

CHARACTERIZING TREES IN CONCEPT LATTICES

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Concept lattices are systems of conceptual clusters, called formal concepts, which are partially ordered by the subconcept/superconcept relationship. Concept lattices are basic structures used in formal concept analysis. In general, a concept lattice may contain overlapping clusters and need not be a tree. On the other hand, tree-like classification schemes are appealing and are produced by several clustering methods. In this paper, we present necessary and sufficient conditions on input data for the output concept lattice to form a tree after one removes its least element. We present these conditions for input data with yes/no attributes as well as for input data with fuzzy attributes. In addition, we show how Lindig's algorithm for computing concept lattices gets simplified when applied to input data for which the associated concept lattice is a tree after removing the least element. The paper also contains illustrative examples.

Keywords: Concept lattice; tree; formal concept; attribute implication.

1. Problem Description

Generating collections of clusters from data is a challenging part of knowledge discovery. Among many methods performing this task, formal concept analysis (FCA) is becoming increasingly popular, see Refs. 7, 10. The main aim of FCA is to extract interesting clusters (called formal concepts) from tabular data along with a partial order of these clusters (called conceptual hierarchy). Formal concepts correspond to maximal rectangles in a data table and are easily interpretable by users. FCA and its methods have been used in two ways. First, as a direct method of data analysis in which case the hierarchically ordered collection of formal concepts extracted from data is presented to a user/expert for further analysis, see e.g. Ref. 7 for such examples of FCA applications. Second, as a data preprocessing method in which case the extracted clusters are used for further processing, see e.g. Ref. 18 for applications of FCA in association rule mining.

Unlike several other clustering and classification techniques,^{1,8} which yield clustering and classification trees, FCA yields diagrams of hierarchically ordered clusters which are richer than trees. Namely, the diagrams are lattices and are called concept lattices. A practical difference is that concept lattices usually contain overlapping clusters. Another difference is that the clusters in FCA are based on sharing of attributes rather than distance. FCA can thus be thought of as a new method of clustering and classification which is substantially different from conventional methods. Although FCA has been justified by many real-world applications already, see e.g. Ref. 7, the following quote from John Hartigan, a leading expert in clustering and classification, is relevant^{1,8}:

“My second remark is about future focus. We pay too much attention to the details of the algorithms. . . . It is more important to think about the purposes of clustering, about the types of clusters we wish to construct, . . . These details are interesting, . . . , but we have plenty of algorithms already. . . . what kinds of families of classes should we be looking for? At present, we think of partitions, trees, sometimes overlapping clusters; these structures are a faint echo of the rich classifications available in everyday language. . . . We must seek sufficiently rich class of structures”

The present paper seeks to contribute to the problem of establishing relationships between FCA and other methods of clustering and classification. Needless to say, this goal requires a long-term effort. In this paper we consider a particular problem. Namely, we present conditions for input data which are necessary and sufficient for the output concept lattice to form a tree after removing its least element. In addition, we present illustrative examples and several remarks on related efforts and future research topics. Note that a related problem, namely, of selecting a tree from a concept lattice by means of constraints using attribute-dependency formulas, was considered in Ref. 5.

Section 2 presents preliminaries. Section 3 presents the main results, illustrative examples, and remarks. Section 4 presents conclusions and an outline of future research.

2. Preliminaries

In this section, we summarize basic notions of formal concept analysis (FCA). An object-attribute data table describing which objects have which attributes can be identified with a triplet $\langle X, Y, I \rangle$ where X is a non-empty set (of objects), Y is a non-empty set (of attributes), and $I \subseteq X \times Y$ is an (object-attribute) relation. In FCA, $\langle X, Y, I \rangle$ is called a formal context. Objects and attributes correspond to table rows and columns, respectively, and $\langle x, y \rangle \in I$ indicates that object x has attribute y (table entry corresponding to row x and column y contains \times or 1; if $\langle x, y \rangle \notin I$ the table entry contains blank symbol or 0). For each $A \subseteq X$ and $B \subseteq Y$ denote by A^\uparrow the subset of Y and by B^\downarrow the subset of X defined by

$$A^\uparrow = \{y \in Y \mid \text{for each } x \in A: \langle x, y \rangle \in I\}, \quad (1)$$

$$B^\downarrow = \{x \in X \mid \text{for each } y \in B: \langle x, y \rangle \in I\}. \quad (2)$$

Described verbally, A^\uparrow is the set of all attributes from Y shared by all objects from A and B^\downarrow is the set of all objects from X sharing all attributes from B . A formal concept in $\langle X, Y, I \rangle$ is a pair $\langle A, B \rangle$ of $A \subseteq X$ and $B \subseteq Y$ satisfying $A^\uparrow = B$ and $B^\downarrow = A$. That is, a formal concept consists of a set A (so-called extent) of objects which fall under the concept and a set B (so-called intent) of attributes which fall under the concept such that A is the set of all objects sharing all attributes from B and, conversely, B is the collection of all attributes from Y shared by all objects from A . Alternatively, formal concepts can be defined as maximal rectangles (submatrices) of $\langle X, Y, I \rangle$ which are full of \times 's: For $A \subseteq X$ and $B \subseteq Y$, $\langle A, B \rangle$ is a formal concept in $\langle X, Y, I \rangle$ iff $A \times B \subseteq I$ and there is no $A' \supset A$ or $B' \supset B$ such that $A' \times B \subseteq I$ or $A \times B' \subseteq I$.

The set $\mathcal{B}(X, Y, I) = \{\langle A, B \rangle \mid A^\uparrow = B, B^\downarrow = A\}$ of all formal concepts in $\langle X, Y, I \rangle$ can be equipped with a partial order \leq (modeling the subconcept-superconcept hierarchy, e.g. $\text{dog} \leq \text{mammal}$) defined by

$$\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle \quad \text{iff} \quad A_1 \subseteq A_2 \quad (\text{iff} \quad B_2 \subseteq B_1). \quad (3)$$

Under \leq , $\mathcal{B}(X, Y, I)$ happens to be a complete lattice, called the concept lattice of $\langle X, Y, I \rangle$, the basic structure of which is described by a so-called main theorem of concept lattices¹⁰:

Theorem 1. (1) *The set $\mathcal{B}(X, Y, I)$ equipped with \leq is a complete lattice where the infima and suprema are given by*

$$\bigwedge_{j \in J} \langle A_j, B_j \rangle = \langle \bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)^\downarrow \rangle, \quad (4)$$

$$\bigvee_{j \in J} \langle A_j, B_j \rangle = \langle (\bigcup_{j \in J} A_j)^\uparrow, \bigcap_{j \in J} B_j \rangle. \quad (5)$$

(2) Moreover, an arbitrary complete lattice $\mathbf{V} = \langle V, \leq \rangle$ is isomorphic to $\mathcal{B}(X, Y, I)$ iff there are mappings $\gamma: X \rightarrow V$, $\mu: Y \rightarrow V$ such that

- (i) $\gamma(X)$ is \bigvee -dense in V , $\mu(Y)$ is \bigwedge -dense in V ;
- (ii) $\gamma(x) \leq \mu(y)$ iff $\langle x, y \rangle \in I$.

Note that a subset $K \subseteq V$ is \bigvee -dense in V (\bigwedge -dense in V) if for every $v \in V$ there is $K' \subseteq K$ such that $v = \bigvee K'$ ($v = \bigwedge K'$). Note also that the operators \uparrow and \downarrow form a so-called Galois connection¹⁰ and that $\mathcal{B}(X, Y, I)$ is in fact the set of all fixed points of \uparrow and \downarrow . That is, \uparrow and \downarrow satisfy the following conditions:

$$A \subseteq A^{\uparrow\downarrow}, \quad (6)$$

$$\text{if } A_1 \subseteq A_2 \text{ then } A_2^{\uparrow} \subseteq A_1^{\uparrow}, \quad (7)$$

$$B \subseteq B^{\downarrow\uparrow}, \quad (8)$$

$$\text{if } B_1 \subseteq B_2 \text{ then } B_2^{\downarrow} \subseteq B_1^{\downarrow}, \quad (9)$$

for each $A, A_1, A_2 \subseteq X$ and $B, B_1, B_2 \subseteq Y$. Furthermore, the composed operators $\uparrow\downarrow: 2^X \rightarrow 2^X$ and $\downarrow\uparrow: 2^Y \rightarrow 2^Y$ are closure operators in X and Y , respectively. As a consequence, $A \subseteq X$ is an extent of some concept in $\mathcal{B}(X, Y, I)$ (i.e., there is $B \subseteq Y$ such that $\langle A, B \rangle \in \mathcal{B}(X, Y, I)$) iff $A = A^{\uparrow\downarrow}$, i.e. A is closed under $\uparrow\downarrow$. Analogously for intents.

Concept lattices are the primary output of formal concept analysis. There is another output of FCA which is equally important, namely, so-called non-redundant bases of attribute implications. An attribute implication is an expression $A \Rightarrow B$ where $A, B \subseteq Y$ with Y being the same set of attributes as above. An attribute implication $A \Rightarrow B$ is called true in $M \subseteq Y$, written $M \models A \Rightarrow B$, if the following condition is satisfied:

$$\text{if } A \subseteq M \text{ then } B \subseteq M.$$

If $M \subseteq Y$ represents a set of attributes of some object x then $M \models A \Rightarrow B$ has the following meaning: “if x has all attributes from A , then x has all attributes from B ”. Thus, attribute implications are particular if-then rules describing dependencies between attributes.

Given a formal context $\langle X, Y, I \rangle$, for each $x \in X$ we define a set I_x of attributes $I_x = \{y \in Y \mid \langle x, y \rangle \in I\}$, i.e. I_x is the set of all attributes of object x in $\langle X, Y, I \rangle$. Notice that I_x corresponds to a row in the data table representing the formal context $\langle X, Y, I \rangle$. An attribute implication $A \Rightarrow B$ is called true in $\langle X, Y, I \rangle$, written $I \models A \Rightarrow B$, iff $I_x \models A \Rightarrow B$ for each $x \in X$. Hence, $I \models A \Rightarrow B$ iff for each object $x \in X$ we have: if x has all attributes from A , then x has all attributes from B .

The set of all attribute implications which are true in $\langle X, Y, I \rangle$ is, along with the concept lattice $\mathcal{B}(X, Y, I)$, the basic output of FCA. Unfortunately, the set of all attribute implications is usually too large and it cannot be presented to users directly. Therefore, we use a special indirect description of all attribute implications being true in $\langle X, Y, I \rangle$. Namely, we select from all the attribute implications in

question a small subset from which the other implications follow. This can be done using the following notions.

Let T be any set of attribute implications. A set $M \subseteq Y$ of attributes is called a model of T , if $M \models A \Rightarrow B$ for each $A \Rightarrow B \in T$. The set of all models of T will be denoted by $\text{Mod}(T)$, i.e.

$$\text{Mod}(T) = \{M \subseteq Y \mid \text{for each } A \Rightarrow B \in T: M \models A \Rightarrow B\}. \quad (10)$$

An attribute implication $A \Rightarrow B$ follows from T ($A \Rightarrow B$ is semantically entailed by T), written $T \models A \Rightarrow B$, if $M \models A \Rightarrow B$ for each $M \in \text{Mod}(T)$. A set T of attribute implications is called complete in $\langle X, Y, I \rangle$ if, for each attribute implication $A \Rightarrow B$, we have

$$T \models A \Rightarrow B \quad \text{iff} \quad I \models A \Rightarrow B,$$

i.e., if the attribute implications which are entailed by T are exactly the attribute implications which are true in $\langle X, Y, I \rangle$. Hence, if T is complete in $\langle X, Y, I \rangle$, then T describes exactly the attribute implications which are true in $\langle X, Y, I \rangle$. This is important especially if T is “reasonably small”. Therefore, we define the following notion. A set T of attribute implications is a non-redundant basis of $\langle X, Y, I \rangle$ if (i) T is complete in $\langle X, Y, I \rangle$ and (ii) no proper subset of T is complete in $\langle X, Y, I \rangle$. Alternatively, a non-redundant basis of $\langle X, Y, I \rangle$ can be described as a complete set of attribute implications such that no implication in the set is entailed by the other implications in that set. There have been proposed algorithms to generate, given $\langle X, Y, I \rangle$, a non-redundant basis of $\langle X, Y, I \rangle$, see e.g. Refs. 9, 10, 13.

For detailed information on formal concept analysis and lattice theory we refer to Refs. 7, 10, 11 where a reader can find theoretical foundations, methods and algorithms, and applications in various areas.

3. Trees in Concept Lattices

In this section we will be interested in concept lattices corresponding to trees. Trees are usually defined as undirected graphs that are acyclic and connected.¹² Since we are going to identify trees in particular ordered sets, we deal with trees as with ordered sets. In particular, a finite partially ordered set $\langle U, \leq \rangle$ will be called a tree if for each $a, b \in U$:

- (i) there is a supremum of a and b in $\langle U, \leq \rangle$, and
- (ii) there is an infimum of a and b in $\langle U, \leq \rangle$ iff a and b are comparable (i.e., iff $a \leq b$ or $b \leq a$).

Obviously, $\langle U, \leq \rangle$ being a tree corresponds to the usual graph-theoretical representation of a rooted tree. The root of $\langle U, \leq \rangle$ is the supremum of all elements from U (which exists in $\langle U, \leq \rangle$ because U is finite). An element $u \in U$ is a direct descendant of $w \in U$ iff $u < w$, and there is no $v \in U$ such that $u < v < w$.

From Theorem 1 it follows that each concept lattice is a complete lattice. Hence, the above-mentioned condition (i) is satisfied for each $\mathcal{B}(X, Y, I)$. On the other hand, (ii) need not be satisfied. It is easily seen that (ii) is satisfied iff $\mathcal{B}(X, Y, I)$ is linearly ordered. So, the whole concept lattice is a tree iff it is linearly ordered, which is not a worthwhile observation because linear trees are a degenerate form of trees and therefore not interesting. Because of the observation we have just made, we turn our attention to trees which form important parts of concept lattices. We focus mainly on trees which appear in $\mathcal{B}(X, Y, I)$ if we remove its least element.

Since concept lattices are complete lattices, each concept lattice $\mathcal{B}(X, Y, I)$ has both the greatest and least element. Namely, $\langle X, X^\uparrow \rangle$ is the greatest element (concept of all objects) of $\mathcal{B}(X, Y, I)$ and $\langle Y^\downarrow, Y \rangle$ is the least one (concept of objects sharing all attributes from Y). If $\langle X, Y, I \rangle$ does not contain an attribute shared by all objects (i.e., a table representing $\langle X, Y, I \rangle$ does not contain a column full of \times 's), which is quite common if $\langle X, Y, I \rangle$ represents real-world data, then $\langle X, X^\uparrow \rangle$ equals $\langle X, \emptyset \rangle$. Analogously, if there is no object sharing all the attributes from Y (i.e., a table representing $\langle X, Y, I \rangle$ does not contain a row full of \times 's), $\langle Y^\downarrow, Y \rangle$ becomes $\langle \emptyset, Y \rangle$.

In what follows we investigate under which conditions $\mathcal{B}(X, Y, I)$ becomes a tree if we remove its least element.

3.1. Formal contexts generating trees

For brevity, let $\mathcal{B}(X, Y, I) - \{\langle Y^\downarrow, Y \rangle\}$ be denoted by $\mathcal{B}^\wedge(X, Y, I)$. Note that if we consider $\mathcal{B}^\wedge(X, Y, I)$, we assume that it is equipped with a partial order which is a restriction of the partial order defined by (3) to elements of $\mathcal{B}^\wedge(X, Y, I)$.

The following assertion characterizes when $\mathcal{B}^\wedge(X, Y, I)$ is a tree in terms of extents of formal concepts.

Theorem 2. *Let $\langle X, Y, I \rangle$ be a formal context. Then $\mathcal{B}^\wedge(X, Y, I)$ is a tree iff, for any concepts $\langle A, B \rangle, \langle C, D \rangle \in \mathcal{B}(X, Y, I)$ at least one of the following is true:*

- (i) $A \subseteq C$ or $C \subseteq A$,
- (ii) $A \cap C \subseteq Y^\downarrow$.

Proof. Let $\mathcal{B}^\wedge(X, Y, I)$ be a tree. Take any concepts $\langle A, B \rangle, \langle C, D \rangle \in \mathcal{B}(X, Y, I)$. If (i) is satisfied for A and C , we are done. Hence, assume that (i) is not satisfied, i.e. we have $A \not\subseteq C$ and $C \not\subseteq A$. From the definition of \leq , see (3), it follows that $\langle A, B \rangle \not\leq \langle C, D \rangle$ and $\langle C, D \rangle \not\leq \langle A, B \rangle$, i.e. formal concepts $\langle A, B \rangle$ and $\langle C, D \rangle$ are incomparable. Therefore, both $\langle A, B \rangle$ and $\langle C, D \rangle$ are in $\mathcal{B}^\wedge(X, Y, I)$. Since $\mathcal{B}^\wedge(X, Y, I)$ is supposed to be a tree, the infimum of $\langle A, B \rangle$ and $\langle C, D \rangle$ does not exist in $\mathcal{B}^\wedge(X, Y, I)$. It means that the infimum of $\langle A, B \rangle$ and $\langle C, D \rangle$ in $\mathcal{B}(X, Y, I)$ is $\langle Y^\downarrow, Y \rangle$ because $\mathcal{B}^\wedge(X, Y, I)$ results from $\mathcal{B}(X, Y, I)$ by removing $\langle Y^\downarrow, Y \rangle$ and $\mathcal{B}(X, Y, I)$ is a complete lattice. Using (4), we get that $Y^\downarrow = A \cap C$, showing (ii).

Conversely, let (i) and (ii) be satisfied for any $\langle A, B \rangle, \langle C, D \rangle \in \mathcal{B}(X, Y, I)$. Take $\langle A, B \rangle, \langle C, D \rangle \in \mathcal{B}^\wedge(X, Y, I)$ such that $\langle A, B \rangle$ and $\langle C, D \rangle$ are incomparable. Such $\langle A, B \rangle$ and $\langle C, D \rangle$ cannot satisfy (i), i.e. we have $A \cap C \subseteq Y^\downarrow$. Hence, using (4), the infimum of $\langle A, B \rangle$ and $\langle C, D \rangle$ in $\mathcal{B}(X, Y, I)$ is the least element of $\mathcal{B}(X, Y, I)$. As a consequence, $\langle A, B \rangle$ and $\langle C, D \rangle$ do not have an infimum in $\mathcal{B}^\wedge(X, Y, I)$, which proves that $\mathcal{B}^\wedge(X, Y, I)$ is a tree. \square

Theorem 2 can also be formulated in terms of intents of formal concepts:

Corollary 1. $\mathcal{B}^\wedge(X, Y, I)$ is a tree iff, for each $\langle A, B \rangle, \langle C, D \rangle \in \mathcal{B}(X, Y, I)$ we either have (i) $B \subseteq D$ or $D \subseteq B$, or (ii) $(B \cup D)^\downarrow \subseteq Y^\downarrow$.

Remark 1. If $\langle Y^\downarrow, Y \rangle$ is equal to $\langle \emptyset, Y \rangle$, i.e. if the table representing $\langle Y^\downarrow, Y \rangle$ does not contain a row full of \times 's (or 1's), then (ii) in Theorem 2 gets simplified to $A \cap C = \emptyset$, i.e. A and C are required to be disjoint.

Example 1. Consider a set of objects $X = \{1, 2, \dots, 14\}$ (objects are denoted by numbers) and a set of attributes $Y = \{g, h, \dots, z\}$. If we consider the formal context $\langle X, Y, I \rangle$ which is represented by the data table in Fig.1 (left) then the corresponding $\mathcal{B}^\wedge(X, Y, I)$, which is depicted in Fig.1 (right), is a tree. The root of the tree represents concept $\langle X, \emptyset \rangle$. The other nodes are numbered and the intents of the corresponding concepts are the following:

1: $\{i, r\}$,	5: $\{m, s, z\}$,	9: $\{g, m, n, q, s, v, z\}$,
2: $\{i, o, r\}$,	6: $\{g, m, n, s, v, z\}$,	10: $\{m, s, x, z\}$,
3: $\{i, l, o, r\}$,	7: $\{g, j, m, n, p, s, t, v, z\}$,	11: $\{h, m, s, x, z\}$,
4: $\{i, r, w\}$,	8: $\{g, j, k, m, n, p, s, t, u, v, z\}$,	12: $\{m, s, y, z\}$.

If two nodes are connected by an edge, the lower concept has a strictly greater intent. Using this observation, we can decorate edges of the tree by attributes being added to intents of lower concepts as it is shown in Fig.1 (right). One can check that all the intents from $\mathcal{B}(X, Y, I)$ satisfy the conditions of Corollary 1.

Now, an important question is whether we can check that $\mathcal{B}^\wedge(X, Y, I)$ is a tree directly from the context $\langle X, Y, I \rangle$, i.e. without computing the set of all concepts first. We shall show that this is indeed possible. We will take advantage of the following notion.

Definition 1. Let $\langle X, Y, I \rangle$ be a formal context. We say that $\langle X, Y, I \rangle$ generates a tree if $\mathcal{B}^\wedge(X, Y, I)$ is a tree.

Recall that due to (2), $\{y\}^\downarrow$ is the set of all objects sharing the attribute y . That is, $\{y\}^\downarrow$ naturally corresponds to a column in the data table representing $\langle X, Y, I \rangle$. Such ‘‘columns’’ will play an important role in the following theorem which characterizes contexts generating trees.

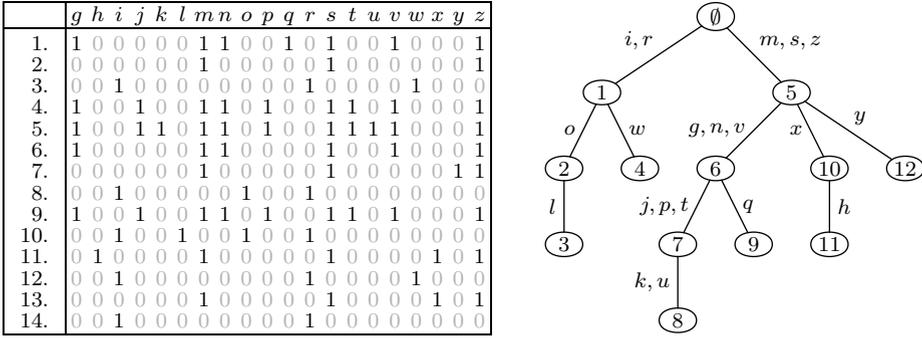


Fig. 1. Formal context generating a tree.

Theorem 3. *Let $\langle X, Y, I \rangle$ be a formal context. Then $\langle X, Y, I \rangle$ generates a tree iff, for any attributes $y_1, y_2 \in Y$, at least one of the following conditions is true:*

- (i) $\{y_1\}^\downarrow \subseteq \{y_2\}^\downarrow$,
- (ii) $\{y_2\}^\downarrow \subseteq \{y_1\}^\downarrow$,
- (iii) $\{y_1\}^\downarrow \cap \{y_2\}^\downarrow \subseteq Y^\downarrow$.

Proof. Assume that $\langle X, Y, I \rangle$ generates a tree, i.e. $\mathcal{B}^\wedge(X, Y, I)$ is a tree. Each pair of the form $\langle \{y\}^\downarrow, \{y\}^{\downarrow\uparrow} \rangle$ is a formal concept from $\mathcal{B}(X, Y, I)$, see Ref. 10. Therefore, Theorem 2 yields that the above conditions (i)–(iii), being particular instances of (i) and (ii) from Theorem 2, are satisfied.

Conversely, suppose that $\langle X, Y, I \rangle$ does not generate a tree. Thus, there are incomparable formal concepts $\langle A_1, B_1 \rangle, \langle A_2, B_2 \rangle \in \mathcal{B}(X, Y, I)$ whose infimum is not equal to the least element of $\mathcal{B}(X, Y, I)$. That is, for $\langle A_1, B_1 \rangle$ and $\langle A_2, B_2 \rangle$ we have $A_1 \not\subseteq A_2$, $A_2 \not\subseteq A_1$, and $A_1 \cap A_2 \not\subseteq Y^\downarrow$. Note that as a consequence we get that $B_1 \not\subseteq B_2$ and $B_2 \not\subseteq B_1$. We now show that we can pick from B_1 and B_2 two attributes violating the above conditions (i)–(iii). Since $B_1 \not\subseteq B_2$, there is $y_1 \in B_1$ such that $y_1 \notin B_2$. Analogously, there is $y_2 \in B_2$ such that $y_2 \notin B_1$ because $B_2 \not\subseteq B_1$. For y_1 and y_2 we can show that (i) is not satisfied. Indeed, from $y_1 \in B_1 = A_1^\uparrow$ and (9) it follows that

$$A_1 \subseteq \{y_1\}^\downarrow. \tag{11}$$

Moreover, $y_2 \notin B_1 = A_1^\uparrow$ gives that there is $x \in A_1$ such that $\langle x, y_2 \rangle \notin I$. Hence, there is $x \in A_1$ such that $x \notin \{y_2\}^\downarrow$, i.e. we get

$$A_1 \not\subseteq \{y_2\}^\downarrow. \tag{12}$$

As an immediate consequence of (11) and (12) we get that $\{y_1\}^\downarrow \not\subseteq \{y_2\}^\downarrow$, i.e. condition (i) is violated. In a symmetric way (i.e., with y_1 and y_2 interchanged), we can also show that (ii) is violated. So, now it remains to show that (iii) cannot be satisfied. But this is now easy to see. From $y_1 \in B_1$, $y_2 \in B_2$, and (9) we get

$A_1 = B_1^\downarrow \subseteq \{y_1\}^\downarrow$ and $A_2 = B_2^\downarrow \subseteq \{y_2\}^\downarrow$ which yield $A_1 \cap A_2 \subseteq \{y_1\}^\downarrow \cap \{y_2\}^\downarrow$. Therefore, from $A_1 \cap A_2 \not\subseteq Y^\downarrow$ it follows that $\{y_1\}^\downarrow \cap \{y_2\}^\downarrow \not\subseteq Y^\downarrow$, showing that (iii) is not satisfied. Altogether, we have shown that if $\langle X, Y, I \rangle$ does not generate a tree, then there are $y_1, y_2 \in Y$ such that none of (i)–(iii) is satisfied. \square

Remark 2. Conditions (i)–(iii) from Theorem 3 say that, roughly speaking, for each two columns of a data table, either one of the columns is contained in the other, or the columns have in common only attributes shared by all objects. In particular, if no row of the data table contains all \times 's (or 1's), the latter condition says that the columns do not have any attributes in common. Note that (i)–(iii) can be checked with asymptotic time complexity $O(n^3)$, where n is the maximum of $|X|$ and $|Y|$.

Theorem 3 can be restated as follows:

Corollary 2. *A formal context $\langle X, Y, I \rangle$ generates a tree iff, for any $y_1, y_2 \in Y$, we either have $\{y_1\}^\downarrow \cap \{y_2\}^\downarrow \in \{\{y_1\}^\downarrow, \{y_2\}^\downarrow\}$, or $\{y_1\}^\downarrow \cap \{y_2\}^\downarrow \subseteq Y^\downarrow$.*

We now turn our attention to a converse problem. Given a tree (defined possibly by its graph-theoretical representation), we wish to find a formal context which generates the tree. First, let us note that for each tree such a context exists. This is, in fact, a consequence of the main theorem of concept lattices. In more detail, consider a graph $\mathbf{G} = \langle V, E \rangle$ which is a tree.¹² We say that edge $e_1 \in E$ is under $e_2 \in E$ (in \mathbf{G}) if \mathbf{G} contains a path $v_1, e_1, \dots, v_2, e_2, \dots$ ending in the root node of \mathbf{G} (for the notions involved, see Ref. 12). We now get the following characterization.

Theorem 4. *Let $\mathbf{G} = \langle V, E \rangle$ be a tree. Define a formal context $\langle E, E, I_{\mathbf{G}} \rangle$ such that $\langle e_1, e_2 \rangle \in I_{\mathbf{G}}$ iff e_1 is under e_2 in \mathbf{G} . Then $\langle E, E, I_{\mathbf{G}} \rangle$ generates a tree which is isomorphic to $\mathbf{G} = \langle V, E \rangle$.*

Proof. Follows from Theorem 1 by standard verification. \square

Example 2. If we return to Example 1 and consider the tree from Fig. 1 (right) as an input tree, we may construct a formal context generating that tree as follows. First, we choose a labeling of edges. For instance, we may choose the labeling as in Fig. 2 (left). Then, a formal context which corresponds to $\langle E, E, I_{\mathbf{G}} \rangle$ from Theorem 4 is given by the data table in Fig. 2 (right). Since we have labeled the edges in a depth-first manner, $\langle E, E, I_{\mathbf{G}} \rangle$ is in a lower-triangular form. By Theorem 4, the tree $\mathcal{B}^\wedge(E, E, I_{\mathbf{G}})$ generated from $\langle E, E, I_{\mathbf{G}} \rangle$ is isomorphic to the initial tree.

3.2. Characterization of trees by attribute implications

In the previous section, we have shown that contexts generating trees can be characterized based on the dependencies between attributes (columns of data tables

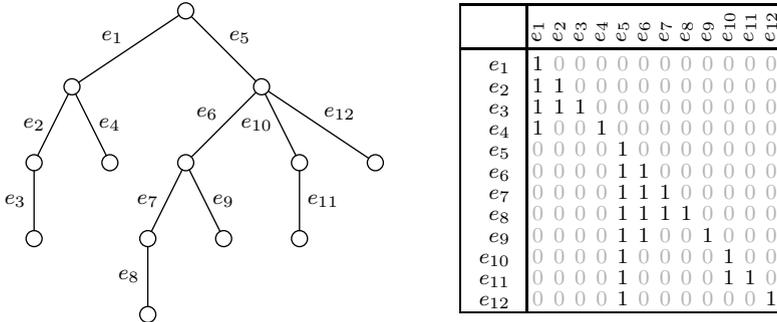


Fig. 2. Tree and its generating formal context.

representing formal contexts). Since attribute dependencies are often expressed by attribute implications, it is tempting to look at trees in a concept lattice from the point of view of attribute implications.

The following assertion characterizes contexts generating trees by means of attribute implications.

Theorem 5. *Let $\langle X, Y, I \rangle$ be a formal context. Then $\langle X, Y, I \rangle$ generates a tree iff, for any attributes $y_1, y_2 \in Y$, at least one of the following is true in $\langle X, Y, I \rangle$:*

- (i) $I \models \{y_1\} \Rightarrow \{y_2\}$,
- (ii) $I \models \{y_2\} \Rightarrow \{y_1\}$,
- (iii) $I \models \{y_1, y_2\} \Rightarrow Y$.

Proof. Note that attribute implications being true in $\langle X, Y, I \rangle$ can be characterized using the operators \uparrow and \downarrow induced by $\langle X, Y, I \rangle$. Namely, one can check that $I \models A \Rightarrow B$ iff, for each $x \in X$, if $A \subseteq \{x\}^\uparrow$ then $B \subseteq \{x\}^\uparrow$ which is iff, for each $x \in X$, if $x \in A^\downarrow$ then $x \in B^\downarrow$ which is true iff $A^\downarrow \subseteq B^\downarrow$, see Ref. 10. Thus, (i) and (ii) are true iff $\{y_1\}^\downarrow \subseteq \{y_2\}^\downarrow$ and $\{y_2\}^\downarrow \subseteq \{y_1\}^\downarrow$, cf. Theorem 3 (i) and (ii). Moreover, (iii) is true iff we have $\{y_1\}^\downarrow \cap \{y_2\}^\downarrow = (\{y_1\} \cup \{y_2\})^\downarrow = \{y_1, y_2\}^\downarrow \subseteq Y^\downarrow$. Using Theorem 3, we finally obtain that $\langle X, Y, I \rangle$ generates a tree iff, for any $y_1, y_2 \in Y$, at least one of (i)–(iii) is true. □

3.3. Algorithms for trees in concept lattices

Trees in concept lattices, as they were introduced in the previous sections, can be computed by algorithms for computing formal concepts. Currently, there have been proposed several algorithms, see e.g. Refs. 7, 10, 16 and a survey paper.¹⁵ Some of the algorithms for FCA get simplified in the case of contexts generating trees.

For instance, Lindig’s algorithm for generating formal concepts gets simplified due to the fact that it is no longer necessary to organize the concepts found in some type of searching structure, because we cannot generate the same concept

Algorithm 1.

<pre> procedure NEIGHBORS (B): $\mathcal{U} := \emptyset$ $Min := Y - B$ for each $y \in Y - B$: $D := (B \cup \{y\})^{\downarrow\uparrow}$ $Increased := D - (B \cup \{y\})$ if $Min \cap Increased = \emptyset$: add D to \mathcal{U} else: remove y from Min return \mathcal{U} </pre>	<pre> procedure GENERATEFROM (B): while $B \neq Y$: $B_{descendant} := NEIGHBORS (B)$ for each $D \in B_{descendant}$: set D_{parent} to B call GENERATEFROM (D) </pre>
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multiple times. Indeed, recall from Ref. 16 that Lindig's algorithm is based on the NEXTNEIGHBORS procedure which, given a concept as its input, generates all its (lower or upper) neighbors. Then, all concepts are computed using a recursive procedure which first uses NEXTNEIGHBORS to compute neighbors of a given concept and then recursively processes all the neighbors to obtain further concepts. During the computation, the original procedure has to ensure that no concept will be processed twice (or multiple times). Therefore, the procedure must organize all found concepts in a suitable searching structure which allows us to check whether a concept has already been found. Needless to say, the searching structure should be efficient because the tests of presence of a concept between the found concepts influences the overall efficiency of the procedure. The searching structure is usually implemented as a searching tree or a hashing table.

In case of a formal context generating a tree, this part of the algorithm need not be implemented at all because the only concept that can be computed multiple times is $\langle Y^\downarrow, Y \rangle$ which is excluded from $\mathcal{B}^\wedge(X, Y, I)$. This allows to design an algorithm which is faster and simpler to implement.

The core of the algorithm for computing the tree is shown in Algorithm 1. The procedure NEIGHBORS accepts, as its argument, an intent which corresponds in our case to a tree node. The output of NEIGHBORS is the set of all descendants of the node. Note that this procedure is the same as NEXTNEIGHBORS. The procedure GENERATEFROM from Algorithm 1 is a procedure that generates the tree nodes together with the edges between them. GENERATEFROM accepts a root of the tree which is given by the intent $\emptyset^{\downarrow\uparrow}$. Given the root, GENERATEFROM recursively generates all nodes. After the procedure finishes, for each node B , B_{parent} will be the parent node of B and the nodes contained in $B_{descendant}$ will be the descendants of B .

3.4. Extension to fuzzy attributes

Formal concept analysis has been extended to data with fuzzy attributes, see e.g. Refs. 3, 17, and also Ref. 4. In this case, attributes are allowed to apply to objects to degrees from a suitable set L of truth degrees such as the unit interval $[0, 1]$. This situation is captured by the notion of a formal fuzzy context, which is a triplet $\langle X, Y, I \rangle$ where X and Y are ordinary sets of objects and attributes and I is a fuzzy relation between X and Y with truth degrees from L . That is, I is a mapping $I : X \times Y \rightarrow L$ assigning to every object $x \in X$ and every attribute $y \in Y$ a degree $I(x, y)$ to which x has y (y applies to x). The set L of truth degrees needs to be equipped with appropriate truth functions of logical connectives. Before proceeding, we recall some necessary notions from fuzzy logic and fuzzy sets. More details can be found in the monographs.^{3,14}

A complete residuated lattice, which is our basic structure of truth degrees, is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$, where $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice, $\langle L, \otimes, 1 \rangle$ is a commutative monoid, and \otimes and \rightarrow satisfy the so-called adjointness property, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.^{3,14} Each $a \in L$ is called a truth degree; \otimes and \rightarrow are (truth functions of) “fuzzy conjunction” and “fuzzy implication”. Complete residuated lattices include structures of truth degrees defined on the real unit interval with \wedge and \vee being minimum and maximum, \otimes being a left-continuous t-norm with the corresponding \rightarrow . Finite residuated lattices represent another important subfamily of complete residuated lattices. A particular finite residuated lattice is the Boolean algebra with $L = \{0, 1\}$ (structure of truth degrees of classical logic). Given \mathbf{L} which serves as a structure of truth degrees, we introduce the usual structural notions: an \mathbf{L} -set (fuzzy set) A in universe U is a mapping $A : U \rightarrow L$, $A(u)$ being interpreted as “the degree to which u belongs to A ”. \mathbf{L}^U denotes the collection of all \mathbf{L} -sets in U . The operations with \mathbf{L} -sets are defined componentwise. For instance, intersection of \mathbf{L} -sets $A, B \in \mathbf{L}^U$ is an \mathbf{L} -set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ ($u \in U$). For each $u \in U$ and $a \in L$, we let $\{^a/u\}$ denote \mathbf{L} -set in U such that $(\{^a/u\})(u) = a$ and, for each $v \neq u$, $(\{^a/u\})(v) = 0$. Binary \mathbf{L} -relations (binary fuzzy relations) between U and V can be thought of as \mathbf{L} -sets in $U \times V$. Given $A, B \in \mathbf{L}^U$, we put $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$.

Let now \mathbf{L} be a complete residuated lattice and let $\langle X, Y, I \rangle$ be a formal fuzzy context with truth degrees from L . For $A \in \mathbf{L}^X$, $B \in \mathbf{L}^Y$ (i.e. A is a fuzzy set of objects, B is a fuzzy set of attributes), we define fuzzy sets $A^\uparrow \in \mathbf{L}^Y$ (fuzzy set of attributes), $B^\downarrow \in \mathbf{L}^X$ (fuzzy set of objects) by

$$\begin{aligned} A^\uparrow(y) &= \bigwedge_{x \in X} (A(x) \rightarrow I(x, y)), \\ B^\downarrow(x) &= \bigwedge_{y \in Y} (B(y) \rightarrow I(x, y)). \end{aligned}$$

Described verbally, A^\uparrow is the fuzzy set of all attributes from Y shared by all objects from A (and similarly for B^\downarrow). A formal fuzzy concept in $\langle X, Y, I \rangle$ is a pair $\langle A, B \rangle$ of $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$ satisfying $A^\uparrow = B$ and $B^\downarrow = A$. That is, a fuzzy concept consists of a fuzzy set A (extent) of objects which fall under the concept and a fuzzy

set B (intent) of attributes which fall under the concept such that A is the fuzzy set of all objects sharing all attributes from B and, conversely, B is the fuzzy set of all attributes from Y shared by all objects from A .

The collection $\mathcal{B}(X, Y, I) = \{\langle A, B \rangle \mid A^\uparrow = B, B^\downarrow = A\}$ of all formal fuzzy concepts in $\langle X, Y, I \rangle$ can be equipped with a partial order \leq defined by

$$\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle \text{ iff } A_1 \subseteq A_2 \text{ (iff } B_2 \subseteq B_1). \quad (13)$$

Note that \uparrow and \downarrow form a so-called fuzzy Galois connection³ and that $\mathcal{B}(X, Y, I)$ is in fact the set of all fixed points of \uparrow and \downarrow . Under \leq , $\mathcal{B}(X, Y, I)$ happens to be a complete lattice, called the fuzzy concept lattice of $\langle X, Y, I \rangle$. The basic structure of fuzzy concept lattices is described by the so-called main theorem of concept lattices.^{3,4}

Remark 3. It can be easily seen that if $L = \{0, 1\}$, the above-introduced notions of a formal fuzzy context, operators \uparrow and \downarrow , formal fuzzy concept, fuzzy concept lattice, etc., can be identified with their ordinary counterparts introduced in Section 2. This way, FCA generalizes ordinary FCA.

Next, we present results characterizing fuzzy concept lattices which are trees (after removing the least element), analogous to the results from Section 3.1.

Theorem 6. *Let $\langle X, Y, I \rangle$ be a formal fuzzy context with truth degrees from a complete residuated lattice \mathbf{L} . Then $\langle X, Y, I \rangle$ generates a tree iff, for any attributes $y_1, y_2 \in Y$ and any $a_1, a_2 \in L$, at least one of the following conditions is true:*

- (i) $\{a_1/y_1\}^\downarrow \subseteq \{a_2/y_2\}^\downarrow$,
- (ii) $\{a_2/y_2\}^\downarrow \subseteq \{a_1/y_1\}^\downarrow$,
- (iii) $\{a_1/y_1\}^\downarrow \cap \{a_2/y_2\}^\downarrow \subseteq Y^\downarrow$ (here, by abuse of notation, Y denotes a full fuzzy set, i.e. $Y(y) = 1$ for any y).

Proof. The proof is based on a reduction theorem from Ref. 2 and Theorem 3. The reduction theorem from Ref. 2 says that the fuzzy concept lattice $\mathcal{B}(X, Y, I)$ is isomorphic to an ordinary concept lattice $\mathcal{B}(X \times L, Y \times L, I^\times)$ where $X \times L$ and $Y \times L$ are the Cartesian products (objects \times truth degrees, and attributes \times truth degrees) and I^\times is an ordinary relation between $X \times L$ and $Y \times L$ defined by

$$\langle \langle x, a \rangle, \langle y, b \rangle \rangle \in I^\times \quad \text{iff} \quad a \otimes b \leq I(x, y).$$

Moreover, Ref. 2 provides “translation formulas” between the fuzzy setting for $\mathcal{B}(X, Y, I)$ and the ordinary setting for $\mathcal{B}(X \times L, Y \times L, I^\times)$. For an ordinary set $C \subseteq U \times L$ and a fuzzy set $D \in \mathbf{L}^U$ denote by $\lceil C \rceil \in \mathbf{L}^U$ a fuzzy set in U and by $\lfloor D \rfloor$ an ordinary subset of U defined by

$$\lceil C \rceil(u) = \bigvee_{\langle u, a \rangle \in C} a, \quad \text{and} \quad \lfloor D \rfloor = \{\langle u, a \rangle \mid a \leq D(u)\}.$$

Then, for the Galois connection \wedge, \vee associated to $\mathcal{B}(X \times L, Y \times L, I^\times)$ we have $B^\vee = \lfloor \lceil B \rceil^\downarrow \rfloor$, see Ref. 2.

Because of the isomorphism, $\mathcal{B}(X, Y, I)$ is a tree iff $\mathcal{B}(X \times L, Y \times L, I^\times)$ is a tree. Therefore, by Theorem 3, $\mathcal{B}(X, Y, I)$ is a tree iff for any $y_1, y_2 \in Y$ and $a_1, a_2 \in L$ we have (i') $\{\langle y_1, a_1 \rangle\}^\vee \subseteq \{\langle y_2, a_2 \rangle\}^\vee$, or (ii') $\{\langle y_2, a_2 \rangle\}^\vee \subseteq \{\langle y_1, a_1 \rangle\}^\vee$, or (iii') $\{\langle y_1, a_1 \rangle\}^\vee \cap \{\langle y_2, a_2 \rangle\}^\vee \subseteq (Y \times L)^\vee$. Now, (i') is equivalent to condition (i) from the present theorem, (ii') is equivalent to (ii), and (iii') is equivalent to (iii). Indeed, for (i') and (i): $\{\langle y_i, a_i \rangle\}^\vee = \lfloor \lceil \{a_i/y_i\}^\downarrow \rceil \rfloor$ and a routine verification shows that $\lfloor \lceil \{a_1/y_1\}^\downarrow \rceil \rfloor \subseteq \lfloor \lceil \{a_2/y_2\}^\downarrow \rceil \rfloor$ iff $\{a_1/y_1\}^\downarrow \subseteq \{a_2/y_2\}^\downarrow$. Due to symmetry, (ii) and (ii') are equivalent, too. For (iii) and (iii'), the proof follows by observing that for fuzzy sets A, B , and C we have that $\lfloor A \rfloor \cap \lfloor B \rfloor \subseteq \lfloor C \rfloor$ is equivalent to $A \cap B \subseteq C$. \square

A similar reasoning can be applied to obtain a generalization of Theorem 5.

4. Conclusions and Future Research

We presented conditions for input data for FCA which are necessary and sufficient for the output concept lattice to form a tree after one removes its least element. Trees are the most common structures which appear in traditional clustering and classification. Future research will focus on establishing connections between FCA and other clustering and classification methods. First, establishing such relationships helps us see the pros and cons, and limits of the respective methods. Second, with the basic relationships established, one can hopefully enrich the respective methods by techniques used in the other methods. The problems we want to address next include the following ones:

- A concept lattice can be seen as consisting of several overlapping trees. What can we say about such a “decomposition” of a concept lattice into trees? What are the relationships between these trees?
- A user of FCA might be interested in a part of a concept lattice rather than in the whole lattice. Particularly, that part might be a tree, but other parts might be interesting as well. The issue of selecting parts of concept lattices by constraints was discussed in Refs. 5, 6. In particular, it can be shown that a tree contained in a concept lattice can be selected by means of a particular closure operator. Constraints which lead to tree-like parts of concept lattices need to be further investigated.
- Investigate connections between concept lattices and decision trees. Both concept lattices and decision trees contain clusters of objects in their nodes. Leafs of a decision tree correspond to particular attribute-concepts. The construction of a decision tree may be thought of as the selection of a particular part from a concept lattice. Containment of decision trees in concept lattices needs to be further investigated.

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