THE BLOCK EXTENSION PROPERTY\(^1\)

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Abstract We show that the block extension property and the principal block extension property are equivalent in permutable varieties, generalizing the result of A. Day [1].

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It is well-known that the Congruence Extension Property (briefly CEP) plays an important role in universal algebra. It was proved by A. Day [1] and G. Grätzer and H. Lakser [2] that a variety \( \mathcal{V} \) satisfies CEP if and only if \( \mathcal{V} \) satisfies the Principal Congruence Extension Property (PCEP). However, for some algebras, CEP or PCEP can be too strong but they could satisfy some weaker congruence conditions.

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Definition. An algebra \( A = \langle A, F \rangle \) satisfies the Block Extension Property (briefly BEP) if for every \( B = \langle B, F \rangle \in \text{Sub } A \), each \( b \in B \) and every \( \theta \in \text{Con } B \) there exists \( \psi \in \text{Con } A \) such that \( [b]_\theta = [b]_\psi \cap B \). \( A \) satisfies the Principal Block Extension Property (briefly PBEP) if for every \( B \in \text{Sub } A \) and every \( x, y, b \in B \) there exists \( \psi \in \text{Con } A \) such that \( [b]_{\theta_B(x,y)} = [b]_\psi \cap B \).

Recall that \( \theta_A(x,y) \) denotes the least congruence on \( A \) containing the pair \( \langle x, y \rangle \). Notice that if \( A \) satisfies PBEP then \( [b]_{\theta_B(a,b)} = [b]_{\theta_A(a,b)} \cap B \) for any \( B \in \text{Sub } A \), \( a, b \in B \). We are going to show that there are algebras satisfying BEP but not CEP.

Example 1. Let \( A = \langle A, \cdot, a, b, c, d \rangle \) be an algebra of type \( \langle 2, 0, 0, 0, 0 \rangle \) with \( A = \{a, b, c, d, x\} \) and the binary operation is given by setting

\[
|   & a & b & c & d & x \\
- & - & - & - & - & - \\
a & c & c & b & d & c \\
b & c & c & c & d & d \\
c & a & a & a & a \\
d & b & b & b & x \\
x & d & d & d & d
\]

Here, \( B = \{a, b, c, d\} \) is the only non-trivial subuniverse of \( A \). There are only two non-trivial congruences on \( B \) given by the partitions

\[
B/\theta_1 = \{\{a, b\}, \{c\}, \{d\}\} \quad \text{and} \quad B/\theta_2 = \{\{a, b\}, \{c, d\}\}.
\]

The algebra \( A \) has only one non-trivial congruence \( \phi \) given by \( A/\phi = \{\{a, b, x\}, \{c, d\}\} \). Hence, \( A \) has BEP:

\[
\{a, b\} = \{a, b, x\} \cap B \\
\{c, d\} = \{c, d\} \cap B \\
\{c\} = \{c\} \cap B \\
\{d\} = \{d\} \cap B
\]

where \( \{c\}, \{d\} \) are blocks of the trivial congruence (i.e. the identity) \( \omega_A \) on \( A \). On the other hand, \( A \) does not satisfy CEP since \( \theta_1 \in \text{Con } B \) has no
extension on \( A \). Let us note that both \( A \) and \( B \) are congruence permutable since \( \text{Con} A \) as well as \( \text{Con} B \) are chains.

Of course, BEP implies PBEP. We show that there are algebras satisfying PBEP but not BEP.

**Example 2.** Let \( A = \{a, b, c, d, x\} \) and \( A = \langle A, \cdot \rangle \) be a groupoid given by the table

\[
\begin{array}{cccc|c}
\cdot & a & b & c & d & x \\
\hline
a & a & a & a & a & b \\
b & a & a & a & a & a \\
c & a & a & a & a & c \\
d & a & a & a & a & d \\
x & b & a & d & c & x \\
\end{array}
\]

We show that \( A \) satisfies PBEP. Evidently, we have to test only the 4- and 3-element subalgebras \( B \) of \( A \). The only 4-element subuniverse of \( A \) is \( B = \{a, b, c, d\} \). It is easy to see that for all \( u, v \in B \),

\[ \theta_B(u, v) = \omega_B \cup \{(u, v), (v, u)\} \]

holds. There are four 3-element subuniverses:

\[ B_1 = \{a, b, c\}, \ B_2 = \{a, b, d\}, \ B_3 = \{a, c, d\}, \ B_4 = \{a, b, x\} \]

For \( i = 1, 2, 3 \), we have \( \theta_{B_i}(u, v) = \omega_{B_i} \cup \{(u, v), (v, u)\} \) for any \( u, v \in B_i \). Furthermore, \( \theta_{B_1}(a, b) \) corresponds to the partition \( \{\{a, b\}, \{x\}\} \), while both \( \theta_{B_4}(a, x) \) and \( \theta_{B_4}(b, x) \) correspond to \( \{\{a, b, x\}\} \). Hence, all the blocks of all principal congruences of the non-trivial subalgebras of \( A \) are all singletons \( \{u\} \) \( (u \in A) \), \( \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \) and \( \{a, b, x\} \) (the last one is a block on \( B_4 \)). To see that each of the blocks can be extended by a block of some \( \psi \in \text{Con} A \), i.e. to see that \( A \) satisfies PBEP, it is enough to consider the following (principal) congruences on \( A \):

\[
\begin{align*}
\theta_A(a, b) & \text{ with the partition } \{\{a, b\}, \{c\}, \{d\}, \{x\}\}, \\
\theta_A(a, c) & \text{ with the partition } \{\{a, c\}, \{b, d\}, \{x\}\}, \\
\theta_A(a, d) & \text{ with the partition } \{\{a, d\}, \{b, c\}, \{x\}\}, \\
\theta_A(c, d) & \text{ with the partition } \{\{c, d\}, \{a\}, \{b\}, \{x\}\}, \\
\theta_A(a, x) & \text{ with the partition } \{\{a, b, c, d, x\}\}.
\end{align*}
\]

However, \( A \) does not satisfy BEP. Indeed, the congruence

\[ \phi = \theta_B(a, b) \lor \theta_B(b, c) \lor \theta_B(a, c) \]

in \( \text{Con} B \) has the block \( \{a, b, c\} \) which cannot be extended (it can be easily shown that in \( \text{Con} A \), \( \theta_A(a, b) \lor \theta_A(b, c) \lor \theta_A(a, c) \) corresponds to the partition \( \{\{a, b, c, d\}, \{x\}\} \)). Therefore, \( A \) satisfies PBEP but not BEP.
An algebra $A$ is **permutable** if $\theta \circ \phi = \phi \circ \theta$ for each $\theta, \phi \in \text{Con} A$. A variety $V$ is **permutable** if each $A \in V$ has this property.

**Lemma.** Let $V$ be a permutable variety, $A = \langle A, F \rangle \in V$ and $S = \langle S, F \rangle$ be a subalgebra of $A$. Suppose $d \in S$, $\phi \in \text{Con} S$ and $\theta \in \text{Con} A$. If $[d]_\theta \cap S \subseteq [d]_\phi$ then $[[d]_\phi]_\theta \cap S = [d]_\phi$ and $[[d]_\phi]_\theta$ is a block of some congruence on the subalgebra $S^\theta = \{ x \in A; [x]_\theta \cap S \neq \emptyset \}$ of $A$.

**Proof.** It is easy to check that $S^\theta$ is a subalgebra of $A$ containing $S$. Since $[d]_\phi \subseteq [[d]_\phi]_\theta$ and $[d]_\phi \subseteq S$ we have $[d]_\phi \subseteq [[d]_\phi]_\theta \cap S$. On the contrary, suppose $a \in [[d]_\phi]_\theta \cap S$ and take $\psi = \theta \cap (S \times S)$. Of course, $\psi \in \text{Con} S$. Then $a \in [[d]_\phi]_\psi = [d]_{\psi \circ \phi}$. Hence, there is $b \in [d]_\phi$ with $\langle a, b \rangle \in \psi$, $\langle b, d \rangle \in \phi$.

In account of permutability of $A$, there exists $x \in S$ such that $\langle a, x \rangle \in \phi$ and $\langle x, d \rangle \in \psi$. Thus $x \in [d]_\psi = [d]_\theta \cap S$. By the assumption of Lemma, $x \in [d]_\phi$ thus $\langle a, x \rangle \in \phi$ implies also $a \in [d]_\phi$.

It remains to show that $[[d]_\phi]_\theta = [d]_{\theta \circ \phi}$ is a block of some $\psi \in \text{Con} S^\theta$. Put $\psi = \theta_{S^\theta} \{ d \} \times [d]_{\theta \circ \phi}$, the least congruence relation on $S^\theta$ containing $\{ d \} \times [d]_{\theta \circ \phi}$. We show $[d]_{\theta \circ \phi} = [d]_\psi$. Clearly, $[d]_{\theta \circ \phi} \subseteq [d]_\psi$. Conversely, let $x \in [d]_\psi$, i.e. $\langle x, d \rangle \in \psi$. By [3], there are unary algebraic functions $\tau_i (\xi) = t_i (\xi, a_1, \ldots, a_m)$ (where $a_i \in S^\theta$, i.e. $\langle a_i, s_i \rangle \in \theta$ for some $s_i \in S$) and $c_i, d_i \in [d]_{\theta \circ \phi}$ (i.e. $\langle c_i, c'_i \rangle \in \theta$, $\langle c'_i, d'_i \rangle \in \phi$, $\langle d_i, d'_i \rangle \in \theta$, $\langle d_i, d' \rangle \in \phi$ for some $c'_i, d'_i \in S$) for $i = 1, \ldots, n$, such that $x = \tau_1 (c_1), \tau_i (c_i) = \tau_{i+1} (c_{i+1}), \tau_n (d_n) = d$. We prove by induction for $i = n, \ldots, 1$, that $\tau_i (c_i) \in [d]_{\theta \circ \phi}$, i.e. in particular $x = \tau_1 (c_1) \in [d]_{\theta \circ \phi}$.

For $i = n$ we have

$$
\tau_n (c_n) = t_n (c_n, a_1, \ldots, a_m) \theta t_n (c'_n, s_1, \ldots, s_m) \phi t_n (d'_n, s_1, \ldots, s_m)
$$

(since $\langle c'_n, d'_n \rangle \in \phi$) and

$$
t_n (d', s_1, \ldots, s_m) \theta t_n (a_1, \ldots, a_n) = d
$$

and from $t_n (d', s_1, \ldots, s_m) \in S$ and $[d]_\theta \cap S \subseteq [d]_\phi$ we have $\langle t_n (d', s_1, \ldots, s_m), d \rangle \in \phi$. To sum up, $\tau_n (c_n) \in [d]_{\theta \circ \phi}$.

Suppose the assertion holds for $i + 1$. Similarly as above one obtains $\langle \tau_i (c_i), \tau_i (d_i) \rangle \in \theta \circ \phi \circ \theta$. Since $\tau_i (d_i) = \tau_{i+1} (c_{i+1})$ and $\langle \tau_{i+1} (c_{i+1}), d \rangle \in \theta \circ \phi$ (inductive assumption), we conclude $\langle \tau_i (c_i), d \rangle \in \theta \circ \phi \circ \theta \circ \phi = \theta \circ \phi \circ \theta \circ \phi$, i.e. $\langle \tau_i (c_i), x \rangle \in \theta \circ \phi$ and $\langle x, d \rangle \in \theta \circ \phi$ for some $x \in S$. Let $p$ be the Mal’cev term [3] of the variety $V$, i.e. $p(z, z, x) = x = p(x, z, z)$. As can be easily verified, $\theta \circ \phi$ is a compatible relation, thus, by $\langle x, x \rangle \in \theta \circ \phi$, it holds
\[ \langle \tau_i(c_i), d \rangle = \langle p(\tau_i(c_i), x, x), p(x, x, d) \rangle \in \theta \circ \phi, \text{ i.e. } \tau_i(c_i) \in [d]_{\theta \circ \phi}, \text{ finishing the proof.} \]

We are ready to generalize the theorem of A. Day [1] for congruence blocks in permutable varieties. The proof is based on the inductive method used in [2].

**Theorem.** Let \( V \) be a permutable variety and \( A \in V \). Then \( A \) satisfies BEP if and only if \( A \) satisfies PBEP.

**Proof.** We need only to show that for each algebra \( A \) of a permutable variety \( V \), PBEP implies BEP. For any \( C \subseteq \text{Sub} \ A \), \( d \in C \), \( \theta \in \text{Con} \ C \), put \( \Theta = \theta_A(\{d\} \times [d]_\theta) \in \text{Con} \ A \), the least congruence containing \( \{d\} \times [d]_\theta \). Let now \( B = \langle B, F \rangle \) be a subalgebra of \( A \). We show that each subalgebra \( C \) of \( A \) satisfying \( B \subseteq C \) has BEP, thus, in particular, \( B \) has BEP. Let \( d \in C \), \( \theta \in \text{Con} \ C \). Since, obviously, \([d]_\theta \subseteq [d]_\theta \cap C \subseteq [d]_\theta \), i.e. \( \langle a, d \rangle \in \Theta \) implies \( \langle a, d \rangle \in \theta \) for each \( a \in C \). By \( \Theta = \bigvee_{x,y \in [d]_\theta} \theta_A(x, y) \) and using the well-known description of the join of congruence relations, we have to prove that

\[
\begin{align*}
\text{holds for each } C \subseteq \text{Sub} \ A \text{ satisfying } B \subseteq C \text{ and every } d \in C, \theta \in \text{Con} \ C. \\
\text{We prove (1) by induction over } n.
\end{align*}
\]

For \( n = 1 \), we have to prove that if \( a \in C, a_1, b_1 \in [d]_\theta, \langle a, d \rangle \in \theta_A(a_1, b_1) \) then \( \langle a, d \rangle \in \theta \). By PBEP, \( [b_1]_{\theta_A(a_1, b_1)} \cap C = [b_1]_{\theta_C(a_1, b_1)} \subseteq [b_1]_{\theta} \). Applying Lemma (put \( d = b_1, \phi = \theta, \theta = \theta_A(a_1, b_1) \)), the fact \( a \in [b_1]_{\theta_A(a_1, b_1)} \cap C \rightarrow [b_1]_{\theta_A(a_1, b_1)} \cap C \) yields \( a \in [b_1]_{\theta} \), i.e. \( \langle a, d \rangle \in \theta \).

Now, we prove the assertion for \( n \) provided it is valid for \( n - 1 \). Put

\[
\theta' = \theta_C(a_1, b_1) \vee \cdots \vee \theta_C(a_n, b_n) \vee \theta_C(a_1, d) \vee \cdots \vee \theta_C(a_n, d)
\]

and \( D = C^{\theta_A(a_n, b_n)} \). By PBEP, \( [b_n]_{\theta_A(a_n, b_n)} \cap C = [b_n]_{\theta_C(a_n, b_n)} \subseteq [b_n]_{\theta'} \). Lemma yields a congruence relation \( \psi \in \text{Con} \ D \) such that \( [b_n]_{\psi} = [b_n]_{\theta_A(a_n, b_n) \circ \theta'} \) and \( [b_n]_{\psi} \cap C = [b_n]_{\theta_A(a_n, b_n) \circ \theta' \circ \psi} \). Clearly, \( [b_n]_{\theta_A(a_n, b_n) \circ \theta'} = [d]_{\theta_A(a_n, b_n) \circ \theta'} \), hence also \( [b_n]_{\psi} = [d]_{\psi} \). Since \( \theta_A(a_n, b_n) \subseteq \psi \) and \( \theta' \subseteq \psi \), it holds \( \theta_A(a_n, b_n) \circ \theta' \subseteq \psi \circ \psi = \psi \). From \( \langle x_{n-1}, d \rangle = \langle x_{n-1}, x_n \rangle \in \theta_A(a_n, b_n) \) and \( \langle d, a_i \rangle \in \theta', \langle d, b_i \rangle \in \theta' \) we therefore conclude \( \langle x_{n-1}, a_i \rangle \in \psi, \langle x_{n-1}, b_i \rangle \in \psi \), i.e. \( a_i, b_i \in [x_{n-1}]_\psi \). It is now immediate that \( A, D, a = x_0, x_1, \ldots, x_{n-1}, \text{ and } \psi \) satisfy the conditions
of (1) for \( n - 1 \), hence \( \langle a, x_{n-1} \rangle \in \psi \) by inductive assumption. Furthermore, \( \langle x_{n-1}, x_n \rangle \in \theta_A(a_n, b_n) \subseteq \psi \), so \( \langle a, d \rangle = \langle a, x_n \rangle \in \psi \). We therefore have \( a \in [d]_\psi \cap C = [d]_\theta \), i.e. \( \langle a, d \rangle \in \theta' \). Finally, \( \theta' \subseteq \theta \) gives \( a \in [d]_\theta \) which has to be proved.

\[ \square \]

**References**

