Knowledge Spaces, Attribute Dependencies, and Graded Knowledge States

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***Abstract*— The present paper deals with dependencies developed within the theory of knowledge spaces. Knowledge spaces represent a new paradigm in psychological approaches to assessment of knowledge. A distinguishing feature of knowledge spaces is their non-numerical character. The aim of the present paper is twofold. First, we bring up several remarks on data dependencies studied within knowledge spaces. Second, we consider the dependencies in a framework which is more general than that of classical knowledge spaces. Namely, we abandon the assumption that a knowledge state is a set of problems/questions which an individual is able to solve. Instead, we assume that a knowledge state is a graded set (fuzzy set) of problems. Our assumption accounts for situations where it is possible that an individual can solve a particular problem partially, rather than just "can solve" or "cannot solve". We propose a definition of dependencies and validity of dependencies in knowledge spaces with graded knowledge states, provide selected properties of the dependencies, and a lemma which serves as a bridge to existing results on so-called fuzzy attribute implications.**

I. Problem Setting

The aim of the present paper is twofold. First, we present several remarks on data dependencies studied within the framework of knowledge spaces. Second, we consider the dependencies in a framework which is more general than the framework of classical knowledge spaces. Namely, we abandon the classical assumption that a knowledge state is a set of problems/questions which an individual is able to solve. Rather, we assume that a knowledge state is a graded set (fuzzy set) of problems. Our assumption accounts for situations where it is possible that an individual can solve a particular problem partially, rather than just "can solve" or "cannot solve" (consider for instance the problem "can understand spoken English" or any problem where the judgment of whether an individual can solve the problem comes naturally in degrees rather than in an yes-or-no way).

Knowledge spaces [17] represent a new paradigm in mathematical psychology, namely, a psychological approach to assessment of knowledge. A distinguishing feature of knowledge spaces is their non-numerical character. Mathematical foundations of knowledge spaces are in the spirit of discrete mathematics, ordered sets, and combinatorics, which makes knowledge spaces considerably different from classical numerical approaches. Note also that knowledge spaces have been used in several real-world psychological studies and that contributions to knowledge spaces appeared in primary psychological journals, see e.g. [18].

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An important role in knowledge spaces is played by dependencies

$$A \Rightarrow B, \tag{1}$$

where A and B are sets of problems/questions, i.e. $A, B \subseteq Y$, where Y is the universe of all problems/questions being considered. In knowledge spaces, a dependency $A \Rightarrow B$ has the following meaning:

$$\begin{array}{c}\textit{if an individual fails on all problems from } A\\ \textit{then he/she fails on all problems from } B.\end{array} \tag{2}$$

Dependencies (1) and reasoning about these dependencies have been studied in several areas of computer science. In databases, these dependencies are called functional dependencies, see e.g. [26]. In formal concept analysis, $A \Rightarrow B$'s are called attribute implications, see e.g. [19]. In data mining, these dependencies are called association rules, see e.g. [29]. Surprisingly, no connections to functional dependencies, attribute implications, or association rules are indicated in [17]. To point out the basic relationship between dependencies (1) and the above-mentioned dependencies, and to consider dependencies (1) in the above-sketched framework which allows to judge the capability to solve problems by degrees (not only yes or no), is the main aim of the present paper.

Section II contains preliminaries from fuzzy sets. Section III presents our results, remarks, and examples. Conclusions and outline of future research are summarized in Section IV.

II. Preliminaries

We use sets of truth degrees equipped with operations (logical connectives) which form a so-called complete residuated lattice with a truth-stressing hedge. A complete residuated lattice with truth-stressing hedge (shortly, a hedge) [24], [25] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{3}$$

for each $a, b, c \in L$; hedge * satisfies

$$1^* = 1, \tag{4}$$

$$a^* \leq a, \tag{5}$$

$$(a \rightarrow b)^* \leq a^* \rightarrow b^*, \tag{6}$$

$$a^{**} = a^*, \tag{7}$$

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for each $a, b \in L$, $a_i \in L$ ($i \in I$). Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true", see [24], [25]. Properties (5)–(7) have natural interpretations, e.g. (5) can be read: "if formula φ is very true, then φ is true", (6) can be read: "if it is very true that φ implies ψ and if φ is very true, then ψ is very true", etc. Note that hedges other than truth-stressing ones like "at least a little bit true" have different properties and are not considered in our paper.

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are:

$$\text{Łukasiewicz:} \quad \begin{aligned} a \otimes b &= \max(a + b - 1, 0), \\ a \rightarrow b &= \min(1 - a + b, 1), \end{aligned} \tag{8}$$

$$\text{Gödel:} \quad \begin{aligned} a \otimes b &= \min(a, b), \\ a \rightarrow b &= \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases} \end{aligned} \tag{9}$$

$$\text{Goguen (product):} \quad \begin{aligned} a \otimes b &= a \cdot b, \\ a \rightarrow b &= \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases} \end{aligned} \tag{10}$$

In applications, we often need a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of L and restrictions of Gödel operations on $[0, 1]$ to L.

Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^* = a$ ($a \in L$); (ii) globalization [28]:

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

A special case of the complete residuated lattice with hedge is the two-element Boolean algebra $\langle \{0, 1\}, \wedge, \vee, \otimes, \rightarrow, ^*, 0, 1 \rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge, \vee, \otimes, \rightarrow$ of $\mathbf{2}$ are the truth functions (interpretations) of the corresponding logical connectives of the classical logic and $0^* = 0$, $1^* = 1$. Note that if we prove an assertion for general $\mathbf{L}$, then, in particular, we obtain a "crisp version" of this assertion for $\mathbf{L}$ being $\mathbf{2}$.

Having $\mathbf{L}$, we define usual notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A : U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". If $U = \{u_1, \ldots, u_n\}$ then A can be denoted by $A = \{^{a_1}/u_1, \ldots, ^{a_n}/u_n\}$ meaning that $A(u_i)$ equals a_i for each $i = 1, \ldots, n$. Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. $\mathbf{2}$-sets (operations with $\mathbf{2}$-sets) can be identified with the ordinary (crisp) sets (operations with ordinary sets) of the naive set theory. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$.

Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A, B) = \textstyle\bigwedge_{u \in U} \bigl(A(u) \rightarrow B(u)\bigr), \tag{12}$$

which generalizes the classical subsethood relation $\subseteq$ (note that unlike $\subseteq$, S is a binary $\mathbf{L}$-relation on $\mathbf{L}^U$). Described verbally, $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$.

In the following we use well-known properties of residuated lattices and fuzzy sets over residuated lattices which can be found, e.g., in [5], [24].

III. Knowledge Spaces, Dependencies, and Graded States

A. Ordinary Knowledge Spaces and Attribute Dependencies

We first recall basic notions regarding knowledge spaces. Let Y be a non-empty set (of problems/questions). A knowledge structure on Y is any family $\mathcal{K}$ of subsets of Y containing $\emptyset$ and Y. Elements $K \in \mathcal{K}$, i.e. subsets of Y, are called knowledge states of $\mathcal{K}$. Sometimes, a term knowledge state just refers to a subset of Y. A knowledge space on Y is a knowledge structure $\mathcal{K}$ on Y which is an interior system in Y. Note that the fact that $\mathcal{K}$ is an interior system in Y means that $\mathcal{K}$ is closed under arbitrary unions, i.e. we have $\bigcup_{j \in J} K_j \in \mathcal{K}$ for any family $\{K_j \in \mathcal{K} \,|\, j \in J\}$.

As mentioned above, elements $y \in Y$ represent problems/questions. A knowledge state $K \in \mathcal{K}$ represents a state of knowledge of an individual who is capable of solving all problems from K.

Consider now an expression $A \Rightarrow B$ where $A, B \subseteq Y$ are sets of problems. The following notion of validity of $A \Rightarrow B$ is used in knowledge spaces: For a subset M of Y, we say that $A \Rightarrow B$ is true in M (or, M satisfies $A \Rightarrow B$) iff

$$A \cap M = \emptyset \text{ implies } B \cap M = \emptyset. \tag{13}$$

The basic meaning is the following. Let M represent a knowledge state of some individual, i.e., M is just the set of all problems from Y the individual can solve. One can easily see that (13) is the meaning of (2), i.e., (13) says that if the individual fails on all problems from A, then he/she fails on all problems from B. In this sense, the individual, or, equivalently, the knowledge state M, satisfies a dependency expressed by $A \Rightarrow B$. Expressions $A \Rightarrow B$ will play the role of formulas in our treatment, subsets M of Y will play the role of semantical structures. Let T be a set of formulas (1). By $\mathrm{Mod}^\circ(T)$ we denote the set of all models of T, i.e.,

$$\begin{aligned} \mathrm{Mod}^\circ(T) \;=\; \{M \subseteq Y \,|\, &\text{for each } A \Rightarrow B \in T : \\ &A \cap M = \emptyset \text{ implies } B \cap M = \emptyset\}. \end{aligned}$$

That is, $\mathrm{Mod}^\circ(T)$ is the set of all subsets M of Y such that M satisfies each $A \Rightarrow B \in T$ in the sense of (13).

Remark 1: Note that our notation as well as terminology regarding dependencies (1) differs from that of [17]. The reason is that we want to be compatible with standard logical notation and terminology.

For an example of pointing out connections between results on dependencies (1) developed within knowledge spaces and other results on dependencies (1), we will use the following theorem which was proven in [17].

Theorem 1: Let T be a set of formulas (1) such that $A \neq \emptyset$ and $B \neq \emptyset$. Then $\mathrm{Mod}^{\circ}(T)$ is a knowledge space on Y.

Remark 2: Note first that the requirement $A \neq \emptyset$ and $B \neq \emptyset$ ensures that $Y \in \mathrm{Mod}^{\circ}(T)$. However, $Y \in \mathrm{Mod}^{\circ}(T)$ follows even from a weaker assumption, namely from the one saying that T does not contain formula of form $\emptyset \Rightarrow B$ with $B \neq \emptyset$. Moreover, one can easily see that $Y \in \mathrm{Mod}^{\circ}(T)$ iff T does not contain formula $\emptyset \Rightarrow B$ with $B \neq \emptyset$. Therefore, the conclusion of Theorem 1 can be inferred from weaker assumptions than those presented in [17].

Note also that the fact that $\mathrm{Mod}^{\circ}(T)$ is closed under arbitrary unions holds true for arbitrary T, i.e., with no restrictions on $A \Rightarrow B$.

We are now going to show that Theorem 1 is an easy consequence of results on attribute implications and well-known results on the relationships between closure systems and interior systems. As mentioned above, formulas (1) are called attribute implications in formal concept analysis, see [19]. For $M \subseteq Y$, an attribute implication is considered true in M if

$$A \subseteq M \text{ implies } B \subseteq M. \tag{14}$$

Denote now for a set T of attribute implications

$$\begin{aligned}\mathrm{Mod}^{\bullet}(T) \quad = \quad & \{M \subseteq Y \mid \text{for each } A \Rightarrow B \in T: \\ & A \subseteq M \text{ implies } B \subseteq M\}.\end{aligned}$$

That is, $\mathrm{Mod}^{\bullet}(T)$ is the set of all subsets of Y such that each $A \Rightarrow B \in T$ is true in M in the sense of (14). The following theorem is well known, see e.g. [19]:

Theorem 2: For any T, $\mathrm{Mod}^{\bullet}(T)$ is a closure system in Y.

That is, $\mathrm{Mod}^{\bullet}(T)$ is closed under arbitrary intersections. Now, as one can easily see, $A \Rightarrow B$ is true in M according to (13) iff $A \Rightarrow B$ is true in $\overline{M}$ according to (14), where $\overline{M} = Y - M$ is the complement of M w.r.t. Y. Therefore, we have

$$M \in \mathrm{Mod}^{\circ}(T) \text{ iff } \overline{M} \in \mathrm{Mod}^{\bullet}(T),$$

i.e. $\mathrm{Mod}^{\bullet}(T)$ contains just the complements of sets from $\mathrm{Mod}^{\circ}(T)$. Due to well-known relationships between closure systems and interior systems, one gets from Theorem 2 that $\mathrm{Mod}^{\circ}(T)$ is an interior system in Y for any T. Namely, it is well known that complements of sets of any closure system form an interior system (and *vice versa*). Moreover, it is easy to see that if T does not contain dependencies $\emptyset \Rightarrow B$ with $B \neq \emptyset$ then $\mathrm{Mod}^{\bullet}(T)$ contains $\emptyset$ and, therefore, $\mathrm{Mod}^{\circ}(T)$ contains Y. Since $\mathrm{Mod}^{\circ}(T)$ always contains $\emptyset$, it follows that $\mathrm{Mod}^{\circ}(T)$ is a knowledge space. This way, one obtains Theorem 1 as a consequence of known results.

Another connection to mention here is the one between $A \Rightarrow B$'s and the recently developed AD-formulas, see e.g. [14]. Namely, $A \Rightarrow B$ s true in M in the sense of given by (13) iff AD-formula $B \sqsubseteq A$ is true in M in the sense of validity of AD-formulas.

Due to restricted scope of this paper, we stop our excursion on the relationship between the semantics of dependencies (1) as used in knowledge spaces and other well-established types of semantics of (1) and leave further results to an upcoming paper.

B. Graded Knowledge States

Let us now consider the concept of a knowledge state from the point of view of a graded approach (fuzzy approach). In the ordinary case, a given problem y either belongs to a given state K or not, indicating that an individual possessing a knowledge state K can or can not solve y, respectively. It might be, however the case that an individual can solve a problem, such as "can understand spoked English" or "can stop a car smoothly", only partially. In this case, it seems more appropriate to consider a knowledge state as a fuzzy set $K \in \mathbf{L}^Y$ of problems with $K(y) \in L$ being interpreted as a degree to which an individual can solve the problem. $K(y) = 1$ and $K(y) = 0$ means that the individual can solve the problem completely and that the individual can not solve the problem at all. This idea leads to the concept of a knowledge space with graded knowledge states which will be presented elsewhere [2].

Obviously, due to the crucial role of dependencies (1) in ordinary knowledge spaces, appropriate dependencies need to be considered in our generalized framework. Our aim in the rest of this paper is to look at this problem.

C. Fuzzy Attribute Implications

In this section, we recall basic concepts related to fuzzy attribute implications which will be needed in the sequel. Due to the limited scope, we only provide definitions and refer the reader to [10]–[13] for details. Recall that a fuzzy attribute implication (FAI) over Y is an expression $A \Rightarrow B$ with both A and B being fuzzy sets in Y, i.e., $A, B \in \mathbf{L}^Y$. For a fuzzy set $M \in \mathbf{L}^Y$, a degree $||A \Rightarrow B||_M$ to which $A \Rightarrow B$ is true in M is defined by

$$||A \Rightarrow B||_M = S(A, M)^* \rightarrow S(B, M). \tag{15}$$

Here, $*$ is a hedge which controls the meaning of $A \Rightarrow B$. Occasionally, we write $||A \Rightarrow B||_M^{\bullet}$ to to distinguish (15) from a different definition of validity of $A \Rightarrow B$ to be introduced later.

For a set T of FAIs, we define a set $\mathrm{Mod}(T)$ of all models of T by

$$\begin{aligned}\mathrm{Mod}(T) \quad = \quad & \{M \in \mathbf{L}^Y \mid ||A \Rightarrow B||_M = 1 \\ & \text{for every } A \Rightarrow B \in T\}.\end{aligned}$$

Again, we occasionally write $\mathrm{Mod}^{\bullet}(T)$.

D. Interior-Based Validity of Fuzzy Attribute Implications

We propose to use FAIs as our dependencies for graded knowledge states. Our choice is motivated by many existing results on fuzzy attribute implications and related topics, see e.g. [10]–[13], and by the close relationship between dependencies (1) and attribute implications in the ordinary case.

For a graded knowledge state $M \in \mathbf{L}^Y$ and a fuzzy attribute implication $A \Rightarrow B$, we define a degree $||A \Rightarrow B||^\circ_M$ to which $A \Rightarrow B$ is true in M by

$$||A \Rightarrow B||^\circ_M = S(A \otimes M, \emptyset)^* \rightarrow S(B \otimes M, \emptyset).$$

Remark 3: (1) $A \otimes M$ is a $\otimes$-based intersection of A and M, i.e., we have $(A \otimes M)(y) = A(y) \otimes M(y)$ for each $y \in Y$; the same for $B \otimes M$.

(2) $S(A \otimes M, \emptyset)$ is a degree to which $A \otimes M$ is a subset of $\emptyset$ (empty fuzzy set), which is equal to the degree to which $A \otimes M$ equals $\emptyset$. Namely, we have $S(C, \emptyset) = (C \approx \emptyset)$, where a degree $(C \approx D)$ to which C equals D is defined by

$$(C \approx D) = \bigwedge_{y \in Y} (C(y) \leftrightarrow D(y)),$$

with $\leftrightarrow$ defined by $(a \leftrightarrow b) = (a \rightarrow b) \wedge (b \rightarrow a)$.

(3) $*$ is a hedge controlling the meaning of $A \Rightarrow B$.

(4) It is easy to see that the ordinary setting is a particular case of our general setting with arbitrary $\mathbf{L}$. Namely, a particular case for $\mathbf{L} = \mathbf{2}$.

(5) We use $||\cdots||^\circ_{\dots}$ to indicate that we deal with an "interior-based" definition of validity, contrary to $||\cdots||^\bullet_{\dots}$ which indicates a "closure-based", see Section III-F.

If $*$ is globalization, $||A \Rightarrow B||^\circ_M = 1$ means that

$$A \otimes M = \emptyset \text{ implies } B \otimes M = \emptyset,$$

i.e., if the intersection of A and M is empty then the intersection of B and M is empty, too. If $*$ is identity, $||A \Rightarrow B||^\circ_M = 1$ means that

$$(A \otimes M \approx \emptyset) \leq (B \otimes M \approx \emptyset),$$

i.e., a degree to which $B \otimes M$ is empty is greater or equal to a degree to which $A \otimes M$ is empty. In general, considering an individual with a graded knowledge state M, $||A \Rightarrow B||^\circ_M$ is a truth degree of "if the individual fails on all problems from A, then the individual fails on all problems from B". Therefore, a verbal description of $||A \Rightarrow B||^\circ_M$ is essentially the same as in the ordinary case.

For a set T of FAIs, we define a set $\mathrm{Mod}^\circ(T)$ of all models of T by

$$\begin{aligned}\mathrm{Mod}^\circ(T) \quad = \quad & \{M \in \mathbf{L}^Y \mid ||A \Rightarrow B||^\circ_M = 1 \\ & \text{for every } A \Rightarrow B \in T\}.\end{aligned}$$

E. Basic Properties of Interior-Based Validity

As in the ordinary case, there is a close relationship between $||A \Rightarrow B||^\circ_M$ and $||A \Rightarrow B||^\bullet_M$. Namely, we have

Lemma 3: For any T and M we have

$$||A \Rightarrow B||^\circ_M = ||A \Rightarrow B||^\bullet_{\overline{M}},$$

where $\overline{M}$, denoted sometimes $M \rightarrow \emptyset$, is a complement of M defined by $\overline{M}(y) = M(y) \rightarrow 0$.

Proof: Follows directly from

$$\begin{aligned} & S(C \otimes M, \emptyset) = \textstyle\bigwedge_{y \in Y}((C \otimes M)(y) \rightarrow 0) = \\ = \quad & \textstyle\bigwedge_{y \in Y}((C(y) \otimes M(y)) \rightarrow 0) = \\ = \quad & \textstyle\bigwedge_{y \in Y}(C(y) \rightarrow (M(y) \rightarrow 0)) = S(C, M \rightarrow \emptyset). \end{aligned}$$

■

Lemma 3 serves as a basic bridge between the "interior-based" and the "closure-based" definition of validity of $A \Rightarrow B$. Let us first consider the following theorem and the use of Lemma 3 to obtain the theorem. Consider a fixed hedge $*$. We might ask the following question. To what extent is the truth degree of $A \Rightarrow B$ in a given M dependent on the truth degrees involved in A, B? That is, to what extent is $||A_1 \Rightarrow B_1||^\circ_M$ similar (close) to $||A_2 \Rightarrow B_2||^\circ_M$ in terms of closeness (similarity) of A_1 to A_2 and B_1 to B_2? And, analogously, to what extent is the truth degree of a given $A \Rightarrow B$ in M dependent on the truth degrees involved in M? That is, to what extent is $||A \Rightarrow B||^\circ_{M_1}$ close (similar) to $||A \Rightarrow B||^\circ_{M_2}$ in terms of closeness (similarity) of M_1 to M_2? An answer is provided by the following theorem.

Theorem 4: For a fixed * we have

$$\begin{aligned} & S(A_1, A_2)^* \otimes S(B_2, B_1) \otimes ||A_1 \Rightarrow B_1||^\circ_M \leq \\ \leq \quad & ||A_2 \Rightarrow B_2||^\circ_M, \\ & S(M_2, M_1)^* \otimes S(M_1, M_2) \otimes ||A \Rightarrow B||^\circ_{M_1} \leq \\ \leq \quad & ||A \Rightarrow B||^\circ_{M_2}. \end{aligned}$$

Proof: We proceed just for the first inequality. First, by Lemma 3, we have $||A_1 \Rightarrow B_1||^\circ_M = ||A_1 \Rightarrow B_1||^\bullet_{M \rightarrow \emptyset}$ and $||A_2 \Rightarrow B_2||^\circ_M = ||A_2 \Rightarrow B_2||^\bullet_{M \rightarrow \emptyset}$. Now, by [12, Theorem 6], $S(A_1, A_2)^* \otimes S(B_2, B_1) \otimes ||A_1 \Rightarrow B_1||^\bullet_{M \rightarrow \emptyset} \leq ||A_2 \Rightarrow B_2||^\bullet_{M \rightarrow \emptyset}$, finishing the proof. ■

Note that Theorem 4 has some natural consequences, for instance, it gives

$$\begin{aligned} ||A \Rightarrow B||^\circ_M & \leq & ||A \cup C \Rightarrow B||^\circ_M, \\ ||A \Rightarrow B||^\circ_M & \geq & ||A \Rightarrow B \cup C||^\circ_M. \end{aligned}$$

As a particular case, this says that if $A \Rightarrow B$ is true then $A \cup C \Rightarrow B$ is true as well, that is, the inference rule

"from $A \Rightarrow B$ infer $A \cup C \Rightarrow B$"

is sound.

F. Interior-Based Models of Fuzzy Attribute Implications

Recall from [8] that an $\mathbf{L}^*$-interior operator in Y is an operator $\mathrm{I} : \mathbf{L}^Y \to \mathbf{L}^Y$ satisfying $\mathrm{I}(A) \subseteq A$; $S(A, B)^* \leq S(\mathrm{I}(A), \mathrm{I}(B))$; $\mathrm{I}(\mathrm{I}(A)) = \mathrm{I}(A)$. An $\mathbf{L}^*$-interior system in Y is a system $\mathcal{S} \subseteq \mathbf{L}^Y$ of fuzzy sets in Y satisfying $\bigcup_{i\in I} A_i \in \mathcal{S}$ for any $A_i \in \mathcal{S}$, i.e., $\mathcal{S}$ is closed under unions, and $a^* \otimes A \in \mathcal{S}$ for any $a \in L$, $A \in \mathcal{S}$, i.e., $\mathcal{S}$ is closed under a^*-multiplication. Recall that $a^* \otimes A$ is a fuzzy set defined by $(a^* \otimes A)(y) = a^* \otimes A(y)$.

The following theorem justifies the term "interior-based" definition we used for $||\cdots||^\circ_{\ldots}$.

Theorem 5: For any T, $\mathrm{Mod}^\circ(T)$ is an L^*-interior system in Y.

Proof: Due to limited scope, we present a sketch only: We need to check that $\mathrm{Mod}^\circ(T)$ is closed under (i) arbitrary unions and (ii) a^*-multiplications.

(i): Let $M_i \in \mathrm{Mod}^\circ(T)$, $i \in I$. We need to verify $\bigcup_{i\in I} M_i \in \mathrm{Mod}^\circ(T)$. That is, we need to check that for any $A \Rightarrow B \in T$ we have $||A \Rightarrow B||^\circ_{\bigcup_{i\in I} M_i} = 1$, i.e. $S(A \otimes \bigcup_{i\in I} M_i, \emptyset)^* \leq S(B \otimes \bigcup_{i\in I} M_i, \emptyset)$. Using adjointness, the latter holds true iff for each $y \in Y$ we have $S(A \otimes \bigcup_{i\in I} M_i, \emptyset)^* \otimes (B \otimes \bigcup_{i\in I} M_i)(y) \leq 0$ which can be shown to hold true iff for each $i \in I$ we have $S(A \otimes \bigcup_{i\in I} M_i, \emptyset)^* \otimes B(y) \otimes M_i(y) \leq 0$. Now, since $S(A \otimes \bigcup_{i\in I} M_i, \emptyset) \leq S(A \otimes M_i, \emptyset)$, it suffices to verify $S(A \otimes M_i, \emptyset)^* \otimes B(y) \otimes M_i(y) \leq 0$. But the last inequality is true since it is equivalent to $||A \Rightarrow B||^\circ_{M_i} = 1$ which follows directly from $M_i \in \mathrm{Mod}^\circ(T)$ (we omit details).

(ii): The proof showing that $M \in \mathrm{Mod}^\circ(T)$ implies $a^* \otimes M \in \mathrm{Mod}^\circ(T)$ is left to a full version of this paper. ■

Remark 4: (1) Note that Theorem 5 can alternatively be obtained using Lemma 3 and results from [13] as follows: As shown in [13], $\mathrm{Mod}^\bullet(T)$ is an $\mathbf{L}^*$-closure system, i.e. we have $\bigcap_{i\in I} M_i \in \mathrm{Mod}^\bullet(T)$ whenever $M_i \in \mathrm{Mod}^\bullet(T)$ for $i \in I$; and $a^* \to M \in \mathrm{Mod}^\bullet(T)$ for any $a \in L$ and $M \in \mathrm{Mod}^\bullet(T)$. Now, if $M_i \in \mathrm{Mod}^\circ(T)$ then Lemma 3 yields $M_i \to \emptyset \in \mathrm{Mod}^\bullet(T)$, and thus $\bigcap_{i\in I}(M_i \to \emptyset) \in \mathrm{Mod}^\bullet(T)$. But since $\bigcap_{i\in I}(M_i \to \emptyset) = (\bigcup_{i\in I} M_i) \to \emptyset$ (details omitted), Lemma 3 gives $\bigcup_{i\in I} M_i \in \mathrm{Mod}^\circ(T)$. Next, if $M \in \mathrm{Mod}^\circ(T)$ then $M \to \emptyset \in \mathrm{Mod}^\bullet(T)$. Therefore, $a^* \to (M \to \emptyset) \in \mathrm{Mod}^\bullet(T)$. Now, since $a^* \to (M \to \emptyset) = (a^* \otimes M) \to \emptyset$, Lemma 3 yields $a^* \otimes M \in \mathrm{Mod}^\circ(T)$.

(2) Note, however, that the result saying that $\mathrm{Mod}^\bullet(T)$ is an $\mathbf{L}^*$-closure system cannot be obtained from Theorem 5 due to the lack of the law of double negation, i.e., lack of $a = (a \to 0) \to 0$ in general (we omit details).

Remark 5: It is easy to see that if each $A \Rightarrow B \in T$ satisfies

$$S(A, \emptyset)^* \leq S(B, \emptyset), \tag{16}$$

then $Y \in \mathrm{Mod}^\circ(T)$. Namely, $S(A \otimes Y, \emptyset)^* = S(A, \emptyset)^* \leq S(B, \emptyset) = S(B \otimes Y, \emptyset)$, i.e. $Y \in \mathrm{Mod}^\circ(T)$. Note that in crisp case, i.e. $\mathbf{L} = \mathbf{2}$, (16) says that T does not contain $\emptyset \Rightarrow B$ with $B \neq \emptyset$, cf. Remark 2.

We are now going to look at the converse statement to Theorem 5. Note that it is proved in [13] that for each $\mathbf{L}^*$-closure system $\mathcal{S}$ in Y there exists a set T of fuzzy attribute implications such that $\mathrm{Mod}^\bullet(T) = \mathcal{S}$. This result and the above-mentioned result from [13] saying that for each T, $\mathrm{Mod}^\bullet(T)$ is an $\mathbf{L}^*$-closure system, show that $\mathbf{L}^*$-closure systems are just models of theories T of FAIs. Note that connections of FAIs to closure-like structures are important both from the theoretical and computational point of view, see [13]. The situation with $\mathrm{Mod}^\circ(T)$ is not so simple. Namely, although we know that each $\mathrm{Mod}^\circ(T)$ is an $\mathbf{L}^*$-interior system, it is not true that for each $\mathbf{L}^*$-interior system $\mathcal{S}$ there is T such that $\mathrm{Mod}^\circ(T) = \mathcal{S}$. The reason for this is that $\mathbf{L}^*$-interior systems $\mathrm{Mod}^\circ(T)$ obey a special property which follows from the following lemma (recall that $\overline{M}$ is defined by $\overline{M}(y) = M(y) \to 0$).

Lemma 6: For any $A, B, M \in \mathbf{L}^Y$ we have

$$||A \Rightarrow B||^\circ_M = ||A \Rightarrow B||^\circ_{\overline{\overline{M}}} = ||\overline{\overline{A}} \Rightarrow B||^\circ_M = ||A \Rightarrow \overline{\overline{B}}||^\circ_M.$$

Proof: Follows directly from definitions. ■

Therefore, we have:

Theorem 7: For any set T of fuzzy attribute implications, the $\mathbf{L}^*$-interior system $\mathrm{Mod}^\circ(T)$ satisfies

$$M \in \mathrm{Mod}^\circ(T) \text{ iff } \overline{\overline{M}} \in \mathrm{Mod}^\circ(T). \tag{17}$$

Realizing this additional property, we can prove the following theorem.

Theorem 8: Let $\mathcal{S}$ be an $\mathbf{L}^*$-interior system in Y satisfying

$$M \in \mathcal{S} \quad \text{iff} \quad \overline{\overline{M}} \in \mathcal{S}. \tag{18}$$

Denote by I the $\mathbf{L}^*$-interior operator corresponding to $\mathcal{S}$. Then for a set

$$T = \{A \Rightarrow \overline{\mathrm{I}(\overline{A})} \mid A \in \mathbf{L}^Y\}$$

of FAIs we have $\mathrm{Mod}^\circ(T) = \mathcal{S}$.

Proof: "$\supseteq$": Let $M \in \mathcal{S}$. We need to show $M \in \mathrm{Mod}^\circ(T)$, i.e., $||A \Rightarrow \overline{\mathrm{I}(\overline{A})}||^\circ_M = 1$, for each $A \in \mathbf{L}^Y$. Since M is a fixpoint of I, and since $\mathrm{I}(\overline{A}) \subseteq \overline{\overline{\mathrm{I}(\overline{A})}}$ we have $S(A \otimes M, \emptyset)^* = S(M, \overline{A})^* \leq S(\mathrm{I}(M), \mathrm{I}(\overline{A})) = S(M, \mathrm{I}(\overline{A})) \leq S(M, \overline{\overline{\mathrm{I}(\overline{A})}}) = S(M \otimes \overline{\mathrm{I}(\overline{A})}, \emptyset)$, proving $||A \Rightarrow \overline{\mathrm{I}(\overline{A})}||^\circ_M = 1$.

"$\subseteq$": Let $M \in \mathrm{Mod}^\circ(T)$. We need to show $M = \mathrm{I}(M)$. Due to (18), it suffices to show $\overline{\overline{M}} = \mathrm{I}(\overline{\overline{M}})$. $M \in \mathrm{Mod}^\circ(T)$ and $\overline{M} \Rightarrow \overline{\mathrm{I}(\overline{\overline{M}})} \in T$ (an instance for $A = \overline{M}$) imply that $||\overline{M} \Rightarrow \overline{\mathrm{I}(\overline{\overline{M}})}||^\circ_M = 1$ and due to Lemma 6, $||\overline{M} \Rightarrow \overline{\mathrm{I}(\overline{\overline{M}})}||^\circ_{\overline{\overline{M}}} = 1$. This means that

$$S(\overline{\overline{M}} \otimes \overline{M}, \emptyset)^* \leq S(\overline{\overline{M}} \otimes \overline{\mathrm{I}(\overline{\overline{M}})}, \emptyset),$$

and since

$$S(\overline{\overline{M}} \otimes \overline{M}, \emptyset)^* = S(\emptyset, \emptyset)^* = 1,$$

we get $\overline{\overline{M}} \otimes \overline{\mathrm{I}(\overline{\overline{M}})} = \emptyset$, i.e., $\overline{\overline{M}} \subseteq \overline{\overline{\mathrm{I}(\overline{\overline{M}})}}$. Taking into account $\overline{\overline{M}} \supseteq \overline{\overline{\mathrm{I}(\overline{\overline{M}})}}$ (which follows from $\overline{\overline{M}} \supseteq \mathrm{I}(\overline{\overline{M}})$), we get

$$\overline{\overline{M}} = \overline{\overline{\mathrm{I}(\overline{\overline{M}})}}. \tag{19}$$

Since $\mathrm{I}(\overline{\overline{M}})$ is a fixpoint of I, (18) yields that $\overline{\overline{\mathrm{I}(\overline{\overline{M}})}}$ is a fixpoint as well, i.e. we have

$$\mathrm{I}(\overline{\overline{\mathrm{I}(\overline{\overline{M}})}}) = \overline{\overline{\mathrm{I}(\overline{\overline{M}})}}. \tag{20}$$

Applying I to (19), and using (20) and (19) we get

$$\mathrm{I}(\overline{\overline{M}}) = \mathrm{I}(\overline{\overline{\mathrm{I}(\overline{\overline{M}})}}) = \overline{\overline{\mathrm{I}(\overline{\overline{M}})}} = \overline{\overline{M}},$$

proving that $\overline{\overline{M}}$ is a fixpoint of I. The proof is finished. ■

Theorem 5, Theorem 7, and Theorem 8 say that models of FAIs in the sense of knowledge spaces are exactly $\mathbf{L}^*$-interior systems satisfying (18).

IV. Conclusions and Future Research

We presented several results regarding dependencies in the area of knowledge spaces. First, we brought up connections between these dependencies and so-called attribute dependencies which are used in formal concept analysis. By means of deriving theorem as a consequence of well-known results, we demonstrated that these connections are useful and shall be taken into considerations in knowledge spaces. Furthermore, we proposed an approach to dependencies in a setting which generalizes the ordinary setting of knowledge spaces in that it allows graded knowledge states. For our approach, we presented selected results among which is a lemma which serves as a bridge to so-called fuzzy attribute implications.

Our future research will focus on the following topics:

- Further study of relationships between dependencies (1) as developed in knowledge spaces, on the one hand, and attribute dependencies and further kinds of related dependencies on the other hand.
- Development of knowledge spaces with graded knowledge states.
- Further study of dependencies in knowledge spaces with graded knowledge states.
- Axiomatic systems dependencies in knowledge spaces with graded knowledge states.

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