THE BLOCK EXTENSION PROPERTY[1]

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Abstract We show that the block extension property and the principal block extension property are equivalent in permutable varieties, generalizing the result of A. Day [1].

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It is well-known that the Congruence Extension Property (briefly CEP) plays an important role in universal algebra. It was proved by A. Day [1] and G. Grätzer and H. Lakser [2] that a variety $\mathcal{V}$ satisfies CEP if and only if $\mathcal{V}$ satisfies the Principal Congruence Extension Property (PCEP). However, for some algebras, CEP or PCEP can be too strong but they could satisfy some weaker congruence conditions.

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Definition. An algebra $\mathcal{A} = \langle A, F\rangle$ satisfies the *Block Extension Property* (briefly BEP) if for every $\mathcal{B} = \langle B, F\rangle \in \operatorname{Sub}\mathcal{A}$, each $b \in B$ and every $\theta \in \operatorname{Con}\mathcal{B}$ there exists $\psi \in \operatorname{Con}\mathcal{A}$ such that $[b]_\theta = [b]_\psi \cap B$. $\mathcal{A}$ satisfies the *Principal Block Extension Property* (briefly PBEP) if for every $\mathcal{B} \in \operatorname{Sub}\mathcal{A}$ and every $x, y, b \in B$ there exists $\psi \in \operatorname{Con}\mathcal{A}$ such that $[b]_{\theta_B(x,y)} = [b]_\psi \cap B$.

Recall that $\theta_A(x, y)$ denotes the least congruence on $\mathcal{A}$ containing the pair $\langle x, y\rangle$. Notice that if $\mathcal{A}$ satisfies PBEP then $[b]_{\theta_B(a,b)} = [b]_{\theta_A(a,b)} \cap B$ for any $\mathcal{B} \in \operatorname{Sub}\mathcal{A}$, $a, b \in B$. We are going to show that there are algebras satisfying BEP but not CEP.

Example 1. Let $\mathcal{A} = \langle A, \cdot, a, b, c, d\rangle$ be an algebra of type $\langle 2, 0, 0, 0, 0\rangle$ with $A = \{a, b, c, d, x\}$ and the binary operation is given by setting

$\cdot$	a	b	c	d	x
a	c	c	b	d	c
b	c	c	c	d	d
c	a	a	a	a	a
d	b	b	b	b	x
x	d	d	d	d	d

Here, $B = \{a, b, c, d\}$ is the only non-trivial subuniverse of $\mathcal{A}$. There are only two non-trivial congruences on $\mathcal{B}$ given by the partitions

$$\begin{aligned} B/\theta_1 &= \{\{a, b\}, \{c\}, \{d\}\} \quad \text{and} \\ B/\theta_2 &= \{\{a, b\}, \{c, d\}\}. \end{aligned}$$

The algebra $\mathcal{A}$ has only one non-trivial congruence ϕ given by $A/\phi = \{\{a, b, x\}, \{c, d\}\}$. Hence, $\mathcal{A}$ has BEP:

$$\begin{aligned} \{a, b\} &= \{a, b, x\} \cap B \\ \{c, d\} &= \{c, d\} \cap B \\ \{c\} &= \{c\} \cap B \\ \{d\} &= \{d\} \cap B \end{aligned}$$

where $\{c\}$, $\{d\}$ are blocks of the trivial congruence (i.e. the identity) ω_A on $\mathcal{A}$. On the other hand, $\mathcal{A}$ does not satisfy CEP since $\theta_1 \in \operatorname{Con}\mathcal{B}$ has no extension on $\mathcal{A}$. Let us note that both $\mathcal{A}$ and $\mathcal{B}$ are congruence permutable since $\operatorname{Con}\mathcal{A}$ as well as $\operatorname{Con}\mathcal{B}$ are chains.

Of course, BEP implies PBEP. We show that there are algebras satisfying PBEP but not BEP.

Example 2. Let $A = \{a, b, c, d, x\}$ and $\mathcal{A} = \langle A, \cdot\rangle$ be a groupoid given by the table

$\cdot$	a	b	c	d	x
a	a	a	a	a	b
b	a	a	a	a	a
c	a	a	a	a	d
d	a	a	a	a	c
x	b	a	d	c	x

We show that $\mathcal{A}$ satisfies PBEP. Evidently, we have to test only the 4- and 3-element subalgebras $\mathcal{B}$ of $\mathcal{A}$. The only 4-element subuniverse of $\mathcal{A}$ is $B = \{a, b, c, d\}$. It is easy to see that $\theta_B(u, v) = \omega_B \cup \{\langle u, v\rangle, \langle v, u\rangle\}$ holds for all $u, v \in B$. There are four 3-element subuniverses: $B_1 = \{a, b, c\}$, $B_2 = \{a, b, d\}$, $B_3 = \{a, c, d\}$, and $B_4 = \{a, b, x\}$. Again, for $i = 1, 2, 3$, we have $\theta_{B_i}(u, v) = \omega_{B_i} \cup \{\langle u, v\rangle, \langle v, u\rangle\}$ for any $u, v \in B_i$. Furthermore, $\theta_{B_4}(a, b)$ corresponds to the partition $\{\{a, b\}, \{x\}\}$, while both $\theta_{B_4}(a, x)$ and $\theta_{B_4}(b, x)$ correspond to $\{\{a, b, x\}\}$. Hence, all the blocks of all principal congruences of the non-trivial subalgebras of $\mathcal{A}$ are all singletons $\{u\}$ $(u \in A)$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, and $\{a, b, x\}$ (the last one is a block on B_4). To see that each of the blocks can be extended by a block of some $\psi \in \operatorname{Con}\mathcal{A}$, i.e. to see that $\mathcal{A}$ satisfies PBEP, it is enough to consider the following (principal) congruences on $\mathcal{A}$:

$$\begin{array}{lll} \theta_A(a,b) & \text{with the partition} & \{\{a,b\},\{c\},\{d\},\{x\}\}, \\ \theta_A(a,c) & \text{with the partition} & \{\{a,c\},\{b,d\},\{x\}\}, \\ \theta_A(a,d) & \text{with the partition} & \{\{a,d\},\{b,c\},\{x\}\}, \\ \theta_A(c,d) & \text{with the partition} & \{\{c,d\},\{a\},\{b\},\{x\}\}, \\ \theta_A(a,x) & \text{with the partition} & \{\{a,b,c,d,x\}\}. \end{array}$$

However, $\mathcal{A}$ does not satisfy BEP. Indeed, the congruence

$$\phi = \theta_B(a, b) \vee \theta_B(b, c) \vee \theta_B(a, c)$$

in $\operatorname{Con}\mathcal{B}$ has the block $\{a, b, c\}$ which cannot be extended (it can be easily shown that in $\operatorname{Con}\mathcal{A}$, $\theta_A(a, b) \vee \theta_A(b, c) \vee \theta_A(a, c)$ corresponds to the partition $\{\{a, b, c, d\}, \{x\}\}$). Therefore, $\mathcal{A}$ satisfies PBEP but not BEP.

An algebra $\mathcal{A}$ is *permutable* if $\theta \circ \phi = \phi \circ \theta$ for each $\theta, \phi \in \operatorname{Con} \mathcal{A}$. A variety $\mathcal{V}$ is *permutable* if each $\mathcal{A} \in \mathcal{V}$ has this property.

Lemma. *Let $\mathcal{V}$ be a permutable variety, $\mathcal{A} = \langle A, F \rangle \in \mathcal{V}$ and $\mathcal{S} = \langle S, F \rangle$ be a subalgebra of $\mathcal{A}$. Suppose $d \in S$, $\phi \in \operatorname{Con} \mathcal{S}$ and $\theta \in \operatorname{Con} \mathcal{A}$. If $[d]_\theta \cap S \subseteq [d]_\phi$ then $[[d]_\phi]_\theta \cap S = [d]_\phi$ and $[[d]_\phi]_\theta$ is a block of some congruence on the subalgebra $S^\theta = \{x \in A;\ [x]_\theta \cap S \neq \emptyset\}$ of $\mathcal{A}$.*

Proof. It is easy to check that $\mathcal{S}^\theta$ is a subalgebra of $\mathcal{A}$ containing S. Since $[d]_\phi \subseteq [[d]_\phi]_\theta$ and $[d]_\phi \subseteq S$ we have $[d]_\phi \subseteq [[d]_\phi]_\theta \cap S$. On the contrary, suppose $a \in [[d]_\phi]_\theta \cap S$ and take $\psi = \theta \cap (S \times S)$. Of course, $\psi \in \operatorname{Con} \mathcal{S}$. Then $a \in [[d]_\phi]_\psi = [d]_{\psi \circ \phi}$. Hence, there is $b \in [d]_\phi$ with $\langle a, b \rangle \in \psi$, $\langle b, d \rangle \in \phi$. In account of permutability of $\mathcal{A}$, there exists $x \in S$ such that $\langle a, x \rangle \in \phi$ and $\langle x, d \rangle \in \psi$. Thus $x \in [d]_\psi = [d]_\theta \cap S$. By the assumption of Lemma, $x \in [d]_\phi$ thus $\langle a, x \rangle \in \phi$ implies also $a \in [d]_\phi$.

It remains to show that $[[d]_\phi]_\theta = [d]_{\theta \circ \phi}$ is a block of some $\psi \in \operatorname{Con} \mathcal{S}^\theta$. Put $\psi = \theta_{S^\theta}(\{d\} \times [d]_{\theta \circ \phi})$, the least congruence relation on $\mathcal{S}^\theta$ containing $\{d\} \times [d]_{\theta \circ \phi}$. We show $[d]_{\theta \circ \phi} = [d]_\psi$. Clearly, $[d]_{\theta \circ \phi} \subseteq [d]_\psi$. Conversely, let $x \in [d]_\psi$, i.e. $\langle x, d \rangle \in \psi$. By [3], there are unary algebraic functions $\tau_i(\xi) = t_i(\xi, a_1, \ldots, a_m)$ (where $a_i \in S^\theta$, i.e. $\langle a_i, s_i \rangle \in \theta$ for some $s_i \in S$) and $c_i, d_i \in [d]_{\theta \circ \phi}$ (i.e. $\langle c_i, c_i' \rangle \in \theta$, $\langle c_i', d \rangle \in \phi$, $\langle d_i, d_i' \rangle \in \theta$, $\langle d_i', d \rangle \in \phi$ for some $c_i', d_i' \in S$) for $i = 1, \ldots, n$, such that $x = \tau_1(c_1)$, $\tau_i(d_i) = \tau_{i+1}(c_{i+1})$, $\tau_n(d_n) = d$. We prove by induction for $i = n, \ldots, 1$, that $\tau_i(c_i) \in [d]_{\theta \circ \phi}$, i.e. in particular $x = \tau_1(c_1) \in [d]_{\theta \circ \phi}$.
For $i = n$ we have

$$\tau_n(c_n) = t_n(c_n, a_1, \ldots, a_m)\,\theta\,t_n(c_n', s_1, \ldots, s_m)\,\phi\,t_n(d_n', s_1, \ldots, s_m)$$

(since $\langle c_n', d_n' \rangle \in \phi$) and

$$t_n(d_n', s_1, \ldots, s_m)\,\theta\,t_n(d_n, a_1, \ldots, a_m) = d$$

and from $t_n(d_n', s_1, \ldots, s_m) \in S$ and $[d]_\theta \cap S \subseteq [d]_\phi$ we have $\langle t_n(d_n', s_1, \ldots, s_m), d \rangle \in \phi$. To sum up, $\tau_n(c_n) \in [d]_{\theta \circ \phi}$.
Suppose the assertion holds for $i + 1$. Similarly as above one obtains $\langle \tau_i(c_i), \tau_i(d_i) \rangle \in \theta \circ \phi \circ \theta$. Since $\tau_i(d_i) = \tau_{i+1}(c_{i+1})$ and $\langle \tau_{i+1}(c_{i+1}), d \rangle \in \theta \circ \phi$ (inductive assumption), we conclude $\langle \tau_i(c_i), d \rangle \in \theta \circ \phi \circ \theta \circ \theta \circ \phi = \theta \circ \phi \circ \theta \circ \phi$, i.e. $\langle \tau_i(c_i), x \rangle \in \theta \circ \phi$ and $\langle x, d \rangle \in \theta \circ \phi$ for some $x \in S$. Let p be the Mal'cev term [3] of the variety $\mathcal{V}$, i.e. $p(z, z, x) = x = p(x, z, z)$. As can be easily verified, $\theta \circ \phi$ is a compatible relation, thus, by $\langle x, x \rangle \in \theta \circ \phi$, it holds

$\langle \tau_i(c_i), d\rangle = \langle p(\tau_i(c_i), x, x), p(x, x, d)\rangle \in \theta \circ \phi$, i.e. $\tau_i(c_i) \in [d]_{\theta\circ\phi}$, finishing the proof. □

We are ready to generalize the theorem of A. Day [1] for congruence blocks in permutable varieties. The proof is based on the inductive method used in [2].

Theorem. *Let $\mathcal{V}$ be a permutable variety and $\mathcal{A} \in \mathcal{V}$. Then $\mathcal{A}$ satisfies BEP if and only if $\mathcal{A}$ satisfies PBEP.*

Proof. We need only to show that for each algebra $\mathcal{A}$ of a permutable variety $\mathcal{V}$, PBEP implies BEP. For any $\mathcal{C} \in \operatorname{Sub}\mathcal{A}$, $d \in C$, $\theta \in \operatorname{Con}\mathcal{C}$, put $\Theta = \theta_A(\{d\} \times [d]_\theta) \in \operatorname{Con}\mathcal{A}$, the least congruence containing $\{d\} \times [d]_\theta$. Let now $\mathcal{B} = \langle B, F\rangle$ be a subalgebra of $\mathcal{A}$. We show that each subalgebra $\mathcal{C}$ of $\mathcal{A}$ satisfying $B \subseteq C$ has BEP, thus, in particular, $\mathcal{B}$ has BEP. Let $d \in C$, $\theta \in \operatorname{Con}\mathcal{C}$. Since, obviously, $[d]_\theta \subseteq [d]_\Theta \cap C$, we need to show $[d]_\Theta \cap C \subseteq [d]_\theta$, i.e. $\langle a, d\rangle \in \Theta$ implies $\langle a, d\rangle \in \theta$ for each $a \in C$. By $\Theta = \bigvee_{x,y\in[d]_\theta} \theta_A(x, y)$ and using the well-known description of the join of congruence relations, we have to prove that

$$\begin{aligned} &a \in C,\ a_i, b_i \in [d]_\theta,\ x_i \in A,\ a = x_0, x_n = d,\ \langle x_{i-1}, x_i\rangle \in \theta_A(a_i, b_i) \\ &(i = 1, \ldots, n) \quad \text{implies} \quad \langle a, d\rangle \in \theta \end{aligned} \tag{1}$$

holds for each $\mathcal{C} \in \operatorname{Sub}\mathcal{A}$ satisfying $B \subseteq C$ and every $d \in C$, $\theta \in \operatorname{Con}\mathcal{C}$. We prove (1) by induction over n.

For $n = 1$, we have to prove that if $a \in C$, $a_1, b_1 \in [d]_\theta$, $\langle a, d\rangle \in \theta_A(a_1, b_1)$ then $\langle a, d\rangle \in \theta$. By PBEP, $[b_1]_{\theta_A(a_1,b_1)} \cap C = [b_1]_{\theta_C(a_1,b_1)} \subseteq [b_1]_\theta$. Applying Lemma (put $d = b_1$, $\phi = \theta$, $\theta = \theta_A(a_1, b_1)$), the fact $a \in [b_1]_{\theta_A(a_1,b_1)\circ\theta} \cap C$ yields $a \in [b_1]_\theta = [d]_\theta$, i.e. $\langle a, d\rangle \in \theta$.

Now, we prove the assertion for n provided it is valid for $n - 1$. Put

$$\theta' = \theta_C(a_1, b_1) \vee \cdots \vee \theta_C(a_n, b_n) \vee \theta_C(a_1, d) \vee \cdots \vee \theta_C(a_n, d)$$

and $D = C^{\theta_A(a_n,b_n)}$. By PBEP, $[b_n]_{\theta_A(a_n,b_n)} \cap C = [b_n]_{\theta_C(a_n,b_n)} \subseteq [b_n]_{\theta'}$. Lemma yields a congruence relation $\psi \in \operatorname{Con}\mathcal{D}$ such that $[b_n]_\psi = [b_n]_{\theta_A(a_n,b_n)\circ\theta'}$ and $[b_n]_\psi \cap C = [b_n]_{\theta_A(a_n,b_n)\circ\theta'}$. Clearly, $[b_n]_{\theta_A(a_n,b_n)\circ\theta'} = [d]_{\theta_A(a_n,b_n)\circ\theta'}$, hence also $[b_n]_\psi = [d]_\psi$. Since $\theta_A(a_n, b_n) \subseteq \psi$ and $\theta' \subseteq \psi$, it holds $\theta_A(a_n, b_n)\circ\theta' \subseteq \psi\circ\psi = \psi$. From $\langle x_{n-1}, d\rangle = \langle x_{n-1}, x_n\rangle \in \theta_A(a_n, b_n)$ and $\langle d, a_i\rangle \in \theta'$, $\langle d, b_i\rangle \in \theta'$ we therefore conclude $\langle x_{n-1}, a_i\rangle \in \psi$, $\langle x_{n-1}, b_i\rangle \in \psi$, i.e. $a_i, b_i \in [x_{n-1}]_\psi$. It is now immediate that A, D, $a = x_0, x_1, \ldots, x_{n-1}$, and ψ satisfy the conditions of (1) for $n-1$, hence $\langle a, x_{n-1}\rangle \in \psi$ by inductive assumption. Furthermore, $\langle x_{n-1}, x_n\rangle \in \theta_A(a_n, b_n) \subseteq \psi$, so $\langle a, d\rangle = \langle a, x_n\rangle \in \psi$. We therefore have $a \in [d]_\psi \cap C = [d]_{\theta'}$, i.e. $\langle a, d\rangle \in \theta'$. Finally, $\theta' \subseteq \theta$ gives $a \in [d]_\theta$ which has to be proved. □

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