GGEN 41007—17/3/2004—ADMINISTRATOR—97872

International Journal of General Systems, 2004 Vol. 00 (0), pp. 1–16

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FUZZY INTERIOR OPERATORS

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(Received 22 October 2003; In final form 21 January 2004)

We study interior operators from the point of view of fuzzy set theory. The present approach generalizes the particular cases studied previously in the literature in two aspects. First, we use complete residuated lattices as structures of truth values generalizing thus several important cases like the classical Boolean case, (left-)continuous t-norms, MV-algebras, BL-algebras, etc. Second, and more importantly, we pay attention to graded subsethood of fuzzy sets, which turns out to play an important role. In the first part, we define, illustrate by examples and study general fuzzy interior operators. The second part is devoted to fuzzy interior operators induced by fuzzy equivalence relations (similarities).

Keywords: Interior operator; Fuzzy set; Fuzzy logic; Fuzzy equivalence

AMS Classification: 03B52; 08B05

1. INTRODUCTION AND PRELIMINARIES

Closure and interior operators on ordinary sets belong to the very fundamental mathematical structures with direct applications, both mathematical (topology, logic, for instance) and extramathematical (e.g. data mining, knowledge representation). In fuzzy set theory, several particular cases as well as general theory of closure operators which operate with fuzzy sets (so called fuzzy closure operators) are studied (Mashour and Ghanim, 1985; Bandler and Kohout, 1988; Bělohlávek, 2001; 2002a,b; Gerla, 2001). Interior operators, however, have appeared in a few studies only (Bandler and Kohout, 1988; Dubois and Prade, 1991; Bodenhofer *et al.*, 2003), and it seems that no general theory of interior operators appeared so far. In ordinary set theory, closure and interior operators on a set are in a bijective correspondence. Namely, recall that a mapping $I : \mathbf{2}^X \to \mathbf{2}^X$ is called an interior operator on X if (1) $I(A) \subseteq A$; (2) $A \subseteq B$ implies $I(A) \subseteq I(B)$; (3) $I(A) = I(I(A))$ for any subsets A and B of X. A closure operator on X is a mapping $C : \mathbf{2}^X \to \mathbf{2}^X$ satisfying (1′) $A \subseteq C(A)$; (2′) $A \subseteq B$ implies $C(A) \subseteq C(B)$; (3′) $C(A) = C(C(A))$ for any subsets A and B of X. It is a well known fact that given an interior operator I and a closure operator C, putting $C_I(A) = \overline{I(\overline{A})}$ and $I_C(A) = \overline{C(\overline{A})}$, C_I is a closure operator and I_C is an interior operator. Moreover, the thus defined mappings are bijective. That is, having developed the theory of

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ISSN 0308-1079 print/ISSN 1563-5104 online
DOI: 10.1080/03081070410001679724

closure operators, one can automatically obtain the theory of interior operators with corresponding "translation rules" transforming true statements about closure operators to true statements about interior operators and vice versa. This is possible, as an easy observation shows, due to the law of double negation (which says that for each set A we have $\overline{\overline{A}} = A$ with $\overline{B}$ denoting the complement of B) which is true in ordinary set theory. In general, however, the law of double negation does not hold in fuzzy set theory. This means that the easy one-to-one way between closure and interior operator is no more at our disposal in fuzzy set theory and that unless developing other (possibly partial) translation rules, one has to develop an appropriate theory of fuzzy interior operators from scratch.

The development of a general theory of fuzzy interior operators is the main purpose of the present paper. Our formal setting is given by complete residuated lattices which we take for the structures of truth values and which represent general structures of which many particular structures discussed in the literature are particular cases. An important aspect of our treatment is that we take into account, in a parametrized manner, graded subsethood of fuzzy sets that is required to be preserved by interior operators. In second section, we study general fuzzy interior operators. Third section is devoted to fuzzy interior operators induced by fuzzy equivalence relations. In the rest of this section, we recall the necessary notions.

Recall that a complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with the least element 0 and the greatest element 1;
(2) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is commutative, associative, and $x \otimes 1 = x$ holds for each $x \in L$;
(3) $\otimes, \rightarrow$ form an adjoint pair, i.e.

$$x \otimes y \leq z \text{ iff } x \leq y \rightarrow z \tag{1}$$

holds for all $x, y, z \in L$.

We say that $\mathbf{L}$ satisfies the law of double negation iff $x = \neg\neg x$ is true in $\mathbf{L}$ where $\neg$ is defined by $\neg x = x \rightarrow 0$ for any $x \in L$.

Residuated lattices play the role of structures of truth values in fuzzy logic. Introduced originally in the study of ideal systems of rings (Ward and Dilworth, 1939), residuated lattices have been introduced into the context of fuzzy logic by Goguen (1967). For logical calculi with truth values in residuated lattices (and special types of residuated lattices), basic properties of residuated lattices, and further references we refer to Höhle (1996), Hájek (1998) and Bělohlávek (2002a,b).

We only recall that the most studied and applied residuated lattices are those defined on the real interval [0,1] (residuated lattices on [0,1] uniquely correspond to left-continuous t-norms). Three most important structures pairs of adjoint operations are the following: the Łukasiewicz one ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), Gödel one ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and product one ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ else). More generally, $\langle [0, 1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ is a complete residuated lattice iff $\otimes$ is a left-continuous t-norm and $a \rightarrow b = \bigvee \{z | a \otimes z \leq b\}$. An example of left-continuous t-norm which is not continuous is the so-called nilpotent minimum defined by $x \otimes y = \min(x, y)$ if $x + y > 1$, $x \otimes y = 0$ if $x + y \leq 1$. Another important set of truth values is the set $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. A special case of the latter algebras is the Boolean algebra $\mathbf{2}$ of classical logic with the support $2 = \{0, 1\}$.

A nonempty subset $K \subseteq L$ is called an $\leq$-filter if for every $a, b \in L$ such that $a \leq b$ it holds that $b \in K$ whenever $a \in K$. An $\leq$-filter K is called a filter if $a, b \in K$ implies $a \otimes b \in K$. Unless otherwise stated, in what follows we denote by $\mathbf{L}$ a complete residuated lattice and by K an $\leq$-filter in $\mathbf{L}$ (both $\mathbf{L}$ and K possibly with indices).

An **L**-set (fuzzy set), see (Zadeh, 1965; Goguen, 1967), A in a universe set X is any map $A : X \to L$. By L^X, we denote the set of all **L**-sets in X. The concept of an **L**-relation is defined obviously. Operations on L extend pointwise to L^X, e.g. $(A \vee B)(x) = A(x) \vee B(x)$ for $A, B \in L^X$. Following common usage, we write $A \cup B$ instead of $A \vee B$, etc. The complement of an **L**-set A is a fuzzy set $\neg A$ defined by $(\neg A)(x) = A(x) \to 0$. Given $A, B \in L^X$, the subsethood degree (Goguen, 1967) $S(A,B)$ of A in B is defined by $S(A,B) = \bigwedge_{x \in X} A(x) \to B(x)$. We write $A \subseteq B$ if $S(A,B) = 1$. Analogously, the equality degree $E(A,B)$ of A and B is defined by $E(A,B) = \bigwedge_{x \in X}(A(x) \leftrightarrow B(x))$. It is immediate that $E(A,B) = S(A,B) \wedge S(B,A)$. By $\{a_1/x_1, \cdots, a_n/x_n\}$ we denote an **L**-set A for which $A(x) = a_i$ if $x = x_i$ $(i = 1, \ldots, n)$ and $A(x) = 0$ otherwise. By $\emptyset$ and X we denote the empty and full **L**-set in X, i.e. $\emptyset(x) = 0$ and $X(x) = 1$ for each $x \in X$.

A binary fuzzy relation R on X is called reflexive if $R(x,x) = 1$; symmetric if $R(x,y) = R(y,x)$; transitive if $R(x,y) \otimes R(y,z) \leq R(x,z)$, for all $x, y, z \in X$. An **L**-equivalence (fuzzy equivalence) is a fuzzy relation which is reflexive, symmetric and transitive.

2. FUZZY INTERIOR OPERATORS

First, we show some natural examples of fuzzy interior operators (and subsequently, we discuss these examples from the point of view of Definition 1 presented below).

Example 1 A fuzzy topology, see Chang (1968) or Liu (1999) for a recent survey on fuzzy topology, in X is a collection $\mathcal{T} \subseteq [0,1]^X$ of fuzzy sets in X (i.e. mappings $A : X \to [0,1]$) satisfying (i) $\emptyset, X \in \mathcal{T}$; (ii) $A \cap B \in \mathcal{T}$ for any $A, B \in \mathcal{T}$; (iii) $\cup_i A_i \in \mathcal{T}$ for any $A_i \in \mathcal{T}$ $(i \in I)$. Fuzzy sets $A \in \mathcal{T}$ are called open in the topology $\mathcal{T}$. To each fuzzy set $A : X \to [0,1]$, one can assign the greatest fuzzy set $I_{\mathcal{T}}(A)$ which is contained in A (i.e. $I_{\mathcal{T}}(A) \subseteq A$). Remark 3 places $I_{\mathcal{T}}$ into the context of fuzzy interior operators defined below.

Example 2 R be a binary relation on X, consider an arbitrary left-continuous t-norm $\otimes$ and the corresponding residuum $\to$. Let us consider the operator $I_R : [0,1]^X \to [0,1]^X$ defined by

$$(I_R(A))(x) = \bigwedge_{y \in X} R(x,y) \to A(y).$$

That is, the degree to which x belongs to $I_R(A)$ equals the degree to which it is true that for each y we have that if x and y are in R then y belongs to A. Two particular cases of I_R are important.

First, if R is a fuzzy equivalence relation (this case will be examined in section "Interior induced by fuzzy equivalences"), then $I_R(A)$ is the so-called lower approximation of A with respect to R in terms of rough set theory of Pawlak when modified to fuzzy setting, see Dubois and Prade (1991) and Pawlak (1991). If R is interpreted as representing some intrinsic indiscernibility on the universe X (which may be due to various limitations), then $I_R(A)$ is the greatest fuzzy set in X which is included in $A(I_R(A) \subseteq A)$ and compatible with the indiscernibility represented by R (in that $I_R(A)(x) \otimes R(x,y) \leq I_R(A)(y)$, i.e. if x belongs to $I_R(A)$ and x and y are indiscernible then y belongs to $I_R(A)$ as well).

Second, if R is reflexive and transitive then I_R is the operator studied in Bodenhofer *et al.* (2003) where it is called an opening operator induced by R. Several properties of interior and closure operators are shown and applications of these operators to so-called ordering-based modifiers are demonstrated in Bodenhofer *et al.* (2003). Remark 5 places I_R into the context of fuzzy interior operators defined below.

In the following, we are going to present the concept of a fuzzy interior operator. We denote by **L** an arbitrary complete residuated lattice and by K an $\leq$-filter in **L**. Moreover, by X we denote some fixed nonempty set.

DEFINITION 1 An $\mathbf{L}_K$-interior operator (fuzzy interior operator) on X is a mapping $I\colon L^X \to L^X$ satisfying

$$I(A) \subseteq A \tag{2}$$

$$S(A_1, A_2) \leq S(I(A_1), I(A_2)) \text{ whenever } S(A_1, A_2) \in K \tag{3}$$

$$I(A) = I(I(A)) \tag{4}$$

for every $A, A_1, A_2 \in L^X$.

Remark 1

(1) If K and $\mathbf{L}$ are obvious, we speak of a fuzzy interior operator. If $K = L$, we omit the subscript K and call I an $\mathbf{L}$-interior operator. The set K plays the role of the set of designated truth values, condition (3) says that the interior preserves also partial subsethood whenever the subsethood degree is designated. Since K is an $\leq$-filter in $\mathbf{L}$, the designated truth values represent, in a sense, sufficiently high truth values. In this view, Eq. (3) reads "if A_1 is almost included in A_2 then $I(A_1)$ is almost included in $I(A_2)$". It is easily seen that each $\mathbf{L}_K$-interior operator on X is an interior operator on the complete lattice $\langle L^X, \subseteq \rangle$ (recall that an interior operator on an ordered set $\langle V, \leq \rangle$ is a map $i: V \mapsto V$ satisfying $i(v) \leq v$, if $u \leq v$ then $i(u) \leq i(v)$, and $i(i(u)) = i(u)$ for each $u, v \in V$).
(2) One easily verifies that for $L = \{0, 1\}$, $\mathbf{L}_K$-interior operators are precisely the ordinary interior operators (no matter what K). Clearly, if $K_1 \subseteq K_2$ then each $\mathbf{L}_{K_2}$-interior operator is an $\mathbf{L}_{K_1}$-interior operator. As we will see, the converse is not true. Note also that for $L = \{0, 1\}$, $\mathbf{L}_{\{1\}}$-interior operators are what is usually known as fuzzy interior operators, see, e.g. Bodenhofer *et al.*, (2003).
(3) We show that for residuated lattices $\mathbf{L}$ where $L = [0, 1]$ with Łukasiewicz, Gödel, and product structures, K is relevant: Take $X = \{x_1, x_2\}$, and define I by $I(A)(x_1) = 0.5$, $I(A)(x_2) = 1$ for $A(x_1) \geq 0.5, A(x_2) = 1$, and $I(A)(x_1) = I(A)(x_2) = 0$ otherwise. An easy inspection shows that I is an $\mathbf{L}_{\{1\}}$-interior operator. However, for A_1, A_2 given by $A_1(x_1) = A_2(x_1) = 0.5$, $A_1(x_2) = 1$, $A_2(x_2) = 0.5$ it holds $S(A_1, A_2) = 0.5 > 0 = S(I(A_1), I(A_2))$ (for all of the three above-mentioned structures). Thus I is not an $\mathbf{L}_{[0.5,1]}$-interior operator.

We define $\mathcal{S}_I = \{A | A = I(A)\}$. Note that it follows from Eq. (4) that $\mathcal{S}_I = \{I(A) | A \in L^X\}$. Indeed Eq. (4) implies that $I(A) \in \mathcal{S}_I$ for each $A \in L^X$. On the other hand, if $A \in \mathcal{S}_I$, then $A = I(A)$ by definition of $\mathcal{S}_I$.

Unless otherwise stated, I denotes an $\mathbf{L}_K$-interior operator.

LEMMA 2 For $A_i \in \mathcal{S}_I$, $i \in I$, we have $\bigcup_{i \in I} A_i = I(\bigcup_{i \in I} A_i)$, i.e. $\mathcal{S}_I$ is closed under union.

Proof For each $i \in I$ we have

$$I(A_i) \subseteq \bigcup_{i \in I} I(A_i),$$

and so

$$I(A_i) = I(I(A_i)) \subseteq I\left(\bigcup_{i \in I} I(A_i)\right)$$

from which we get

$$\bigcup_{i \in I} (I(A_i)) \subseteq I\left(\bigcup_{i \in \mathrm{I}} I(A_i)\right).$$

The converse inequality follows from Eq. (2). □

LEMMA 3 For $A \in \mathcal{S}_I$ and $a \in K$ we have

$$I(a \otimes A) = a \otimes A \tag{5}$$

Proof For $A \in \mathcal{S}_I$ we have $A = I(A)$. Therefore, we have to show $I(a \otimes I(A)) = a \otimes I(A)$. $I(a \otimes I(A)) \subseteq a \otimes I(A)$ follows from Eq. (2).

Conversely we have $I(a \otimes I(A)) \supseteq a \otimes I(A)$ iff for any $x \in X$ we have

$$a \otimes I(A)(x) \leq I(a \otimes I(A))(x)$$

iff

$$a \leq I(A)(x) \rightarrow I(a \otimes I(A))(x)$$

iff

$$a \leq S(I(A), I(a \otimes I(A)))$$

which is true. Indeed, it is easily seen that $a \leq S(I(A), a \otimes I(A))$ and since $a \in K$, the $\leq$-filter property of K yields $S(I(A), a \otimes I(A)) \in K$. By Eq. (3) we thus have

$$a \leq S(I(A), a \otimes I(A)) \leq S(I(I(A)), I(a \otimes I(A))) = S(I(A), I(a \otimes I(A)))$$

completing the proof. □

THEOREM 4 $I: L^X \rightarrow L^X$ is an $\mathbf{L}_K$-interior operator on X iff it satisfies Eq. (2) and the following condition:

$$S(I(A_1), A_2) \leq S(I(A_1), I(A_2)) \text{ whenever } S(I(A_1), A_2) \in K. \tag{6}$$

Proof Suppose Eqs. (2)–(4) hold. If $S(I(A_1), A_2) \in K$ then by Eqs. (3) and (4) we have $S(I(A_1), A_2) \leq S(I(I(A_1)), I(A_2)) = S(I(A_1), I(A_2))$, i.e. Eq. (6) holds.

Conversely, let Eqs. (2) and (6) hold. Suppose $S(A_1, A_2) \in K$. By Eq. (2), $I(A_1) \subseteq A_1$, whence $S(A_1, A_2) \leq S(I(A_1), A_2) \leq S(I(A_1), I(A_2))$, proving Eq. (3). Since $1 \in K$, Eq. (6) yields $1 = S(I(A), I(A)) \leq S(I(A), I(I(A)))$, and so $I(A) \subseteq I(I(A))$. Since the converse inclusion holds by Eq. (2), we conclude Eq. (4). □

DEFINITION 5 A system $\mathcal{S} = \{A_I \in L^X | i \in I\}$ is called closed under S_K-unions iff for each $A \in L^X$ it holds that

$$\bigcup_{i \in I, S(A_i, A) \in K} S(A_i, A) \otimes A_i \in \mathcal{S}$$

where

$$\left(\bigcup_{i \in I, S(A_i, A) \in K} S(A_i, A) \otimes A_i \right)(x) = \bigvee_{i \in I, S(A_i, A) \in K} S(A_i, A) \otimes A_i(x)$$

for each $x \in X$. A system closed under S_K-unions will be called an $\mathbf{L}_K$-interior system.

Loosely speaking, $\mathcal{S}$ is closed under S_K-unions iff for each fuzzy set A in X, the union of all $A_i \in \mathcal{S}$ which are almost included in A, belongs to $\mathcal{S}$.

Remark 2

(1) We have

$$\bigcup_{i \in I, S(A_i, A) \in \{1\}} S(A_i, A) \otimes A_i = \bigcup_{i \in I, A_i \subseteq A} A_i.$$

Therefore, $\mathcal{S}$ is a **2**-interior system iff for each $A \subseteq X$ it holds $\bigcup_{A_i \subseteq A} A_i \in \mathcal{S}$. It is well known that the last condition is equivalent to the fact that $\mathcal{S}$ is closed under arbitrary unions.

(2) In general, being closed under arbitrary unions is a weaker condition than being closed under S_K-unions. Indeed, let $\mathcal{S}$ be closed under S_K-unions. To show that $\mathcal{S}$ is closed under arbitrary unions, it suffices to show that

$$\bigvee_{j\in J} A_j(x) = \bigvee_{i\in I, S\left(A_i, \bigcup_{j\in J} A_j\right)\in K} S\left(A_i, \bigcup_{j\in J} A_j\right) \otimes A_i(x)$$

holds for any $J \subseteq I$. The inequality $\leq$ is true since for each $j' \in J$ we have $S\left(A_{j'}, \bigcup_{j\in J} A_j\right) \otimes A_{j'}(x) = 1 \otimes A_{j'}(x) = A_{j'}(x)$. The converse inequality holds iff

$$\bigvee_{j\in J} A_j(x) \geq S\left(A_i, \bigcup_{j\in J} A_j\right) \otimes A_i(x)$$

for each $i \in I$ such that $S(A_i, \bigcup_{j\in J} A_j) \in K$ which is equivalent to

$$A_i(x) \otimes \left(\bigwedge_{y\in X} A_i(y) \to \left(\bigvee_{j\in J} A_j(y)\right)\right) \leq \bigvee_{j\in J} A_j(x)$$

which is true because

$$A_i(x) \otimes \left(\bigwedge_{y\in X} A_i(y) \to \left(\bigvee_{j\in J} A_j(y)\right)\right) \leq \quad A_i(x) \otimes \left(A_i(x) \to \left(\bigvee_{j\in J} A_j(x)\right)\right) \leq \bigvee_{j\in J} A_j(x).$$

On the other hand, put $X = \{x\}$, take the Łukasiewicz structure with $\mathbf{L} = \{0, 1/2, 1\}$, $K = L$, $\mathcal{S} = \{\{0/x\}, \{1/x\}\}$, and $A = \{(1/2)/x\}$. Then $\mathcal{S}$ is clearly closed under arbitrary unions but not under S_K-unions since $\bigcup_{i\in I, s(A_i,A)\in K} S(A_i, A) \otimes A_i = A \notin \mathcal{S}$.

The next theorem shows that closedness under S_K-unions is equivalent to closedness under unions of "K-cut" $\mathbf{L}$-sets of $\mathcal{S}$.

Theorem 6 $\mathcal{S} = \{A_i \in L^X | i \in I\}$ is an $\mathbf{L}_K$-interior system iff for any $a_i \in K$, $i \in I$, we have

$$\bigcup_{a_i\in K} (a_i \otimes A_i) \in \mathcal{S}.$$

Proof Let $\bigcup_{a_i\in K}(a_i \otimes A_i) \in \mathcal{S}$, and put $a_i = S(A_i, A)$ for $S(A_i, A) \in K$ and $a_i = 0$ otherwise. Then $\bigcup_{i\in I, S(A_i,A)\in K} S(A_i, A) \otimes A_i = \bigcup_{a_i\in K} a_i \otimes A_i \in \mathcal{S}$, showing that $\mathcal{S}$ is an $\mathbf{L}_K$-interior system.

Conversely, let $\mathcal{S}$ be an $\mathbf{L}_K$-interior system. Take $a_i \in L$ and put $A = \bigcup_{a_i\in K} a_i \otimes A_i$. We have to show that $A \in \mathcal{S}$. It suffices to show $\bigcup_{i\in I, S(A_i,A)\in K}(S(A_i, A) \otimes A_i) = A$. The fact

$$\bigcup_{i\in I, S(A_i,A)\in K} (S(A_i, A) \otimes A_i) \subseteq A$$

is shown in Lemma 9. For the converse inclusion, observe first that if $a_j \in K$ then $S(A_j, A) \in K$. Indeed, by $\leq$-filter property of K it suffices to show that $a_j \leq S(A_j, A)$. This holds iff for each $x \in X$ we have

$$a_j \leq \left(A_j(x) \to \bigvee_{a_i\in K} (a_i \otimes A_i(x))\right)$$

i.e. $a_j \otimes A_j(x) \leq (\bigvee_{a_i \in K}(a_i \otimes A_i(x)))$ which is clear. Now, the converse inclusion to be proved holds true iff for each $x \in X$ we have

$$\bigcup_{a_i \in K} a_i \otimes A_i(x) \leq \bigvee_{i \in I,\, S(A_i,A) \in K} S(A_i, A) \otimes A_i(x)$$

iff for each $a_j \in K$ it holds

$$a_j \leq A_j(x) \rightarrow \bigvee_{i \in I, S(A_i,A) \in K} S(A_i, A) \otimes A_i(x)$$

which holds since by the above observation

$$a_j \leq S(A_j,A) \leq A_j(x) \rightarrow S(A_j,A) \otimes A_j(x) \leq A_j(x) \rightarrow \bigvee_{i \in I, S(A_i,A) \in K} S(A_i,A) \otimes A_i(x). \qquad \square$$

We have an immediate corollary.

COROLLARY 7 A system $\mathcal{S}$ which is closed under arbitrary unions is an $\mathbf{L}_K$-interior system iff for each $a \in K$ and $A \in \mathcal{S}$ it holds $a \otimes A \in \mathcal{S}$.

Remark 3 Consider now Example 1 from the point of view of Theorem 6 (and Corollary 7). One can easily see that $\mathcal{T}$ is a fuzzy topology iff $\mathcal{T}$ is an $\mathbf{L}_{\{1\}}$-interior system (for any structure $\mathbf{L}$ with $L = [0, 1]$) which is closed under finite intersections and contains X. Moreover, it is worth mentioning that in Lowen (1976), a stronger notion of fuzzy topology is studied in that instead of requiring $\emptyset$, $X \in \mathcal{T}$, the author requires that for each $a \in [0, 1]$, $\mathcal{T}$ contains the constant fuzzy set c_a defined by $c_a(x) = a$ for each $x \in X$ (note that $c_0 = \emptyset$ and $c_1 = X$). We can easily see that $\mathcal{T}$ is a fuzzy topology in the above-mentioned stronger sense iff $\mathcal{T}$ is an $\mathbf{L}_L$-interior system for $\mathbf{L}$ with $L = [0, 1]$ and $a \otimes b = a \wedge b$, which is closed under finite intersections and contains X. Indeed, if $\mathcal{T}$ is a fuzzy topology then for any $A \in \mathcal{T}$ and $a \in L$ we have $c_a \in \mathcal{T}$ (by the stronger definition of fuzzy topology) and so $a \otimes A = a \wedge A \in \mathcal{T}$ by closedness of $\mathcal{T}$ under finite intersections. Corollary 7 then yields that $\mathcal{T}$ is an $\mathbf{L}_L$-interior system which is closed under finite intersections and contains X. On the other hand, if $\mathcal{T}$ is an $\mathbf{L}_L$-interior system which is closed under finite intersections and contains X then since $X \in \mathcal{T}$, Corollary 7 yields $c_a = a \wedge X \in \mathcal{T}$ for any $a \in L$, i.e. $\mathcal{T}$ is a fuzzy topology.

The following theorem shows another way to obtain the interior in an $\mathbf{L}_K$-interior system.

THEOREM 8 Let $\mathcal{S} = \{A_i \in L^X | i \in I\}$ be an $\mathbf{L}_K$-interior system. Then for each $A \in L^X$ we have

$$\bigcup_{i \in I, S(A_i,A) \in K} S(A_i, A) \otimes A_i = \bigcup_{i \in I, A_i \subseteq A} A_i.$$

Proof Clearly,

$$\bigcup_{i \in I, S(A_i,A) \in K} S(A_i, A) \otimes A_i \supseteq \bigcup_{i \in I, S(A_i,A) = 1} S(A_i, A) \otimes A_i = \bigcup_{i \in I, A_i \subseteq A} A_i.$$

On the other hand, it is easy to check that $\bigcup_{i \in I, S(A_i,A) \in K} S(A_i, A) \otimes A_i \subseteq A$. Since $\mathcal{S}$ is an $\mathbf{L}_K$-interior system, we have $\bigcup_{i \in I, S(A_i,A) \in K} S(A_i, A) \otimes A_i \in \mathcal{S}$, which immediately gives

$$\bigcup_{i \in I, S(A_i,A) \in K} S(A_i, A) \otimes A_i \subseteq \bigcup_{i \in I, A_i \subseteq A} A_i. \qquad \square$$

LEMMA 9 Let $\mathcal{S} = \{A_i | i \in I\}$ be an $\mathbf{L}_K$-interior system, K be a filter in $\mathbf{L}$. Then $I_{\mathcal{S}}\colon L^X \to L^X$ defined by

$$I_{\mathcal{S}}(A)(x) = \bigvee_{i \in I, S(A_i, A) \in K} (S(A_i, A) \otimes A_i(x))$$

is an $\mathbf{L}_K$-interior operator. Moreover, for $A \in L^X$ it holds $A \in \mathcal{S}$ iff $A = I_{\mathcal{S}}(A)$.

Proof We check (2)–(4).

(2): We have to show $I_{\mathcal{S}}(A)(x) \leq A(x)$ for each $x \in X$ which holds true iff for each $i \in I$ such that $S(A_i, A) \in K$ we have $S(A_i, A) \otimes A_i(x) \leq A(x)$, i.e. $A_i(x) \otimes \bigwedge_{y \in X}(A_i(y) \to A(y)) \leq A(x)$, which holds because of

$$A_i(x) \otimes \bigwedge_{y \in X}(A_i(y) \to A(y)) \leq A_i(x) \otimes (A_i(x) \to A(x)) \leq A(x).$$

(3): Suppose $S(A_1, A_2) \in K$. We have to show

$$S(A_1, A_2) \leq S(I_{\mathcal{S}}(A_1), I_{\mathcal{S}}(A_2))$$

which is equivalent to the fact that for each $x \in X$ we have $S(A_1, A_2) \leq I_{\mathcal{S}}(A_1)(x) \to I_{\mathcal{S}}(A_2)(x)$, i.e. by adjointness,

$$I_{\mathcal{S}}(A_1)(x) \otimes S(A_1, A_2) \leq I_{\mathcal{S}}(A_2)(x)$$

i.e.

$$\left(\bigvee_{i \in I, S(A_i, A) \in K} S(A_j, A_1) \otimes A_j(x) \right) \otimes S(A_1, A_2) \leq I_{\mathcal{S}}(A_2)(x)$$

which is true iff for each $j \in I$ with $S(A_1, A_j) \in K$ we have

$$S(A_j, A_1) \otimes A_j(x) \otimes S(A_1, A_2) \leq I_{\mathcal{S}}(A_2)(x)$$

which is true. Indeed, since

$$S(A_j, A_1) \otimes S(A_1, A_2) \leq S(A_j, A_2),$$

$S(A_j, A_1) \in K$, $S(A_1, A_2) \in K$, and the filter property of K yield $S(A_j, A_2) \in K$, and we have

$$S(A_j, A_1) \otimes A_j(x) \otimes S(A_1, A_2) \leq S(A_j, A_2) \otimes A_j(x)$$

$$\leq \bigvee_{i \in I, S(A_i, A) \in K} S(A_i, A_2) \otimes A_i(x) = I_{\mathcal{S}}(A_2)(x).$$

(4): Clearly, we only have to show $I_{\mathcal{S}}(A) \subseteq I_{\mathcal{S}}(I_{\mathcal{S}}(A))$. Since $I_{\mathcal{S}}(A) \in \mathcal{S}$, there is some $j \in I$ such that $A_j = I_{\mathcal{S}}(A)$. We, therefore, have

$$I_{\mathcal{S}}(I_{\mathcal{S}}(A))(x) = \bigvee_{i \in I, S(A_i, I_{\mathcal{S}}(A)) \in K} (S(A_i, I_{\mathcal{S}}(A)) \otimes A_i(x))$$

$$\geq S(I_{\mathcal{S}}(A), I_{\mathcal{S}}(A)) \otimes I_{\mathcal{S}}(A)(x) = I_{\mathcal{S}}(A)(x).$$

We now show that $A \in \mathcal{S}$ iff $A = I_{\mathcal{S}}(A)$. Indeed, if $A = A_j \in \mathcal{S}$ then $I_{\mathcal{S}}(A_j) \subseteq A_j$ as proved above.

Conversely,

$$I_{\mathcal{S}}(A_j)(x) = \bigvee_{i \in I, S(A_i, A_j) \in K} (S(A_i, A_j) \otimes A_i(x)) \geq (S(A_j, A_j) \otimes A_j(x)) = A_j(x),$$

i.e. $A_j \subseteq I_{\mathcal{S}}(A_j)$. If $A = I_{\mathcal{S}}(A)$, then $A \in \mathcal{S}$ by the definition of the $\mathbf{L}_K$-interior system, completing the proof. □

LEMMA 10 Let $I : L^X \to L^X$ be an $\mathbf{L}_K$-interior operator. Then $\mathcal{S}_I = \{A \in L^X | A = I(A)\}$ is an $\mathbf{L}_K$-interior system.

Proof Let $\mathcal{S}_I = \{A_i | i \in I\}$. We have to show that for each $A \in L^X$ we have $\bigvee_{i \in I, S(A_i, A) \in K} S(A_i, A) \otimes A_i \in \mathcal{S}_I$. To this end it clearly suffices to show

$$\bigvee_{i \in I, S(A_i, A) \in K} S(A_i, A) \otimes A_i = I(A). \tag{7}$$

On the one hand, since $S(I(A), A) = 1 \in K$, we have

$$\bigvee_{i \in I, S(A_i, A) \in K} S(A_i, A) \otimes A_i(x) \geq S(I(A), A) \otimes I(A)(x) = I(A)(x).$$

On the other hand

$$\bigvee_{i \in I, S(A_i, A) \in K} S(A_i, A) \otimes A_i(x) \leq I(A)(x)$$

iff for each $i \in I$ such that $S(A_i, A) \in K$ it holds $S(A_i, A) \otimes A_i(x) \leq I(A)(x)$. This is, indeed, true since

$$\begin{aligned} S(A_i, A) \otimes A_i(x) &\leq S(I(A_i), I(A)) \otimes I(A_i)(x) = I(A_i)(x) \otimes \bigwedge_{y \in X} I(A_i)(y) \to I(A(y)) \\ &\leq I(A_i)(x) \otimes (I(A_i)(x) \to I(A(x))) \leq I(A)(x). \end{aligned}$$

To sum up, Eq. (7) is proved. □

THEOREM 11 Let I be an $\mathbf{L}_K$-interior operator on X, $\mathcal{S}$ be an $\mathbf{L}_K$-interior system on X, K be a filter in $\mathbf{L}$. Then $\mathcal{S}_I$ is an $\mathbf{L}_K$-interior system, $I_{\mathcal{S}}$ is an $\mathbf{L}_K$-interior operator on X, and we have $I = I_{\mathcal{S}_I}$ and $\mathcal{S} = \mathcal{S}_{I_{\mathcal{S}}}$, i.e. the mappings $I \mapsto \mathcal{S}_I$ and $\mathcal{S} \mapsto I_{\mathcal{S}}$ are mutually inverse.

Proof By Lemma 9 and Lemma 10 it remains to prove $I = I_{\mathcal{S}_I}$, i.e. that for any $A \in L^X$, $x \in X$,

$$I(A)(x) = \bigvee_{A' \in L^X, A' = I(A'), S(A', A) \in K} S(A', A) \otimes A'(x).$$

The inequality $\geq$ holds iff for each $A' \in L^X$ such that $A' = I(A')$ and $S(A', A) \in K$ we have $S(A', A) \otimes A'(x) \leq I(A)(x)$ which is true since $S(A', A) \otimes A'(x) \leq S(I(A'), I(A)) \otimes I(A')(x) \leq I(A)(x)$.

Conversely, putting $A' = I(A)$ we get

$$S(I(A), A) \otimes I(A)(x) = 1 \otimes I(A)(x) = I(A)(x),$$

verifying the $\leq$-part. □

3. INTERIOR INDUCED BY FUZZY EQUIVALENCES

For a fuzzy equivalence $\approx$ on X denote by $I_{\approx}$ the operator $I_{\approx} : L^X \to L^X$ defined by

$$I_{\approx}(A)(x) = \bigwedge_{y \in X} (x \approx y) \to A(y) \tag{8}$$

DEFINITION 12 Let I be an **L**-interior operator on X. By x^I we denote the intersection of all open **L**-sets A for which $A(x) = 1$.

LEMMA 13 For a fuzzy equivalence $\approx$, $x^{I_\approx}$ is open and we have $x^{I_\approx} = [x]_\approx$.

Proof First we show that $I_\approx([x]_\approx) = [x]_\approx$:

$$I_\approx([x]_\approx)(y) = \bigwedge_{z\in X}(y \approx z) \to [x]_\approx(z) = \bigwedge_{z\in X}(y \approx z) \to (x \approx z).$$

It suffices to show that $\bigwedge_{z\in X}(y \approx z) \to (x \approx z) = (x \approx y)$. On the one hand, $\bigwedge_{z\in X}(y \approx z) \to (x \approx z) \leq (y \approx y) \to (x \approx y) = 1 \to (x \approx y) = (x \approx y)$. On the other hand, $(x \approx y) \leq \bigwedge_{z\in X}(y \approx z) \to (x \approx z)$ iff for each $z \in X$ we have $(x \approx y) \leq (y \approx z) \to (x \approx z)$ which is true due to adjointness and transitivity.

Furthermore, if A is another open **L**-set with $A(x) = 1$ then $1 = A(x) = I_\approx(A(x)) = \bigwedge_{y\in X}(x \approx y) \to A(y)$, from which we get that for any $y \in X$ we have $(x \approx y) \leq A(y)$, i.e. $[x]_\approx \subseteq A$, completing the proof. □

COROLLARY 14 We have

$$(x \approx y) = \bigwedge_{A\in \mathcal{S}_{I_\approx}, A(x)=1} A(y).$$

Remark 4 In general, it is not true that x^I is open. For take $X = \{x_1, x_2, x_3\}$, $L = \{0, 1\}$ and define I by $I(A) = A$ for $A = X$, $A = \{1/x_1, 1/x_2, 0/x_3\}$, $A = \{1/x_1, 0/x_2, 1/x_3\}$, and $I(A) = \emptyset$ otherwise. An easy inspection shows that $x_1^I = \{1/x_1, 0/x_2, 0/x_3\}$, but it is not open.

Recall that an **L**-set is said to be compatible with $\approx$ if for any $x,y \in X$ it holds $A(x) \otimes (x \approx y) \leq A(y)$.

LEMMA 15 $A = I_\approx(A)$ iff A is compatible with $\approx$. Thus $\mathcal{S}_{I_\approx} = L^{\langle X,\approx\rangle}$.

Proof Let $A = I_\approx(A)$. Then $A(x) \otimes (x \approx y) = I_\approx(A) \otimes (x \approx y) = (\bigwedge_{z\in X}(x \approx z) \to A(z)) \otimes (x \approx y) \leq (x \approx y) \otimes ((x \approx y) \to A(y)) \leq A(y)$.

Conversely, let A be compatible with $\approx$. Then $A(x) \leq (x \approx y) \to A(y)$ for any $y \in X$, thus $A(x) \leq \bigwedge_{y\in X}(x \approx y) \to A(y) = I_\approx(A)(x)$. □

LEMMA 16 Let $\approx$ be an **L**-equivalence. Then the mapping $I_\approx$ defined by Eq. (8) is an **L**-interior operator satisfying, moreover,

$$I_\approx\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} I_\approx(A_i) \tag{9}$$

$$I_\approx(x^{I_\approx})(y) = I_\approx(y^{I_\approx})(x) \tag{10}$$

$$I_\approx(A)(x) = \bigwedge_{y\in X} x^{I_\approx}(y) \to A(y) \tag{11}$$

for any $A_i \in L^X$, $i \in I$, $x, y, z \in X$.

Proof We have $I_\approx(A)(x) = \bigwedge_{y\in X}(x \approx y) \to A(y) \leq (x \approx x) \to A(x) = 1 \to A(x) = A(x)$, thus $I_\approx(A) \subseteq A$, proving Eq. (2).

Equation (3) is true iff for each $x \in X$ and every $A, B \in L^X$ we have $S(A, B) \leq I_\approx(A)(x) \to I_\approx(B)(x)$ which is equivalent to $I_\approx(A)(x) \otimes S(A, B) \leq I_\approx(B)(x)$.

The last inequality is true. Indeed,

$$\begin{aligned} I_{\approx}(A)(x) \otimes S(A,B) &= \left(\bigwedge_{y\in X} (x \approx y) \to A(y) \right) \otimes \left(\bigwedge_{y\in X} A(y) \to B(y) \right) \\ &\leq \bigwedge_{y\in X} (x \approx y) \to B(y) \\ &= I_{\approx}(B)(x), \end{aligned}$$

using $(a \to b) \otimes (b \to c) \leq a \to c$.

Equation (4) is true iff for any $x \in X$ we have

$$I_{\approx}(A)(x) \leq I_{\approx}(I_{\approx}(A))(x) = \bigwedge_{y\in X} \left((x \approx y) \to \bigwedge_{z\in X} ((y \approx z) \to A(z)) \right)$$

which holds iff for each $u, v \in X$ we have

$$I_{\approx}(A)(x) \otimes (x \approx u) \otimes (u \approx v) \leq A(v).$$

The last inequality is true since

$$\begin{aligned} I_{\approx}(A)(x) \otimes (x \approx u) \otimes (u \approx v) &= (x \approx u) \otimes (u \approx v) \otimes \bigwedge_{y\in X} (x \approx y) \to A(y) \\ &\leq (x \approx v) \otimes ((x \approx v) \to A(v)) \\ &\leq A(v). \end{aligned}$$

For Eq. (9) we have

$$\begin{aligned} I_{\approx}\left(\bigcap_{i\in I} A_i \right)(x) &= \bigwedge_{y\in X} \left((x \approx y) \to \bigwedge_{i\in I} A_i(y) \right) \\ &= \bigwedge_{y\in X} \bigwedge_{i\in I} ((x \approx y) \to A_i)(y) \\ &= \left(\bigcap I_{\approx}(A_i) \right)(x). \end{aligned}$$

Using Lemma 13 and the fact, that $\approx$ is symmetric we conclude Eq. (10). Equation (11) follows directly from definition of $I_{\approx}$ and from Lemma 13.

Remark 5 Lemma 16 shows that for R being a fuzzy equivalence, I_R from Example 2 is an $\mathbf{L}_K$-interior operator satisfying, moreover, some natural additional conditions. An inspection of the proof of Lemma 16 shows that if R is reflexive and transitive then except for Eq. (10), all properties mentioned in Lemma 16 are valid as well.

LEMMA 17 Let I be an $\mathbf{L}$-interior operator on X that satisfies Eqs. (9)–(11). For $x, y \in X$ put

$$x \approx_I y = I(x^I)(y).$$

Then $\approx_I$ is an $\mathbf{L}$-equivalence.

Proof First let us show that for an **L**-interior operator I satisfying Eqs. (9)–(11), x^I is open for each $x \in X$. Putting $I = \emptyset$, Eq. (9) implies

$$I(X) = I\left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I} I(A_i) = X.$$

That is, X is open. Since $X(x) = 1$ and, due to Eq. (9), any intersection of open **L**-sets is again open, x^I is open for each $x \in X$ (it is the intersection of the non-empty collection of all open A's with $A(x) = 1$).Therefore, x^I is open and $x \approx_I y = I(x^I)(y) = x^I(y)$.

Now, $(x \approx_I x) = I(x^I)(x) = x^I(x) = 1$, i.e. $\approx_I$ is reflexive.

Symmetry of $\approx_I$ follows from Eq. (10).

We prove transitivity of $\approx_I$, using idempotency of I we get $I(A) \subseteq I(I(A))$, i.e. $I(A)(y) \leq I(I(A))(y)$. Using Eq. (11) we get

$$I(A)(y) \leq \bigwedge_{z\in X} y^I(z) \to I(A)(z),$$

i.e. for each $z \in X$ we have

$$I(A)(y) \otimes y^I(z) \leq I(A)(z).$$

Putting $A = x^I$ we obtain

$$I(x^I)(y) \otimes y^I(z) \leq I(x^I)(z).$$

Since $y^I = I(y^I)$ we have

$$I(x^I)(y) \otimes I(y^I)(z) \leq I(x^I)(z),$$

i.e.

$$(x \approx_I y) \otimes (y \approx_I z) \leq (x \approx_I z),$$

showing that $\approx_I$ is transitive.

THEOREM 18 The mappings sending $\approx$ to $I_\approx$, and I to $\approx_I$, as defined in Lemmas 16 and 17, are mutually inverse mappings between the set of all **L**-equivalences on X and the set of all **L**-interior operators on X satisfying Eqs. (9)–(11).

Proof By Lemmas 16 and 17, we have to check that $\approx = \approx_{I_\approx}$ and $I = I_{\approx_I}$: We have to prove $(x \approx y) = (x \approx_{I_\approx} y)$, for any $x, y \in X$, which is true. Indeed using Lemma 13 we have $(x \approx_{I_\approx} y) = I_\approx(x^{I_\approx})(y) = (x \approx y)$. Using Eq. (11) we have $I(A) = I_{\approx_I}(A)$. □

3.1. Law of Double Negation

As mentioned in "Introduction and preliminaries" section, there is a duality in the ordinary case between the notion of closure operator and that of an interior operator. The duality can be justified using the law of double negation (i.e. a rule $\neg\neg\, a = a$ valid in the structure of truth values). We are going to show that providing the law of double negation, conditions (9)–(11) characterizing fuzzy interior operator induced by fuzzy equivalences can be obtained using conditions from Belohlávek (2003) characterizing fuzzy closure operators induced by fuzzy equivalences.

We need to recall the following definition (Belohlávek, 2001).

DEFINITION 19 An $\mathbf{L}_K$-closure operator (fuzzy closure operator) on a set X is a mapping $C\colon L^X \to L^X$ satisfying

$$A \subseteq C(A) \tag{12}$$

$$S(A_1, A_2) \leq S(C(A_1), C(A_2)) \text{ whenever } S(A_1, A_2) \in K \tag{13}$$

$$C(A) = C(C(A)) \tag{14}$$

for every $A, A_1, A_2 \in L^X$.

As in case of interior operators, we speak of $\mathbf{L}$-closure operators if $K = L$.

The following fact is well-known and easy to see Belohlávek (2002a,b).

LEMMA 20 For any $A, B \in L^X$ we have $S(A, B) \leq S(\neg B, \neg A)$. Provided the law of double negation, we have, moreover, $S(A, B) = S(\neg B, \neg A)$.

LEMMA 21 Let I be a an $\mathbf{L}_K$-interior operator on X. Then C_I defined for any $A \in L^X$ by $C_I(A) = \neg I(\neg A)$ is an $\mathbf{L}_K$-closure operator on X.

Proof In order to prove Eq. (12), i.e. $A \subseteq \neg I(\neg A)$, we use Eq. (2) to obtain $I(\neg A) \subseteq \neg A$ from which we get $\neg I(\neg A) \supseteq \neg\neg A \supseteq A$.

To prove Eq. (13), suppose $S(A, B) \in K$. We have

$$S(C_I(A), C_I(B)) = S(\neg I(\neg A), \neg I(\neg B)) \geq S(I(\neg B), I(\neg A)) \geq S(\neg B, \neg A) \geq S(A, B),$$

using Eq. (3) and Lemma 20, verifying Eq. (13).

Finally, we have to prove Eq. (14), i.e. $\neg I(\neg\neg I(\neg A) = \neg I(\neg A)$, which is true. Indeed, using $\neg\neg\neg a = \neg a$ for any $a \in L$ and $I(\neg A) \subseteq \neg A$ we have $\neg\neg I(\neg A) \subseteq \neg A$. Furthermore, using Eq. (3) we have $I(\neg\neg I(\neg A)) \subseteq I(\neg A)$ which implies $\neg I(\neg\neg I(\neg A)) \supseteq \neg I(\neg A)$.

Conversely, using $a \leq \neg\neg a$ for any $a \in L$ we have $I(\neg A) \subseteq \neg\neg I(\neg A)$, and by Eqs. (3) and (4) we have $I(\neg A) \subseteq I(\neg\neg I(\neg A))$ which implies $\neg I(\neg A) \supseteq \neg I(\neg\neg I(\neg A))$, i.e. Eq. (14) holds.

LEMMA 22 Let $\mathbf{L}$ satisfy the law of double negation. If C is an $\mathbf{L}_K$-closure operator on X, then I_C defined for any $A \in L^X$ by $I_C(A) = \neg C(\neg A)$ is an $\mathbf{L}_K$-interior operator on X.

Proof The proof is analogous to proof of Lemma 21; one needs to use $\neg\neg a \leq a$. □

Remark 6 An inspection of the proof of Lemma 21 shows that in general, without the assumption of the law of double negation, if C is an $\mathbf{L}_K$-closure operator then I_C satisfies Eq. (13) but does not have to satisfy Eqs. (12) and (14). To see an example, take $L = [0, 1]$, with Gödel structure on [0,1] (i.e. $a \otimes b = \min(a, b)$), $X = \{x_1, x_2\}$, and define C by $C(A)(x_1) = 0$, $C(A)(x_2) = 0.5$ for $A(x_1) = 0$, $A(x_2) \leq 0.5$, and $C(A)(x_1) = C(A)(x_2) = 1$ otherwise. Then $\mathbf{L}$ does not satisfy the law of double negation since for Gödel structure we have $\neg a = 1$ for $a = 0$ and $\neg a = 0$ for $a > 0$. Furthermore, C is an $\mathbf{L}_{\{1\}}$-closure operator, which is not an $\mathbf{L}_{[0.5,1]}$-closure operator. Taking $A = \{0.2/x_1, 0.8/x_2\}$, we have $\neg A = \{0/x_1, 0/x_2\}$, $C(\neg A) = \{0/x_1, 0.5/x_2\}$, $\neg C(\neg A) = \{1/x_1, 0/x_2\} = I_C(A)$, but $I_C(I_C(A)) = I_C(\{1/x_1, 0/x_2\}) = \neg C(\{0/x_1, 1/x_2\}) = \neg \{1/x_1, 1/x_2\} = \{0/x_1, 0/x_2\} \neq I_C(A)$.

LEMMA 23 Let $\mathbf{L}$ satisfy the law of double negation. Then $I_{C_I} = I$ and $C_{I_C} = C$.

Proof We have $I_{C_I}(A) = \neg C_I(\neg A) = \neg\neg I(\neg\neg A) = I(A)$, verifying $I_{C_I} = I$. The part for C_{I_C} is analogous. □

We need the following theorem (Bělohlávek, 2002a,b).

PROPOSITION 24 Let C be an **L**-closure operator on X that satisfies

$$C\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} C(A_i) \tag{15}$$

$$C(\{a/x\}) = a \otimes C(\{1/x\}) \tag{16}$$

$$C(\{1/x\})(y) = C(\{1/y\})(x). \tag{17}$$

For $x, y \in X$, put

$$(x \approx_C y) = C(\{1/x\})(y).$$

Then $\approx_C$ is an **L**-equivalence on X.

LEMMA 25 Let $\approx$ be an **L**-equivalence. Then the mapping $I_{\approx}$ defined by Eq. (8) is an **L**-interior operator satisfying, moreover, Eq. (9),

$$I_{\approx}(\neg\ \{a/x\})(y) = a \rightarrow I_{\approx}(\neg\ \{1/x\})(y) \tag{18}$$

$$I_{\approx}(\neg\ \{1/x\})(y) = I_{\approx}(\neg\ \{1/y\})(x) \tag{19}$$

for any $A_i \in L^X$, $i \in I$, $x,y \in X$, $a \in L$.

Proof From Lemma 16 we know that $I_{\approx}$ is an **L**-interior operator satisfying Eq. (9).Eq. (18) is true since

$$\begin{aligned} I_{\approx}(\neg\ \{a/x\})(y) &= \bigwedge_{z\in X} (y \approx z) \rightarrow (\neg\ \{a/x\})(z) \\ &= (y \approx x) \rightarrow (a \rightarrow 0) \\ &= a \rightarrow ((y \approx x) \rightarrow 0) \\ &= a \rightarrow I_{\approx}(\neg\ \{1/x\})(y) \end{aligned}$$

Finally Eq. (19) follows from symmetry of $\approx$ since

$$I_{\approx}(\neg\ \{1/x\})(y) = (y \approx x) \rightarrow 0 = (x \approx y) \rightarrow 0 = I_{\approx}(\neg\ \{1/y\})(x).$$ □

LEMMA 26 Let L satisfy the law of double negation. Let I be an **L**-interior operator on X that satisfies Eqs. (9), (18) and (19). For $x,y \in X$ put

$$(x \approx_I y) = \neg\ I(\neg\ \{1/x\})(y).$$

Then $\approx_I$ is an **L**-equivalence on X.

Proof We have

$$(x \approx_I y) = \neg\ I(\neg\ \{1/x\})(y) = C_I(\{1/x\})(y).$$

By Lemma 21 and Proposition 24, it suffices to show that C_I satisfies Eqs. (15)–(17).

Notice that Eq. (9) is equivalent to $\neg\ I_{\approx}\left(\neg\ \left(\bigcup_{i\in I} A_i\right)\right) = \bigcup_{i\in I}(\neg\ I_{\approx}(\neg\ A_i))$. Indeed, $\neg\ \left(\bigcup \neg\ A_i\right) = \bigcap A_i$ is always satisfied, and because L satisfy the property of double negation, we have $\neg\ (\bigcap \neg\ A_i) = \bigcup A_i$. Therefore Eq. (15) holds.

The law of double negation implies $\neg(a \to b) = a \otimes \neg b$. Indeed, $a \to b = a \to ((b \to 0) \to 0) = (a \otimes (b \to 0)) \to 0$. Therefore, Eq. (18) implies Eq. (16). Equation (17) follows directly from Eq. (19). □

THEOREM 27 Let L satisfy the law of double negation. The mappings sending $\approx$ to $I_{\approx}$, and I to $\approx_I$, as defined by Eq. (8) and Lemma 26, are mutually inverse mappings between the set of all **L**-equivalences on X and the set of all **L**-interior operators on X satisfying Eqs. (9), (18) and (19).

Proof By Lemmas 25 and 26, we have to check that $\approx = \approx_{I_{\approx}}$ and $I = I_{\approx_I}$.

First, we check $\approx = \approx_{I_{\approx}}$, we have to prove $(x \approx y) = (x \approx_{I_{\approx}} y)$ for any $x, y \in X$.

Due to the law of double negation, it is sufficient to verify

$$(x \approx y) \to 0 = \bigwedge_{z \in X} (y \approx z) \to (\neg \{1/x\})(z)$$

which is true. Indeed,

$$\bigwedge_{z \in X} (y \approx z) \to (\neg \{1/x\})(z) \le (y \approx x) \to (\neg \{1/x\})(x) = (x \approx y) \to 0.$$

The converse inequality, i.e.

$$\bigwedge_{z \in X} (y \approx z) \to (\neg \{1/x\})(z) \ge (x \approx y) \to 0$$

holds true iff for any $z \in X$ we have

$$(y \approx z) \to (\neg \{1/x\})(z) \ge (x \approx y) \to 0.$$

Using $(y \approx z) \to (\neg\{1/x\})(z) = (y \approx z) \to ((\{1/x\})(z) \to 0) = ((y \approx z) \otimes (\{1/x\})(z)) \to 0$, it remains to check

$$(y \approx z) \otimes (\{1/x\})(z) \le (x \approx y)$$

which is true. Indeed, for $z = x$ we obtain $(x \approx y) \le (x \approx y)$ and for $z \ne x$ we obtain $0 \le (x \approx y)$.

Second, we check $I = I_{\approx_I}$. Using $A(x) = \bigcup_{y \in X} \{A(y)/y\}(x) = \neg \bigcap_{y \in X} \neg \{A(y)/y\}(x)$, we have

$$\begin{aligned} I(A)(x) &= \bigwedge_{y \in X} I(\neg \{\neg A(y)/y\})(x) \\ &= \bigwedge_{y \in X} \neg A(y) \to I(\neg \{1/y\}(x)) \\ &= \bigwedge_{y \in X} \neg I(\neg \{1/y\}(x)) \to A(y) \\ &= I_{\approx_I}(A)(x) \end{aligned}$$

completing the proof. □

Remark 7 For the general case of a structure of truth values which does not satisfy the law of double negation, we do not know if there are some conditions characterizing fuzzy interior operator induced by fuzzy equivalences that can be derived from Eqs. (15)–(17).

Acknowledgements

Supported by grant no. B1137301 of the GA AV ČR and partly by project Kontakt 2003-1 "Algebraic tools and methods for non-classical logic".

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