International Journal of General Systems
Vol. 00, No. 00, February 2011, 1–12

Sup-t-norm and inf-residuum are a single type of relational equations

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(*Received 00 Month 200x; final version received 00 Month 200x*)

We show that the sup-t-norm and inf-residuum types of fuzzy relational equations, considered in the literature as two different types, are in fact two particular instances of a single, more general type of equations. We demonstrate that several pairs of corresponding results on the sup-t-norm and inf-residuum types of equations are simple consequences of single results regarding the more general type of equations. We also show that the new type of equations subsumes other types of equations such as equations with constraints on solutions examples of which are fuzzy relational equations whose solutions are required to be crisp (ordinary) relations.

Keywords: fuzzy logic, fuzzy relational equation, inf-residuum product, sup-t-norm product, sup-preserving aggregation structure

1. Motivation and preliminaries

1.1 *Motivation*

Since Sanchez's seminal paper (Sanchez 1976), fuzzy relational equations played an important role in fuzzy set theory and its applications, see e.g. (De Baets 2000, Di Nola *et al.* 1989, Gottwald 1993, 2002, Klir and Yuan 1995) for overviews. Tracing the contributions to fuzzy relational equations, one can see that two basic types of fuzzy relational equations are involved. One is based on the sup-t-norm product and the other on the inf-residuum product of fuzzy relations. These types of equations have completely different meanings and are both important. The literature on fuzzy relational equations treats these two types of equations, naturally, as two distinct types and, accordingly, provides results and solution methods separately for each of these two types, see e.g. (De Baets 2000, Di Nola *et al.* 1989, Gottwald 1993, Klir and Yuan 1995). An inspection of these results and their proofs reveals that even though they are not apparently dual, they follow a similar scheme.

The question that is the main subject of the present paper is whether the sup-t-norm and the inf-residuum relational equations are indeed two different types of equations. We answer this question in negative.

1.2 *Contributions of this paper*

This paper is a continuation of (Belohlavek 2010b) where it is shown that the sup-t-norm and inf-residuum products of relations are in fact one type of product.

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ISSN: 0308-1079 print/ISSN 1563-5104 online

DOI: 10.1080/0308107YYxxxxxxxx
http://www.informaworld.com

In this paper, we develop the consequences of this result for fuzzy relational equations. Namely, we show using a framework proposed in Belohlavek (2010a) and developed in (Belohlavek 2010b) that the sup-t-norm and the inf-residuum relational equations are two particular instances of a single type of a more general equation. Moreover, we demonstrate by several examples, that the pairs of well-known results regarding solvability of sup-t-norm and inf-residuum relational equations are simple consequences of single results developed in the above-mentioned framework. In addition, we show that the new type of equations subsumes other types of equations such as equations with constraints on solutions examples of which are fuzzy relational equations whose solutions are required to be crisp relations.

1.3 *Preliminaries*

When working with fuzzy relational equations, one needs a set L of truth degrees and (truth functions of) conjunction and implication, denoted by $\otimes$ and $\rightarrow$, respectively. One usually takes $L = [0,1]$ (real unit interval), a continuous or at least left-continuous t-norm $\otimes$ (such as the Łukasiewicz t-norm given by $a \otimes b = \max(0, a+b-1)$), and its residuum $\rightarrow$ (such as the Łukasiewicz residuum given by $a \rightarrow b = \min(1, 1-a+b)$) (Klement *et al.* 2000). In what follows, the reader may indeed safely assume that $L = [0,1]$, $\wedge$ and $\vee$ are min and max (with an infinite number of arguments, $\bigwedge$ and $\bigvee$ denote infima and suprema in $[0,1]$), and that $\otimes$ is a left-continuous t-norm and $\rightarrow$ its residuum. However, we develop the results in a more general framework, namely we assume that the scale of truth degrees forms a complete residuated lattice (Belohlavek 2002, Goguen 1967, Hajek 1998), i.e. an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$$

such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$); and $\otimes$ and $\rightarrow$ satisfy the adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

for each $a, b, c \in L$ ($\leq$ denotes the lattice order). Residuated lattices are used in several areas of mathematics, notably in mathematical fuzzy logic. In fuzzy logic, elements a of L are called truth degrees.

Given $\mathbf{L}$, we define the usual notions. An L-set (fuzzy set, graded set) A in a universe U is a mapping $A\colon U \rightarrow L$; L^U denotes the collection of all L-sets in U; operations with L-sets are defined componentwise (e.g., the intersection of L-sets $A_i \in L^U$, $i \in I$, is defined by $(\bigcap_{i\in I} A_i)(u) = \bigwedge_{i\in I} A_i(u)$ for each $u \in U$); for L-sets A and B in universe U, we put

$$A \subseteq B \text{ if and only if } A(u) \leq B(u) \text{ for each } u \in U, \tag{2}$$

in which case we say that A is included in B.

A fuzzy relation R between sets X and Y is a fuzzy set in the Cartesian product $X \times Y$, i.e. $R \in L^{X\times Y}$. The sup-t-norm and inf-residuum products of fuzzy relations $R \in L^{X\times Y}$ $S \in L^{Y\times Z}$ are denoted by $R \circ S$ and $R \triangleleft S$ and are defined for every

$x \in X$ and $y \in Y$ by

$$(R \circ S)(x, z) = \bigvee_{y \in Y} (R(x, y) \otimes S(y, z)), \tag{3}$$

$$(R \triangleleft S)(x, z) = \bigwedge_{y \in Y} (R(x, y) \to S(y, z)). \tag{4}$$

A fuzzy relational equation is an equation of the form

$$U * S = T$$

where $S \in L^{Y \times Z}$ and $T \in L^{X \times Z}$ are given fuzzy relations and $*$ is a product of relations ($\circ$, $\triangleleft$, or possibly other product). The goal is to determine an (unknown) fuzzy relation U for which $U * S = T$. In an obvious way, equations of the form $R * U = T$ and systems of fuzzy relational equations are defined.

2. Sup-preserving aggregation and general product

2.1 *Sup-preserving aggregation structures*

We need the following concept (Belohlavek 2010b), see also (Belohlavek and Vychodil 2005, Krajci 2005).

Definition 2.1: A *sup-preserving aggregation structure* (aggregation structure, for short) is a quadruple $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \square \rangle$, where $\mathbf{L}_i = \langle L_i, \leq_i \rangle$ $(i = 1, 2, 3)$ are complete lattices and $\square : L_1 \times L_2 \to L_3$ is a function which commutes with suprema in both arguments.

(a) The operations in $\mathbf{L}_i$ are denoted as usual, adding subscript i. That is, the infima, suprema, the least, and the greatest element in $\mathbf{L}_2$ are denoted by $\bigwedge_2$, $\bigvee_2$, 0_2, and 1_2, respectively; the same for $\mathbf{L}_1$ and $\mathbf{L}_3$.

(b) Commuting of $\square$ with suprema in both arguments means that for any $a, a_j \in L_1$ $(j \in J)$, $b, b_{j'} \in L_2$ $(j' \in J')$,

$$(\bigvee_{1 j \in J} a_j) \square b = \bigvee_{3 j \in J} (a_j \square b) \text{ and } a \square (\bigvee_{2 j' \in J'} b_{j'}) = \bigvee_{3 j' \in J'} (a \square b_{j'}). \tag{5}$$

Since the supremum of the empty set is the least element, commuting with suprema implies that

$$0_1 \square a_2 = 0_3 \text{ and } a_1 \square 0_2 = 0_3. \tag{6}$$

(c) It follows from the well-known relationship between commuting with suprema and left-continuity (Belohlavek 2002, Klement *et al.* 2000) that $\langle \langle [0, 1], \leq \rangle, \langle [0, 1], \leq \rangle, \langle [0, 1], \leq \rangle, \square \rangle$ is an aggregation structure if and only if the projections $x \mapsto x \square b$ and $y \mapsto a \square y$ are non-decreasing left-continuous functions on $[0, 1]$ for which $0 \square b = a \square 0 = 0$, for all $a, b \in [0, 1]$.

Define operations $\circ_\square : L_1 \times L_3 \to L_2$ and ${}_\square\!\circ : L_3 \times L_2 \to L_1$ (residua of $\square$) by

$$a_1 \circ_\square a_3 = \bigvee\nolimits_2 \{a_2 \mid a_1 \square a_2 \leq_3 a_3\}, \tag{7}$$

$$a_3 \,{}_\square\!\circ\, a_2 = \bigvee\nolimits_1 \{a_1 \mid a_1 \square a_2 \leq_3 a_3\}. \tag{8}$$

Note that due to (6), $a_1 \circ_\square a_3$ and $a_3 \,{}_\square\!\circ\, a_2$ are defined for every a_1, a_2, a_3.

The following example is important for our considerations. Other examples appear in (Belohlavek 2010b).

Example 2.2 Let $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ be a complete residuated lattice. Consider the following aggregation structures. In both cases, $L_i = L$ and $\leq_i$ is either $\leq$ or the dual of $\leq$ (i.e. $\leq_i = \leq$ or $\leq_i = \leq^{-1}$).

(a) Let $\mathbf{L}_1 = \langle L, \leq\rangle$, $\mathbf{L}_2 = \langle L, \leq\rangle$, $\mathbf{L}_3 = \langle L, \leq\rangle$, and let $\Box$ be $\otimes$. Then, as is well known, $\Box$ commutes with suprema in both arguments. Furthermore,

$$a_1 \circ_\Box a_3 = \bigvee\{a_2 \mid a_1 \otimes a_2 \leq a_3\} = a_1 \rightarrow a_3$$

and, similarly, $a_3 \,{}_\Box\!\circ\, a_2 = a_2 \rightarrow a_3$.

(b) Let $\mathbf{L}_1 = \langle L, \leq\rangle$, $\mathbf{L}_2 = \langle L, \leq^{-1}\rangle$, and $\mathbf{L}_3 = \langle L, \leq^{-1}\rangle$, let $\Box$ be $\rightarrow$. Then, $\Box$ commutes with suprema in both arguments. Namely, the conditions (5) for commuting with suprema in this case become

$$(\textstyle\bigvee_{j\in J} a_j) \rightarrow b = \bigwedge_{j\in J}(a_j \rightarrow b) \quad \text{and} \quad a \rightarrow (\bigwedge_{j\in J} b_j) = \bigwedge_{j\in J}(a \rightarrow b_j)$$

which are well-known properties of residua. In this case, we have

$$a_1 \circ_\Box a_3 = \bigwedge\{a_2 \mid a_1 \rightarrow a_2 \geq a_3\} = a_1 \otimes a_3,$$
$$a_3 \,{}_\Box\!\circ\, a_2 = \bigvee\{a_1 \mid a_1 \rightarrow a_2 \geq a_3\} = a_3 \rightarrow a_2.$$

The following theorems show some properties of aggregation structures we need.

Theorem 2.3 (Belohlavek 2010b)

$$a_1 \Box a_2 \leq_3 a_3 \;\textit{iff}\; a_2 \leq_2 a_1 \circ_\Box a_3 \;\textit{iff}\; a_1 \leq_1 a_3 \,{}_\Box\!\circ\, a_2, \quad (9)$$
$$a_1 \Box (a_1 \circ_\Box a_3) \leq_3 a_3,\;\; (a_3 \,{}_\Box\!\circ\, a_2) \Box a_2 \leq_3 a_3, \quad (10)$$
$$a_2 \leq_2 a_1 \circ_\Box (a_1 \Box a_2),\;\; a_1 \leq_1 (a_1 \Box a_2) \,{}_\Box\!\circ\, a_2, \quad (11)$$
$$a_1 \leq_1 a_3 \,{}_\Box\!\circ (a_1 \circ_\Box a_3),\;\; a_2 \leq_2 (a_3 \,{}_\Box\!\circ\, a_2) \circ_\Box a_3. \quad (12)$$

In addition, $\circ_\Box$ is antitone in the first and isotone in the second argument, and ${}_\Box\!\circ$ is isotone in the first and antitone in the second argument (Belohlavek 2010b).

2.2 *General product*

The general product of fuzzy relations we need is defined as follows.

Definition 2.4 (Belohlavek 2010b) For an aggregation structure $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \Box\rangle$, and fuzzy relations $R \in L_1^{X\times Y}$ and $S \in L_2^{Y\times Z}$, let a fuzzy relation $R \boxdot S \in L_3^{X\times Z}$ be defined by

$$(R \boxdot S)(x, z) = \bigvee{}_{3\, y\in Y}(R(x, y) \Box S(y, z)). \quad (13)$$

The following example shows that $\boxdot$ generalizes both $\circ$ (sup-t-norm product) and $\triangleleft$ (inf-residuum product).

Example 2.5 (Belohlavek 2010b)(a) For the setting of Example 2.2 (a),

$$R \boxdot S = R \circ S.$$

(b) For the setting of Example 2.2 (b),

$$R \boxdot S = R \triangleleft S.$$

Furthermore (Belohlavek 2010b), for $R \in L_1^{X \times Y}$ and $S \in L_3^{Y \times Z}$, let $R \triangleleft_{\square} S \in L_2^{X \times Z}$ be defined by

$$(R \triangleleft_{\square} S)(x, z) = \bigwedge\nolimits_{2\, y \in Y} (R(x, y) \circ_{\square} S(y, z)). \tag{14}$$

For $R \in L_3^{X \times Y}$, $S \in L_2^{Y \times Z}$, let $R\, {}_{\square}\!\triangleleft\, S \in L_1^{X \times Z}$ be defined by

$$(R\, {}_{\square}\!\triangleleft\, S)(x, z) = \bigwedge\nolimits_{1\, y \in Y} (R(x, y)\, {}_{\square}\!\circ\, S(y, z)). \tag{15}$$

3. Single equations

We now consider the problems of fuzzy relational equations involving composition $\boxdot$, namely, equations

$$U \boxdot S = T, \text{ and} \tag{16}$$

$$R \boxdot U = T. \tag{17}$$

Remark 1:

(a) Due to certain duality constructions mentioned in (Belohlavek 2010b), we may ignore equations $U \triangleleft_{\square} S = T$ and $R \triangleleft_{\square} U = T$ (as well as $U\, {}_{\square}\!\triangleleft\, S = T$ and $R\, {}_{\square}\!\triangleleft\, U = T$) because these may be considered as the above equations involving $\boxdot$.

(b) Note also that due to the following construction, it is sufficient to consider only one of equations (16) and (17). For an aggregation structure $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \square \rangle$, consider the tuple $\langle \mathbf{L}_2, \mathbf{L}_1, \mathbf{L}_3, \square^d \rangle$ where $\square^d : L_2 \times L_1 \to L_3$ is defined by $a_2 \square^d a_1 = a_1 \square a_2$. $\langle \mathbf{L}_2, \mathbf{L}_1, \mathbf{L}_3, \square^d \rangle$ is an aggregation structure again (Belohlavek 2010b). An easy observation shows that solutions to a fuzzy relational equation $R \boxdot U = T$ for a given $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \square \rangle$, i.e. equation of the form (16), may be identified with solutions to $U^{-1} \boxdot^d R^{-1} = T^{-1}$ for $\langle \mathbf{L}_2, \mathbf{L}_1, \mathbf{L}_3, \square^d \rangle$, i.e. equation of the form (17). For convenience, however, we consider both (16) and (17).

Theorem 3.1: *Let $R \in L_1^{X \times Y}$, $S \in L_2^{Y \times Z}$, and $T \in L_3^{X \times Z}$ be fuzzy relations. Then*

(1) $U \boxdot S = T$ has a solution iff $T\, {}_{\square}\!\triangleleft\, S^{-1}$ is its solution,
(2) $R \boxdot U = T$ has a solution iff $R^{-1} \triangleleft_{\square} T$ is its solution.

Proof: From (10) we have

$$\Big(\bigwedge\nolimits_{1 z \in Z} T(x, z)\, {}_{\square}\!\circ\, S^{-1}(z, y)\Big) \square S(y, z)$$
$$\leq_3 \big(T(x, z)\, {}_{\square}\!\circ\, S(y, z)\big) \square S(y, z) \leq_3 T(x, z).$$

Therefore

$$\bigvee\nolimits_{3 y \in Y} \Big(\bigwedge\nolimits_{1 z \in Z} T(x, z)\, {}_{\square}\!\circ\, S^{-1}(z, y)\Big) \square S(y, z) \leq_3 T(x, z),$$

which means that $(T\,{}_{\square}\!\lhd S^{-1}) \boxdot S \subseteq_3 T$. Note that we define $\subseteq_i$ based on $\leq_i$, i.e. for L_i-sets A and B in universe V we put $A \subseteq_i B$ iff $A(v) \leq_i B(v)$ for every $v \in V$.

Moreover, if R is a solution of $U \boxdot S = T$ then due to (9) and properties of infima, we get the following claims:

$$R(x,y) \,\square\, S(y,z) \leq_3 T(x,z),$$
$$R(x,y) \leq_1 T(x,z) \,{}_{\square}\!\circ\, S(y,z) = T(x,z) \,{}_{\square}\!\circ\, S^{-1}(z,y),$$
$$R(x,y) \leq_1 \bigwedge\nolimits_{1 z\in Z} T(x,z) \,{}_{\square}\!\circ\, S^{-1}(z,y).$$

Therefore, $R \subseteq_1 T\,{}_{\square}\!\lhd S^{-1}$. We thus have

$$T = R \boxdot S \subseteq_3 (T\,{}_{\square}\!\lhd S^{-1}) \boxdot S \subseteq_3 T$$

showing the fact that $T\,{}_{\square}\!\lhd S^{-1}$ is a solution of the equation $U \boxdot S = T$. The second assertion can be proven analogously. □

The following theorem shows the set of all solutions of an equation forms a semilattice with a greatest element.

Theorem 3.2: *If an equation $U \boxdot S = T$ is solvable then the set of all of its solutions along with $\subseteq_1$ forms a complete join-semilattice with the greatest element $T\,{}_{\square}\!\lhd S^{-1}$. If an equation $R \boxdot U = T$ is solvable then the set of all of its solutions along with $\subseteq_2$ forms a complete join-semilattice with the greatest element $R^{-1} \lhd_{\square} T$.*

Proof: Suppose R_i, $i \in I$, are solutions of $U \boxdot S = T$. From (5) we can easily see that for all $x \in X$, $z \in Z$

$$\bigvee\nolimits_{3 y\in Y} \big(\bigvee\nolimits_{1 i\in I} R_i(x,y)\big) \,\square\, S(y,z)$$
$$= \bigvee\nolimits_{3 y\in Y} \big(\bigvee\nolimits_{3 i\in I} R_i(x,y) \,\square\, S(y,z)\big)$$
$$= \bigvee\nolimits_{3 i\in I} T(x,z) = T(x,z).$$

Therefore, $\bigcup_{1 i\in I} R_i$ is a solution of $U \boxdot S = T$ as well. In addition, in the proof of Theorem 3.1 we have shown that $R_i \subseteq_1 T\,{}_{\square}\!\lhd S^{-1}$ for all $i \in I$, i.e. $T\,{}_{\square}\!\lhd S^{-1}$ is the greatest solution. The proof of the second claim is similar. □

Now, the well-known solvability criteria for sup-t-norm and inf-residuum fuzzy relational equations (see (Belohlavek 2002, Gottwald 1993, Klir and Yuan 1995)) can be easily proven as corollaries of the general results above. Indeed, they result as particular cases of the general results by taking appropriate sup-preserving aggregation structures.

Corollary 3.3: *An equation $U \circ S = T$ is solvable iff $(S \lhd T^{-1})^{-1}$ is its solution. An equation $R \circ U = T$ is solvable iff $R^{-1} \lhd T$ is its solution.*

Proof: For the setting of Example 2.2 (a), i.e. $\square = \otimes$, equations $U \boxdot S = T$ and $R \boxdot U = T$ become $U \circ S = T$ and $R \circ U = T$, respectively. Moreover, ${}_{\square}\!\circ = \leftarrow$ and $\circ_{\square} = \rightarrow$. The fuzzy relations from Theorem 3.1 therefore become

$$(T\,{}_{\square}\!\lhd S^{-1})(x,y) = \bigwedge\nolimits_{1 z\in Z} T(x,z) \leftarrow S^{-1}(z,y)$$
$$= \bigwedge\nolimits_{1 z\in Z} S(y,z) \rightarrow T^{-1}(z,x) = (S \lhd T^{-1})^{-1}(x,y),$$

and

$$(R^{-1} \lhd_{\Box} T)(y,z) = \bigwedge\nolimits_{2 x\in X} R^{-1}(y,x) \to T(x,z)$$
$$= (R^{-1} \lhd T)(y,z).$$

□

Corollary 3.4: *An equation $U \lhd S = T$ is solvable iff $T \lhd S^{-1}$ is its solution. An equation $R \lhd U = T$ is solvable iff $R^{-1} \circ T$ is its solution.*

Proof: For the setting of Example 2.2 (b), i.e. $\boxdot = \lhd$, ${}_{\Box}\lhd = \lhd$, and $\lhd_{\Box} = \circ$, the assertion follows directly from Theorem 3.1. □

Corollary 3.5: *If an equation $U \circ S = T$ is solvable then the set of all of its solutions along with $\subseteq$ forms a complete join-semilattice with the greatest element $(S \lhd T^{-1})^{-1}$. If an equation $R \circ U = T$ is solvable then the set of all of its solutions along with $\subseteq$ forms complete a meet-semilattice with the least element $R^{-1} \lhd T$.*

Proof: Directly from Theorem 3.2 and Corollary 3.3. □

Corollary 3.6: *If an equation $U \lhd S = T$ is solvable then the set of all of its solutions along with $\subseteq$ forms a complete join-semilattice with the greatest element $T \lhd S^{-1}$. If an equation $R \lhd U = T$ is solvable then the set of all of its solutions along with $\subseteq$ forms a complete meet-semilattice with the least element $R^{-1} \circ T$.*

Proof: Directly from Theorem 3.2 and Corollary 3.4. □

4. Systems of equations

We now turn to systems of fuzzy relational equations. As in the case of single equations, it is sufficient to deal with systems $\{U \boxdot S_j = T_j \mid j \in J\}$ only. For convenience again, we also consider systems $\{R_j \boxdot U = T_j \mid j \in J\}$.

We say that a relation R is solution of a system $\{U \boxdot S_j = T_j \mid j \in J\}$ if $R \boxdot S_j = T_j$ for all $j \in J$; similarly for the second type.

Theorem 4.1: *Let $R_j \in L_1^{X\times Y}$, $S_j \in L_2^{Y\times Z}$, and $T_j \in L_3^{X\times Z}$ be fuzzy relations for $j \in J$. Then*

(1) a system $\varepsilon_1 = \{U \boxdot S_j = T_j \mid j \in J\}$ has a solution iff $\bigcap_{1 j\in J} T_j \,{}_{\Box}\!\lhd S_j^{-1}$ is its solution,
(2) a system $\varepsilon_2 = \{R_j \boxdot U = T_j \mid j \in J\}$ has a solution iff $\bigcap_{2 j\in J} R_j^{-1} \lhd_{\Box} T_j$ is its solution.

Proof: Suppose ε_1 is solvable and R is its solution. Then $R \boxdot S_j = T_j$ for each $j \in J$. From the proof of Theorem 3.1 we get $R \subseteq_1 T_k \,{}_{\Box}\!\lhd S_k^{-1}$ for each $k \in J$. So $R \subseteq_1 \bigcap_{1 k\in J} T_k \,{}_{\Box}\!\lhd S_k^{-1}$, and for every $j \in J$ we have

$$\begin{aligned} T_j(x,z) &= \bigvee\nolimits_{3 y\in Y} R(x,y) \,\Box\, S_j(y,z) \\ &\le_3 \bigvee\nolimits_{3 y\in Y} \big(\bigwedge\nolimits_{1 k\in J, z\in Z} T_k(x,z) \,{}_{\Box}\!\circ\, S_k^{-1}(z,y)\big) \,\Box\, S_j(y,z) \\ &\le_3 \bigvee\nolimits_{3 y\in Y} \big(\bigwedge\nolimits_{1 z\in Z} T_j(x,z) \,{}_{\Box}\!\circ\, S_j^{-1}(z,y)\big) \,\Box\, S_j(y,z) \\ &\le_3 \bigvee\nolimits_{3 y\in Y} \big(T_j(x,z) \,{}_{\Box}\!\circ\, S_j(y,z)\big) \,\Box\, S_j(y,z) \le_3 T_j(x,z). \end{aligned}$$

Hence,

$$T_j \subseteq_3 (\bigcap_{1k \in J} T_k \,{}_{\square}\!\!\lhd S_k^{-1}) \boxdot S_j \subseteq_3 T_j$$

for every $j \in J$ proving that $\bigcap_{1k \in J} T_k \,{}_{\square}\!\!\lhd S_k^{-1}$ is a solution of ε_1. The proof ε_2 is similar. □

The structure of solutions of systems of equation is described by the following theorem.

Theorem 4.2: *If a system $\{U \boxdot S_j = T_j \mid j \in J\}$ is solvable then the set of all of its solutions along with $\subseteq_1$ forms a complete join-semilattice with the greatest element $\bigcap_{1 j \in J} T_j \,{}_{\square}\!\!\lhd S_j^{-1}$. If a system $\{R_j \boxdot U = T_j \mid j \in J\}$ is solvable then the set of all of its solutions along with $\subseteq_2$ forms a complete join-semilattice with the greatest element $\bigcap_{2 j \in J} R_j^{-1} \lhd_{\square} T_j$.*

Proof: Similar to that of Theorem 3.2. □

The following corollaries are analogous to the case of single equations.

Corollary 4.3: *A system $\{U \circ S_j = T_j \mid j \in J\}$ is solvable iff $\bigcap_{j \in J} (S_j \lhd T_j^{-1})^{-1}$ is its solution. A system $\{R_j \circ U = T_j \mid j \in J\}$ is solvable iff $\bigcap_{j \in J} R_j^{-1} \lhd T_j$ is its solution.*

Corollary 4.4: *A system $\{U \lhd S_j = T_j \mid j \in J\}$ is solvable iff $\bigcap_{j \in J} T_j \lhd S_j^{-1}$ is its solution. A system $\{R_j \lhd U = T_j \mid j \in J\}$ is solvable iff $\bigcup_{j \in J} R_j^{-1} \circ T_j$ is its solution.*

Corollary 4.5: *If a system $\{U \circ S_j = T_j \mid j \in J\}$ is solvable then a set of all its solutions along with $\subseteq$ forms a join-semilattice such that $\bigcap_{j \in J} (S_j \lhd T_j^{-1})^{-1}$ is the greatest element. If a system $\{R_j \circ U = T_j \mid j \in J\}$ is solvable then a set of all its solutions along with $\subseteq$ forms a join-semilattice such that $\bigcap_{j \in J} R_j^{-1} \lhd T_j$ is the greatest element.*

Corollary 4.6: *If a system $\{U \lhd S_j = T_j \mid j \in J\}$ is solvable then a set of all its solutions along with $\subseteq$ forms a join-semilattice such that $\bigcap_{j \in J} T_j \lhd S_j^{-1}$ is the greatest element. If a system $\{R_j \lhd U = T_j \mid j \in J\}$ is solvable then a set of all its solutions along with $\subseteq$ forms a meet-semilattice such that $\bigcup_{j \in J} R_j^{-1} \circ T_j$ is the least element.*

5. Fuzzy relational equations with constraints as a further particular type

In this section we show that the new, general type of fuzzy relational equations subsumes other types of fuzzy relational equations which may be of interest. As an example, we show that it subsumes fuzzy relational equations with constraints on solutions requiring that solutions be crisp (ordinary) relations. This is possible by taking the aggregation structure presented in the following example (cf. (Belohlavek 2010b)).

Example 5.1 Let $L_1 = \{0, 1\}$, $L_2 = [0, 1]$, $L_3 = [0, 1]$, let $\leq_1$, $\leq_2$, $\leq_3$ be the usual total orders on L_1, L_2, and L_3, respectively. Let $\square$ be defined by $a_1 \square a_2 = \min(a_1, a_2)$. Then $\langle \mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3, \square \rangle$ is an aggregation structure in which

$$0 \circ_{\square} a_3 = 1, \;\; 1 \circ_{\square} a_3 = a_3,$$

and

$$a_3 \,{}_{\Box}{\circ}\, a_2 = \begin{cases} 0 & \text{for } a_2 > a_3, \\ 1 & \text{for } a_2 \leq a_3. \end{cases}$$

Suppose we are interested in solutions of a sup-t-norm fuzzy relational equation $U \circ S = T$ that is a crisp relation, that is, $U(x, y) = 0$ or $U(x, y) = 1$ for every x, y. Results regarding such constrained fuzzy relational equations are covered by the results regarding the general type of equations. Namely, we have the following theorem.

Theorem 5.2: *Let $S \in [0,1]^{Y \times Z}$ and $T \in [0,1]^{X \times Z}$ be fuzzy relations. Consider the aggregation structure from Example 5.1. The solutions of $U \boxdot S = T$ are exactly the crisp solutions of $U \circ S = T$ (for any t-norm $\otimes$).*

Proof: Immediately from definitions taking into account that $a \Box b = a \otimes b$ for every t-norm $\otimes$. □

In a similar way, one may obtain crisp of otherwise constrained solutions of other types of fuzzy relational equations and their systems.

Example 5.3 Consider the aggregation structure from Example 2.2(a) with $\Box = \otimes$ being the Łukasiewicz t-norm. Suppose we need to find a solution of an equation $U \circ S = T$, where S and T are represented by matrices $\begin{pmatrix} 0.4 \\ 0.8 \end{pmatrix}$ and $\begin{pmatrix} 0.5 \\ 0.3 \end{pmatrix}$, respectively. One can check, using Corollary 3.3, that this equation is solvable and one of its solution is $(S \triangleleft T^{-1})^{-1}$ whose matrix is $\begin{pmatrix} 1 & 0.7 \\ 0.9 & 0.5 \end{pmatrix}$. But there is no crisp solution of $U \circ S = T$. This follows from Theorem 3.1 and 5.2 by checking that for the aggregation structure from Example 5.1, $T \,{}_{\Box}{\triangleleft}\, S^{-1}$ is not a solution of $U \boxdot S = T$.

On the other hand, an equation $U \circ S = T$, where

$$S = \begin{pmatrix} 0.4 & 0.9 \\ 0.7 & 0.5 \\ 0.8 & 0.6 \end{pmatrix},$$

$$T = \begin{pmatrix} 0.8 & 0.9 \\ 0.8 & 0.6 \end{pmatrix},$$

has a crisp solution

$$U = T \,{}_{\Box}{\triangleleft}\, S^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

6. Conclusions

The main result of this paper consists in showing that the well-known pairs of results regarding sup-t-norm and inf-residuum types of fuzzy relational equations are consequences of single results regarding a general type of fuzzy relational equations developed in the framework of sup-preserving aggregation structures.

Another benefit of working within the general framework of aggregation structures is the possibility of obtaining interesting results in a simple way as consequences of results obtained within the general framework. This is illustrated in this paper by an example concerning solvability of fuzzy relational equations under the requirement that solutions be crisp relations.

Several other issues regarding solutions of fuzzy relational equations, including the issue of approximate solutions, remain open for future research.

7. Acknowledgment

Supported by Grant No. 202/10/0262 of the Czech Science Foundation and by grant No. MSM 6198959214 of the Czech Ministry of Education. The paper is an extended version of the NAFIPS 2011 contribution “Reducing sup-t-norm and inf-residuum to a single type of fuzzy relational equations” by E. Bartl and R. Belohlavek.

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