S we can read that, for instance, hauler h_2 (second row) is rather expensive (entry 0.75 in the second column), the packages are sometimes delayed but its other services are perfect. Suppose matrix T is the following one:

$$T = \begin{pmatrix} 0.50 & 0.75 & 0.25 & 0.00 & 0.25 & 0.00 \\ 0.25 & 0.75 & 1.00 & 0.00 & 0.25 & 1.00 \\ 0.75 & 0.75 & 0.00 & 0.75 & 0.00 & 0.00 \\ 0.50 & 0.75 & 1.00 & 0.00 & 0.25 & 1.00 \\ 0.75 & 0.75 & 1.00 & 0.75 & 0.25 & 1.00 \end{pmatrix}.$$

According to T, producer p_5 (the last row) provides bad services to its customers except for y_5.

In order to find a binary solution of the equation $U \circ S = T$ we need to solve $U \circ^* S = T$ for $*$ being globalization. The equation $U \circ^* S = T$ is solvable and one of its solutions is

$$U = (S \triangleleft T^{-1})^{-1} = \begin{pmatrix} 1.00 & 1.00 & 0.00 & 0.25 \\ 0.75 & 1.00 & 1.00 & 0.25 \\ 0.75 & 1.00 & 0.00 & 1.00 \\ 1.00 & 1.00 & 1.00 & 0.25 \\ 1.00 & 1.00 & 1.00 & 1.00 \end{pmatrix}.$$

According to the results from the previous section, a binary solution of our problem is therefore given by the following matrix:

$$U^* = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

From this result one can read that p_5 uses haulers h_3 and h_4 to deliver goods to customers and the like. Suppose that a new delivery company h_5 wants to estimate its value compared to established companies $h_1, \ldots, h_4$. This can be done provided an assessment of h_5 by means of the attributes $y_1, \ldots, y_6$ is available (it can be obtained by a customer survey if it is already in operation, or by an expert evaluation of its business strategy). Suppose for example that the values of the attributes for h_5 are given by the following vector:

$$(0.75\ 0.75\ 0.25\ 0.00\ 0.25\ 0.00).$$

Then, if producer p_5 switches to haulers h_4 and h_5 instead of using h_3 and h_4, the switch improves its performance on y_3 and y_5 from degrees 1 to 0.25 and from 1 to 0, respectively, the performance on other attributes remaining the same. This can be seen from the following computation:

$$(0\ 0\ 0\ 1\ 1) \circ \begin{pmatrix} 0.50 & 0.00 & 0.25 & 0.00 & 0.25 & 0.00 \\ 0.25 & 0.75 & 0.00 & 0.00 & 0.00 & 0.00 \\ 0.00 & 0.75 & 1.00 & 0.00 & 0.25 & 1.00 \\ 0.75 & 0.25 & 0.00 & 0.75 & 0.00 & 0.00 \\ 0.75 & 0.75 & 0.25 & 0.00 & 0.25 & 0.00 \end{pmatrix}$$

$$= (0.75\ 0.75\ \mathbf{0.25}\ 0.75\ 0.25\ \mathbf{0.00}).$$

The new company can use this information to persuade p_5 to accept its services. This example, however simple, shows that constrained solutions of fuzzy relations equations can help reveal interesting information.

4. Conclusions

We presented an approach to obtain criteria regarding constrained solutions of fuzzy relational equations. The approach consists in modifying the definitions of composition of fuzzy relations by inserting (intensifying or relaxing) linguistic hedges and interpreting the hedges by appropriate truth functions that represent the constraints. Further issues regarding constrained solutions, in particular the structure of constrained solutions and approximate constrained solutions, are topics for future research.

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References

[1] M. Baaz, Infinite-valued Gödel logics with 0–1 projections and relativizations, GÖDEL ’96—Logical Foundations of Mathematics, Computer Sciences and Physics, Lecture Notes in Logic, vol. 6, Springer-Verlag, 1996, pp. 23–33.
[2] W. Bandler, L.J. Kohout, Semantics of implication operators and fuzzy relational products, Int. J. Man-Mach. Stud. 12 (1980) 89–116.
[3] W. Bandler, L.J. Kohout, Fuzzy relational products as a tool for analysis and synthesis of the behaviour of complex natural and artificial systems, in: P.P. Wang, S.K. Chang (Eds.), Fuzzy Sets: Theory and Applications to Policy Analysis and Information Systems, Plenum Press, New York, 1980, pp. 341–367.
[4] R. Belohlavek, T. Funioková, V. Vychodil, Fuzzy closure operators with truth stressers, Logic J. IGPL 13 (5) (2005) 503–513.
[5] R. Belohlavek, V. Vychodil, Reducing the size of fuzzy concept lattices by hedges, in: Proceedings of FUZZ-IEEE, Reno, Nevada, 2005, pp. 663–668.
[6] R. Belohlavek, V. Vychodil, Attribute implications in a fuzzy setting, in: R. Missaoui, J. Schmid (Eds.), Proceedings of ICFCA 2006, Lecture Notes in Artificial Intelligence, vol. 3874, 2006, pp. 45–60.
[7] R. Belohlavek, V. Vychodil, Fuzzy concept lattices constrained by hedges, J. Adv. Comput. Intell. Intell. Inf. 11 (6) (2007) 536–545.
[8] B. De Baets, Analytical solution methods for fuzzy relational equations, in: D. Dubois, H. Prade (Eds.), The Handbook of Fuzzy Set Series, vol. 1, Academic Kluwer Publications, Boston, 2000, pp. 291–340.
[9] A. Di Nola, E. Sanchez, W. Pedrycz, S. Sessa, Fuzzy Relation Equations and Their Applications to Knowledge Engineering, Kluwer, 1989.
[10] S. Gottwald, Fuzzy Sets and Fuzzy Logic. Foundations of Applications—From a Mathematical Point of View, Vieweg, Wiesbaden, 1993.
[11] S. Gottwald, A Treatise on Many-Valued Logics, Research Studies Press, Baldock, Hertfordshire, England, 2001.
[12] S. Gottwald, Calculi of information granules: fuzzy relational equations, in: W. Pedrycz, A. Skowron, V. Kreinovich (Eds.), Handbook of Granular Computing, John Wiley & Sons, Ltd, Chichester, UK, 2002doi: 10.1002/9780470724163.ch11.
[13] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer, Dordrecht, 1998.
[14] P. Hájek, On very true, Fuzzy Sets Syst. 124 (2001) 329–333.
[15] K. Hirota, W. Pedrycz, Solving fuzzy relational equations through logical filtering, Fuzzy Sets Syst. 81 (1996) 355–363.
[16] K. Hirota, W. Pedrycz, Specificity shift in solving fuzzy relational equations, Fuzzy Sets Syst. 106 (1999) 211–220.
[17] G.J. Klir, B. Yuan, Fuzzy Sets and Fuzzy Logic. Theory and Applications, Prentice-Hall, 1995.
[18] E. Sanchez, Resolution of composite fuzzy relation equations, Inf. Control 30 (1) (1976) 38–48.
[19] G. Takeuti, S. Titani, Globalization of intuitionistic set theory, Ann. Pure Appl. Logic 33 (1987) 195–211.
[20] V. Vychodil, Truth-depressing hedges and BL-logic, Fuzzy Sets Syst. 157 (2006) 2074–2090.
[21] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning—I, Inf. Sci. 8 (3) (1975) 199–249.