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Knowledge spaces with graded knowledge states ☆,☆☆

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ABSTRACT

Knowledge spaces represent a framework for assessment of knowledge with solid theoretical foundations, methodology, software tools, and practical applications. The underlying assumption in knowledge spaces is that a knowledge state of an individual is represented by a set of items which the individual has mastered. In this paper, we propose an extension of the theory of knowledge spaces which accounts for gradedness of knowledge states. Namely, we assume that a knowledge state is represented by a fuzzy (graded) set with degrees representing levels to which an individual has mastered the items. If 0 and 1 are the only degrees, our approach coincides with that of ordinary knowledge spaces. We develop basic concepts and results in the graded setting including bases of graded knowledge states and their computation and a logic of partial failure with its completeness theorem. We also present an illustrative example. The main aim of this paper is to demonstrate mathematical and computational feasibility of knowledge spaces with graded knowledge states.

1. Introduction and preliminaries

1.1. Introduction

The concept of a knowledge space [10] provides a theoretical framework for assessment of knowledge with advanced foundations and existing software, see e.g. [11–14,21]. Fundamental to knowledge spaces is the concept of a knowledge state. By definition, a knowledge state of an individual is a set of items the individual has mastered, e.g. a set of questions an individual is capable of answering. The framework of knowledge spaces assumes that an item can only be classified as mastered or not mastered. Such an assumption is appropriate if items are elementary questions for which the answers can clearly be classified as "yes" (1) and "no" (0). However, the assumption excludes the possibility of modeling explicitly the possibly intermediate degrees (levels) of mastering an item, of which "mastered" and "not mastered" are the boundary cases. An example of an item of this sort is the question "When was the printing press invented?" Namely, the mastery of this item may be assessed by assigning degrees to the answers to this question, e.g. assigning 1 to 1440, 0.9 to 1450, 0 to 1830, etc., because it is 1450 is close enough to 1440 which is the year Johannes Guttenberg is believed to have invented the printing press. In addition, assigning degrees rather than "yes" or "no" only is appropriate if the items are reasonably complex questions (topics of knowledge) such as mastery of past tense in English rather than atomic questions such as "2 + 9 = ?" In order to develop a framework suitable for handling such items, a natural approach is to generalize the concept of a knowledge state to that of a graded knowledge state in that every item y belongs to a graded knowledge state K to a

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certain degree $K(y)$. $K(y) = 0$ and $K(y) = 1$ mean that item y is not mastered at all and that y is fully mastered, respectively. Intermediate degrees, such as $K(y) = 0.7$, indicate a partial mastery of y. That is, one replaces the two-element set $\{0,1\}$, representing {"not mastered", "mastered"} which is implicitly used in ordinary knowledge spaces, by a more general scale L of degrees which includes 0 and 1 but also possibly other degrees which represent partial mastery. Further details of such extension are, however, not obvious.

The aim of this paper is to propose a framework in which partial mastery of items and graded knowledge states can be handled mathematically as well as computationally. We assume that the scale L of degrees is partially ordered and bounded from below and above by 0 ("not mastered") and 1 ("mastered"). Particular examples of L are the real unit interval $L = [0,1]$ or a finite chain $L = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ representing "very bad", "bad", "neutral", "good", "very good" or the like, but in general, the degrees from L need not be numbers and need not be linearly ordered. This emphasizes a non-numerical character of knowledge spaces—an important feature which is retained in our approach. In particular, we assume that the scale L of degrees is a complete residuated lattice, i.e. a complete lattice equipped with certain aggregation operations. Residuated lattices are the main structures used in mathematical fuzzy logic [17,19] and make it possible to represent mathematically a manipulation of degrees to which items are mastered. If $L = \{0,1\}$, the residuated lattice coincides with the two-element Boolean algebra, and the concepts and results of our approach become the concepts and results of the ordinary theory of knowledge spaces.

A related approach was presented in [25]. The author in [25] argues that the assumption that every problem is solved either correctly or incorrectly is too restrictive. He presents motivations supported by his research on psychological models of problem solving processes and proposes a generalization of knowledge spaces in which the two-element set $\{0,1\}$ used in ordinary knowledge spaces is replaced by a linearly ordered, possibly unbounded, scale. In this setting, he generalizes the concepts of a knowledge state and knowledge space and generalizes some results concerning surmise relations and surmise functions. The main distinction between our approach and that of [25] is that in addition to the degrees, we also consider aggregation operations acting on degrees, while [25] does not. Aggregation of degrees is implicitly present in the ordinary knowledge spaces and is accomplished by means of logical connectives. As a simple example, if q belongs to knowledge states K_1 as well as knowledge state K_2, i.e. $K_1(q) = 1$ and $K_2(q) = 1$, one uses the truth function $\wedge$ of conjunction to infer $(K_1 \cap K_2)(q) = 1 \wedge 1 = 1$, i.e. that q belongs to the intersection of K_1 and K_2. We use the operations of a complete residuated lattices to aggregate degrees, most importantly a t-norm $\otimes$ and its residuum $\rightarrow$ which play the role of many-valued conjunction and implication. This enables us to generalize the concepts and results of ordinary knowledge spaces so that degrees are consistently taken into account.

Note that our approach is different from probabilistic knowledge spaces, see e.g. in [10]. The main aim of probabilistic knowledge spaces is to handle situations in which an individual may commit an error due to carelessness as well as situations in which an individual guesses a correct answer (rather than answers the question correctly because of mastery of the question). In both of these cases a knowledge state does not reflect the actual state of knowledge. Probabilistic knowledge spaces attempt to solve this problem by introducing a probability distribution p on a collection of all states. The value $p(K)$ for a knowledge state K then measures a likeliness that a sampled individual is actually in the knowledge state K. Note however that in the framework of probabilistic knowledge spaces all questions from K can be still regarded as being mastered in a yes-or-no way only. Therefore, while probabilistic knowledge spaces address the above-described uncertainty, our approach addresses the vagueness (gradedness) of the notion "correct answer" when responding to questions.

The paper is organized as follows. In Section 1.2, we present preliminaries regarding residuated lattices and basic notions on sets and relations with degrees of membership in residuated lattices. In Section 2 we present the basic concepts of knowledge spaces in a graded setting. Then we turn in detail to two important issues of knowledge spaces, namely bases and dependencies. First, we develop atoms and bases in a graded setting with focus on algorithms in Section 3. Second, in Section 4 we develop dependencies in knowledge spaces with graded knowledge states including a calculus for reasoning with such dependencies with two versions of its completeness theorem. An illustrative example is presented in Section 5. Conclusions are presented in Section 6.

1.2. Preliminaries

Knowledge spaces. The reader is supposed to be familiar with the theory of knowledge spaces; we refer to [10] for relevant background.

Residuated lattices. We assume that degrees form a bounded partially ordered set L which is a complete lattice. Furthermore, we assume that L is equipped with certain aggregation operators which are known from mathematical fuzzy logic [17,19,20]. In particular, we assume that the scale L of degrees forms a complete residuated lattice [20] i.e. an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$$

such that $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1 \rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leqslant c \quad \text{iff} \quad a \leqslant b \rightarrow c \tag{1}$$

for each $a, b, c \in L$. Residuated lattices are used in several areas of mathematics, notably in mathematical fuzzy logic. In fuzzy logic, elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) many-valued conjunction and implication. Furthermore, we use a unary function $*$ on L, called a truth-stressing hedge (shortly, a hedge), which satisfies

$$1^* = 1, \tag{2}$$
$$a^* \leqslant a, \tag{3}$$
$$(a \rightarrow b)^* \leqslant a^* \rightarrow b^*, \tag{4}$$
$$a^{**} = a^* \tag{5}$$

for each $a, b \in L$. Hedge * can be interpreted as a (truth function of) logical connective "very true", see [20]. A common choice of $\mathbf{L}$ is a structure with $L = [0,1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are:

$$\text{Łukasiewicz}: \begin{array}{l} a \otimes b = \max(a+b-1, 0), \\ a \rightarrow b = \min(1-a+b, 1), \end{array} \tag{6}$$

$$\text{Gödel}: \begin{array}{l} a \otimes b = \min(a, b), \\ a \rightarrow b = \begin{cases} 1 & \text{if } a \leqslant b, \\ b & \text{otherwise}, \end{cases} \end{array} \tag{7}$$

$$\text{Goguen}: \begin{array}{l} a \otimes b = a \cdot b, \\ a \rightarrow b = \begin{cases} 1 & \text{if } a \leqslant b, \\ \frac{b}{a} & \text{otherwise}. \end{cases} \end{array} \tag{8}$$

Another common choice is a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of L and restrictions of Gödel operations from $[0,1]$ to L. Two boundary cases of hedges are (i) identity, i.e. $a^* = a (a \in L)$; (ii) globalization [20]:

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise}. \end{cases} \tag{9}$$

A special case of a complete residuated lattice is the two-element Boolean algebra $\langle \{0,1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of classical logic. That is, the operations $\wedge$, $\vee$, $\otimes$, $\rightarrow$ of $\mathbf{2}$ are the truth functions of the corresponding logical connectives of classical logic.

L-*Sets and* **L**-*Relations*. Given a complete residuated lattice $\mathbf{L}$, we define usual notions [4,18,20]: an $\mathbf{L}$-set (fuzzy set, graded set) A in a universe U is a mapping $A: U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". If $U = \{u_1, \ldots, u_n\}$ then A can be denoted by $A = \{^{a_1}/u_1, \ldots, {^{a_n}}/u_n\}$ meaning that $A(u_i)$ equals a_i for each $i = 1, \ldots, n$. Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. $\mathbf{2}$-sets and operations with $\mathbf{2}$-sets can be identified with ordinary sets and operations with ordinary sets, respectively. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$. Given $A, B \in \mathbf{L}^U$, we define a degree $S(A,B)$ to which A is included in B by

$$S(A,B) = \bigwedge\nolimits_{u \in U} (A(u) \rightarrow B(u)). \tag{10}$$

Note that (10) generalizes the ordinary subsethood relation $\subseteq$. Described verbally, $S(A,B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leqslant B(u)$ for each $u \in U$. For $a \in L$ and $A \in \mathbf{U}$, an a-cut of A is an ordinary subset of U defined by ${}^aA = \{u \in U | A(u) \geqslant a\}$.

In the following we use well-known properties of residuated lattices and fuzzy sets over residuated lattices which can be found, e.g., in [4,18,20].

2. Graded knowledge states: basic concepts

We assume that Y is a finite set of items, such as questions, problems, problem areas, but there are several other reasonable interpretations, see [10]. Our aim in this section is to introduce the basic concepts of knowledge spaces in a graded setting and show that in the more general setting, they retain their main properties known from the classic setting. We assume that $\mathbf{L}$ is a complete residuated lattice and that $*$ is a hedge on $\mathbf{L}$. $\mathbf{L}$ plays the role of the scale of degrees; $*$ plays the role of a parameter which will be explained later. Most likely, a reasonable choice would be a finite chain for $\mathbf{L}$ and identity or globalization for $*$, but we keep the notions as general as possible.

Definition 1. For a given Y and $\mathbf{L}$, a *graded knowledge state* is an $\mathbf{L}$-set K in Y. For $y \in Y$, $K(y)$ is interpreted as the degree to which an individual with knowledge state K has mastered item y. A *graded knowledge structure* is any family $\mathcal{K} \subseteq \mathbf{L}^Y$ of graded knowledge states which contains $\emptyset$ and Y.

Note that some theorems remain valid even when the assumption $Y \in \mathcal{K}$ is replaced by a weaker assumption requiring the existence of $K \in \mathcal{K}$ such that $K(y) > 0$ for all $y \in Y$. Since this is only a technical matter, we do not discuss it further. In the ordinary setting, a knowledge space is a knowledge structure which is closed under a union of knowledge states, i.e. an interior system in Y. An appropriate counterpart to the concept of an interior system was introduced in [5]. An $\mathbf{L}^*$-interior system in Y (with $*$ being a truth-stressing hedge whose meaning will be explained later) is a system which is closed under union and under $\otimes$-multiplication. Hence, we define:

Definition 2. A *graded knowledge space* is a graded knowledge structure $\mathcal{K}$ which forms an $\mathbf{L}^*$-interior system, see [5], that is, a system $\mathcal{K}$ of $\mathbf{L}$-sets in Y satisfying: (i) if $K_i \in \mathcal{K}$, $i \in I$, then $\bigcup_{i\in I} K_i \in \mathcal{K}$ (closedness under union); (ii) if $K \in \mathcal{K}$ and $a \in L$, then $a^* \otimes K \in \mathcal{K}$ (closedness under $\otimes$-multiplication by a^*).

Closedness under union says that if K_i are feasible knowledge states then $\bigcup_{i\in I} K_i$ is a feasible knowledge state as well. In the particular case of $I = \{1,2\}$, this means that if it is conceivable that a person is in the knowledge state K_1 (i.e. the person knows the topics from K_1 in that he knows every topic y at least to degree $K_1(y)$) and it is conceivable that the person is in the knowledge state K_2, then it is conceivable that the person is in the knowledge state $K_1 \cup K_2$ (the person knows the topics from both K_1 and K_2). This is a natural requirement and we adopt it from the ordinary case [10]. Closedness under multiplication can be seen as a condition saying that if it is conceivable that a person is in a knowledge state K then it is conceivable that the person is in a new knowledge state, namely $b \otimes K$, that results by decreasing every degree $K(y)$ to $b \otimes K(y)$. Namely, note that $b \otimes K$ is defined by $(b \otimes K)(y) = b \otimes K(y)$ and that $b \otimes K(y) \leqslant K(y)$. Therefore, closedness under multiplication says that when we decrease a conceivable knowledge state using multiplication by a truth degree, we get a conceivable knowledge state. In fact, our condition of closedness under multiplication is a bit more technically involved in that we multiply only by $b \in L$ that are of the form $b = a^*$ for some $a \in L$, not by arbitrary $b \in L$. The reason is a technical one in that we wish to have a general definition to cover two important cases and using the hedge $*$ as a parameter allows us to do it. Namely, the two important cases result by taking identity (case 1) and globalization (case 2) as the hedge $*$, see Section 1.2. For identity, $a^* = a$, and thus closedness under multiplication says that a multiplication of K by a degree a is a knowledge state for any degree $a \in L$. For globalization, $a^* = 1$ for $a = 1$ and $a^* = 0$ for $a < 1$, and thus closedness under multiplication is a vacuous condition since $1 \otimes K = K$ and $0 \otimes K = \emptyset$. Hence, if $*$ is globalization then a graded knowledge space is a graded knowledge structure closed under union. Therefore, using a hedge as a parameter in Definition 2, we cover two plausible definitions of a knowledge space, one requiring closedness under union and closedness under multiplication by any truth degree (if the hedge is identity) and the other requiring closedness under union only (if the hedge is globalization), by a single concept. Note that Definition 2 subsumes also other particular definitions, corresponding to hedges other than globalization and identity, but globalization and identity are the two main cases we consider. Note also that closedness under multiplication is trivially satisfied in the ordinary case and that in the framework with grades its validation is a matter of further experimental study.

Example 1. Let $Y = \{a,b,c\}$ be a set of questions. Let $\mathbf{L}$ be a structure with $L = \{0, 0.25, 0.5, 0.75, 1\}$, and $\otimes$, $\to$ being Łukasiewicz operations. The family $\mathcal{K} = \{K_1, \ldots, K_5\}$, where $K_1 = \emptyset$, $K_2 = \{{}^{0.25}/b\}$, $K_3 = \{{}^{0.75}/a, {}^{0.25}/c\}$, $K_4 = \{{}^{0.75}/a, {}^{0.25}/b, {}^{0.25}/c\}$, $K_5 = \{{}^{1}/a, {}^{1}/b, {}^{1}/c\}$, is graded knowledge structure. Moreover, if $*$ is globalization then $\mathcal{K}$ is a graded knowledge space. On the other hand, if $*$ is identity then $\mathcal{K}$ does not form a graded knowledge space because $0.75 \otimes K_4 = \{{}^{0.5}/a\} \notin \mathcal{K}$.

For the same Y and $\mathbf{L}$ as in the previous example, let $*$ be identity. Then $\mathcal{K} = \{K_1, \ldots, K_6\}$ consisting of $K_1 = \emptyset$, $K_2 = \{{}^{0.25}/b\}$, $K_3 = \{{}^{0.25}/a, {}^{0.25}/b, {}^{0.25}/c\}$, $K_4 = \{{}^{0.5}/a, {}^{0.5}/b, {}^{0.5}/c\}$, $K_5 = \{{}^{0.75}/a, {}^{0.75}/b, {}^{0.75}/c\}$, $K_6 = \{{}^{1}/a, {}^{1}/b, {}^{1}/c\}$, is a graded knowledge space.

Remark 1. It can be easily checked that if $L = \{0, 1\}$, the notions of a graded knowledge state, graded knowledge structure, and a graded knowledge space coincide with the ordinary notions. Namely, if $L = \{0, 1\}$, a graded knowledge state is just a characteristic function of a knowledge state, etc. Note also that closedness under multiplication is satisfied for free if $L = \{0, 1\}$.

Definition 3. Let $\mathcal{K}$ be a graded knowledge structure on Y. We define a binary $\mathbf{L}$-relation $\approx_{\mathcal{K}}$ on Y by

$$p \approx_{\mathcal{K}} r = \bigwedge_{K \in \mathcal{K}} K(p) \leftrightarrow K(r).$$

Since $\leftrightarrow$ is a (truth function of) many-valued equivalence, $K(p) \leftrightarrow K(r)$ is a degree which represents (logical) closeness of $K(p)$ and $K(r)$. For example, with Łukasiewicz operations on $L = [0,1]$, $K(p) \leftrightarrow K(r) = 1 - |K(p) - K(r)|$. $p \approx_{\mathcal{K}} r$ can thus be seen as a truth degree of "for every feasible knowledge state, p has been mastered iff r has been mastered".

Theorem 4. *$\approx_{\mathcal{K}}$ is an $\mathbf{L}$-equivalence, i.e. $p \approx_{\mathcal{K}} p = 1$ (reflexivity); $p \approx_{\mathcal{K}} q = q \approx_{\mathcal{K}} p$ (symmetry); and $(p \approx_{\mathcal{K}} q) \otimes (q \approx_{\mathcal{K}} r) \leqslant (p \approx_{\mathcal{K}} r)$ (transitivity).*

Proof. The proof follows from well-known results on inducing **L**-equivalences by systems of **L**-sets, see e.g. [4]. □

One can check that for Łukasiewicz operations on $L = [0,1]$, the function $d(p,q) = 1 - (p \approx_{\mathcal{K}} q)$ is a quasi-metric on Y; for Gödel operations on $L = [0,1]$, the function $d(p,q) = 1 - (p \approx_{\mathcal{K}} q)$ is a quasi-ultrametric on Y. If $L = \{0,1\}$ (the case of ordinary knowledge spaces), then $\approx_{\mathcal{K}}$ is the characteristic function of an equivalence relation on Y.

In ordinary knowledge spaces, classes of $\approx_{\mathcal{K}}$ are called *notions*. In our setting, notions are fuzzy sets of items. Particularly, a notion corresponding to item $p \in Y$ is the $\approx_{\mathcal{K}}$-class of p, i.e. the fuzzy set $[p]_{\approx_{\mathcal{K}}} \in L^Y$ defined for every $r \in Y$ by

$$[p]_{\approx_{\mathcal{K}}}(r) = p \approx_{\mathcal{K}} r.$$

We call a graded knowledge structure $\mathcal{K}$ *discriminative* if ${}^1[p]_{\approx_{\mathcal{K}}} = \{p\}$ for all $p \in Y$, where ${}^1[p]_{\approx_{\mathcal{K}}} = \{r \in Y | p \approx_{\mathcal{K}} r = 1\}$ is the 1-cut of $[p]_{\approx_{\mathcal{K}}}$, see Section 1.2. One can easily see that $\mathcal{K}$ is discriminative iff $\approx_{\mathcal{K}}$ is an **L**-equality, i.e. if $p \approx_{\mathcal{K}} r = 1$ implies $p = q$.

As in the ordinary case, every graded knowledge structure can be transformed in a canonical way into a discriminative one. Namely, for a graded knowledge structure $\mathcal{K}$, put

$$Y^{\diamondsuit} = \left\{[p]_{\approx_{\mathcal{K}}} \in L^Y | p \in Y\right\},$$

$$\mathcal{K}^{\diamondsuit} = \left\{K^{\diamondsuit} \in L^{Y^{\diamondsuit}} | K \in \mathcal{K}\right\}, \quad \text{where} \quad K^{\diamondsuit}([p]_{\approx_{\mathcal{K}}}) = K(p).$$

Then $\mathcal{K}^{\diamondsuit}$ is graded knowledge structure on $Y^{\diamondsuit}$, called a *discriminative reduction* of $\mathcal{K}$:

Theorem 5. *$\mathcal{K}^{\diamondsuit}$ is a discriminative knowledge structure.*

Proof. Immediately from definitions. □

It is easily seen that the discriminative reduction is analogous to the procedure for obtaining a metric from a given pseudometric. It is well-known (see e.g. [4, Theorem 7.54]) that if $L = [0,1]$ and $\otimes$ is a continuous Archimedean t-norm, there is a one-to-one correspondence between fuzzy equivalences and pseudometrics. In such case, the discriminative reduction coincides, modulo the one-to-one correspondence, with the procedure for getting metric from a pseudometric.

Example 2. Consider the graded knowledge structure $\mathcal{K}$ on Y, with $Y = \{a,b,c,d,e,f,g\}$ and $\mathcal{K} = \{K_1, \ldots, K_7\}$ described by Table 1. $\mathcal{K}$ is not discriminative because, for example, columns a, c and f are the same. The items of the discriminative reduction $\mathcal{K}^{\diamondsuit}$ of $\mathcal{K}$ are ${}^1[a]_{\approx_{\mathcal{K}}} = \{a,c,f\}$, ${}^1[b]_{\approx_{\mathcal{K}}} = \{b,e\}$, $[d]_{\approx_{\mathcal{K}}} = \{d\}$, and $[g]_{\approx_{\mathcal{K}}} = \{g\}$. Table 2 shows the graded states of $\mathcal{K}^{\diamondsuit}$.

Another crucial concept from knowledge spaces is that of a surmise relation. In a graded setting, surmise relations naturally come with degrees:

Definition 6. Let $\mathcal{K}$ be a graded knowledge structure on Y. For $K \in \mathcal{K}$ and $p \in Y$, put $\mathcal{K}_p(K) = K(p)$. A binary **L**-relation $\rightsquigarrow_{\mathcal{K}}$ on Y defined by

Table 1
Discriminative reduction.

	a	b	c	d	e	f	g
K_1	0	0	0	0	0	0	0
K_2	0.4	0.2	0.4	0.8	0.2	0.4	0.7
K_3	0.1	0	0.1	0.1	0	0.1	0.9
K_4	1	0.8	1	0	0.8	1	0.1
K_5	0	0.3	0	0.8	0.3	0	0.3
K_6	0.3	0.9	0.3	1	0.9	0.3	0.3
K_7	1	1	1	1	1	1	1

Table 2
Discriminative reduction.

	$[a]_{\approx_{\mathcal{K}}}$	$[b]_{\approx_{\mathcal{K}}}$	$[d]_{\approx_{\mathcal{K}}}$	$[g]_{\approx_{\mathcal{K}}}$
$K_1^{\diamondsuit}$	0	0	0	0
$K_2^{\diamondsuit}$	0.4	0.2	0.8	0.7
$K_3^{\diamondsuit}$	0.1	0	0.1	0.9
$K_4^{\diamondsuit}$	1	0.8	0	0.1
$K_5^{\diamondsuit}$	0	0.3	0.8	0.3
$K_6^{\diamondsuit}$	0.3	0.9	1	0.3
$K_7^{\diamondsuit}$	1	1	1	1

$$p{\rightsquigarrow}_{\mathcal{K}}r = S(\mathcal{K}_r, \mathcal{K}_p)$$

is called the *surmise* **L**-relation of $\mathcal{K}$.

Note that $S(\mathcal{K}_r, \mathcal{K}_p)$ is the subsethood degree of $\mathcal{K}_r$ in $\mathcal{K}_p$ defined by (10). The meaning of surmise relation is: $p{\rightsquigarrow}_{\mathcal{K}}r$ is the truth degree of proposition "every individual mastering problem r is also able to master problem p". Therefore, $p{\rightsquigarrow}_{\mathcal{K}}r$ can be seen as a degree to which problem p is a prerequisite for problem r. Again, if $L = \{0, 1\}$, ${\rightsquigarrow}_{\mathcal{K}}$ is the characteristic function of the ordinary surmise relation.

We have the following description of ${\rightsquigarrow}_{\mathcal{K}}$.

Theorem 7. *Let $\mathcal{K}$ be a graded knowledge structure on Y.*

(i) *${\rightsquigarrow}_{\mathcal{K}}$ is an **L**-quasi-order on Y, i.e. $p{\rightsquigarrow}_{\mathcal{K}}p = 1$ and $(p{\rightsquigarrow}_{\mathcal{K}}q) \otimes (q{\rightsquigarrow}_{\mathcal{K}}r) \leqslant (p{\rightsquigarrow}_{\mathcal{K}}r)$;*
(ii) *${\rightsquigarrow}_{\mathcal{K}}$ is compatible w.r.t. $\approx_{\mathcal{K}}$, i.e. $(p_1 \approx_{\mathcal{K}} p_2) \otimes (r_1 \approx_{\mathcal{K}} r_2) \otimes (p_1{\rightsquigarrow}_{\mathcal{K}}r_1) \leqslant (p_2{\rightsquigarrow}_{\mathcal{K}}r_2)$;*
(iii) *if $\mathcal{K}$ on Y is discriminative then ${\rightsquigarrow}_{\mathcal{K}}$ satisfies antisymmetry, i.e. $(p_1{\rightsquigarrow}_{\mathcal{K}}p_2) \wedge (p_2{\rightsquigarrow}_{\mathcal{K}}p_1) \leqslant (p_1 \approx_{\mathcal{K}} p_2)$, and hence, ${\rightsquigarrow}_{\mathcal{K}}$ is a partial **L**-order on $\langle Y, \approx_{\mathcal{K}}\rangle$.*

Proof

(i) is known from fuzzy set theory from the results on inducing **L**-relations from systems of **L**-sets, see e.g. [4].
(ii) For all $p_1, p_2, r_1, r_2 \in Y$ it holds

$$\begin{aligned}(p_1 \approx_{\mathcal{K}} p_2) \otimes (r_1 \approx_{\mathcal{K}} r_2) \otimes (p_1{\rightsquigarrow}_{\mathcal{K}}r_1) &= \bigwedge_{K\in\mathcal{K}}(K(p_1) \leftrightarrow K(p_2)) \otimes \bigwedge_{K\in\mathcal{K}}(K(r_1) \leftrightarrow K(r_2)) \otimes \bigwedge_{K\in\mathcal{K}}(K(r_1) \to K(p_1))\\ &\leqslant \bigwedge_{K\in\mathcal{K}}(K(p_1) \leftrightarrow K(p_2)) \otimes (K(r_1) \leftrightarrow K(r_2)) \otimes (K(r_1) \to K(p_1))\\ &\leqslant \bigwedge_{K\in\mathcal{K}}(K(r_2) \to K(r_1)) \otimes (K(r_1) \to K(p_1)) \otimes (K(p_1) \to K(p_2))\\ &\leqslant \bigwedge_{K\in\mathcal{K}}(K(r_2) \to K(p_2)) = p_2{\rightsquigarrow}_{\mathcal{K}}r_2.\end{aligned}$$

(iii) It suffices to check the antisymmetry of ${\rightsquigarrow}_{\mathcal{K}}$. For every $p, r \in Y$,

$$(p{\rightsquigarrow}_{\mathcal{K}}r) \wedge (r{\rightsquigarrow}_{\mathcal{K}}p) \leqslant \bigwedge_{K\in\mathcal{K}}(K(r) \to K(p)) \wedge (K(p) \to K(r)) = \bigwedge_{K\in\mathcal{K}}(K(p) \leftrightarrow K(r)) = p \approx_{\mathcal{K}} r,$$

finishing the proof. □

Note that (ii) of Theorem 7 can be interpreted as "if p_1 cannot be discriminated from p_2 and r_1 cannot be discriminated from r_2 and p_1 is a prerequisite for r_1 then p_2 is a prerequisite for r_2".

Substructures and subspaces in the graded setting have the usual properties. For an **L**-set $K \in \mathbf{L}^Y$, we define a restriction $K|_U \in \mathbf{L}^U$ of K to $U \subseteq Y$ by $K|_U(y) = K(y)$ for $y \in U$. Furthermore, we define an extension $K|^V \in \mathbf{L}^V$ of K to $V \supseteq Y$ by $K|^V(y) = K(y)$ for $y \in Y$ and $K|^V(y) = 0$ for $y \notin Y$. Then we get:

Theorem 8. *Let $\mathcal{K}$ be a graded knowledge structure on Y, and $U \subseteq Y$. Then $\mathcal{H} = \{H \in \mathbf{L}^U \mid H = K|_U, K \in \mathcal{K}\}$ is a graded knowledge structure on U. Moreover, if $\mathcal{K}$ is a graded knowledge space on Y, then $\mathcal{H}$ is a graded knowledge space on U as well.*

Proof. The first claim follows from the obvious fact that $\emptyset, U \in \mathcal{H}$. Closedness under union: we suppose $H_i \in \mathcal{H}$ for $i \in I$, i.e. $H_i = K_i|_U$ where $K_i \in \mathcal{K}$. Therefore, $\bigvee_{i\in I}H_i(y) = \bigvee_{i\in I}K_i(y)$ for all $y \in U$, i.e. $\bigcup_{i\in I}H_i = \bigcup_{i\in I}K_i|_U = (\bigcup_{i\in I}K_i)|_U$. Since $\bigcup_{i\in I}K_i \in \mathcal{K}$ we get $\bigcup_{i\in I}H_i \in \mathcal{H}$.

Closedness under multiplication: analogously. □

3. Bases

In ordinary knowledge spaces, bases are small sets of knowledge states which determine knowledge spaces by means of a span.

Definition 9. For $\mathcal{F} \subseteq \mathbf{L}^Y$, we put

$$\mathrm{span}_*(\mathcal{F}) = \left\{K \in \mathbf{L}^Y \mid K = \bigcup_{F\in\mathcal{F}} a_F^* \otimes F \text{ for some } a_F \in L\right\}.$$

A *basis* of a graded knowledge space $\mathcal{K}$ is a family $\mathcal{B} \subseteq \mathbf{L}^Y$ of **L**-sets in Y such that

(i) $\mathcal{B} \subseteq \mathcal{K}$,
(ii) $\text{span}_*(\mathcal{B}) = \mathcal{K}$,
(iii) $\mathcal{B}$ is a minimal family satisfying (i) and (ii). That is, if $\mathcal{F}$ satisfies (i) and (ii), and if $\mathcal{F} \subseteq \mathcal{B}$, then $\mathcal{F} = \mathcal{B}$.

The aim of the results in this section is to provide algorithms for a construction of a basis and a construction of a knowledge space from a basis. For this purpose, we assume that **L** is a finite chain (linearly ordered) and that $*$ is globalization. We write $\text{span}(\mathcal{F})$ instead of $\text{span}_*(\mathcal{F})$. Therefore, $\text{span}(\mathcal{F}) = \{K \in \mathbf{L}^Y \mid K = \bigcup_{F \in \mathcal{F}'} F, \mathcal{F}' \subseteq \mathcal{F}\}$. As we will see, the assumptions (linearity and finiteness of **L**, $*$ being globalization) make it possible to generalize the notions and arguments from ordinary knowledge spaces, which is a special case for $L = \{0,1\}$, in an easy way.

Remark 2. For general **L** and $*$, the situation is complex. For example, it can be easily shown that if $*$ is identity, a basis of $\mathcal{K}$ is not unique, cf. Theorem 10. We leave the study of bases for general **L** and $*$ to future research.

Theorem 10. *A basis $\mathcal{B}$ of a graded knowledge space $\mathcal{K}$ on Y is the least (with respect to set inclusion) family of* **L***-sets satisfying conditions (i) and (ii) from Defintion 9. That is, $\mathcal{B} \subseteq \mathcal{F}$ for all $\mathcal{F} \subseteq L^Y$ which satisfy (i) and (ii) from Definition 9.*

Proof. Let $\mathcal{F} \subseteq L^Y$ satisfy (i) and (ii) from Definition 9. Suppose $K \in \mathcal{B} \setminus \mathcal{F}$. Because $\text{span}(\mathcal{F}) = \mathcal{K}$, we have $K = \bigcup_{F \in \mathcal{F}'} F$ for some $\mathcal{F}' \subseteq \mathcal{F}$. But $\mathcal{B}$ is a basis, so $K = \bigcup_{F \in \mathcal{F}'} \bigcup_{B \in \mathcal{B}_F} B$ for suitable $\mathcal{B}_F \subseteq \mathcal{B}$. Since $B \subseteq F$ for all $B \in \mathcal{B}_F$ and $F \subset K$ for all $F \in \mathcal{F}'$, we get $B \subset K$. That is, $B \neq K$. Hence, the state K belongs to $\mathcal{B}$ and is a union of some states from $\mathcal{B}$ which are different from K. But then, $\mathcal{B} \setminus \{K\} \subset \mathcal{B}$ still satisfies (i) and (ii) from Definition 9 which is a contradiction to the fact that $\mathcal{B}$ is a basis. □

As a result, any graded knowledge space has at most one basis.

Definition 11. Let $\mathcal{K}$ be a graded knowledge space on Y, $y \in Y$, and $0 < a \in L$. An *atom at* $\langle y,a \rangle$ is a minimal state $K \in \mathcal{K}$ such that $K(y) = a$. A state $K \in \mathcal{K}$ is called an *atom* if K is an atom at $\langle y,a \rangle$ for some $y \in Y$ and $a \in L$.

Example 3

(a) Let $\mathcal{K} = \{K_1, K_2, K_3, K_4\}$ be a graded knowledge space on Y, where $Y = \{a,b,c\}$, $K_1 = \emptyset$, $K_2 = \{^{0.9}/a, {}^{0.3}/c\}$, $K_3 = \{^{0.2}/b, {}^{0.3}/c\}$, $K_4 = \{^{0.9}/a, {}^{0.2}/b, {}^{0.3}/c\}$. States K_2 and K_3 are both atoms at $\langle c, 0.3 \rangle$.
(b) Let $\mathcal{K} = \{K_1, K_2, K_3, K_4\}$ be a graded knowledge space on Y, where $Y = \{a,b,c\}$, $K_1 = \emptyset$, $K_2 = \{^{0.9}/a, {}^{0.3}/c\}$, $K_3 = \{^{0.2}/b\}$, $K_4 = \{^{0.9}/a, {}^{0.2}/b, {}^{0.3}/c\}$. Then the state K_2 is an atom at $\langle a, 0.9 \rangle$ and at $\langle c, 0.3 \rangle$.

The next two assertions explain the role of atoms.

Lemma 12. *Let $\mathcal{K}$ be a graded knowledge space on Y. Then a state $K \in \mathcal{K}$ is an atom iff $K \in \mathcal{F}$ for any family of states $\mathcal{F}$ satisfying $K = \bigcup \mathcal{F}$.*

Proof. Let $K \in \mathcal{K}$ be an atom at $\langle y, a \rangle$ such that $K = \bigcup \mathcal{F}$ for some $\mathcal{F} \subseteq \mathcal{K}$. Since **L** is a finite chain, there exists a state $K' \in \mathcal{F}$ such that $K'(y) = a$. As $K = \bigcup \mathcal{F}$ and $K' \in \mathcal{F}$, we have $K' \subseteq K$. Clearly, K is equal to K' because K is a minimal state which $K(y) = a$, and thus $K \in \mathcal{F}$. On the other hand, consider a state K which is not an atom, i.e. K is not an atom at any $\langle y,a \rangle$. Then, for each $y \in Y$ and $0 < a \in L$ such that $K(y) = a$ there exists a state $K' \subset K$ such that $K'(y) = a$. For a family $\mathcal{F}$ of these states we get $K = \bigcup \mathcal{F}$, but $K \notin \mathcal{F}$. □

Theorem 13. *The set of all atoms of a graded knowledge space $\mathcal{K}$ on Y is a basis of $\mathcal{K}$.*

Proof. Let $\mathcal{A}$ be the set of all atoms of $\mathcal{K}$, let $\mathcal{B}$ be a basis of $\mathcal{K}$. We prove that $\mathcal{B} = \mathcal{A}$. $\mathcal{B} \subseteq \mathcal{A}$: Suppose that $K \in \mathcal{B}$ is not an atom. Then for any $y \in Y$ such that $K(y) > 0$ there exists an atom $K_y \subset K$ such that $K_y(y) = K(y)$. Therefore, $K = \bigcup_{y \in Y} K_y$. Every K_y, and hence also K, is a union of some states from $\mathcal{B}$. As $K_y \subset K$, there states are different from K. Therefore, $K \in \mathcal{B}$ is a union of states from $\mathcal{B}$ which are different from K. This is a contradiction to the assumption that $\mathcal{B}$ is a basis. $\mathcal{A} \subseteq \mathcal{B}$: Let $K \in \mathcal{A}$. Since $\mathcal{B}$ is a basis, $K = \bigcup \mathcal{F}$ for some $\mathcal{F} \subseteq \mathcal{B}$. Lemma 12 yields $K \in \mathcal{F}$. Hence, $K \in \mathcal{B}$.

We now present an algorithm which constructs of a basis of a graded knowledge space $\mathcal{K}$. Let $Y = \{y_1, y_2, \ldots, y_n\}$ and $|\mathcal{K}| = m$. Let us order the knowledge states lexicographically: for $K, L \in \mathcal{K}$, put $K < L$ iff there exists $i \in \{1, 2, \ldots, n-1\}$ such that $K(y_j) = L(y_j)$ for all $j \leqslant i$ and $K(y_{i+1}) < L(y_{i+1})$. We construct an $m \times n$ table, with columns labeled by items y_j and rows labeled by states K_i. Denote by $T(k)$ the table which we obtain in step k and by $T_{ij}(k)$ the (i,j)-entry of table $T(k)$. The algorithm follows:

Algorithm 1. Construction of basis

Input: $Y = \{y_1, \ldots, y_n\}$, lexicographically ordered states $K_1, \ldots, K_m$ of $\mathcal{K}$
Output: basis $\mathcal{B}$ of $\mathcal{K}$
  $T_{ij} = K_i(y_j)$ (ith row of T coincides with ith state)
  for $k = 2$ to m **do**
    for $j = 1$ to n **do**
      if $(T_{kj} \neq 0)$ **and** (there is $2 \leqslant p < k$: $T_{pj} = T_{kj}$ and $K_p \subset K_k$) **then**
        $T_{kj} = 0$
      end if
    end for
  end for
rows of T having at least one non-zero entry represent basis states

Table 3
Construction of basis.

	a	b	c	$T_{ij}(1)$	1	2	3
K_1	0	0	0	1	0	0	0
K_2	0	0.2	0	2	0	0	0.6
K_3	0	0.2	0.8	3	0	0.2	0
K_4	0	0	0.6	4	0	0.2	0.6
K_5	0	0.2	0.6	5	0	0.2	0.8
K_6	0	0.3	0.1	6	0	0.3	0.1
K_7	0	0.3	0.8	7	0	0.3	0.6
K_8	0	0.3	0.6	8	0	0.3	0.8
K_9	0.9	0.2	0.8	9	0.9	0	0.7
K_{10}	0.9	0	0.7	10	0.9	0.2	0.7
K_{11}	0.9	0.2	0.7	11	0.9	0.2	0.8
K_{12}	0.9	0.3	0.8	12	0.9	0.3	0.7
K_{13}	0.9	0.3	0.7	13	0.9	0.3	0.8

Table 4
Construction of basis.

p	$T_{pj}(k) = T_{kj}(k)$ $K_p \subset K_k$		
	$j = 1$	$j = 2$	$j = 3$
$k = 4$	–	3	2
$k = 5$	–	3	–
$k = 7$	–	6	2
$k = 8$	–	6	5
$k = 10$	9	3	9
$k = 11$	9	3	5
$k = 12$	9	6	9
$k = 13$	9	6	5

Table 5
Construction of basis.

$T_{ij}(13)$	1	2	3
1	0	0	0
2	0	0	0.6
3	0	0.2	0
4	0	0	0
5	0	0	0.8
6	0	0.3	0.1
7	0	0	0
8	0	0	0
9	0.9	0	0.7
10	0	0	0
11	0	0	0
12	0	0	0
13	0	0	0

Example 4. Let $\mathcal{K} = \{K_1, \ldots, K_{13}\}$ be a graded knowledge space on $Y = \{a,b,c\}$ depicted in Table 3 (left). Table 3 (right) contains $T(1)$, i.e. the knowledge states are ordered lexicograpxically. Table 4 demonstrates a run of the algorithm (only for steps k for which $T(k)$ was updated by 0). For example, for $k = 10$ and question $b \in Y$ (i.e., $j = 2$) the algorithm found $p = 3$ (i.e., state K_3) such that $T_{3,2}(10) = T_{10,2}(10)$ and $K_3 \subset K_{10}$. Table 5 describes the output of the algorithm. Therefore, the basis computed by the algorithm consists of states $\{^{0.6}/c\}$, $\{^{0.2}/b\}$, $\{^{0.8}/c\}$, $\{^{0.3}/b, {}^{0.1}/c\}$, and $\{^{0.9}/a, {}^{0.7}/c\}$.

One can easily verify that the algorithm is correct, i.e. constructs a basis. Clearly, its complexity is $\mathcal{O}(m \cdot n)$, where m is number of states and n is number of items.

Next, we describe an algorithm which generates a graded knowledge space from its basis $\mathcal{B}$. Note that $K \cup B_i$ is obtained by $(K \cup B_i)(y) = K(y) \vee B_i(y)$, and that $D \subseteq K$ means $D(y) \leqslant K(y)$ for every $y \in Y$, see Section 1.2.

Algorithm 2. Construction of graded knowledge space

Input: set of knowledge states $\mathcal{B} = \{B_1, \ldots, B_n\}$ on Y
Output: graded knowledge space $\mathcal{K}$ on Y such that $\mathcal{B}$ is its basis
  $\mathcal{K}_1 = \{\emptyset, B_1\}$
  for $i = 2$ to n **do**
    $\mathcal{L} = \emptyset$
    for all $K \in \mathcal{K}_{i-1}$ **do**
      if $(B_i \not\subseteq K)$ **and** $(D \subseteq K \cup B_i \Rightarrow D \subseteq K$ for all $D \in \{B_1, \ldots, B_{i-1}\})$
      then
        $\mathcal{L} \leftarrow \mathcal{L} \cup \{K \cup B_i\}$
    end if
    $\mathcal{K}_i = \mathcal{K}_{i-1} \cup \mathcal{L}$
  end for
  $\mathcal{K} = \mathcal{K}_n$

Example 5. Let $Y = \{a,b,c\}$ and let $\mathcal{B}$ be a basis, with $B_1 = \{^{0.2}/b\}$, $B_2 = \{^{0.6}/c\}$, $B_3 = \{^{0.1}/b, {}^{0.1}/c\}$, $B_4 = \{^{0.9}/a, {}^{0.7}/c\}$. Then $\mathcal{K}_1 = \{\emptyset, \{^{0.2}/b\}\}$. The steps of our algorithm are described in Table 6. The knowledge space obtained by the algorithm contains $\emptyset$, $\{^{0.2}/b\}$, and the states which are described in the right column of the Table 6.

One can check that the algorithm is correct and that complexity is $\mathcal{O}(m \cdot n^2 \cdot k)$, where m is the number of items, and n is number of basis states, and k cardinality of set $\mathcal{K}$ of all states of the graded knowledge space $\mathcal{K}$ which the algorithm produces.

4. Dependencies

In the case of ordinary knowledge spaces, dependencies between sets of questions play a fundamental role, see [10]. Dependencies in knowledge spaces can be thought of as expressions (formulas) $A \Rightarrow B$ with $A, B \subseteq Y$ being sets of questions. The meaning of $A \Rightarrow B$ is the following: "if a person cannot answer any question from A, then the person cannot answer any question from B". Formally, $A \Rightarrow B$ is considered true in a set $M \subseteq Y$ (a person's knowledge state) if $A \cap M = \emptyset$ implies $B \cap M = \emptyset$. Every knowledge space can be described by a set of its dependencies. Namely, if one considers a knowledge space $\mathcal{K}$ and the set T of its dependencies (i.e. all dependencies that hold true in every state $K \in \mathcal{K}$), then $\mathcal{K}$ is just the set of all

Table 6
Construction of graded knowledge space.

i	B_i			$K \in \mathcal{K}_{i-1}$			$K \cup B_i$			$L \in \mathcal{L}$		
	a	b	c	a	b	c	a	b	c	a	b	c
2	0	0	0.6	0	0	0	0	0	0.6	0	0	0.6
				0	0.2	0	0	0.2	0.6	0	0.2	0.6
3	0	0.1	0.1	0	0	0	0	0.1	0.1	0	0.1	0.1
				0	0.2	0	0	0.2	0.1	0	0.2	0.1
				0	0	0.6	0	0.1	0.6	0	0.1	0.6
				0	0.2	0.6	0	0.2	0.6			
4	0.9	0	0.7	0	0	0	0.9	0	0.7			
				0	0.2	0	0.9	0.2	0.7			
				0	0	0.6	0.9	0	0.7	0.9	0	0.7
				0	0.2	0.6	0.9	0.2	0.7	0.9	0.2	0.7
				0	0.1	0.1	0.9	0.1	0.7			
				0	0.2	0.1	0.9	0.2	0.7			
				0	0.1	0.6	0.9	0.1	0.7	0.9	0.1	0.7

models of T (i.e. all sets $M \subseteq Y$ in which every dependency from T is true). There exist algorithms for querying experts, by means of asking about dependencies in a non-redundant way, which yield a (description of a) knowledge space.

Therefore, there is a logic behind the concept of a knowledge space. The semantic structures of this logic are knowledge states, the formulas are the above-described dependencies, and the systems of models of theories in this logic are just knowledge spaces. In this section, we introduce such dependencies in the graded setting and develop a logic for reasoning with such dependencies. Note that the logic fits the general framework of Pavelka's abstract logic [24]. Unless stated otherwise, we assume that $\mathbf{L}$ is an arbitrary complete residuated lattice and that $*$ is a hedge on $\mathbf{L}$.

The dependencies which we consider in the graded setting are expressions $A \Rightarrow B$, where A and B are $\mathbf{L}$-sets of questions from Y. We consider the following condition for $A \Rightarrow B$ to be true in a person's graded knowledge state $M \in \mathbf{L}^Y$: if the intersection of A and M is empty then the intersection of B and M is empty. In a graded setting, validity of $A \Rightarrow B$ in M is a matter of degree. An appropriate definition which takes the degrees into account is the following.

Definition 14. For an $\mathbf{L}$-set $M \in \mathbf{L}^Y$ and $A \Rightarrow B$ ($A, B \in \mathbf{L}^Y$), the *degree* $||A \Rightarrow B||_M$ *to which* $A \Rightarrow B$ *is true in* M is defined by

$$||A \Rightarrow B||_M = S(A \otimes M, \emptyset)^* \to S(B \otimes M, \emptyset).$$

Note that $S(C,D)$ denotes the degree to which C is included in D, defined by (10), and $C \otimes D$ is the intersection of C and D defined by $(C \otimes D)(y) = C(y) \otimes D(y)$. Using the basic principles of fuzzy logic [20] $||A \Rightarrow B||_M$ can be interpreted as the truth degree of "if $A \otimes M$ is included in the empty set (i.e. $A \otimes M$ is empty) then $B \otimes M$ is included in the empty set (i.e. $B \otimes M$ is empty)". Therefore, Definition 14 generalizes the concept of validity of $A \Rightarrow B$ to graded setting. This is, indeed, also easy to see when $L = \{0,1\}$, i.e. in the case which can be identified with that of ordinary knowledge spaces, in which $||A \Rightarrow B||_M = 1$ just means that the emptiness of the intersection of A and M implies the emptiness of the intersection of B and M. If M is a graded knowledge state of person p, $||A \Rightarrow B||_M$ is the degree of the following proposition: "if p fails on all questions from A then p fails on all questions from B".

Remark 3. Note that $*$ plays the role of a parameter in our definition. If $*$ is identity, then due to $a \to b = 1$ iff $a \leqslant b$, $||A \Rightarrow B||_M = 1$ ($A \Rightarrow B$ is fully true) means

$$S(A \otimes M, \emptyset) \leqslant S(B \otimes M, \emptyset),$$

i.e. the degree to which $B \otimes M$ is empty is greater than or equal to the degree to which $A \otimes M$ is empty. If $*$ is globalization then $||A \Rightarrow B||_M = 1$ means

$$\text{if } A \otimes M = \emptyset \text{ then } B \otimes M = \emptyset,$$

i.e. $B \otimes M$ is empty whenever $A \otimes M$ is empty. These two particular cases of $*$ represent two possible natural ways to define the meaning of dependencies in the graded setting.

Remark 4. Note that a related semantics of expressions $A \Rightarrow B$ was studied in a series of papers, see e.g. an overview paper [9].

For a set T of dependencies, we define a set Mod (T) of all *models* of T (i.e. knowledge states in which all dependencies from T are fully true) by

$$\mathrm{Mod}(T) = \left\{M \in \mathbf{L}^Y \,|\, ||A \Rightarrow B||_M = 1\right\}.$$

We now develop a logic for reasoning with dependencies $A \Rightarrow B$ for the semantics given by Definition 14. We use the framework of so-called abstract logic, see [20], which is a rather general logic framework for semantics in which formulas are evaluated to degrees. Our logic consists of the following *deduction rules*:

$$\begin{array}{lll} (Ax) & \text{infer} & A \cup B \Rightarrow A, \\ (Cut) & \text{from} & A \Rightarrow B, B \cup C \Rightarrow D \quad \text{infer} \quad A \cup C \Rightarrow D, \\ (Mul) & \text{from} & A \Rightarrow B \quad \text{infer} \quad c^* \otimes A \Rightarrow c^* \otimes B, \\ (dncB) & \text{from} & A \Rightarrow B \quad \text{infer} \quad A \Rightarrow \overline{\overline{B}} \end{array}$$

for each $A, B, C, D \in \mathbf{L}^Y$, and $c \in L$. Note that $\overline{B}$ (complement of B) is defined by $\overline{B}(y) = B(y) \to 0$ and thus $\overline{\overline{B}}(y) = (B(y) \to 0) \to 0$. Notice that (dncB) is not trivial because the law of double negation does not hold in general residuated lattices. We always have $B \subseteq \overline{\overline{B}}$ but it may happen that $B \subset \overline{\overline{B}}$. *Provability* (*syntactic entailment*) is defined as usual. That is, given a set T of dependencies,

$$T \vdash A \Rightarrow B$$

denotes that $A \Rightarrow B$ is provable from T, meaning that $A \Rightarrow B$ can be obtained by application of rules (Ax)–(dncB) to dependencies from T and to dependencies which result this way. *Semantic entailment* is defined as follows:

$$T \models A \Rightarrow B$$

(T entails $A \Rightarrow B$) means that every model of T is a model of $A \Rightarrow B$.

A set T of dependencies is called *semantically closed* if it contains all dependencies which are entailed by T, i.e. if $T = \{A \Rightarrow B \mid \|A \Rightarrow B\|_T = 1\}$. T is called *syntactically closed* if it contains all dependencies which are provable from T, i.e. if $T = \{A \Rightarrow B \mid T \vdash A \Rightarrow B\}$.

First, we establish that every semantically closed T is syntactically closed. This corresponds to soundness of our logic. A deduction rule "from $\varphi_1,\ldots,\varphi_n$ infer φ" is *sound* if for each $M \in \mathrm{Mod}(\{\varphi_1,\ldots,\varphi_n\})$ we have $M \in \mathrm{Mod}(\{\varphi\})$, i.e. φ is true in every knowledge state in which all $\varphi_1,\ldots,\varphi_n$ are true.

Lemma 15. *Rules (Ax)–(dncB) are sound.*

Proof. (Ax): We have to check that $\|A \cup B \Rightarrow A\|_M = 1$ for all $M \in \mathbf{L}^Y$, i.e. that $S((A \cup B) \otimes M, \emptyset)^* \leqslant S(A \otimes M, \emptyset)$ for all $M \in \mathbf{L}^Y$. This inequality is true since

$$S((A \cup B) \otimes M, \emptyset)^* = (S(A \otimes M, \emptyset) \wedge S(B \otimes M, \emptyset))^* \leqslant S(A \otimes M, \emptyset)^* \leqslant S(A \otimes M, \emptyset).$$

(Cut): Let $M \in \mathrm{Mod}(\{A \Rightarrow B, B \cup C \Rightarrow D\})$, so we have $S(A \otimes M, \emptyset)^* \leqslant S(B \otimes M, \emptyset)$ and $S((B \cup C) \otimes M, \emptyset)^* \leqslant S(D \otimes M, \emptyset)$. As a consequence,

$$\begin{aligned} S((A \cup C) \otimes M, \emptyset)^* &= S((A \cup C) \otimes M, \emptyset)^{**} \leqslant (S(A \otimes M, \emptyset)^* \wedge S(C \otimes M, \emptyset)^*)^* \leqslant (S(B \otimes M, \emptyset) \wedge S(C \otimes M, \emptyset))^* \\ &= S((B \cup C) \otimes M, \emptyset)^* \leqslant S(D \otimes M, \emptyset), \end{aligned}$$

proving $M \in \mathrm{Mod}(\{A \cup C \Rightarrow D\})$.

(Mul): We have $M \in \mathrm{Mod}(\{A \Rightarrow B\})$ iff $S(A \otimes M, \emptyset)^* \leqslant S(B \otimes M, \emptyset)$ iff $(B \otimes M)(y) \otimes S(A \otimes M, \emptyset)^* \leqslant 0$ for all $y \in Y$. We need to prove that $M \in \mathrm{Mod}(\{c^* \otimes A \Rightarrow c^* \otimes B\})$ which holds true iff $c^* \otimes (B \otimes M)(y) \otimes S(c^* \otimes (A \otimes M), \emptyset)^* \leqslant 0$ for all $y \in Y$. For each $y \in Y$ we have

$$\begin{aligned} c^* \otimes (B \otimes M)(y) \otimes S(c^* \otimes (A \otimes M), \emptyset)^* &= c^* \otimes (B \otimes M)(y) \otimes \left(\bigwedge_{y \in Y} (c^* \to ((A \otimes M)(y) \to 0)) \right)^* \\ &= c^* \otimes (B \otimes M)(y) \otimes \left(c^* \to \bigwedge_{y \in Y} ((A \otimes M)(y) \to 0) \right)^* \\ &\leqslant c^* \otimes (B \otimes M)(y) \otimes (c^{**} \to S(A \otimes M, \emptyset)^*) \leqslant (B \otimes M)(y) \otimes S(A \otimes M, \emptyset) \leqslant 0. \end{aligned}$$

(dncB): Let M be a model of $A \Rightarrow B$. Due to [1, Lemma 6], $\|A \Rightarrow B\|_M = \|A \Rightarrow \overline{\overline{B}}\|_M$, hence M is a model of $A \Rightarrow \overline{\overline{B}}$. □

As a consequence:

Lemma 16. *If T is semantically closed then T is syntactically closed.*

Proof. It is easy to see that it suffices to verify the following claim: For each deduction rule "from $\varphi_1,\ldots,\varphi_n$ infer φ", i.e. every one of (Ax)–(dncB), if $\varphi_1,\ldots,\varphi_n \in T$ then $\varphi \in T$. Let thus $\varphi_1,\ldots,\varphi_n \in T$. Since $\{\varphi_1,\ldots,\varphi_n\} \subseteq T$, every model M of T is a model of $\{\varphi_1,\ldots,\varphi_n\}$. As the rule is sound, M is a model of φ. Hence, φ is true in every model of T. Due to semantic closedness of T, we get $\varphi \in T$. □

The following lemma is crucial for the proof of completeness (note that the restriction of finiteness can be overcome by generalizing the notion of a proof; however, finiteness of both Y and L is not too restrictive from practical point of view):

Lemma 17. *Let Y and L be a finite. If T is syntactically closed then T is semantically closed.*

Proof. First, note that the following deduction rules are derivable from (Ax) and (Cut), see [7]: (Ref) "infer $A \Rightarrow A$", (Add) "from $A \Rightarrow B$ and $A \Rightarrow C$ infer $A \Rightarrow B \cup C$", (Pro) "from $A \Rightarrow B \cup C$ infer $A \Rightarrow B$", and (Tra) "from $A \Rightarrow B$ and $B \Rightarrow C$ infer $A \Rightarrow C$".

Suppose T is syntactically closed. We want to show that T is semantically closed, i.e. $\{A \Rightarrow B \mid \|A \Rightarrow B\|_T = 1\} \subseteq T$. We prove this by showing that if $A \Rightarrow B \notin T$ then $A \Rightarrow B \notin \{A \Rightarrow B \mid \|A \Rightarrow B\|_T = 1\}$, i.e. if $A \Rightarrow B \notin T$ then there is $M \in \mathrm{Mod}(T)$ such that $M \notin \mathrm{Mod}(\{A \Rightarrow B\})$. Let $A \Rightarrow B \notin T$, and $M = \overline{A^+}$ where A^+ is the largest one for which $A \Rightarrow A^+ \in T$. Such A^+ exists. Indeed, consider $S = \{C \mid A \Rightarrow C \in T\}$, then $S \neq \emptyset$ because $A \Rightarrow A \in T$ by (Ref), S is finite by finiteness of Y and L, and for $A \Rightarrow C_1,\ldots,A \Rightarrow C_n \in T$, we have $A \Rightarrow \bigcup_{i=1}^{n} C_i \in T$ by a repeated use of (Add). Further, we need to check (a) and (b).

(a) $\overline{A^+}$ is not model of $A \Rightarrow B$, i.e. $\|A \Rightarrow B\|_{\overline{A^+}} \neq 1$. By contradiction, if

$$1 = \|A \Rightarrow B\|_{\overline{A^+}} = S(A \otimes \overline{A^+}, \emptyset)^* \to S(B \otimes \overline{A^+}, \emptyset) = 1^* \to S(B \otimes \overline{A^+}, \emptyset) = S(B, \overline{\overline{A^+}}),$$

then $B \subseteq \overline{\overline{A^+}}$. Due to closedness of T under (dncB), $A \Rightarrow A^+ \in T$ implies $A \Rightarrow \overline{\overline{A^+}} \in T$. Since $B \subseteq \overline{\overline{A^+}}$, (Pro) gives $A \Rightarrow B \in T$, a contradiction to $A \Rightarrow B \notin T$.

(b) $\overline{A^+}$ is model of T, i.e. $\|C \Rightarrow D\|_{\overline{A^+}} = 1$ for every $C \Rightarrow D \in T$. Since $A \Rightarrow \overline{\overline{A^+}} \in T$, $A^+ \subseteq \overline{\overline{A^+}}$, and A^+ is the largest one for which $A \Rightarrow A^+ \in T$, we have $\overline{\overline{A^+}} = A^+$. Therefore $S(C \otimes \overline{A^+}, \emptyset)^* \to S(D \otimes \overline{A^+}) = S(C, \overline{A^+})^* \to S(D, \overline{A^+}) = S(C, A^+)^* \to S(D, A^+)$. Let $C \Rightarrow D \in T$. Then we need to show $S(C,A^+)^* \to S(D,A^+) = 1$ which is equivalent to $S(C,A^+)^* \otimes D \subseteq A^+$. To see this, it is sufficient to show that $A \Rightarrow S(C,A^+)^* \otimes D \in T$. Note that we have (b1) $A \Rightarrow A^+ \in T$, (b2) $A^+ \Rightarrow S(C,A^+)^* \otimes C \in T$, and (b3) $S(C,A^+)^* \otimes C \Rightarrow S(C,A^+)^* \otimes D \in T$. Indeed, $A \Rightarrow A^+ \in T$ by definition of A^+; $A^+ \Rightarrow S(C,A^+)^* \otimes C \in T$ since as $S(C,A^+)^* \otimes C \subseteq A^+$, $A^+ \Rightarrow S(C,A^+)^* \otimes C$ in an instance of (Ax); and $S(C,A^+)^* \otimes C \Rightarrow S(C,A^+)^* \otimes D \in T$ by (Mul) applied to $C \Rightarrow D \in T$. Now, $A \Rightarrow S(C,A^+)^* \otimes D \in T$ follows by (Tra) applied twice to (b1), (b2), and (b3). □

Theorem 18 (ordinary completeness). *Let Y and L be a finite. Then*

$$T \vdash A \Rightarrow B \quad \text{iff} \quad T \models A \Rightarrow B.$$

Proof. Denote by $syn(T)$ the least syntactically closed set of dependencies which contains T. It can be easily shown that $syn(T) = \{A \Rightarrow B | T \vdash A \Rightarrow B\}$. Furthermore, denote by $sem(T)$ the least semantically closed set of dependencies which contains T. Again, it can be easily shown that $sem(T) = \{A \Rightarrow B | \|A \Rightarrow B\|_T = 1\}$. To prove the claim, we need to show $syn(T) = sem(T)$. As $syn(T)$ is syntactically closed, it is also semantically closed by Lemma 17 which means $sem(syn(T)) \subseteq syn(T)$. Therefore, by $T \subseteq syn(T)$ we get $sem(T) \subseteq sem(syn(T)) \subseteq syn(T)$. In a similar manner we get $syn(T) \subseteq sem(T)$, showing $syn(T) = sem(T)$. The proof is complete. □

It is worth noting that while Theorem 18 considers only bivalent consequence, consequence comes naturally in degrees in a graded setting. Namely, a *degree* $\|A \Rightarrow B\|_T$ *to which T entails* $A \Rightarrow B$ is naturally defined by

$$\|A \Rightarrow B\|_T = \bigwedge_{M \in \mathrm{Mod}(T)} \|A \Rightarrow B\|_M,$$

i.e. the infimum of truth degrees of $A \Rightarrow B$ over all models of T. Clearly, $T \models A \Rightarrow B$ just means $\|A \Rightarrow B\|_T = 1$. Such a graded notion of entailment can be captured syntactically, namely by defining a *degree* $|A \Rightarrow B|_T$ *to which* $A \Rightarrow B$ *is provable from T* as

$$|A \Rightarrow B|_T = \bigvee \{c \in L | T \vdash A \Rightarrow c \otimes B\}.$$

Note that $c \otimes B$ is a fuzzy set defined by $(c \otimes B)(y) = c \otimes B(y)$. Then we obtain:

Theorem 19 (graded completeness). *Let Y and L be a finite. Then*

$$|A \Rightarrow B|_T = \|A \Rightarrow B\|_T.$$

Proof. Observe that $c \leqslant \|A \Rightarrow B\|_M$ is equivalent to $\|A \Rightarrow c \otimes B\|_M = 1$ (easy calculation). As a result,

$$\|A \Rightarrow B\|_T = \bigvee \{c \in L | \|A \Rightarrow c \otimes B\|_T = 1\}.$$

Then use Theorem 18. □

5. Illustrative example

In this section we present an illustrative example demonstrating the concepts introduced above. We assume that a set $Y = \{y_1, y_2, y_3\}$ of items represents mastering of particular topics of English grammar. Particularly, y_1 represents "can use past tense", y_2 represents "can use irregular verbs", and y_3 represents "can use conditional sentences". Obviously, mastering such topics is a matter of degree rather than a yes-or-no matter. In our example, we use a five element Łukasiewicz chain, i.e. a particular finite linearly ordered scale of truth degrees, see Section 1.2. Thus, for a person's knowledge state K, $K(y_2) = 0$ means that the person is not able to use irregular verbs at all, $K(y_2) = 0.25$ means the person is bad in using irregular verbs, $K(y_2) = 0.75$ means the person can use irregular verbs fairly well, and so on.

In order to obtain graded knowledge space, we continue by asking an expert about particular dependencies among topics y_1, y_2, and y_3. An expert (English teacher) might think of dependencies in the following way. If a student is not able to answer any question regarding to past tense then he must fail in all questions where irregular verbs are involved as well. Such an implication can be modeled by the dependency $\{{}^1/y_1\} \Rightarrow \{{}^1/y_2\}$. Similarly, if a student is not capable of using past tense then he is not capable of using any sentences with conditionals, i.e. we get dependency $\{{}^1/y_1\} \Rightarrow \{{}^1/y_3\}$. Moreover, an expert might come up with the following restriction. If a student does not know irregular verbs then he is by and large, not able to construct sentences including past tense (perhaps just those that contain regular verbs). Therefore, the expert may suggest the dependency $\{{}^1/y_2\} \Rightarrow \{{}^{0.75}/y_1\}$. In this way we have acquired the set of dependencies

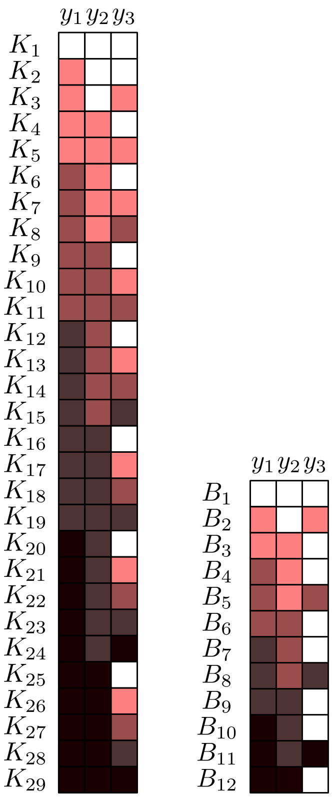

Fig. 1. Graded knowledge space from Section 5 and its basis.

$$T = \{\{{}^{1}/y_1\} \Rightarrow \{{}^{1}/y_2\}, \{{}^{1}/y_2\} \Rightarrow \{{}^{0.75}/y_1\}, \{{}^{1}/y_1\} \Rightarrow \{{}^{1}/y_3\}\},$$

using which we can generate a graded knowledge space. The resulting graded knowledge space is depicted on the left hand side in Fig. 1. The rows represent graded knowledge states $K_1, \ldots, K_{29}$, the columns represent topics y_1, y_2, y_3, and the colors of particular cells correspond to the degrees 0 (cell □) 0.25 (cell ■) 0.5 (cell ■) 0.75 (cell ■) 1 (cell ■). The basis of this graded knowledge space contains 12 states, namely the states $B_1, \ldots, B_{12}$ depicted in the right part of Fig. 1.

6. Conclusions and further issues

We presented an attempt to generalize basic components of the theory of knowledge spaces from its bivalent setting to a graded one. The generalization enables us to apply knowledge spaces to situations in which mastering of an item such as a question or topic is a matter of degree rather than a yes-or-no matter. We developed basic concepts and results for the graded setting with emphasis on bases of knowledge spaces and related algorithmic problems, and on a logic of dependencies in graded knowledge spaces. A conclusion which can be drawn from our paper is that the basic ideas of knowledge spaces are mathematically and computationally feasible even in the more general setting where one allows degrees in place of just yes-or-no assessment.

Further research is to be directed toward a development of further foundational topics of knowledge spaces with graded knowledge states and toward experimental evaluation of the proposed notions. A particular interesting topic is the analysis of sensitivity of the methods of knowledge spaces with graded states [3,23]. In addition, since the formalism of knowledge spaces is closely related to that of formal concept analysis [15], it is desirable to connect the developments in knowledge spaces with graded knowledge states to the developments in formal concept analysis of data with fuzzy attributes, see e.g. [4,6,9,22].

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