

# Congruence classes in regular varieties

RADIM BĚLOHLÁVEK

*Institute for Research and Applications of Fuzzy Modeling, University of  
Ostrava, Bráfova 7, 701 03 Ostrava, Czech Republic, e-mail:  
belohlav@osu.cz*

and

*Department of Computer Science, Technical University of Ostrava,  
tř. 17. listopadu, 708 33 Ostrava-Poruba, Czech Republic,*

IVAN CHAJDA

*Department of Algebra and Geometry, Palacký University Olomouc,  
Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: chajda@risc.upol.cz*

## Abstract

Characterizations of congruence classes of algebras of regular varieties are presented. The problem of deciding whether a given subset of an algebra of regular variety is a congruence class is shown to be solvable in polynomial time.

**1991 Mathematics Subject Classification:** 08A30,  
08A40, 08B05, 68Q25

It has been proved by A. I. Maľcev [6] that a nonempty subset  $C \subseteq A$  of the support of an algebra  $\mathcal{A} = (A, F)$  is a class of some congruence relation on  $\mathcal{A}$  if and only if

$$\text{either } \tau(C) \cap C = \emptyset \quad \text{or} \quad \tau(C) \subseteq C$$

for any unary polynomial  $\tau$  of  $\mathcal{A}$ . This characterization, whatever useful, is not much efficient. In [1], the authors found a simple characterization of congruence classes of algebras from varieties which are both regular and permutable. They also showed that the decision problem of being a congruence class for algebras from a given regular and permutable variety is solvable in polynomial time. In this paper we give a characterization of congruence classes of algebras from regular varieties.

Recall that an algebra  $\mathcal{A} = (A, F)$  is *regular* if  $\theta = \Phi$  for  $\theta, \Phi \in \text{Con } \mathcal{A}$  whenever they have a congruence class in common.  $\mathcal{A}$  is *n-permutable* if  $\theta \circ \phi \circ \theta \circ \dots = \phi \circ \theta \circ \phi \circ \dots$  ( $n$  factors in both relational products) for every  $\theta, \phi \in \text{Con } \mathcal{A}$ . A variety  $\mathcal{V}$  is regular or *n-permutable* if each  $\mathcal{A} \in \mathcal{V}$  has this property.

Regular varieties have been characterized independently by B. Csákány, G. Grätzer and R. Wille in 1970s. For our purposes we present a Malcev condition which is rather similar to that one of R. Wille (cf. Theorem 6.11 in [8]).

**Theorem 1.** *A variety  $\mathcal{V}$  is regular if and only if there exist a positive integer  $n$ , ternary terms  $t_1, \dots, t_n$ , and 5-ary terms  $p_1, \dots, p_n$  such that*

$$\begin{aligned} t_i(x, x, z) &= z \quad \text{for } i = 1, \dots, n \\ x &= p_1(t_1(x, y, z), z, x, y, z) \\ p_i(z, t_i(x, y, z), x, y, z) &= p_{i+1}(t_{i+1}(x, y, z), z, x, y, z) \quad i = 1, \dots, n-1 \\ y &= p_n(t_n(x, y, z), z, x, y, z) . \end{aligned}$$

*Proof:* Let  $\mathcal{V}$  be a regular variety,  $F_{\mathcal{V}}(x, y, z) \in \mathcal{V}$  be a free algebra generated by  $x, y$  and  $z$ , let further  $\theta = \theta(x, y)$ ,  $C = [z]_{\theta}$ . For  $\theta(x, y)$  and  $\theta(C \times \{z\})$  have the class  $C$  in common, it follows from regularity that  $\theta(x, y) = \theta(C \times \{z\})$ . We have therefore  $\langle x, y \rangle \in \theta(C \times \{z\})$ . The compactness of congruence lattice implies that there is a finite subset  $\{d_1, \dots, d_k\} \subseteq C$  such that  $\langle x, y \rangle \in \theta(\{d_1, \dots, d_k\} \times \{z\})$ . By Malcev lemma, there are  $e_1, \dots, e_m \in F_{\mathcal{V}}(x, y, z)$  and  $(2+m)$ -ary terms  $q_1, \dots, q_n$  such that  $x = q_1(d_{j_1}, z, \vec{e})$ ,  $q_i(z, d_{j_i}, \vec{e}) = q_{i+1}(d_{j_{i+1}}, z, \vec{e})$  for  $i = 1, \dots, n-1$ , and  $y = q_n(d_{j_n}, z, \vec{e})$  where  $j_i \in \{1, \dots, k\}$ . Clearly,  $q_i(u, v, \vec{e}) = p_i(u, v, x, y, z)$  and  $d_{j_i} = t_i(x, y, z)$ ,  $i = 1, \dots, n$ , which are the required terms.

Conversely, let  $\mathcal{V}$  satisfy the listed identities, let  $\mathcal{A} \in \mathcal{V}$ . To prove regularity of  $\mathcal{A}$  it is enough to prove that each  $\theta \in \text{Con } \mathcal{A}$  with some singleton class  $\{c\}$  is the identity relation  $\omega$ . Let then  $\theta \in \text{Con } \mathcal{A}$ ,  $\{c\}$  be a class of  $\theta$ ,  $\langle a, b \rangle \in \theta$ . Thus  $\langle t_i(a, b, c), c \rangle = \langle t_i(a, b, c), t_i(a, a, c) \rangle \in \theta$ , i.e.

$t_i(a, b, c) \in \{c\}$ , i.e.  $t_i(a, b, c) = c$ . We conclude

$$\begin{aligned} a &= p_1(t_1(a, b, c), c, a, b, c) = p_1(c, c, a, b, c) = \dots = \\ &= p_n(c, c, a, b, c) = p_n(c, t_n(a, b, c), a, b, c) = b, \end{aligned}$$

hence  $\theta = \omega$ . □

**Theorem 2.** *Let the variety  $\mathcal{V}$  be regular and  $p_1, \dots, p_n$  be terms of Theorem 1. Then  $\mathcal{V}$  is  $(n+1)$ -permutable.*

*Proof:* Put  $q_i(x, y, z) = p_i(t_i(x, y, z), t_i(y, z, z), x, z, z)$ . The identities

$$\begin{aligned} x &= q_1(x, y, y) \\ q_i(x, x, y) &= q_{i+1}(x, y, y) \quad i = 1, \dots, n-1 \\ y &= q_n(x, x, y) \end{aligned}$$

are easy to verify. Hence, by [5],  $\mathcal{V}$  is  $(n+1)$ -permutable. □

**Theorem 3.** *Let  $\mathcal{V}$  be a regular variety, and  $t_1, \dots, t_n$  be the terms of Theorem 1. Let  $\mathcal{A} = (A, F) \in \mathcal{V}$  and  $\emptyset \neq C \subseteq A$ . The following conditions are equivalent:*

(1)  *$C$  is a class of some  $\theta \in \text{Con } \mathcal{A}$ .*

(2) (i) *for each  $m$ -ary  $f \in F$ ,  $a_j, b_j \in A$ ,  $j = 1, \dots, m$ ,  $c \in C$ , it holds*

$$\&_{i=1}^n t_i(a_j, b_j, c) \in C \Rightarrow \&_{i=1}^n t_i(f(a_1, \dots, a_m), f(b_1, \dots, b_m), c) \in C ;$$

(ii) *if  $a, b, d \in A$  then*

$$\&_{i=1}^n (t_i(a, b, c) \in C \& t_i(b, d, c) \in C) \Rightarrow \&_{i=1}^n t_i(a, d, c) \in C ;$$

(iii) *if  $a \in A$ ,  $c, d \in C$ , then  $t_i(d, c, c) \in C$  for  $i = 1, \dots, n$ , and*

$$\&_{i=1}^n t_i(a, c, c) \in C \Rightarrow a \in C .$$

*Proof:* Let  $\mathcal{A} \in \mathcal{V}$ ,  $\emptyset \neq C \subseteq A$ ,  $c \in C$  and let (i), (ii) and (iii) hold. Let  $\theta_C$  be a binary relation on  $A$  defined by

$$(*) \quad \langle x, y \rangle \in \theta_C \quad \text{iff} \quad t_1(x, y, c) \in C, \dots, t_n(x, y, c) \in C .$$

Since  $t_i(x, x, c) = c \in C$ ,  $\theta_C$  is reflexive. Compatibility and transitivity of  $\theta_C$  follow from the conditions (i) and (ii), respectively. Applying Theorem 2 we conclude that  $\mathcal{V}$  is  $(n+1)$ -permutable. By [3], each reflexive, transitive and compatible relation in a  $(n+1)$ -permutable variety is a congruence relation, hence  $\theta_C \in \text{Con } \mathcal{A}$ .

Let  $x \in [c]_{\theta_C}$ . Then  $\langle x, c \rangle \in \theta_C$  and, by (\*),  $t_i(x, c, c) \in C$  for  $i = 1, \dots, k$ . From (iii) it follows  $x \in C$ . Conversely, let  $x \in C$ . Then by (iii) we get  $t_i(x, c, c) \in C$ ,  $i = 1, \dots, k$ . By (\*) this implies  $\langle x, c \rangle \in \theta_C$ , i.e.  $x \in [c]_{\theta}$ . Hence,  $C = [c]_{\theta}$ .

Conversely, let  $C \subseteq A$  be a class of some  $\theta \in \text{Con } \mathcal{A}$  and  $c \in C$ . If  $a_j, b_j \in A$  and  $t_i(a_j, b_j, c) \in C$  ( $j = 1, \dots, m$ ,  $i = 1, \dots, n$ ) and if  $f \in F$  is  $m$ -ary then then  $\langle t_i(a_j, b_j, c), c \rangle \in \theta$  and, by Theorem 1, we have

$$\begin{aligned} a_j &= p_1(t_1(a_j, b_j, c), c, a_j, b_j, c) \theta p_1(c, t_1(a_j, b_j, c), a_j, b_j, c) = \\ &= p_2(t_2(a_j, b_j, c), c, a_j, b_j, c) \theta p_2(c, t_2(a_j, b_j, c), a_j, b_j, c) = \\ &\quad \vdots \\ &= p_n(t_n(a_j, b_j, c), c, a_j, b_j, c) \theta p_n(c, t_n(a_j, b_j, c), a_j, b_j, c) = b_j, \end{aligned}$$

hence  $\langle a_j, b_j \rangle \in \theta$ . From compatibility of  $\theta$  it follows

$$\begin{aligned} &\langle t_i(f(a_1, \dots, a_m), f(b_1, \dots, b_m), c), c \rangle = \\ &= \langle t_i(f(a_1, \dots, a_m), f(b_1, \dots, b_m), c), t_i(f(b_1, \dots, b_m), f(b_1, \dots, b_m), c), c) \rangle \in \theta, \end{aligned}$$

i.e.  $t_i(f(a_1, \dots, a_m), f(b_1, \dots, b_m), c) \in [c]_{\theta} = C$ . Hence, (i) holds.

If  $t_i(x, y, c) \in C$ ,  $t_i(y, z, c) \in C$  ( $i = 1, \dots, n$ ), then as in the previous case,  $\langle x, y \rangle \in \theta$ ,  $\langle y, z \rangle \in \theta$ , hence,  $\langle x, z \rangle \in \theta$ . Therefore,  $\langle t_i(x, z, c), c \rangle = \langle t_i(x, z, c), t_i(z, z, c) \rangle \in \theta$ , i.e.  $t_i(x, z, c) \in [c]_{\theta} = C$ , proving (ii).

If  $t_i(a, c, c) \in C$  ( $i = 1, \dots, n$ ), then again  $\langle a, c \rangle \in \theta$ , i.e.  $a \in C$ . If  $c, d \in C$  then  $\langle c, d \rangle \in \theta$ , and thus  $\langle t_i(d, c, c), c \rangle = \langle t_i(d, c, c), t_i(d, d, c) \rangle \in \theta$ , i.e.  $t_i(d, c, c) \in C$ . We have proved (iii).  $\square$

Let us turn to computational aspects of our problem. Computational properties of universal algebra are of recent interest, see e.g. [2]. Recall first that by a time complexity, see [4], of an algorithm it is meant a function  $f : N \mapsto N$  such that every problem of size  $n$  will be solved after at most  $f(n)$  number of (computational) steps. The class of all problems for which there is an deterministic algorithm (nondeterministic algorithm) of polynomial time complexity ( $f(n)$  is a polynomial) is denoted by  $P$  ( $NP$ ). Algorithms of polynomial time complexities are considered as practically

usable, algorithms of greater complexities (e.g. exponential) are considered as unusable. The class  $P$  is therefore the class of tractable problems. In our case, we are given a class  $\mathcal{K}$  of algebras. We face the following decision problem: For an algebra  $\mathcal{A} \in \mathcal{K}$  and a subset  $C \subseteq A$ , decide whether  $C$  is a congruence class. Suppose that one evaluation step consists in the evaluation of one term. Denote the problem  $p_{\mathcal{K}}$ . It has been shown in [1] that in general  $p_{\mathcal{K}} \in NP$  but for a regular and permutable variety  $\mathcal{V}$ ,  $p_{\mathcal{V}} \in P$ . The following theorem shows that being a regular variety for  $\mathcal{K}$  is sufficient for the problem to belong to  $P$ .

**Theorem 4.** *If  $\mathcal{V}$  is a regular variety then  $p_{\mathcal{V}} \in P$ .*

*Proof:* Let  $\emptyset \neq C \subseteq A$ ,  $\mathcal{A} = \langle A, F \rangle \in \mathcal{V}$ . Denote further  $F = \{f_1, \dots, f_k\}$ ,  $l = \text{card } C$ ,  $m = \text{card } A$ , and let  $\sigma(f)$  denote the arity of  $f \in F$ . To check whether  $C$  is a class of some congruence relation on  $\mathcal{A}$  we can use Theorem 3, i.e. we have to test the conditions (i), (ii) and (iii) of (2). Consider first condition (i). We choose  $f \in F$  ( $k$  choices) and  $a_j, b_j \in A$  ( $m^2$  choices). For this choice we have to test the implication. The test of the antecedent consists of  $n$  steps. The test of the consequent part consists of  $n m^{2(\sigma(f)-1)}$  steps (there are  $m^{2(\sigma(f)-1)}$  possible substitutions for the arguments of  $f$ ). Since the choices are independent we have  $\sum_{i=1}^k m^2 (n + n m^{2(\sigma(f_i)-1)})$  computational steps altogether. Similarly, to test the conditions (ii) and (iii) we have to perform  $m^2 l (2n + n)$  and  $m l^2 (n + (n + 1))$  steps, respectively. For a given variety, the derived expressions are polynomials. Since the overall number of steps is given by the sum of the expressions the assertion is proved.  $\square$

**Remark.** The proof of the foregoing theorem gives a polynomial algorithm solving our problem for  $\mathcal{K}$  being a regular variety. The time complexity of the algorithm is

$$n \sum_{i=1}^k m^{2\sigma(f_i)} + m^2 n(3l + k) + ml^2(2n + 1) .$$

Note that this algorithm is of the same asymptotic complexity as that one for regular and permutable varieties based on Theorem 1 of [1].

Recall the following concept, see e.g. [1]. If  $p(x_1, \dots, x_n, y_1, \dots, y_m)$  is an  $(m + n)$ -ary term of an algebra  $\mathcal{A} = (A, F)$  and  $C \subseteq A$  we say that  $C$  is *y-closed under p* if  $p(a_1, \dots, a_n, c_1, \dots, c_m) \in C$  for every  $a_1, \dots, a_n \in A$

and  $c_1, \dots, c_m \in C$ . The following theorem presents a characterization of special congruence classes of regular varieties by means of  $y$ -closeness.

**Theorem 5.** *Let  $\mathcal{V}$  be a regular variety, let  $\mathcal{A} = (A, F) \in \mathcal{V}$  and  $\emptyset \neq C \subseteq A$ ,  $c \in C$ . Let  $C$  satisfy  $\&_{i=1}^n (t_i(x, y, c) \in C \& t_i(y, z, c) \in C) \Rightarrow \&_{i=1}^n t_i(x, z, c) \in C$ . Then  $C$  is a congruence class of some  $\theta \in \text{Con } \mathcal{A}$  if and only if the following conditions are satisfied:*

- (a)  $C$  is  $y$ -closed under
 
$$t_i(t_j(f(p_l(y_1, y, x_1, x'_1, y), \dots, p_l(y_m, y, x_m, x'_m, y)), f(x'_1, \dots, x'_n), y),$$

$$t_j(f(p_l(y'_1, y', x_1, x'_1, y'), \dots, p_l(y'_m, y', x_m, x'_m, y')), f(x'_1, \dots, x'_n), y'), y^*)$$
 for each  $m$ -ary  $f \in F$ ,  $i, j, l = 1, \dots, n$ ;
- (b)  $C$  is  $y$ -closed under  $r_i(y_1, y_2) = t_i(y_1, y_2, y_2)$ ,  $i = 1, \dots, n$ ;
- (c) for  $x \in A$ ,  $y, y_1 \in C$  it holds: if  $\&_{i=1}^n t_i(x, y, y) \in C$  then  $p_1(y_1, y, x, y, y) \in C$ .

*Proof:* Let  $C$  satisfy the condition of the theorem. Let furthermore  $C$  be a class of some congruence relation,  $C = [c]_\theta$ ,  $c \in C$ . One may easily check that  $C$  is  $y$ -closed under the terms of (a) and (b). To check (c), let  $x \in A$ ,  $y, y_1 \in C$ ,  $\&_{i=1}^n t_i(x, y, y) \in C$ . Similarly as in Theorem 3 one may show that

$$p_1(y_1, y, x, y, y) \theta p_1(t_1(x, y, y), y, x, y, y) \theta \dots \theta p_n(y, t_n(x, y, y), x, y, y) .$$

But  $p_n(y, t_n(x, y, y), x, y, y) = y \in C$ , hence (c) is satisfied.

Conversely, let  $C$  satisfying the condition of the theorem fulfil (a), (b), (c). Define a binary relation  $\theta_C$  on  $A$  by (\*) of the proof of Theorem 3. We show that  $\theta_C$  is a congruence relation having  $C$  as a class. Reflexivity of  $\theta_C$  is obvious, transitivity follows from the condition of the theorem. To show compatibility of  $\theta_C$ , let  $f \in F$  be  $m$ -ary,  $\langle x_i, x'_i \rangle \in \theta_C$ , i.e.  $t_j(x_i, x'_i, c) \in C$ ,

$i = 1, \dots, m, j = 1, \dots, n$ . From Theorem 1 and (a) we obtain

$$\begin{aligned}
& \langle t_j \left( f(x_1, \dots, x_m), f(x'_1, \dots, x'_m), c \right), \\
& \quad t_j \left( f(p_1(c, t_1(x_1, x'_1, c), x_1, x'_1, c), \dots, p_1(c, t_1(x_m, x'_m, c), x_m, x'_m, c)), \right. \\
& \quad \quad \left. f(x'_1, \dots, x'_m), c \right) \rangle = \\
& \langle t_j \left( f(p_1(t_1(x_1, x'_1, c), c, x_1, x'_1, c), \dots, p_1(t_1(x_m, x'_m, c), c, x_m, x'_m, c)), \right. \\
& \quad \quad \left. f(x'_1, \dots, x'_m), c \right), \\
& \quad t_j \left( f(p_1(c, t_1(x_1, x'_1, c), x_1, x'_1, c), \dots, p_1(c, t_1(x_m, x'_m, c), x_m, x'_m, c)), \right. \\
& \quad \quad \left. f(x'_1, \dots, x'_m), c \right) \rangle \in \theta_C \\
& \quad \parallel \\
& \langle t_j \left( f(p_2(t_2(x_1, x'_1, c), c, x_1, x'_1, c), \dots, p_2(t_2(x_m, x'_m, c), c, x_m, x'_m, c)), \right. \\
& \quad \quad \left. f(x'_1, \dots, x'_m), c \right), \\
& \quad t_j \left( f(p_2(c, t_2(x_1, x'_1, c), x_1, x'_1, c), \dots, p_2(c, t_2(x_m, x'_m, c), x_m, x'_m, c)), \right. \\
& \quad \quad \left. f(x'_1, \dots, x'_m), c \right) \rangle \in \theta_C \\
& \quad \parallel \\
& \quad \vdots \\
& \quad \parallel \\
& \langle t_j \left( f(p_n(t_n(x_1, x'_1, c), c, x_1, x'_1, c), \dots, p_n(t_n(x_m, x'_m, c), c, x_m, x'_m, c)), \right. \\
& \quad \quad \left. f(x'_1, \dots, x'_m), c \right), \\
& \quad t_j \left( f(p_n(c, t_n(x_1, x'_1, c), x_1, x'_1, c), \dots, p_n(c, t_n(x_m, x'_m, c), x_m, x'_m, c)), \right. \\
& \quad \quad \left. f(x'_1, \dots, x'_m), c \right) \rangle = \\
& \langle t_j \left( f(p_n(t_n(x_1, x'_1, c), c, x_1, x'_1, c), \dots, p_n(t_n(x_m, x'_m, c), c, x_m, x'_m, c)), \right. \\
& \quad \quad \left. f(x'_1, \dots, x'_m), c \right), \\
& \quad t_j \left( f(x'_1, \dots, x'_m), f(x'_1, \dots, x'_m), c \right) \rangle = \\
& \langle t_j \left( f(p_n(t_n(x_1, x'_1, c), c, x_1, x'_1, c), \dots, p_n(t_n(x_m, x'_m, c), c, x_m, x'_m, c)), \right. \\
& \quad \quad \left. f(x'_1, \dots, x'_m), c \right), c \rangle \in \theta_C .
\end{aligned}$$

By transitivity of  $\theta_C$  we have  $\langle t_j \left( f(x_1, \dots, x_m), f(x'_1, \dots, x'_m), c \right), c \rangle \in \theta_C$ . From (\*) we conclude  $\langle f(x_1, \dots, x_m), f(x'_1, \dots, x'_m) \rangle \in \theta_C$ , verifying compatibility. By [3],  $\theta_C$  is a congruence relation. It remains to show  $C = [c]_{\theta}$ . If  $\langle a, c \rangle \in \theta_C$  then  $\&_{i=1}^n t_i(a, c, c) \in C$  by (\*), and  $a = p_1(t_1(a, c, c), c, a, c, c) \in C$  by (c), hence  $[c]_{\theta} \subseteq C$ . Let conversely  $d \in C$ . Then  $r_i(d, c) = t_i(d, c, c) \in C, i = 1, \dots, n$ , by (b), and thus  $\langle d, c \rangle \in \theta_C$  proving  $C \subseteq [c]_{\theta}$ .  $\square$

## References

- [1] R. Bělohlávek, I. Chajda: A polynomial characterization of congruence classes. *Algebra Univ.* **37**(1997), 235–242.
- [2] S. Burris: Computers and universal algebra: some directions. *Algebra Univ.* **34**(1995), 61–71.
- [3] I. Chajda, J. Rachůnek: Relational characterizations of permutable and  $n$ -permutable varieties. *Czech. Math. J.* **33**(1983), 505–508.
- [4] M. Garey, D. Johnson: *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, New York, 1979.
- [5] J. Hagemann, A. Mitschke: On  $n$ -permutable congruences. *Algebra Univ.* **3**(1973), 8–12.
- [6] A. I. Maľcev: On the general theory of algebraic systems (Russian). *Matem. Sbornik* **35**(1954), 3–20.
- [7] H. Werner: A Maľcev condition on admissible relations. *Algebra Univ.* **3**(1973), 263.
- [8] R. Wille: *Kongruenzklassengeometrien*. Lecture Notes in Math., vol. 113, Springer-Verlag, Berlin—New York, 1970.