Similarity and Fuzzy Tolerance Spaces

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Abstract

The paper studies tolerance relations from the point of view of fuzzy logic and fuzzy set theory. Particularly, we focus on the study of natural groupings induced by a fuzzy tolerance, relations induced by fuzzy tolerances, representational issues, and an algorithm for generating all fuzzy tolerance classes.

Keywords: Similarity, tolerance relation, fuzzy logic, tolerance class.

1 Introduction: similarity and fuzzy tolerances

Similarity is perhaps the most important phenomenon accompanying human perception and reasoning. On the one hand, similarity allows humans to think on an abstract level. Instead of storing individual elements (events, objects encountered in the past, etc.), people store classes of similar elements. Disregarding individual elements and the subsequent reasoning with similarity-based classes of these elements makes it possible to reason with complex evidence. On the other hand, similarity allows humans to act when they face a situation not encountered previously. Usually, people try to act as in a known situation which is most similar to the actual one.

Among the various formal models of similarity, two particular approaches seem to be dominant. First, similarity of objects from a given universe set X is considered as a binary relation S on X. That is, $S \subseteq X \times X$ consists of pairs $\langle x, y\rangle$ $(x, y \in X)$ which are considered similar. In order to reflect basic intuitive feelings about similarity, one usually requires that S be reflexive, symmetric, and sometimes also transitive (see Remark 3.4). This approach was studied mostly in the connection to linguistics [27], algebraic structures (so-called congruence relations [8], and compatible tolerance relations [9]), and several other systems where modelling similarity as a binary relation proved to be feasible [1].

Second, similarity is considered as a mapping $S : X \times X \to \mathbf{R}^+$ assigning to every $x, y \in X$ a non-negative real number $S(x, y)$ (or some more general value) called usually the similarity index between x and y. The higher $S(x, y)$, the more similar x and y are. There have been proposed several formulas for $S(x, y)$ depending on the nature of the objects x and y. The formulas take into account the data (attributes) with which x and y are described (the data can be numerical, ordinal, nominal, binary, or a combination of these), [7, 21, 29, 25].

The first approach may be called a qualitative one, while the second one may be called quantitative. The approaches are, in a sense, mutually complementary. The qualitative approach allows one to perform reasoning governed by natural language statements like ‘if x has a given property and x and y are similar then y has the property as well’. The quantitative approach directly reflects an appealing intuition of gradedness of similarity: various pairs $\langle x, y\rangle$ of elements are pairwise similar to various degrees $S(x, y)$, some pairs are more similar than others. On the other hand, the qualitative approach does not enable us to speak of degrees of similarity other than 'similar' and possibly 'not similar'. The quantitative approach gives no way to evaluate composed natural language statements about similarities; for instance, the meaning of 'x is similar to y and y is similar to z' is not clear.

A solution to this situation is offered by so-called fuzzy approach and fuzzy logic. Fuzzy equivalence relations, proposed originally by Zadeh [32], have been suggested for modelling similarity (see Section 3). Fuzzy equivalence relations share both of the advantageous properties of the two above-described approaches. First, natural language statements about similarity of elements can be evaluated in an appropriate fuzzy logic. Second, by definition, a fuzzy equivalence assigns various degrees to various pairs of elements. This gives fuzzy equivalences a numerical character (if the scale of truth degrees involved is numerical). A detailed study of fuzzy equivalence relations and their applications to fuzzy relational modelling can be found in [5, 10], see also [4, 14, 17, 18, 22]. A fuzzy equivalence relation $\approx$ on a set X is a binary fuzzy relation assigning to any $x, y \in X$ the truth degree $(x \approx y)$ out of some scale L of truth degrees and satisfying $(x \approx x) = 1$ (reflexivity), $(x \approx y) = (y \approx x)$ (symmetry), and $(x \approx y) \otimes (y \approx z) \leq (x \approx z)$ (transitivity; $\otimes$ is a fuzzy conjunction connective). While transitivity seems to be counterintuitive in the bivalent case, it is natural under the fuzzy approach, see Remark 3.4.

One important aspect makes transitivity different from both reflexivity and symmetry. Namely, the formulation of transitivity involves a logical connective of conjunction ($\otimes$). Therefore, to check whether a binary fuzzy relation, which is, for example, represented by a matrix containing similarity degrees, is a fuzzy equivalence, one has to tell the conjunction $\otimes$. While this is not a problem in the bivalent case (there is only one conjunction), in the fuzzy case there are several conjunction operations. A particular fuzzy relation may satisfy the transitivity with respect to some conjunction while it may fail to satisfy transitivity with respect to another one. There have been several approaches to evaluate similarity degrees (indexes) of objects of various kinds. While these approaches, possibly after an appropriate scaling, yield a reflexive and symmetric fuzzy relation, transitivity is usually not addressed. Though this may be either because transitivity is not relevant in the particular context or because the relevance of transitivity was not recognized, an important problem arises of studying binary fuzzy relations which are both reflexive and symmetric (and need not be transitive). A further reason for the investigation of reflexive and symmetric fuzzy relations is the following. A binary fuzzy relation is transitive if and only if a formula representing the statement 'if x and y are similar and y and z are similar then x and z are similar' has truth degree 1 (is fully true); see Remark 3.1. Unlike the bivalent case, such a formula may have several other degrees in the fuzzy case which are different from 1. Satisfying transitivity thus means requiring that the above formula has truth degree at least 1, which is one of the two possible extreme requirements. The other one is to require that the formula have a truth degree at least 0, which is, in fact, an empty requirement. Therefore, if we think of the definition of a fuzzy equivalence this way, one extreme leads to the notion of fuzzy equivalence itself while the other one leads to fuzzy relations, which are reflexive and symmetric. Analogously to the bivalent case, we call such relations fuzzy tolerances. The study of fuzzy tolerances is the main objective of the present paper.

The content of the paper is as follows. In Section 2, we recall necessary notions from fuzzy logic and fuzzy sets. Section 3 is devoted to fuzzy tolerance relations. Definitions and basic examples are present in Section 3.1. Section 3.2 treats collections of elements induced by a fuzzy tolerance. Also, it shows a connection between the degree of transitivity of a fuzzy tolerance and properties of its classes. Section 3.3 studies some relations on X induced by a fuzzy tolerance on X. Various natural properties of fuzzy tolerance spaces are investigated in Section 3.4. Section 3.5 deals briefly with selected representational issues. In Section 3.6, we present an algorithm for generating all classes of a fuzzy tolerance relation.

2 Preliminaries

We pick complete residuated lattices as the structures of truth values. Complete residuated lattices, first introduced in the 1930s in ring theory, were introduced into the context of fuzzy logic by Goguen [15, 16]. Various logical calculi were investigated using residuated lattices or particular types of residuated lattices. Thorough information about the role of residuated lattices in fuzzy logic can be obtained in [17, 18, 26]. Recall that a (complete) residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is a commutative and associative binary operation on L satisfying $a \otimes 1 = a$), and $\otimes$, $\to$ form an adjoint pair, i.e. $a \otimes b \leq c$ if and only if $a \leq b \to c$ is valid for each $a, b, c \in L$. In the following, **L** denotes an arbitrary complete residuated lattice (with L being the universe set of **L**). All properties of complete residuated lattices used in the sequel are well known and can be found in [5]. Note that particular types of residuated lattices (distinguishable by identities) include Boolean algebras, Heyting algebras, algebras of Girard's linear logic, MV-algebras, Gödel algebras, product algebras, and more generally, BL-algebras [18, 20].

Of particular interest are complete residuated lattices defined on the real unit interval $[0, 1]$ or on some subchain of $[0, 1]$. It can be shown [5] that $\mathbf{L} = \langle [0, 1], \min, \max, \otimes, \to, 0, 1\rangle$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and $\to$ is defined by $a \to b = \max\{c \mid a \otimes c \leq b\}$. A t-norm is a binary operation on $[0, 1]$ which is associative, commutative, monotone, and has 1 as its neutral element, and hence, captures the basic properties of conjunction. A t-norm is called left-continuous if, as a real function, it is left-continuous in both arguments. Most commonly used are continuous t-norms, the basic three being the Łukasiewicz t-norm (given by $a \otimes b = \max(a + b - 1, 0)$ with the corresponding residuum $a \to b = \min(1 - a + b, 1)$), the minimum (also called Gödel) t-norm ($a \otimes b = \min(a, b)$, $a \to b = 1$ if $a \leq b$ and $= b$ else), and the product t-norm ($a \otimes b = a \cdot b$, $a \to b = 1$ if $a \leq b$ and $= b/a$ else). In the following, we denote the complete residuated lattices on $[0, 1]$ given by the Łukasiewicz, minimum, and product operations by $[0, 1]_{\text{Ł}}$, $[0, 1]_{\min}$, $[0, 1]_{\Pi}$, respectively. It can be shown [23] that each continuous t-norm is composed of the three above-mentioned t-norms by a simple construction (ordinal sum). Any finite subchain of $[0, 1]$ containing both 0 and 1, equipped with restrictions of the minimum t-norm and its residuum, is a complete residuated lattice. Furthermore, the same holds true for any equidistant finite chain $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ equipped with restrictions of Łukasiewicz operations. The only residuated lattice on the two-element chain $\{0, 1\}$ (with $0 < 1$) has the classical conjunction operation as $\otimes$ and classical implication operation as $\to$. That is, the two-element residuated lattice is the two-element Boolean algebra of classical logic.

A fuzzy set with truth degrees from a complete residuated lattice **L** (also simply an **L**-set) in a universe set X is any mapping $A\colon X \to L$, $A(x) \in L$ being interpreted as the truth value of 'x belongs to A'.

REMARK 2.1
Strictly speaking, since A is a mapping from X to L, the operations on L do not matter. Thus, it would be more appropriate to speak of an L-set instead of an **L**-set and to leave the structure on L open. However, speaking of an **L**-set enables us to define the structure on L in a concise way. Defining the structure on L is necessary in the case of some notions generalized from the point of view of fuzzy approach (like the notion of an equivalence) since the operations on L appear in the definition. Bearing this in mind, we will use both **L**-set and L-set and more generally both **L**-$\cdots$ and L-$\cdots$ where $\cdots$ denotes the appropriate notion in question. If there is no danger of misunderstanding, we may even speak of a fuzzy set only.

Analogously, an n-ary $\mathbf{L}$-relation on a universe set X is an $\mathbf{L}$-set in the universe set X^n, e.g. a binary relation R on X is a mapping $R\colon X \times X \to L$. A singleton is a fuzzy set $\{\,{}^{a}\!/x\}$ for which $\{\,{}^{a}\!/x\}(x) = a$ and $\{\,{}^{a}\!/x\}(y) = 0$ for $y \neq x$. A fuzzy set A is called normal if $A(x) = 1$ for some $x \in X$. For $a \in L$, the a-cut of a fuzzy set $A \in L^X$ is the ordinary subset ${}^{a}A = \{x \in X \mid A(x) \geq a\}$ of X. For $\mathbf{L}$-sets A and B in X we define $(A \approx B) = \bigwedge_{x \in X}(A(x) \leftrightarrow B(x))$ (degree of equality of A and B) and $S(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$ (degree of subsethood of A in B). Note that $\leftrightarrow$ is defined by $a \leftrightarrow b = (a \to b) \wedge (b \to a)$. Clearly, $(A \approx B) = S(A, B) \wedge S(B, A)$. Furthermore, we write $A \subseteq B$ (A is a subset of B) if $S(A, B) = 1$, i.e. for each $x \in X$, $A(x) \leq B(x)$. $A \subset B$ means $A \subseteq B$ and $A \neq B$. The set of all $\mathbf{L}$-sets in X will be denoted by $\mathbf{L}^X$ (or L^X if the operations do not matter). Note that the operations of $\mathbf{L}$ induce the corresponding operations on L^X. For example, we have intersection $\bigcap$ on L^X induced by the infimum $\bigwedge$ of $\mathbf{L}$ by $(\bigcap_{i \in I} A_i)(x) = \bigwedge_{i \in I} A_i(x)$, etc.

3 Fuzzy tolerances and fuzzy tolerance spaces

3.1 Definition and examples

Consider the following properties of a binary $\mathbf{L}$-relation E on a set X:

$$E(x, x) = 1 \tag{3.1}$$

$$E(x, y) = E(y, x) \tag{3.2}$$

$$E(x, y) \otimes E(y, z) \leq E(x, z) \tag{3.3}$$

$$E(x, y) = 1 \quad \text{implies} \quad x = y. \tag{3.4}$$

E is called reflexive, symmetric, transitive, or strict if it satisfies (3.1), (3.2), (3.3), or (3.4), respectively.

REMARK 3.1
(1) Let φ be a logical formula $e(\xi, \nu) \& e(\nu, \zeta) \Rightarrow e(\xi, \zeta)$, i.e. φ expresses transitivity. Let the connectives $\&$ and $\Rightarrow$ be modelled by $\otimes$ and $\to$ from some residuated lattice $\mathbf{L}$. For an interpretation assigning to e a binary $\mathbf{L}$-relation E on X, and assigning to variables ξ, ν, and ζ elements x, y, and z from X, respectively, the truth degree of φ is $E(x, y) \otimes E(y, z) \to E(x, z)$, by standard semantics of predicate fuzzy logic [18]. Since $a \to b = 1$ is equivalent to $a \leq b$ in any residuated lattice, the truth degree of φ is 1 if and only if $E(x, y) \otimes E(y, z) \leq E(x, z)$. Therefore, transitivity says that the truth degree of a transitivity formula φ is 1. Analogously, reflexivity and symmetry say that the truth degree of formulas $e(\xi, \xi)$ and $e(\xi, \nu) \Leftrightarrow e(\nu, \xi)$, respectively, are 1.

(2) We can see that unlike transitivity, checking reflexivity or symmetry does not refer to any logical connective. That is, whether or not a binary $\mathbf{L}$-relation E is reflexive (symmetric) does not depend on logical connectives on L.

(3) For transitivity, however, given two complete residuated lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ with a common set of truth degrees, i.e. $L_1 = L_2$, but with different conjunction connectives, i.e. $\otimes_1 \neq \otimes_2$, it may happen that a fuzzy relation is transitive with respect to $\otimes_1$ but not with respect to $\otimes_2$ (see Example 3.3). It is also clear that if $\otimes_1 \leq \otimes_2$ (in that $\otimes_1(a, b) \leq \otimes_2(a, b)$ for all $a, b \in L$), then if E is transitive with respect to $\otimes_2$, it is also transitive with respect to $\otimes_1$.

DEFINITION 3.2
An $\mathbf{L}$-*tolerance* (fuzzy tolerance) is a reflexive and symmetric binary $\mathbf{L}$-relation. An $\mathbf{L}$-*equivalence* (fuzzy equivalence) is a binary $\mathbf{L}$-relation which is reflexive, symmetric, and transitive. A strict $\mathbf{L}$-equivalence is called an $\mathbf{L}$-*equality*.

$\sim_1$	x	y	z	u
x	1	0.9	0.8	1
y	0.9	1	0.8	0.9
z	0.8	0.8	1	0.8
u	1	0.9	0.8	1

$\sim_2$	x	y	z	u
x	1	0.9	0.8	1
y	0.9	1	0.75	0.9
z	0.8	0.75	1	0.8
u	1	0.9	0.8	1

$\sim_3$	x	y	z	u
x	1	0.9	0.8	1
y	0.9	1	0.7	0.9
z	0.8	0.7	1	0.8
u	1	0.9	0.8	1

$\sim_4$	x	y	z	u
x	1	0.9	0.8	1
y	0.9	1	0.6	0.9
z	0.8	0.6	1	0.8
u	1	0.9	0.8	1

TABLE 1. Fuzzy tolerances from Example 3.3

In what follows, we usually denote a fuzzy tolerance by $\sim$ and a fuzzy equivalence by $\approx$, and use the infix notation. Thus, the truth degree $E(x,y)$ will be denoted by $x \sim y$ or $x \approx y$.

Since whether an **L**-relation is an **L**-tolerance does not depend on the operations of **L**, one could speak of L-tolerances instead (cf. Remark 2.1). On the other hand, being an **L**-equivalence depends on $\otimes$ of **L**.

EXAMPLE 3.3

Take $L = [0,1]$. Tab. 1 shows four examples of fuzzy tolerances on $X = \{x,y,z,u\}$.

$\sim_1$ is a $[0,1]$-tolerance, which is moreover a $[0,1]_{\min}$-equivalence, $[0,1]_{\Pi}$-equivalence, and $[0,1]_{\mathrm{L}}$-equivalence.

$\sim_2$ is a $[0,1]$-tolerance, which is moreover a $[0,1]_{\Pi}$-equivalence and a $[0,1]_{\mathrm{L}}$-equivalence, but is not a $[0,1]_{\min}$-equivalence. That $\sim_2$ is not a $[0,1]_{\min}$-equivalence follows from $(y \sim x) \otimes (x \sim z) = \min(0.9, 0.8) = 0.8 \not\leq 0.75 = (y \sim z)$.

$\sim_3$ is a $[0,1]$-tolerance, which is a $[0,1]_{\mathrm{L}}$-equivalence but is neither a $[0,1]_{\min}$-equivalence, nor a $[0,1]_{\Pi}$-equivalence.

$\sim_4$ is a $[0,1]$-tolerance which is neither a $[0,1]_{\min}$-equivalence, nor a $[0,1]_{\Pi}$-equivalence, nor a $[0,1]_{\mathrm{L}}$-equivalence.

REMARK 3.4

There has been much debate regarding whether reflexivity, symmetry, and transitivity as described by the corresponding logical formulas (or verbally) are appropriate properties of similarity [28, 31]. Reflexivity seems to be mostly agreed upon. There is an argument against symmetry. It is being pointed out that 'A is similar to B' is not the same as 'B is similar to A' (e.g. A being a father and B being a son). However, if one is interested in a relation describing 'A and B are similar to each other', symmetry seems to be an obvious property. Transitivity of similarity has been a point of disagreement. One usually argues against transitivity as follows. If similarity were transitive then any two colours would be similar. For we may suppose that two colours with sufficiently close wavelengths are similar. Now, for any two colours A and B we may find a chain $A = A_1$, A_2, ..., $A_n = B$, of colours such that A_i and A_{i+1} are similar. Using transitivity, A and B are similar. On the other hand, if the transitivity condition is formulated verbally (i.e. 'if x and y are similar, and if y and z are similar then x and z are similar'), it seems plausible. The solution to this puzzle lies in the fact that similarity, by its nature, is a graded (fuzzy) notion. If we look at the meaning of transitivity in a fuzzy setting, we find it quite natural. For example, if $E(x,y) = 0.8$ (x and y are similar in degree 0.8) and $E(y,z) = 0.8$ (y and z are similar in degree 0.8) then x and z have to be similar at least in degree $0.8 \otimes 0.8$. Thus, in the case of the product conjunction, transitivity forces $E(x,z) \geq 0.8 \otimes 0.8 = 0.64$ which seems to reflect intuition in a reasonable way.

To sum up, we observed above that, first, conditions expressing reflexivity and symmetry do not refer to logical connectives on the set of truth degrees. Second, reflexivity and symmetry are commonly accepted properties of similarity. This does not mean that there cannot be any logical connectives involved. It simply means that when an expert evaluates similarity, i.e. when specifying the set of truth degrees used and the particular truth degrees $E(x,y)$ (the $|X| \times |X|$-matrix), only the natural conditions that the matrix has 1 on its diagonal (reflexivity) and that the matrix is symmetric (symmetry of fuzzy relation) have to be met. The expert does not have to consider the question of the choice of a conjunction connective and to check whether the specified matrix represents a transitive relation. Specified this way, the fuzzy relation may be subject to methods of further analysis of fuzzy tolerance relations. These methods may make use of logical connectives in any convenient way.

Following the common usage from the ordinary case, we introduce the following notions.

DEFINITION 3.5
An **L**-*tolerance space* is a pair $\mathbf{X} = \langle X, \sim \rangle$ where $\sim$ is an **L**-tolerance on X. An **L**-tolerance space for which $\sim$ is, moreover, transitive is called an **L**-*equivalence space*.

EXAMPLE 3.6
Suppose we have a collection X of objects and a collection $\{A_1, \ldots, A_n\}$ of their fuzzy attributes with truth degrees in $L = [0,1]$. That is, for each $x \in X$, $A_i(x)$ is the degree to which the fuzzy attribute A_i applies to x. A direct generalization of similarity indices from the ordinary case, i.e. when attributes A_i are crisp [21], yields fuzzy relations s_i on X ($i = 1, \ldots, 6$), with $s_i(x,y)$ representing the degree of similarity between x and y:

$$s_1(x,y) = \frac{\sum_{i=1}^{n} A_i(x) \leftrightarrow A_i(y)}{n}$$

$$s_2(x,y) = \frac{\sum_{i=1}^{n} A_i(x) \wedge A_i(y)}{\sum_{i=1}^{n} A_i(x) \vee A_i(y)}$$

$$s_3(x,y) = \frac{\sum_{i=1}^{n} A_i(x) \leftrightarrow A_i(y)}{n + \sum_{i=1}^{n}[(A_i(x) \leftrightarrow A_i(y)) \to 0]}$$

$$s_4(x,y) = \frac{\sum_{i=1}^{n} A_i(x) \wedge A_i(y)}{\sum_{i=1}^{n} A_i(x) \vee A_i(y) + \sum_{i=1}^{n}[(A_i(x) \leftrightarrow A_i(y)) \to 0]}$$

$$s_5(x,y) = \frac{\sum_{i=1}^{n} A_i(x) \leftrightarrow A_i(y)}{n - \frac{1}{2} \cdot \sum_{i=1}^{n}[(A_i(x) \leftrightarrow A_i(y)) \to 0]}$$

$$s_6(x,y) = \frac{\sum_{i=1}^{n} A_i(x) \wedge A_i(y)}{\sum_{i=1}^{n} A_i(x) \vee A_i(y) - \frac{1}{2} \cdot \sum_{i=1}^{n}[(A_i(x) \leftrightarrow A_i(y)) \to 0]}$$

One can easily see that each of $s_1, \ldots, s_6$ is a fuzzy tolerance on X. Note that with crisp attributes A_i, the above formulas are extensively used in clustering, information retrieval and data mining [2, 12, 19].

EXAMPLE 3.7
Let N be the set of all normal fuzzy sets in X, i.e. $N = \{A \in L^X \mid A(x) = 1 \text{ for some } x\}$. For an **L**-relation $\sim$ on N defined by

$$A \sim B = \bigvee_{x \in X} A(x) \otimes B(x),$$

$\langle N, \sim \rangle$ is an **L**-tolerance space.

EXAMPLE 3.8
Each ordinary tolerance relation can be considered as a **2**-tolerance relation, which is the way fuzzy tolerance relations generalize ordinary tolerance relations.

3.2 Induced groupings: preclasses, classes and a base

One of the most important issues related to a similarity phenomenon is that of natural groupings of elements induced by the similarity. The investigation of such groupings induced by fuzzy tolerance relations is the subject of the present section.

DEFINITION 3.9
For a fuzzy tolerance space $\langle X, \sim \rangle$, a fuzzy set A in X is called an *(**L**-)preclass of* $\sim$ (or $\sim$-preclass) if A is normal and

$$A(x) \otimes A(y) \leq (x \sim y) \qquad \text{for all } x, y \in X. \tag{3.5}$$

The set of all **L**-preclasses of $\sim$ will be denoted by $\mathrm{Precl}_{\mathbf{L}}(\sim)$ (or simply $\mathrm{Precl}(\sim)$).

DEFINITION 3.10
An *(**L**)-cover* of X is a subset $\mathcal{C} \subseteq L^X$, that

(i) for each $A \in \mathcal{C}$ there is $x \in X$ with $A(x) = 1$;
(ii) for each $x \in X$ there is $A \in \mathcal{C}$ with $A(x) = 1$.

REMARK 3.11
(i) For two structures $\mathbf{L}_1$ and $\mathbf{L}_2$ of truth degrees with $L_1 = L_2$ and conjunctions $\otimes_1$ and $\otimes_2$ respectively, we have that if $\otimes_1 \leq \otimes_2$ then each $\mathbf{L}_2$-preclass of $\sim$ is also an $\mathbf{L}_1$-preclass of $\sim$.
(ii) It follows from the definition of a preclass that if $(x \sim_1 y) \leq (x \sim_2 y)$ for all $x, y \in X$ then each **L**-preclass of $\sim_1$ is also an **L**-preclass of $\sim_2$.
(iii) Note that for $L = [0, 1]$ and $\otimes = \min$, some relationships between fuzzy tolerance relations and special systems of preclasses are studied in [30]. A few of the relationships are particular cases of the properties presented in our paper. Except for that, there is almost no overlap between our paper and [30].

THEOREM 3.12
For any **L**-tolerance $\sim$ on X we have

$$(x \sim y) = \bigvee_{A \in \mathrm{Precl}(\sim)} A(x) \otimes A(y). \tag{3.6}$$

Moreover, $\mathrm{Precl}(\sim)$ is an **L**-cover of X.

PROOF. It follows from the definition of a preclass that

$$\bigvee_{A \in \mathrm{Precl}(\sim)} A(x) \otimes A(y) \leq (x \sim y).$$

For the converse inequality, take a fuzzy set A with $A(x) = 1$, $A(y) = (x \sim y)$, $A(z) = 0$ for $x \neq z \neq y$. Then A is a preclass and

$$(x \sim y) = A(x) \otimes A(y) \leq \bigvee_{A \in \mathrm{Precl}(\sim)} A(x) \otimes A(y),$$

proving (3.6). That $\mathrm{Precl}(\sim)$ is an **L**-cover follows easily from the fact that $\{\,{}^{1}\!/x\}$ is a preclass for each $x \in X$. ∎

As any $\{ {}^1/x\}$ is a preclass, preclasses are, in general, not very interesting. The collection of all preclasses contains non-interesting fuzzy sets and is, in a sense, redundant.

DEFINITION 3.13
A fuzzy set A is called an *(**L**-)class of* $\sim$ (or $\sim$-class) if A is a maximal **L**-preclass of $\sim$, i.e. for any $A' \in \mathrm{Precl}_{\mathbf{L}}$, if $A \subset A'$ then $A' \notin \mathrm{Precl}_{\mathbf{L}}$.

The set of all **L**-classes of $\sim$ will be denoted by $\mathrm{Cl}_{\mathbf{L}}(\sim)$ (or simply $\mathrm{Cl}(\sim)$). The following is a useful criterion for checking whether a fuzzy set is a class (we elaborate the proof in Section 3.5.2).

LEMMA 3.14
A normal fuzzy set A is an **L**-class of $\sim$ iff for each $x \in X$ we have

$$A(x) = \bigwedge_{y \in X} A(y) \to (x \sim y). \tag{3.7}$$

THEOREM 3.15
Let $\sim$ be an **L**-tolerance on X. Then

(i) every preclass is contained in some class;
(ii) $\mathrm{Cl}(\sim)$ is an **L**-cover of X.

PROOF. (i) Consider a chain $\mathcal{C} = \{A_i \mid i \in I, A_i \subseteq A_j, \text{ for } i \leq j\}$ of preclasses. We claim that $\bigcup \mathcal{C}$ is itself a preclass. Indeed,

$$\begin{aligned} & (\bigcup \mathcal{C})(x) \otimes (\bigcup \mathcal{C})(y) = \\ = \quad & \bigvee_i A_i(x) \otimes \bigvee_j A_j(y) = \bigvee_{i,j} A_i(x) \otimes A_j(y) \leq \\ \leq \quad & \bigvee_{i,j} A_{max(i,j)}(x) \otimes A_{max(i,j)}(y) \leq x \sim y. \end{aligned}$$

Furthermore, $\bigcup \mathcal{C}$ is normal since each $A \in \mathcal{C}$ is normal. This means that any chain $\mathcal{C}$ of preclasses is bounded from above (by a preclass $\bigcup \mathcal{C}$). Applying the Zorn lemma, any preclass is contained in some maximal preclass, i.e. in a class.
(ii) is an easy consequence of Theorem 3.12 and (i). ∎

For any $x \in X$ we define a fuzzy set $\mathrm{Cl}_x(\sim)$ in $\mathrm{Cl}(\sim)$ by

$$[\mathrm{Cl}_x(\sim)](A) \;=\; A(x),$$

for any class $A \in \mathrm{Cl}(\sim)$. So, we have ${}^1\mathrm{Cl}_x(\sim) = \{A \in \mathrm{Cl}(\sim) \mid A(x) = 1\}$. Furthermore, for $x \in X$, we denote by $[x]_\sim$ a fuzzy set in X defined by

$$[x]_\sim(y) \;=\; (x \sim y).$$

Clearly, $\{[x]_\sim \mid x \in X\}$ is an **L**-cover of X.

REMARK 3.16
In general, $[x]_\sim$ need not be a preclass (see Theorem 3.22). In fact we have that an **L**-tolerance $\sim$ is an **L**-equivalence if for each $x \in X$, $[x]_\sim$ is a preclass. This is easily seen since $[x]_\sim$ is a preclass iff for each $y, z \in X$ we have $[x]_\sim(y) \otimes [x]_\sim(z) \leq (y \sim z)$, i.e. $(x \sim y) \otimes (x \sim z) \leq (y \sim z)$.

LEMMA 3.17
For any **L**-tolerance $\sim$ we have $[x]_\sim = \bigcup {}^1\mathrm{Cl}_x(\sim)$; moreover, for each $y \in X$, there exists $A \in {}^1\mathrm{Cl}_x(\sim)$ such that $(x \sim y) = A(y)$.

PROOF. We have to show that for each $y \in X$ we have

$$\bigvee_{A \in {}^1\mathrm{Cl}_x(\sim)} A(y) = (x \sim y).$$

First, $\bigvee_{A \in {}^1\mathrm{Cl}_x(\sim)} A(y) \leq (x \sim y)$ is true iff for each $A \in {}^1\mathrm{Cl}_x(\sim)$ we have $A(y) \leq (x \sim y)$ which is true. Indeed, as A is a preclass and $A(x) = 1$ we have $A(y) = A(x) \otimes A(y) \leq (x \sim y)$. For the converse inequality, consider A' defined by $A'(x) = 1$, $A'(y) = (x \sim y)$, and $A'(z) = 0$ otherwise. A' is a preclass and so, due to Theorem 3.15 (i), there is a class A with $A' \subseteq A$. Clearly, $A(x) = 1$, i.e. $A \in {}^1\mathrm{Cl}_x(\sim)$. Since A itself is a preclass, we have $A(x) \otimes A(y) \leq (x \sim y)$. To sum up, $(x \sim y) = A'(y) \leq A(y) = A(x) \otimes A(y) \leq (x \sim y)$. This shows both the required equality and the existence of $A \in {}^1\mathrm{Cl}_x(\sim)$ with $[x]_\sim(y) = A(y)$. ∎

THEOREM 3.18
For any **L**-tolerance $\sim$ we have

(i) for any $x, y \in X$,

$$(x \sim y) = \bigvee_{A \in \mathrm{Cl}(\sim)} A(x) \otimes A(y); \tag{3.8}$$

(ii) $[x]_\sim = \bigcup \mathrm{Cl}_x(\sim)$.

PROOF. (ii): The $\geq$ inequality follows from (3.6) and the fact that each class is a preclass. The $\leq$ inequality follows from Lemma 3.17.

(ii): Recall that $(\bigcup \mathrm{Cl}_x(\sim))(y) = \bigvee_{A \in \mathrm{Cl}(\sim)} (\mathrm{Cl}_x(\sim)(A) \otimes A(y))$. The assertion follows from the definition of $\mathrm{Cl}_x(\sim)$ and from (3.8). ∎

As mentioned in Remark 3.1 (1), $\sim$ is an **L**-equivalence iff the formula φ (cf. Remark 3.1 (1)) expressing transitivity of $\sim$ has truth degree $\mathrm{Tra}(\sim) = 1$ (i.e. is fully true). Note that $\mathrm{Tra}(\sim)$ is given by

$$\mathrm{Tra}(\sim) = \bigwedge_{x,y,z \in X} ((x \sim y) \otimes (y \sim z)) \to (y \sim z).$$

In general, $\mathrm{Tra}(\sim)$ can be any value from L. $\mathrm{Tra}(\sim)$ can be considered as the degree of transitivy of $\sim$: The higher $\mathrm{Tra}(\sim)$, the 'more transitive' is $\sim$. The following theorem describes $\mathrm{Tra}(\sim)$ in terms of classes of $\sim$.

THEOREM 3.19
For an **L**-tolerance $\sim$ on X we have

$$\mathrm{Tra}(\sim) = \bigwedge_{A,B \in \mathrm{Cl}(\sim)} (\bigvee_{x \in X} A(x) \otimes B(x)) \to (A \approx B).$$

Therefore, $\sim$ is an **L**-equivalence iff for any $A, B \in \mathrm{Cl}(\sim)$ we have $A(x) \otimes B(x) \leq (A \approx B)$ for each $x \in X$.

PROOF. We have to show that

$$\bigwedge_{A,B\in \mathrm{Cl}(\sim)} (\bigvee_{x\in X} A(x)\otimes B(x)) \to (A \approx B) = \bigwedge_{x,y,z\in X} ((x\sim y)\otimes(y\sim z)) \to (x\sim z).$$

First, we prove the '$\leq$'-inequality. We have

$$\bigwedge_{A,B\in \mathrm{Cl}(\sim)} (\bigvee_{x\in X} A(x)\otimes B(x)) \to (A \approx B) \leq \bigwedge_{x,y,z\in X} ((x\sim y)\otimes(y\sim z)) \to (x\sim z)$$

iff for any $x', y', z' \in X$,

$$\bigwedge_{A,B\in \mathrm{Cl}(\sim)} (\bigvee_{x\in X} A(x)\otimes B(x)) \to (A \approx B) \leq ((x'\sim y')\otimes(y'\sim z')) \to (x'\sim z')$$

i.e. iff for any $x', y', z' \in X$,

$$(x'\sim y')\otimes(y'\sim z')\otimes \bigwedge_{A,B\in \mathrm{Cl}(\sim)} (\bigvee_{x\in X} A(x)\otimes B(x)) \to (A \approx B) \leq (x'\sim z').$$

Let us now consider classes R and S such that $R(x') = 1$, $R(y') = x' \sim y'$, $S(z') = 1$ and $S(y') = y' \sim z'$ (they exist because of Lemma 3.17). Now we have

$$\begin{aligned}
& (x'\sim y')\otimes(y'\sim z')\otimes \bigwedge_{A,B\in \mathrm{Cl}(\sim)} (\bigvee_{x\in X} A(x)\otimes B(x)) \to (A \approx B) \leq \\
\leq\ & (x'\sim y')\otimes(y'\sim z')\otimes(R(y')\otimes S(y') \to (R\approx S)) \leq \\
\leq\ & (x'\sim y')\otimes(y'\sim z')\otimes((x'\sim y')\otimes(y'\sim z') \to (R\approx S)) \\
\leq\ & R\approx S = \bigwedge_{x\in X} R(x) \leftrightarrow S(x) \leq \\
\leq\ & R(z') \leftrightarrow S(z') = R(z') \leftrightarrow 1 = R(z') = R(z')\otimes R(x') \leq (x'\sim z').
\end{aligned}$$

For the second inequality $\geq$ we have

$$\bigwedge_{A,B\in \mathrm{Cl}(\sim)} (\bigvee_{x\in X} A(x)\otimes B(x)) \to (A \approx B) \geq \bigwedge_{x,y,z\in X} ((x\sim y)\otimes(y\sim z)) \to (x\sim z)$$

iff for any classes A, B and any $u, v \in X$ we have

$$\bigwedge_{x,y,z\in X} ((x\sim y)\otimes(y\sim z)) \to (x\sim z) \leq A(u)\otimes B(u) \to (A(v) \leftrightarrow B(v)),$$

i.e.

$$A(u)\otimes B(u)\otimes \bigwedge_{x,y,z\in X} ((x\sim y)\otimes(y\sim z)) \to (x\sim z) \leq (A(v) \leftrightarrow B(v)).$$

Because of symmetry we will only show that

$$A(u)\otimes B(u)\otimes \bigwedge_{x,y,z\in X} ((x\sim y)\otimes(y\sim z)) \to (x\sim z) \leq (A(v) \to B(v)),$$

i.e. that

$$A(v) \otimes A(u) \otimes B(u)) \otimes \bigwedge_{x,y,z \in X} ((x \sim y) \otimes (y \sim z)) \to (x \sim z) \leq B(v).$$

Applying Lemma 3.14 to B, i.e. $B(v) = \bigwedge_{w \in X} B(w) \to (v \sim w)$, we have to show

$$A(v) \otimes A(u) \otimes B(u)) \otimes \bigwedge_{x,y,z \in X} ((x \sim y) \otimes (y \sim z)) \to (x \sim z) \leq \bigwedge_{w \in X} B(w) \to (v \sim w),$$

i.e. for any $w \in X$ we have to prove

$$B(w) \otimes A(v) \otimes A(u) \otimes B(u)) \otimes \bigwedge_{x,y,z \in X} ((x \sim y) \otimes (y \sim z)) \to (x \sim z) \leq v \sim w$$

which is true. Indeed,

$$\begin{aligned} & B(w) \otimes A(v) \otimes A(u) \otimes B(u) \otimes \bigwedge_{x,y,z \in X} ((x \sim y) \otimes (y \sim z)) \to (x \sim z) \leq \\ \leq \quad & (v \sim u) \otimes (u \sim w) \otimes \bigwedge_{x,y,z \in X} ((x \sim y) \otimes (y \sim z)) \to (x \sim z) \leq \\ \leq \quad & (v \sim u) \otimes (u \sim w) \otimes ((v \sim u) \otimes (u \sim w) \to (v \sim w)) \leq v \sim w. \end{aligned}$$

■

REMARK 3.20
In words, Theorem 3.19 says that the degree to which $\sim$ is transitive equals the truth degree of the statement 'for any class A, B, if there is an x which belongs to both A and B then A equals B'.

Recall [5] that an **L**-partition of X is a collection Π of **L**-sets which is an **L**-cover of X and, moreover, for each $A, B \in \Pi$ and each $x \in X$, $A(x) \otimes B(x) \leq (A \approx B)$. We have the following theorem.

PROPOSITION 3.21 ([5])
Let θ be an **L**-equivalence in X, Π be an **L**-partition in X. Define a binary **L**-relation θ_Π in X by

$$\theta_\Pi(x, y) = A_x(y)$$

where $A_x \in \Pi$ is such that $A_x(x) = 1$, and put $\Pi_\theta = \{[x]_\theta \mid x \in X\}$. Then (1) Π_θ is an **L**-partition of X; (2) θ_Π is an **L**-equivalence in X; and (3) $\theta = \theta_{\Pi_\theta}$ and $\Pi = \Pi_{\theta_\Pi}$.

Therefore, we have

THEOREM 3.22
An **L**-tolerance $\sim$ is an **L**-equivalence iff $\mathrm{Cl}(\sim) = \{[x]_\sim \mid x \in X\}$.

PROOF. Let $\sim$ be an **L**-equivalence. Then for any $x \in X$, $[x]_\sim$ is a preclass, see Remark 3.16 . Moreover, each $[x]_\sim$ is a maximal preclass. For suppose there is a preclass A such that $[x]_\sim \subset A$, i.e. $[x]_\sim(y) < A(y)$ for some $y \in X$. Then $A(x) = 1$ and so $A(x) \otimes A(y) = A(y) > [x]_\sim(y) = (x \sim y)$, a contradiction with the assumption that A is a preclass. Therefore, $\mathrm{Cl}(\sim) = \{[x]_\sim \mid x \in X\}$. Conversely, if $\mathrm{Cl}(\sim) = \{[x]_\sim \mid x \in X\}$, we obtain $(x \sim y) \geq (x \sim z) \otimes (z \sim y)$ using (3.8), i.e. $\sim$ is an **L**-equivalence. ■

A system of $\sim$-classes still may be redundant with respect to the capability of restoring the $\sim$. A natural notion removing the redundancy is that of a base.

DEFINITION 3.23
An *(**L**-)base* of an **L**-tolerance space $\langle X, \sim\rangle$ is a minimal family $\mathcal{B} \subseteq \mathrm{Cl}(\sim)$ satisfying

$$(x \sim y) = \bigvee_{A \in \mathcal{B}} A(x) \otimes A(y), \tag{3.9}$$

for any $x, y \in X$. That is, if $\mathcal{B}' \subseteq \mathrm{Cl}(\sim)$ and $\mathcal{B}' \subset \mathcal{B}$, then $\mathcal{B}'$ does not satisfy (3.9).

THEOREM 3.24
Let $\sim$ be an **L**-equivalence on X. Suppose **L** satisfies that if $1 = \bigvee K$ for some $K \subseteq L$ then either $1 \in K$ or $|K| > |X|$. Then $\{[x]_\sim \mid x \in X\}$ is the only base of $\langle X, \sim\rangle$.

PROOF. By Theorem 3.22, **L**-classes of $\sim$ are exactly of the form $[x]_\sim$. We show that any base $\mathcal{B}$ must contain each $[x]_\sim$. Indeed, let $\mathcal{B} \subseteq \{[y]_\sim \mid y \in X\}$ be a base and $[x]_\sim \notin \mathcal{B}$. Denote $K = \{A(x) \otimes A(x) \mid A \in \mathcal{B}\}$. As $(x \sim x) = 1$, (3.9) implies that

$$1 = \bigvee_{A \in \mathcal{B}} A(x) \otimes A(x) = \bigvee K.$$

Since $[x]_\sim \notin \mathcal{B}$ we have $A(x) < 1$ and thus $A(x) \otimes A(x) < 1$ for any $A \in \mathcal{B}$, i.e. $1 \notin K$. Furthermore, $|K| \leq |X|$ which contradicts the condition of the theorem. ∎

REMARK 3.25
There are two important cases where the condition of Theorem 3.24 is met. First, **L** being a finite chain and, second, **L** being a chain and X being finite.

EXAMPLE 3.26
Consider the fuzzy tolerance $\sim_3$ from Example 3.3. For $[0,1]_{\min}$ we have two bases. The first one consists of classes $\{\,{}^{0.9}/x,\ {}^{1}/y,\ {}^{0.7}/z,\ {}^{0.9}/u\}$, $\{\,{}^{0.8}/x,\ {}^{0.7}/y,\ {}^{1}/z,\ {}^{0.8}/u\}$, and $\{\,{}^{1}/x,\ {}^{0.9}/y,\ {}^{0.7}/z,\ {}^{1}/u\}$. The second one consists of classes $\{\,{}^{0.9}/x,\ {}^{1}/y,\ {}^{0.7}/z,\ {}^{0.9}/u\}$, $\{\,{}^{0.8}/x,\ {}^{0.7}/y,\ {}^{1}/z,\ {}^{0.8}/u\}$, and $\{\,{}^{1}/x,\ {}^{0.7}/y,\ {}^{0.8}/z,\ {}^{1}/u\}$. For $[0,1]_\Pi$ we have infinitely many bases each consisting of three classes: $\{\,{}^{0.9}/x,\ {}^{1}/y,\ {}^{0.7}/z,\ {}^{0.9}/u\}$, $\{\,{}^{0.8}/x,\ {}^{0.7}/y,\ {}^{1}/z,\ {}^{0.8}/u\}$, and $\{\,{}^{1}/x,\ {}^{k}/y,\ \frac{0.7}{k}/z,\ {}^{1}/u\}$ where $0.9 \leq k \leq \frac{7}{8}$. Finally for $[0,1]_{\mathrm{L}}$, there is only one base consisting of classes $\{\,{}^{1}/x,\ {}^{0.9}/y,\ {}^{0.8}/z,\ {}^{1}/u\}$, $\{\,{}^{0.9}/x,\ {}^{1}/y,\ {}^{0.7}/z,\ {}^{0.9}/u\}$, and $\{\,{}^{0.8}/x,\ {}^{0.7}/y,\ {}^{1}/z,\ {}^{0.8}/u\}$.

REMARK 3.27
In Section 3.6, we present an algorithm for generating all classes of a fuzzy tolerance relation. However, we do not know of any way to generate bases of a fuzzy tolerance other than starting from the collection of all classes, removing classes and checking whether the remaining collection still restores the fuzzy tolerance until we get a base. To look for more efficient ways seems to be an interesting problem.

We know from the previous results that both the system of all $\sim$-preclasses and the system of all $\sim$-classes form a cover of X. The next theorem gives a necessary and sufficient condition for a covering of X to be the system of all classes of some fuzzy tolerance relation in the case **L** is a finite chain.

THEOREM 3.28
Let $\mathbf{L}$ be a finite chain. A cover $\mathcal{A}$ of X is a system of all classes of some fuzzy tolerance $\sim$ iff $\mathcal{A}$ satisfies

(i) for any $A \in \mathcal{A}$ and any system $\mathcal{B} = \{A_y \in \mathcal{A} \mid y \in X\}$ we have

$$\bigwedge_{y \in X} A(y) \to (A_y(x) \otimes A_y(y)) \leq A(x),$$

for any $x \in X$;

(ii) if a fuzzy set $B \in L^X$ is not contained in any $A \in \mathcal{A}$ (i.e. $B \not\subseteq A$ for each $A \in \mathcal{A}$) then there exist $x, y \in X$ such that

$$\{\, {}^{B(x)}/x,\ {}^{B(y)}/y\} \otimes \{\, {}^{B(x)}/x,\ {}^{B(y)}/y\} \not\subseteq A \otimes A,$$

for any $A \in \mathcal{A}$.

PROOF. Note first that for a fuzzy set $C \in L^U$, $C \otimes C$ is a fuzzy set defined by $(C \otimes C)(u) = C(u) \otimes C(u)$ for any $u \in U$.
'$\Rightarrow$': Let $\mathcal{A} = \mathrm{Cl}(\sim)$ for some fuzzy tolerance $\sim$, i.e. $\mathcal{A}$ is the system of all $\sim$-classes. We verify conditions (i) and (ii).
(i): Suppose that for some $x \in X$ we have $\bigwedge_{y \in X} A(y) \to (A_y(x) \otimes A_y(y)) \not\leq A(x)$ for some $A \in \mathcal{A}$ and $A_y \in \mathcal{A}$ ($y \in X$). Put $C := A \cup \{\, {}^{\bigwedge_{y \in X} A(y) \to (A_y(x) \otimes A_y(y))}/x\}$. We show that C is a $\sim$-preclass, which is a contradiction because A is a $\sim$-class and $A \subset C$. In order to show that C is a $\sim$-preclass, we need to show that $C(y_1) \otimes C(y_2) \leq (y_1 \sim y_2)$ for any $y_1, y_2 \in X$. We have to distinguish three cases:
(a) $y_1 \neq x \neq y_2$, then $C(y_1) \otimes C(y_2) = A(y_1) \otimes A(y_2) \leq (y_1 \sim y_2)$, because A is a $\sim$-class.
(b) $y_1 = x = y_2$, then $C(y_1) \otimes C(y_2) = C(x) \otimes C(x) \leq 1 = (x \sim x) = (y_1 \sim y_2)$.
(c) $y_1 \neq x = y_2$, then $C(y_1) \otimes C(y_2) = A(y_1) \otimes C(x) =$

$$\begin{aligned}
&= A(y_1) \otimes (A(x) \vee \bigwedge_{y \in X} A(y) \to (A_y(x) \otimes A_y(y))) = \\
=\ & A(y_1) \otimes A(x) \vee [A(y_1) \otimes \bigwedge_{y \in X} A(y) \to (A_y(x) \otimes A_y(y))] \leq \\
\leq\ & A(y_1) \otimes A(x) \vee [A(y_1) \otimes (A(y_1) \to A_{y_1}(x) \otimes A_{y_1}(y_1))] \leq \\
\leq\ & A(y_1) \otimes A(x) \vee A_{y_1}(x) \otimes A_{y_1}(y_1)) \leq (y_1 \sim x) \vee (y_1 \sim x) = (y_1 \sim x)
\end{aligned}$$

since both A and A_{y_1} are $\sim$-classes.
(ii): If $B \not\subseteq A$ for any $A \in \mathcal{A}$, then B is not a $\sim$-preclass. Indeed, $\mathcal{A}$ contains all $\sim$-classes and so if B is a $\sim$-preclass, there would exist $A \in \mathcal{A}$ such that $B \subseteq A$. Therefore, there exist $x, y \in X$ with $B(x) \otimes B(y) \not\leq (x \sim y)$. This implies that $\{\, {}^{B(x)}/x,\ {}^{B(y)}/y\}^2 \not\subseteq A^2$ for any $A \in \mathcal{A}$, because if there is $A \in \mathcal{A}$ such that $\{\, {}^{B(x)}/x,\ {}^{B(y)}/y\}^2 \subseteq A^2$ it would imply $B(x) \otimes B(y) \leq A(x) \otimes A(y) \leq (x \sim y)$ which is a contradiction to $B(x) \otimes B(y) \not\leq (x \sim y)$.
'$\Leftarrow$': Conversely, let $\mathcal{A}$ satisfy (i) and (ii). Introduce a fuzzy relation $\sim$ on X by

$$(x \sim y) = \bigvee_{A \in \mathcal{A}} A(x) \otimes A(y).$$

We show that $\mathcal{A} = \mathrm{Cl}(\sim)$, i.e. that $\mathcal{A}$ is the system of all $\sim$-classes.

(a) Any $A \in \mathcal{A}$ is a $\sim$-class: It is obvious that any $A \in \mathcal{A}$ is a $\sim$-preclass. Suppose that for some $A \in \mathcal{A}$, A is not a $\sim$-class, i.e. there exists a $\sim$-preclass $B \supset A$, i.e. there exists $x \in X$ such that $B(x) > A(x)$. For any $y \in X$ we have $A(y) \otimes A(x) < B(y) \otimes B(x) \leq (x \sim y) = \bigvee_{A' \in \mathcal{A}} A'(x) \otimes A'(y) = A_y(x) \otimes A_y(y)$ for some $A_y \in \mathcal{A}$, because $\mathbf{L}$ is a finite chain.

Consider now A (mentioned above) and a system $\mathcal{B} = \{A_y \mid y \in X\} \subseteq \mathcal{A}$ (A_ys are those for which we have $\bigvee_{A' \in \mathcal{A}} A'(x) \otimes A'(y) = A_y(x) \otimes A_y(y)$ above). Therefore, we have $A(y) \otimes B(x) \leq A_y(x) \otimes A_y(y)$, i.e. $B(x) \leq A(y) \to A_y(x) \otimes A_y(y)$ for any $y \in X$, i.e. $B(x) \leq \bigwedge_{y \in X} A(y) \to A_y(x) \otimes A_y(y)$. Using (i), we have $\bigwedge_{y \in X} A(y) \to A_y(x) \otimes A_y(y) \leq A(x)$ which gives $B(x) \leq A(x)$, a contradiction to $B(x) > A(x)$.

(b) Any $\sim$-class belongs to $\mathcal{A}$: If there exists some $\sim$-class $B \notin \mathcal{A}$, then $B \not\subseteq A$ for any $A \in \mathcal{A}$ (otherwise we have either $B \subset A$, i.e. B is not a $\sim$-class because A is a $\sim$-preclass, or $B = A$ which is a contradiction to $B \notin \mathcal{A}$). Using (ii), there exist $x, y \in X$ such that $\{ {}^{B(x)}/x, {}^{B(y)}/y\}^2 \not\subseteq A^2$ for any $A \in \mathcal{A}$. But since B is a $\sim$-class we have $B(x) \otimes B(y) \leq (x \sim y) = \bigvee_{A \in \mathcal{A}} A(x) \otimes A(y) = A'(x) \otimes A'(y)$ for some $A' \in \mathcal{A}$ because $\mathbf{L}$ is a finite chain, and this is a contradiction to $\{ {}^{B(x)}/x, {}^{B(y)}/y\}^2 \not\subseteq A'^2$. ■

REMARK 3.29

Note that for $\mathbf{L} = \mathbf{2}$ (ordinary case) Theorem 3.28 is equivalent to the following well-known theorem characterizing systems of tolerance classes in the ordinary case [9]: a cover $\mathcal{A}$ of X is a system of all classes of some tolerance $\sim$ on X iff (i') for $A \in \mathcal{A}$ and $\mathcal{B} \subseteq \mathcal{A}$, $A \subseteq \bigcup \mathcal{B}$ implies $\bigcap \mathcal{B} \subseteq A$, and (ii') if for some $B \subseteq X$ we have $B \not\subseteq A$ for any $A \in \mathcal{A}$ then there are $x, y \in B$ such that for any $A \in \mathcal{A}$ we have $\{x, y\} \not\subseteq A$.

Indeed, we show that (i) ⇒ (i'): Let $A \in \mathcal{A}, \mathcal{B} \subseteq \mathcal{A}, A \subseteq \bigcup \mathcal{B}$. Then for any $y \in A$ there exists $A_y \in \mathcal{B}$ such that $y \in A_y$. We want to show $\bigcap \mathcal{B} \subseteq A$. If $x \in \bigcap \mathcal{B}$, then $x \in A'$ for any $A' \in \mathcal{B}$, i.e. for any y we have $x \in A_y$. Then $\bigwedge_{y \in X} A(y) \to (A_y(x) \otimes A_y(y)) = 1$, and using (i) we get $A(x) = 1$ which implies $\bigcap \mathcal{B} \subseteq A$.

(i) ⇒ (i'): Let us have $A, A_y \subseteq X$ ($y \in X$) and suppose $\bigwedge_{y \in X} A(y) \to (A_y(x) \otimes A_y(y)) = 1$. Then for any $y \in A$ we have $y \in A_y \in \mathcal{A}$. Putting $\mathcal{B} = \{A_y \mid y \in X, y \in A\}$ we get $A \subseteq \bigcup \mathcal{B}$. From $\bigwedge_{y \in X} A(y) \to (A_y(x) \otimes A_y(y)) = 1$ it follows $x \in \bigcap \mathcal{B}$. Using (i'), we have $x \in A$, i.e. $A(x) = 1$.

(ii) ⇔ (ii') follows from the fact that for a crisp A we have $\{x, y\} \subseteq A$ iff $\{x, y\}^2 \subseteq A^2$.

3.3 Induced relations

DEFINITION 3.30

For an $\mathbf{L}$-tolerance $\sim$, the $\mathbf{L}$-relation $\sim^-$ is defined by

$$(x \sim^- y) = \bigwedge_{z \in X} (x \sim z) \leftrightarrow (z \sim y). \tag{3.10}$$

Fuzzy sets $[x]_{\sim^-}$ are called *kernels of* $\sim$.

Therefore, $x \sim^- y$ is the degree to which it is true that x and y are related to the same elements of X.

LEMMA 3.31

$\sim^-$ is an $\mathbf{L}$-equivalence relation on X.

PROOF. Consider system $\mathcal{S} = \{[z]_\sim \mid z \in X\}$. Using [5, Theorem 4.39] we have that $\approx_\mathcal{S}$ defined by $x \approx_\mathcal{S} y = \bigwedge_{A\in\mathcal{S}} A(x) \leftrightarrow A(y)$ is an **L**-equivalence. Noticing that $(x \approx_\mathcal{S} y) = (x \sim^- y)$ completes the proof. ∎

THEOREM 3.32
For an **L**-tolerance $\sim$ on X and $x, y \in X$ we have

(i) $(x \sim^- y) \leq (x \sim y)$;
(ii) $(x \sim^- y) = \bigwedge_{A\in\mathrm{Cl}(\sim)} A(x) \leftrightarrow A(y)$;
(iii) $(x \sim^- y) = \mathrm{Cl}_x(\sim) \approx \mathrm{Cl}_y(\sim)$.

PROOF. (i): We have $(x \sim^- y) = \bigwedge_{z\in X}(x \sim z) \leftrightarrow (z \sim y) \leq (x \sim y) \leftrightarrow (y \sim y) = (x \sim y)$.
(ii): First, we show:

$$(x \sim^- y) \leq \bigwedge_{A\in\mathrm{Cl}(\sim)} A(x) \leftrightarrow A(y),$$

which holds iff for each $A \in \mathrm{Cl}(\sim)$ we have both $(x \sim^- y) \leq A(x) \to A(y)$ and $(x \sim^- y) \leq A(y) \to A(x)$. Due to symmetry, we check only the first of the two inequalities. Using (3.8), the fact that A is a class, and (3.7), we have

$$\begin{aligned}
& (x \sim^- y) = \bigwedge_{z\in X} (x \sim z) \leftrightarrow (z \sim y) \leq \\
\leq\ & \bigwedge_{z\in X} (x \sim z) \to (z \sim y) = \\
=\ & \bigwedge_{z\in X} \Big(\bigvee_{B\in\mathrm{Cl}(\sim)} [B(x) \otimes B(z)]\Big) \to (z \sim y) \leq \\
\leq\ & \bigwedge_{z\in X} [A(x) \otimes A(z)] \to (z \sim y) = \\
=\ & \bigwedge_{z\in X} A(x) \to [A(z) \to (z \sim y)] = \\
=\ & A(x) \to \bigwedge_{z\in X} [A(z) \to (z \sim y)] = \\
=\ & A(x) \to A(y).
\end{aligned}$$

Conversely,

$$\bigwedge_{A\in\mathrm{Cl}(\sim)} A(x) \leftrightarrow A(y) \leq x \sim^- y$$

holds iff for each $z \in X$ we have both

$$\bigwedge_{A\in\mathrm{Cl}(\sim)} A(x) \leftrightarrow A(y) \leq (x \sim z) \to (z \sim y)$$

and

$$\bigwedge_{A\in\mathrm{Cl}(\sim)} A(x) \leftrightarrow A(y) \leq (z \sim y) \to (x \sim z).$$

	$\sim^-_{\min}$				$\sim^-_{\Pi}$				$\sim^-_{L}$			
	x	y	z	u	x	y	z	u	x	y	z	u
x	1	0.7	0.7	1	1	$\frac{7}{8}$	$\frac{7}{9}$	1	1	0.9	0.8	1
y	0.7	1	0.7	0.7	$\frac{7}{8}$	1	0.7	$\frac{7}{8}$	0.9	1	0.7	0.9
z	0.7	0.7	1	0.7	$\frac{7}{9}$	0.7	1	$\frac{7}{9}$	0.8	0.7	1	0.8
u	1	0.7	0.7	1	1	$\frac{7}{8}$	$\frac{7}{9}$	1	1	0.9	0.8	1

TABLE 2. Fuzzy equivalences from Example 3.34

Again, due to symmetry, we proceed only for the first case. Using (3.8), $(\bigvee_i a_i) \to b = \bigwedge_i (a_i \to b)$, $\bigvee_i (a \to b_i) \leq a \to (\bigvee_i b_i)$, and $a \to b \leq (a \otimes c) \to (b \otimes c)$, we have

$$\begin{aligned} & (x \sim z) \to (z \sim y) = \\ = \; & \Big(\bigvee_{A \in \mathrm{Cl}(\sim)} A(x) \otimes A(z)\Big) \to \Big(\bigvee_{C \in \mathrm{Cl}(\sim)} C(z) \otimes C(y)\Big) \geq \\ \geq \; & \bigwedge_{A \in \mathrm{Cl}(\sim)} \bigvee_{C \in \mathrm{Cl}(\sim)} (A(x) \otimes A(z) \to C(z) \otimes C(y)) \geq \\ \geq \; & \bigwedge_{A \in \mathrm{Cl}(\sim)} A(x) \otimes A(z) \to A(z) \otimes A(y) \geq \\ \geq \; & \bigwedge_{A \in \mathrm{Cl}(\sim)} A(x) \to A(y) \geq \\ \geq \; & \bigwedge_{A \in \mathrm{Cl}(\sim)} A(x) \leftrightarrow A(y). \end{aligned}$$

(iii): Using (ii) we have $\mathrm{Cl}_x(\sim) \approx \mathrm{Cl}_y(\sim) =$

$$= \bigwedge_{A \in \mathrm{Cl}(\sim)} (\mathrm{Cl}_x(\sim)(A) \leftrightarrow \mathrm{Cl}_y(\sim)(A)) = \bigwedge_{A \in \mathrm{Cl}(\sim)} (A(x) \leftrightarrow A(y)) = (x \sim^- y).$$

■

COROLLARY 3.33
For an **L**-tolerance $\sim$ we have

$$[x]_{\sim^-} \subseteq \bigcap \mathrm{Cl}_x(\sim) \subseteq \bigcap{}^1 \mathrm{Cl}_x(\sim). \tag{3.11}$$

PROOF. Using Theorem 3.32, we have for any $y \in X$, that $\bigcap \mathrm{Cl}_x(\sim)(y) = \bigwedge_{A \in \mathrm{Cl}(\sim)} (\mathrm{Cl}_x(\sim)(A) \to A(y)) = \bigwedge_{A \in \mathrm{Cl}(\sim)} (A(x) \to A(y)) \geq \bigwedge_{A \in \mathrm{Cl}(\sim)} (A(x) \leftrightarrow A(y)) = (x \sim^- y) = [x]_{\sim^-}(y)$.

$\bigcap \mathrm{Cl}_x(\sim) \subseteq \bigcap{}^1 \mathrm{Cl}_x(\sim)$ follows by definition. ■

EXAMPLE 3.34
Consider the $[0,1]$-tolerance $\sim_3$ from Example 3.3. For $[0,1]_{\min}$, $[0,1]_{\Pi}$, $[0,1]_{\mathrm{L}}$, the corresponding $[0,1]_{\min}$-equivalence $\sim^-_{\min}$, $[0,1]_{\Pi}$-equivalence $\sim^-_{\Pi}$, $[0,1]_{\mathrm{L}}$-equivalence $\sim^-_{L}$ (that is, $\sim^-_{*}$ is $\sim^-$ for $[0,1]_*$ being the structure of truth degrees) are depicted in Table 2.

REMARK 3.35
For any **L**-tolerance $\sim$, $\otimes_1 \leq \otimes_2$ implies $\sim_1^- \geq \sim_2^-$. Since $x \otimes y = \bigwedge\{z \mid x \leq y \to z\}$, $x \to y = \bigvee\{z \mid x \otimes z \leq y\}$ (see [5]), we have $\otimes_1 \leq \otimes_2$ iff $\to_2 \leq \to_1$. Therefore, if $\otimes_1 \leq \otimes_2$, then

$$(x \sim_1^- y) = \bigwedge_{z \in X} (x \sim z) \leftrightarrow_1 (z \sim y) \geq \bigwedge_{z \in X} (x \sim z) \leftrightarrow_2 (z \sim y) = (x \sim_1^- y).$$

DEFINITION 3.36
For an **L**-tolerance $\sim$ on X we define fuzzy relations $\sim^*$ and $\sim^{**}$ on the family $X/_{\sim^-}$ of all kernels of $\sim$ by

$$[x]_{\sim^-} \sim^* [y]_{\sim^-} = \bigwedge_{u,v \in X} (x \sim^- u) \otimes (v \sim^- y) \to u \sim v$$

and

$$[x]_{\sim^-} \sim^{**} [y]_{\sim^-} = \bigvee_{u,v \in X} (x \sim^- u) \otimes (u \sim v) \otimes (v \sim^- y).$$

THEOREM 3.37
Let $\langle X, \sim \rangle$ be any **L**-tolerance space.Then

(i) $\langle X/_{\sim^-}, \sim^* \rangle$ and $\langle X/_{\sim^-}, \sim^{**} \rangle$ are **L**-tolerance spaces.
(ii) for all $x, y \in X$,

$$[x]_{\sim^-} \sim^* [y]_{\sim^-} \; = \; x \sim y \; = \; [x]_{\sim^-} \sim^{**} [y]_{\sim^-}.$$

PROOF. (i): Follows from definition of $\sim^*$, resp. $\sim^{**}$.
(ii): First we show $[x]_{\sim^-} \sim^* [y]_{\sim^-} = x \sim y$ by checking both inequalities.

$$\begin{aligned} [x]_{\sim^-} \sim^* [y]_{\sim^-} &= \bigwedge_{u,v} (x \sim^- u) \otimes (v \sim^- y) \to u \sim v \leq \\ \leq \quad & (x \sim^- x) \otimes (y \sim^- y) \to x \sim y = x \sim y. \end{aligned}$$

For the converse inequality we have

$$[x]_{\sim^-} \sim^* [y]_{\sim^-} = \bigwedge_{u,v} (x \sim^- u) \otimes (v \sim^- y) \to u \sim v \geq x \sim y$$

iff for all u, v,

$$(x \sim^- u) \otimes (v \sim^- y) \to u \sim v \geq x \sim y,$$

i.e. iff

$$(x \sim y) \otimes (x \sim^- u) \otimes (v \sim^- y) \leq u \sim v \tag{3.12}$$

which is true since

$$\begin{aligned} & (x \sim y) \otimes [\bigwedge_a (x \sim a) \leftrightarrow (a \sim u)] \otimes [\bigwedge_b (v \sim b) \leftrightarrow (b \sim y)] \leq \\ \leq \quad & (x \sim y) \otimes [(x \sim y) \to (y \sim u)] \otimes [(y \sim u) \to (u \sim v)] \leq \\ \leq \quad & (y \sim u) \otimes [(y \sim u) \to (u \sim v)] \leq u \sim v. \end{aligned}$$

	$\sim^+_{\min}$				$\sim^+_{\Pi}$				$\sim^+_{L}$			
	x	y	z	u	x	y	z	u	x	y	z	u
x	1	0.9	0.8	1	1	0.9	0.8	1	1	0.9	0.8	1
y	0.9	1	0.8	0.9	0.9	1	0.72	0.9	0.9	1	0.7	0.9
z	0.8	0.8	1	0.8	0.8	0.72	1	0.8	0.8	0.7	1	0.8
u	1	0.9	0.8	1	1	0.9	0.8	1	1	0.9	0.8	1

TABLE 3. Fuzzy transitive closure from Example 3.38

For the equality $[x]_{\sim^-} \sim^{**} [y]_{\sim^-} = x \sim y$ we have

$$[x]_{\sim^-} \sim^{**} [y]_{\sim^-} = \bigvee_{u,v} (x \sim^- u) \otimes (u \sim v) \otimes (v \sim^- y) \geq$$
$$\geq \quad (x \sim^- x) \otimes (x \sim y) \otimes (y \sim^- y) = x \sim y.$$

Conversely, we have

$$(x \sim^- y) = \bigvee_{u,v} (x \sim^- u) \otimes (u \sim v) \otimes (v \sim^- y) \leq x \sim y$$

iff for any $u, v, \in X$ we have

$$(x \sim^- u) \otimes (u \sim v) \otimes (v \sim^- y) \leq x \sim y,$$

which is true. It suffices to put x equal to u and y equal to v, and we obtain (3.12), which is true. ∎

For an **L**-tolerance $\sim$, $\sim^-$ is an **L**-equivalence contained in $\sim$. On the other hand, one can consider **L**-equivalence relations containing $\sim$. Recall [5] that an **L**-transitive closure of a binary **L**-relation $\sim$ on X is a binary **L**-relation $\sim^+$ on X defined by

$$(x \sim^+ y) = \bigvee_{x_i \in X} (x \sim x_1) \otimes (x_1 \sim x_2) \otimes \cdots \otimes (x_n \sim y)$$

and that $\sim^+$ is the least transitive **L**-relation on X containing $\sim$.

It follows from Theorem 3.22 that $\mathrm{Cl}(\sim^+) = \{[x]_{\sim^+} \mid x \in X\}$. In the following we call the classes of the **L**-similarity space $\langle X, \sim^+ \rangle$ its *components*.

EXAMPLE 3.38
Consider the $[0,1]$-tolerance $\sim_3$ from Example 3.3. For $[0,1]_{\min}$, $[0,1]_{\Pi}$, $[0,1]_{\mathrm{L}}$, the corresponding $[0,1]_{\min}$-transitive closure $\sim^+_{\min}$, $[0,1]_{\Pi}$-transitive closure $\sim^+_{\Pi}$, $[0,1]_{\mathrm{L}}$-transitive closure $\sim^+_{L}$ (that is, $\sim^+_*$ is $\sim^+$ for $[0,1]_*$ being the structure of truth degrees) are depicted in Table 3.

Given a fuzzy tolerance $\sim$, we introduced **L**-equivalence relations $\sim^-$ and $\sim^+$ which can be thought of as the lower and the upper approximation of $\sim$. Recall that a binary **L**-relation E is said to be compatible with a binary **L**-relation R on X if we have $E(x,x') \otimes E(y,y') \otimes R(x,y) \leq R(x',y')$ for each $x, y, x', y' \in X$, i.e. if it is true that if x is E-related to x', y is E-related to y', and x and y are R-related then x' and y' are R-related as well.

LEMMA 3.39
Let $\sim$ be an **L**-tolerance on X. Then $\sim^-$ is compatible with $\sim$. Furthermore, $\sim^-$ is the largest reflexive binary **L**-relation which is compatible with $\sim$. Moreover, $\sim$ is compatible with $\sim^+$.

PROOF. First, we show that $\sim^-$ is compatible with $\sim$ and that any reflexive **L**-relation compatible with $\sim$ is contained in $\sim^-$. We have

$$\begin{aligned} & (x \sim^- x') \otimes (y \sim^- y') \otimes (x \sim y) = \\ = \quad & (x \sim y) \otimes \bigwedge_{z \in X} ((x \sim z) \leftrightarrow (z \sim x')) \otimes \bigwedge_{z} ((y \sim z) \leftrightarrow (z \sim y')) \leq \\ \leq \quad & (x \sim y) \otimes ((x \sim y) \rightarrow (y \sim x')) \otimes ((y \sim x') \rightarrow (x' \sim y')) \leq \\ \leq \quad & (x' \sim y'), \end{aligned}$$

proving compatibility of $\sim^-$ with $\sim$. If $\approx$ is a reflexive **L**-relation on X compatible with $\sim$ then we have $(x \approx y) \otimes (z \approx z) \otimes (x \sim z) \leq (y \sim z)$, i.e. $(x \approx y) = (x \approx y) \otimes (z \approx z) \leq (x \sim z) \rightarrow (y \sim z)$ for each $z \in Z$. Analogously we get $(x \approx y) = (x \approx y) \otimes (z \approx z) \leq (y \sim z) \rightarrow (x \sim z)$ for each $z \in Z$. Putting these inequalities together, we obtain $(x \approx y) \leq \bigwedge_{z \in Z} ((x \sim z) \leftrightarrow (z \sim y)) = (x \sim^- y)$ proving the first part.

Second, $\sim$ is compatible with $\sim^+$ since $(x \sim x') \otimes (y \sim y') \otimes (x \sim^+ y) \leq (x \sim^+ x') \otimes (y \sim^+ y') \otimes (x \sim^+ y) \leq (x' \sim^+ y')$, by transitivity of $\sim^+$. The proof is finished. ∎

From Lemma 3.31, Theorem 3.32 (i), Lemma 3.39, and the above-mentioned property of a transitive closure, we get the following theorem showing the role of $\sim^-$ and $\sim^+$.

THEOREM 3.40
Let $\sim$ be an **L**-tolerance on X.

(i) $\sim^-$ is an **L**-equivalence compatible with $\sim$ which is contained in $\sim$. Moreover, $\sim^-$ is the greatest **L**-equivalence which is compatible with $\sim$.

(ii) $\sim^+$ is the least **L**-equivalence containing $\sim$. Moreover, $\sim$ is compatible with $\sim^+$.

Therefore, $\sim^-$ is the best lower bound of $\sim$ in the sense of being an **L**-equivalence compatible with $\sim$ and $\sim^+$ is the best upper bound of $\sim$ in the sense of being an **L**-equivalence.

REMARK 3.41
(1) $\sim^-$ is not necessarily the greatest **L**-equivalence contained in $\sim$. As a counterexample from the ordinary case, consider $X = \{x, y, z, w\}$, $R = \{\langle x, y \rangle, \langle x, w \rangle, \langle y, z \rangle, \langle y, w \rangle, \langle z, w \rangle\}$ and $\sim = R \cup R^{-1} \cup \mathrm{id}_X$. Then $\sim^-$ has classes $\{x\}, \{y, z\}, \{w\}$. However, the equivalence θ given by classes $\{x, y, w\}$ and $\{z\}$ is larger then $\sim^-$ and still contained in $\sim$.

(2) In general, the fact that a binary fuzzy relation E is compatible with a binary fuzzy relation R has a natural interpretation if E is some 'small' fuzzy relation representing an underlying similarity or indistinguishability on the universe. Namely, compatibility then says that R 'respects' E. From this point of view, the facts that $\sim^-$ is compatible with $\sim$ and that $\sim$ is compatible with $\sim^+$ as presented in Theorem 3.40 are natural since we have both $\sim^- \subseteq \sim$ and $\sim \subseteq \sim^+$, i.e. $\sim^-$ and $\sim$ can be viewed as the underlying similarity relations, respectively.

3.4 *Properties of fuzzy tolerance spaces*

DEFINITION 3.42
For an **L**-tolerance space $\langle X, \sim \rangle$, an **L**-relation $\overline{\sim}$ on X, defined by

$$x \overline{\sim} y = \begin{cases} (x \sim y) \rightarrow 0 & if\ x \neq y \\ 1 & if\ x = y \end{cases}$$

is called an **L**-*quasi-complement* of $\sim$.

	$\overline{\sim}_{\min}$				$\overline{\sim}_L$			
	x	y	z	u	x	y	z	u
x	1	0	0	0	1	0.1	0.2	0
y	0	1	0	0	0.1	1	0.3	0.1
z	0	0	1	0	0.2	0.3	1	0.2
u	0	0	0	1	0	0.1	0.2	1

TABLE 4. Fuzzy quasi-complement from Example 3.44

The following assertion follows directly from the definition.

LEMMA 3.43
The **L**-*quasi-complement* of $\sim$ is an **L**-tolerance on X.

EXAMPLE 3.44
Consider the $[0,1]$-tolerance $\sim_3$ from Example 3.3. For $[0,1]_{\min}$, $[0,1]_{\Pi}$, $[0,1]_{\mathrm{L}}$, the corresponding $[0,1]_{\min}$-quasi-complement $\overline{\sim}_{\min}$, $[0,1]_{\mathrm{L}}$-quasi-complement $\overline{\sim}_L$ are depicted in Table 4 ($[0,1]_{\Pi}$-quasi-complement $\overline{\sim}_{\Pi}$ is equal to $[0,1]_{\min}$-quasi-complement $\overline{\sim}_{\min}$).

DEFINITION 3.45
Let $\langle X, \sim\rangle$ be an **L**-tolerance space.

(i) The degree $\mathrm{simp}(\sim)$ to which $\langle X, \sim\rangle$ is simple is defined by

$$\mathrm{simp}(\sim) = \bigwedge_{x\in X} (\{1/x\} \approx [x]_{\sim^-}).$$

(ii) The degree $\mathrm{reg}(\sim)$ to which $\langle X, \sim\rangle$ is regular is defined by

$$\mathrm{reg}(\sim) = \bigwedge_{x\in X} S(\bigcap{}^1 \mathrm{Cl}_x(\sim), [x]_{\sim^-}).$$

(iii) The degree $\mathrm{con}(\sim)$ to which $\langle X, \sim\rangle$ is connected is defined by

$$\mathrm{con}(\sim) = \bigwedge_{x,y\in X} \bigvee_{z_1,\ldots,z_n\in X} (x \sim z_1) \otimes (z_1 \sim z_2) \otimes \ldots \otimes (z_n \sim y).$$

We say that an **L**-tolerance space is *simple* (*regular*, *connected*) if $\mathrm{simp}(\sim) = 1$ ($\mathrm{reg}(\sim) = 1$, $\mathrm{con}(\sim) = 1$).

REMARK 3.46
Properties simp, reg, and con describe natural features of a tolerance space. The meaning of simp, reg, and con, can be obtained by translating the defining formulas into their verbal descriptions. So, the degree $\mathrm{simp}(\sim)$ to which a tolerance space is simple is the degree to which the assertion 'for each $x \in X$: the $\sim^-$-class of x is a singleton consisting of x'; the $\mathrm{reg}(\sim)$ is the degree to which 'for each $x \in X$: the intersection of all $\sim$-classes to which x belongs is contained in the $\sim^-$-class of x' is true; $\mathrm{con}(\sim)$ is the degree to which 'for each $x, y \in X$, there exist $z_1, \ldots, z_n \in X$ such that x and z_1 are related by $\sim$, z_1 and z_2 are related by $\sim$, …, z_n and y are related by $\sim$' is true.

EXAMPLE 3.47
(1) For **L**-tolerance spaces corresponding to $\sim_3$ from Example 3.3 we have

$$\mathrm{simp}(\sim_{\min}) = \mathrm{simp}(\sim_{\Pi}) = 0 \qquad \mathrm{simp}(\sim_L) = 0.1;$$

$\sim_{SRC}$	x	y	z	u
x	1	1	0	0
y	1	1	1	0
z	0	1	1	1
u	0	0	1	1

TABLE 5. An **L**-tolerance from Example 3.47

$$\mathrm{reg}(\sim_{\min}) = 0.7 \qquad \mathrm{reg}(\sim_{\Pi}) = \frac{7}{9} \qquad \mathrm{reg}(\sim_{L}) = 1;$$
$$\mathrm{con}(\sim_{\min}) = 0.8 \qquad \mathrm{con}(\sim_{\Pi}) = 0.72 \qquad \mathrm{con}(\sim_{L}) = 0.7.$$

An example of an **L**-tolerance space which is simple, regular, and connected (for arbitrary **L**) is shown in Table 5.

(2) One can easily see that for each fuzzy tolerance $\sim$, $\langle X/\sim^{-}, \sim^{*}\rangle$ is simple iff $\langle X/\sim^{-}, \sim^{**}\rangle$ is simple iff $\langle X, \sim\rangle$ is simple.

DEFINITION 3.48
Let $\langle X, \sim\rangle$ be an **L**-tolerance space, $A \in L^X$.

(i) The degree $\mathrm{abs}(A)$ to which A is absorbent is defined by

$$\mathrm{abs}(A) = \bigwedge_{y\in X} \bigvee_{x\in X} (x \sim y) \otimes A(x).$$

(ii) The degree $\mathrm{stab}(A)$ to which A is stable is defined by

$$\mathrm{stab}(A) = \bigwedge_{x,y\in X, x\neq y} A(x) \otimes A(y) \to ((x \sim y) \to 0).$$

(iii) The degree $\mathrm{is}(x)$ to which $x \in X$ is isolated is defined by

$$\mathrm{is}(x) = (\{1/x\} \approx [x]_{\sim}).$$

We say that an **L**-set A is *absorbent* (*stable*) if $\mathrm{abs}(A) = 1$ ($\mathrm{stab}(A) = 1$). Analogously $x \in X$ is called *isolated* if $\mathrm{is}(x) = 1$.

REMARK 3.49
$\mathrm{abs}(A)$ is the degree to which 'for each $y \in X$ there is some x from A which is $\sim$-related to y' is true; $\mathrm{stab}(A)$ is the degree to which 'for every distinct $x, y \in X$: if both x and y belong to A then x and y are not $\sim$-related' is true; $\mathrm{is}(x)$ is the degree to which 'the $\sim$-class of x is a singleton consisting of x' is true.

THEOREM 3.50
Let $\langle X, \sim\rangle$ be an **L**-tolerance space, $A, B \in L^X$. Then

(i) $\mathrm{abs}(\emptyset) = 0$ and X is absorbent;

(ii) $S(A, B) \leq \mathrm{abs}(A) \to \mathrm{abs}(B)$; therefore, if $A \subseteq B$, then $\mathrm{abs}(A) \leq \mathrm{abs}(B)$;

(iii) $\mathrm{abs}(A) \vee \mathrm{abs}(B) \leq \mathrm{abs}(A \cup B)$.

PROOF. (i): We have

$$\bigwedge_{y\in X}\bigvee_{x\in X}(x\sim y)\otimes\emptyset(x)=\bigwedge_{y\in X}\bigvee_{x\in X}(x\sim y)\otimes 0=0,$$

and

$$\bigwedge_{y\in X}\bigvee_{x\in X}(x\sim y)\otimes X(x)=\bigwedge_{y\in X}\bigvee_{x\in X}(x\sim y)\otimes 1=1.$$

(ii): We have $S(A,B)\leq \mathrm{abs}(A)\rightarrow \mathrm{abs}(B)$ iff $S(A,B)\otimes\mathrm{abs}(A)\leq\mathrm{abs}(B)$ which holds iff for any $y\in X$

$$S(A,B)\otimes\bigwedge_{z\in X}\bigvee_{x\in X}(x\sim z)\otimes A(x)\leq\bigvee_{x\in X}(x\sim y)\otimes B(x),$$

which is true. Indeed,

$$S(A,B)\otimes\bigwedge_{z\in X}\bigvee_{x\in X}(x\sim z)\otimes A(x)\leq S(A,B)\otimes\bigvee_{x\in X}(x\sim y)\otimes A(x)=$$

$$=\bigvee_{x\in X}(x\sim y)\otimes A(x)\otimes S(A,B)\leq\bigvee_{x\in X}(x\sim y)\otimes A(x)\otimes(A(x)\rightarrow B(x))\leq$$

$$\leq\bigvee_{x\in X}(x\sim y)\otimes B(x).$$

(iii) follows directly from (ii). Since $A,B\subseteq A\cup B$, (ii) implies $\mathrm{abs}(A),\mathrm{abs}(B)\leq\mathrm{abs}(A\cup B)$, therefore $\mathrm{abs}(A)\vee\mathrm{abs}(B)\leq\mathrm{abs}(A\cup B)$. ■

THEOREM 3.51
Let $\langle X,\sim\rangle$ be an **L**-tolerance space, $A\in L^X$ be normal. Then A is maximal stable iff A is a $\overline{\approx}$-class.

PROOF. Notice first, that A is stable iff for each $x\neq y$ we have $A(x)\otimes A(y)\leq(x\sim y)\rightarrow 0$. Since $A(x)\otimes A(x)\leq 1$, a moment's reflection shows that A is stable iff it is a preclass of $\overline{\approx}$. The result now follows immediately. ■

REMARK 3.52
If A is not normal, Theorem 3.51 does not hold. Namely, we can have a maximal stable fuzzy set A which is not normal, and thus not even a preclass. Consider $X=\{x,y\}$, the Łukasiewicz structure on $[0,1]$, let $\sim$ by given by $x\sim y=1$ (the rest is determined uniquely), and take $A=\{\,{}^{0.5}/x,\,{}^{0.5}/y\}$. One easily checks that A is stable. Stability requires that $A(x)\otimes A(y)\leq(x\sim y)\rightarrow 0=0$, i.e. $A(x)\otimes A(y)=0$. From this we get that A is maximal stable since for any $A'\supset A$ we have $A'(x)\otimes A'(y)>0$. However, A is not normal.

THEOREM 3.53
For an **L**-tolerance space $\langle X,\sim\rangle$ and any $A\in L^X$, $x\in X$, we have

$$\mathrm{abs}(A)\otimes\mathrm{is}(x)\leq A(x).$$

PROOF. We have

$$\mathrm{abs}(A)\otimes\mathrm{is}(x)=(\bigwedge_{u\in X}\bigvee_{v\in X}A(v)\otimes(u\sim v))\otimes(\{1/x\}\approx[x]_{\sim})\leq$$

$$\leq\quad(\bigvee_{v\in X}A(v)\otimes(x\sim v))\otimes(\bigwedge_{y\in X}(x\sim y)\rightarrow\{1/x\}(y))=$$

$$
\begin{aligned}
&= \bigvee_{v\in X} [A(v)\otimes(x\sim v)\otimes(\bigwedge_{y\in X}(x\sim y)\to\{1/x\}(y))] \leq \\
&\leq \bigvee_{v\in X} A(v)\otimes(x\sim v)\otimes[(x\sim v)\to\{1/x\}(v))] \leq \\
&\leq \bigvee_{v\in X} A(v)\otimes\{1/x\}(v) = A(x).
\end{aligned}
$$

∎

In words, Theorem 3.53 says that if A is absorbent and x is isolated then x must belong to A.

COROLLARY 3.54
For an **L**-tolerance space $\langle X,\sim\rangle$ and any $A\in L^X$ we have

(i) If $x\in X$ is isolated, then $\mathrm{abs}(A)\leq A(x)$.
(ii) If $x\in X$ is isolated and A is absorbent, then $A(x)=1$.

THEOREM 3.55
For an **L**-tolerance space $\langle X,\sim\rangle$ and any $A,B\in L^X$ we have

(i) $S(A,B)^2\leq \mathrm{stab}(B)\to\mathrm{stab}(A)$, therefore $A\subseteq B$ implies $\mathrm{stab}(A)\geq\mathrm{stab}(B)$;
(ii) $\mathrm{abs}(X\setminus A)\leq\bigwedge_{x\in X}(A(x)\otimes\mathrm{is}(x))\to 0$.

PROOF. (i): By adjointness, we have to show $S(A,B)^2\otimes\mathrm{stab}(B)\leq\mathrm{stab}(A)$. We have

$$
\begin{aligned}
& S(A,B)^2\otimes\mathrm{stab}(B) = \\
&= S(A,B)^2\otimes\bigwedge_{x,y\in X,x\neq y}(B(x)\otimes B(y))\to(x\overline{\sim}y) \leq \\
&\leq \bigwedge_{x,y\in X}(A(x)\to B(x))\otimes(A(y)\to B(y))\otimes\bigwedge_{x,y\in X,x\neq y}(B(x)\otimes B(y))\to(x\overline{\sim}y) \leq \\
&\leq \bigwedge_{x,y\in X,x\neq y}(A(x)\to B(x))\otimes(A(y)\to B(y))\otimes(B(y)\to(B(x)\to(x\overline{\sim}y)) \leq \\
&\leq \bigwedge_{x,y\in X,x\neq y}(A(x)\to B(x))\otimes(A(y)\to(B(x)\to(x\overline{\sim}y)) = \\
&= \bigwedge_{x,y\in X,x\neq y}(A(x)\to B(x))\otimes(B(x)\to(A(y)\to(x\overline{\sim}y)) \leq \\
&\leq \bigwedge_{x,y\in X,x\neq y}(A(x)\to(A(y)\to(x\overline{\sim}y)) = \\
&= \bigwedge_{x,y\in X,x\neq y}(A(x)\otimes A(y))\to(x\overline{\sim}y)) = \\
&= \mathrm{stab}(A).
\end{aligned}
$$

(ii): Applying Theorem 3.53 to $X\setminus A$ we have for any $x\in X$

$$\mathrm{abs}(X\setminus A)\otimes\mathrm{is}(x)\leq(X\setminus A)(x)=(A(x)\to 0),$$

i.e.

$$\mathrm{abs}(X\setminus A)\leq(A(x)\otimes\mathrm{is}(x))\to 0),$$

which implies

$$\mathrm{abs}(X \setminus A) \leq \bigwedge_{x \in X} (A(x) \otimes \mathrm{is}(x)) \to 0.$$

∎

3.5 Representations of fuzzy tolerances and their classes

3.5.1 Representation by a-cuts

For a fuzzy tolerance $\sim$ on X and a truth degree $a \in L$, the a-cut ${}^a{\sim} = \{\langle x, y\rangle \mid (x \sim y) \geq a\}$ is a natural relation derived from a. Namely, ${}^a{\sim}$ is an ordinary relation containing pairs of elements which are related by $\sim$ (e.g. similar) in degree at least as high as the specified threshold a. Each ${}^a{\sim}$ is an ordinary tolerance relation. In fact, fuzzy tolerances can be seen as special systems of ordinary tolerances.

Recall [5] that a system $\mathcal{S} = \{A_a \subseteq X \mid a \in L\}$ is called **L**-nested if (1) $a \leq b$ implies $A_b \subseteq A_a$ and (2) for each $x \in X$, the set $\{a \mid x \in A_a\}$ has a greatest element. It is well known that there is a bijective correspondence between **L**-sets in X and **L**-nested systems of subsets of X. Namely, for an **L**-set A, the system $\mathcal{S}_A = \{{}^aA \mid a \in L\}$ of all a-cuts of A is **L**-nested. Furthermore, for an **L**-nested system $\mathcal{S}$, $A_{\mathcal{S}}(x) = \bigvee_{x \in A_a} a$ defines an **L**-set in X and we have $A = A_{\mathcal{S}_A}$ and $\mathcal{S} = \mathcal{S}_{A_{\mathcal{S}}}$. A moment's reflection shows that for a binary **L**-relation R on X, R is reflexive (symmetric) if and only each ${}^aR$ ($a \in L$) is reflexive (symmetric). Thus, we have the following theorem.

THEOREM 3.56
For an **L**-tolerance $\sim$ on X, $\sim_{\mathcal{S}} = \{{}^a{\sim} \mid a \in L\}$ is an **L**-nested system of ordinary tolerance (i.e. reflexive and symmetric) relations on X. If $\mathcal{S} = \{\sim_a \mid A \in L\}$ is an **L**-nested system of tolerance relations on X then $\sim_{\mathcal{S}}$ defined by

$$(x \sim_{\mathcal{S}} y) = \bigvee \{a \in L \mid \langle x, y\rangle \in \sim_a\}$$

is an **L**-tolerance. Moreover, the above introduced mappings sending $\sim$ to $\mathcal{S}_\sim$ and $\mathcal{S}$ to $\sim_{\mathcal{S}}$ are mutually inverse.

A natural problem arises as to whether an analogous cut-decomposition can be performed for preclasses and classes.

LEMMA 3.57
If $\otimes$ is $\wedge$, then $A \in \mathrm{Precl}(\sim)$ iff for each $a \in L$: ${}^aA \in \mathrm{Precl}({}^a{\sim})$.

PROOF. '⇒': $x, y \in {}^aA$ implies $a \leq A(x), A(y)$, i.e. $a \leq A(x) \wedge A(y)$, but we have $A(x) \wedge A(y) \leq (x \sim y)$, i.e. $\langle x, y\rangle \in {}^a{\sim}$.

'⇐': We have to show $A(x) \wedge A(y) \leq (x \sim y)$. Put $a = A(x) \wedge A(y)$. Since ${}^aA$ is a preclass of ${}^a{\sim}$, $x, y \in {}^aA$ implies $\langle x, y\rangle \in {}^a{\sim}$, i.e. $(x \sim y) \geq a = A(x) \wedge A(y)$. ∎

EXAMPLE 3.58
The analogous assertion does not hold for classes. Consider $\sim_4$ from Example 3.3 and its class $\{1/x, 0.6/y, 0.8/z, 1/u\}$. Taking a-cut for $a = 0.9$, we obtain $\{x, u\}$, but this is only preclass, not a class of ${}^{0.9}{\sim}_4$.

LEMMA 3.59
For any $a \in L$, $x \in X$ we have

$${}^a[x]_\sim = [x]_{{}^a\sim}.$$

PROOF. Directly by definition,

$$^a[x]_\sim = \{y \mid x \sim y \geq a\} = [x]_{^a\sim}.$$

■

3.5.2 $\sim$-classes as formal fuzzy concepts

We are going to show a connection between fuzzy tolerance classes and so-called fuzzy concept lattices. We will show that classes are, in a sense, exactly normal fuzzy concepts. This observation yields an algorithm (provided all involved entities are finite) for generating all classes of a given **L**-tolerance, see Section 3.6.

Concept lattices are the basic structures employed by so-called Formal Concept Analysis [13]. Basically, the aim of formal concept analysis is to identify all interesting clusters (so-called formal concepts) hidden in data which has the form of a table describing objects and their attributes. The basic method was extended for the purpose of analysing data with fuzzy attributes [5]. The basic notions are as follows. Let I be an **L**-relation between X and Y, $I(x,y)$ being interpreted as the degree to which object x has attribute y. Introduce operators ${}^{\uparrow_I} : L^X \to L^Y$ and ${}^{\downarrow_I} : L^Y \to L^X$ by

$$A^{\uparrow_I}(y) = \bigwedge_{x \in X} A(x) \to I(x,y)$$

and

$$B^{\downarrow_I}(x) = \bigwedge_{y \in Y} B(y) \to I(x,y),$$

for $A \in L^X$, $B \in L^Y$. Denote

$$\mathcal{B}\,(X,Y,I) = \{\langle A,B\rangle \mid A \in L^X,\ B \in L^Y,\ A^{\uparrow_I} = B, B^{\downarrow_I} = A\}.$$

Note that $A^{\uparrow_I}$ is interpreted as the fuzzy set of all atributes shared by all objects from A, and dually for $B^{\downarrow_I}$. Furthermore, introduce a relation $\leq$ on $\mathcal{B}\,(X,Y,I)$ by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \quad \text{(or, equivalently, } B_2 \subseteq B_1).$$

Then $\mathcal{B}\,(X,Y,I)$, equipped with $\leq$, is a complete lattice, called an **L**-concept lattice. Each $\langle A,B\rangle \in \mathcal{B}\,(X,Y,I)$ is called a formal **L**-concept—it is naturally interpreted as a fuzzy concept hidden in the input data $\langle X,Y,I\rangle$. A is interpreted as the collection of all objects covered by the concept, B is interpreted as the collection of all attributes covered by the concept. The whole conception is due to so-called Port-Royal logic, for details see [13, 5].

Recall furthermore, that ${}^{\uparrow_I}$ and ${}^{\downarrow_I}$ form a so-called **L**-Galois connection between X and Y [3]. This means that $S(A_1,A_2) \leq S(A_2^{\uparrow}, A_1^{\uparrow})$, $S(B_1,B_2) \leq S(B_2^{\downarrow}, B_1^{\downarrow})$, $A \subseteq (A^{\uparrow})^{\downarrow}$, $B \subseteq (B^{\downarrow})^{\uparrow}$ holds for every $A, A_1, A_2 \in L^X$, $B, B_1, B_2 \in L^Y$. Conversely, if ${}^{\uparrow}$ and ${}^{\downarrow}$ form an **L**-Galois connection between X and Y then $I_{\langle\uparrow,\downarrow\rangle}(x,y) = \{\,{}^1/x\}^{\uparrow}(y)$ (which equals $\{\,{}^1/y\}^{\downarrow}(x)$) defines an **L**-relation. It holds true that the mappings sending I to $\langle{}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ and $\langle{}^{\uparrow}, {}^{\downarrow}\rangle$ to $I_{\langle\uparrow,\downarrow\rangle}$ are mutually inverse bijections between the set of all binary **L**-relations in X and the set of all **L**-Galois connections between X and Y.

THEOREM 3.60

Let $\sim$ be a binary **L**-relation on X, $\langle{}^{\uparrow}, {}^{\downarrow}\rangle$ be the corresponding **L**-Galois connection between X and X. Then $\sim$ is an **L**-tolerance iff $\langle{}^{\uparrow}, {}^{\downarrow}\rangle$ satisfies

$$\{\,{}^1/x\}^{\uparrow}(x) = 1 \qquad \text{and} \qquad {}^{\uparrow} = {}^{\downarrow} \tag{3.13}$$

for each $x \in X$. Therefore, there is a bijective correspondence between **L**-tolerances on X and **L**-Galois connections between X and X which satisfy (3.13).

PROOF. The result follows from previous discussion observing that $\{ {}^1/x\}^{\uparrow}(x) = 1$ is equivalent to $(x \sim x) = 1$ and that $^{\uparrow} = {}^{\downarrow}$ is equivalent to symmetry of $\sim$. Indeed, we have $\{ {}^1/x\}^{\uparrow}(x) = 1 \rightarrow (x \sim x) = (x \sim x)$. Moreover, if $^{\uparrow} = {}^{\downarrow}$ then $I_{\langle\uparrow,\downarrow\rangle}(x,y) = \{ {}^1/x\}^{\uparrow}(y) = \{ {}^1/y\}^{\downarrow}(x) = \{ {}^1/y\}^{\uparrow}(x) = I_{\langle\uparrow,\downarrow\rangle}(y,x)$. Conversely, if I is symmetric then $^{\uparrow} = {}^{\downarrow}$ by definition of $^{\uparrow}$ and $^{\downarrow}$. ∎

It follows easily that if $\sim$ is an **L**-tolerance on X then for the corresponding **L**-concept lattice $\mathcal{B}(X,X,\sim)$ we have that the set $\{A \mid \langle A,B\rangle \in \mathcal{B}(X,X,\sim)$ for some $B\}$ of first components equals the set $\{B \mid \langle A,B\rangle \in \mathcal{B}(X,X,\sim)$ for some $A\}$ of second components of all fuzzy concepts.

We now present the proof of Lemma 3.14.

Proof of Lemma 3.14 First, let us observe that for any preclass A we have

$$A(y) \leq \bigwedge_{x\in X} A(x) \rightarrow (x \sim y).$$

Indeed, the definition of a preclass yields that $A(x) \otimes A(y) \leq (x \sim y)$ for each $x, y \in X$, from which we have $A(y) \leq A(x) \rightarrow (x \sim y)$ whence $A(y) \leq \bigwedge_{x\in X} A(x) \rightarrow (x \sim y)$. Now, suppose that A is a class. If $A(y) = \bigwedge_{x\in X} A(x) \rightarrow (x \sim y)$ is not the case, there is $v \in X$ with

$$A(v) < \bigwedge_{y\in X} A(y) \rightarrow (v \sim y).$$

Let us define a fuzzy set B by

$$B(y) = \begin{cases} A(y) & \text{if } y \neq v \\ \bigwedge_{y\in X, y\neq v} B(y) \rightarrow (v \sim y) & \text{for } y = v. \end{cases}$$

We show that B is a preclass, i.e. we show $B(x) \otimes B(y) \leq (x \sim y)$ for any $x, y \in X$. Distinguishing by cases, we have (1) for $x \neq v \neq y$,

$$B(x) \otimes B(y) = A(x) \otimes A(y) \leq (x \sim y) \tag{3.14}$$

since A is a preclass; (2) for $x \neq v, y = v$,

$$B(x) \otimes B(y) \leq B(x) \otimes (B(x) \rightarrow (x \sim y)) \leq (x \sim y); \tag{3.15}$$

(3) for $x = y = v$,

$$B(v) \otimes B(v) \leq 1 = (v \sim v). \tag{3.16}$$

Since A is normal, i.e. $A(x) = 1$ for some $x \in X$, $A \subseteq B$ implies that B is normal as well. Therefore, A is properly contained in B which is itself a preclass. A is therefore not a class. We proved that a class A satisfies (3.7).

Conversely, we show that if A satisfies (3.7), it is a class. The fact that A is a preclass, i.e. $A(x) \otimes A(y) \leq (x \sim y)$, follows directly from (3.7) using adjointness. If A is not a class then there is a preclass B such that $A \subset B$, i.e. $A(x) \leq B(x)$ for each $x \in X$ and $A(v) < B(v)$ for some $v \in X$. We thus have

$$\begin{aligned} B(v) > A(v) = \bigwedge_{y\in Y} A(y) \rightarrow (v \sim y) \geq \\ \geq \bigwedge_{y\in Y} B(y) \rightarrow (v \sim y). \end{aligned}$$

On the other hand, since B is a preclass, $B(v) \leq \bigwedge_{y \in Y} B(y) \to (v \sim y)$ (see the beginning of the proof), which is a contradiction. $\square$

Lemma 3.14 provides an important insight into the relationship of classes of a fuzzy tolerance $\sim$ and the fuzzy concept lattice $\mathcal{B}(X, X, \sim)$, i.e. the fuzzy concept lattice $\mathcal{B}(X, Y, I)$ with $X = Y$ and $I = \sim$. Indeed, A is a $\sim$-class iff (by Lemma 3.14) A is a normal fuzzy set satisfying $A(x) = \bigwedge_{y \in X} A(y) \to (x \sim y)$ iff (by definition of ${}^{\uparrow\sim}$ and ${}^{\downarrow\sim}$, and by Theorem 3.60) A is a normal fuzzy set satisfying $A = A^{\uparrow\sim}$ and $A = A^{\downarrow\sim}$ iff (by definition) A is a normal fuzzy set satisfying $\langle A, A \rangle \in \mathcal{B}(X, X, \sim)$. Therefore, we have the following theorem.

THEOREM 3.61
For a fuzzy tolerance $\sim$, a fuzzy set $A \in L^X$ is a $\sim$-class iff A is normal and $\langle A, A \rangle$ is a formal fuzzy concept from $\mathcal{B}(X, X, \sim)$.

3.6 Algorithm for generating all fuzzy tolerance classes

Classes of fuzzy tolerance $\sim$ represent interesting patterns. If a fuzzy tolerance represents similarity of elements of the universe X then, by definition, a fuzzy tolerance class is a maximal grouping of elements from X which are pairwise similar. The classes can be therefore seen as natural similarity-based clusters. It might be of interest to generate all the fuzzy tolerance classes.

Theorem 3.61 shows a relationship, interesting by itself, that can be utilized in the problem of generating all $\sim$-classes. Note that the definition of a $\sim$-class does not provide us with an efficient way to generate all $\sim$-classes. If one wants to use the definition directly, one has to generate all fuzzy sets A in X and test the conditions of the definition. This takes exponential time (there are $|L|^{|X|}$ fuzzy sets in X). In [6], however, an algorithm for generating all formal fuzzy concepts is present. The algorithm avoids testing all pairs $\langle A, B \rangle$ of fuzzy sets A and B. Since $\sim$-classes can be identified with special formal fuzzy concepts of $\mathcal{B}(X, X, \sim)$, the algorithm from [6] can be used to generate all $\sim$-classes. In the following, we present the resulting algorithm for generating all $\sim$-classes of a fuzzy tolerance relation.

Suppose $X = \{1, 2, \ldots, n\}$; $L = \{0 = a_1 < a_2 < \cdots < a_k = 1\}$ (the assumption that L is linearly ordered is in fact not essential). Put

$$(i, j) \leq (r, s) \quad \text{iff} \quad i < r \quad \text{or} \quad i = r, a_j \geq a_s.$$

In the following, we do not distinguish between $X \times L$ and $\{1, \ldots, n\} \times \{1, \ldots, k\}$, i.e. we denote $(i, a_j) \in X \times L$ also simply by (i, j). The following lemma is obvious.

LEMMA 3.62
$\leq$ is a total order on $X \times L$.

For $A \in L^X$, $(i, j) \in X \times L$, put

$$A \oplus (i, j) := ((A \cap \{1, 2, \ldots, i-1\}) \cup \{ {}^{a_j}/i \})^{\downarrow\sim\uparrow\sim}.$$

Here, $A \cap \{1, 2, \ldots, i-1\}$ is the intersection of a fuzzy set A and the ordinary set $\{1, 2, \ldots, i-1\}$, i.e. $(A \cap \{1, 2, \ldots, i-1\})(x) = A(x)$ for $x < i$ and $(A \cap \{1, 2, \ldots, i-1\})(x) = 0$ otherwise. Furthermore, for $B, C \in L^X$, put

$$B <_{(i,j)} C \quad \text{iff} \quad B \cap \{1, \ldots, i-1\} = C \cap \{1, \ldots, i-1\} \text{ and } B(i) < C(i) = a_j.$$

Finally,

$$B < C \quad \text{iff} \quad B <_{(i,j)} C \quad \text{for some } (i, j).$$

LEMMA 3.63
$<$ is a strict total order on L^X.

PROOF. By easy verification (see [6]). ∎

Note that the strict order $<$ on L^X is the usual lexicographic order.

We call an extent of a fuzzy context $\langle X, X, \sim\rangle$ each fuzzy set A in X such that $\langle A, B\rangle \in \mathcal{B}(X, X, \sim)$ for some B. By $\mathrm{Ext}(\sim)$ we denote the set of all extents of $\langle X, X, \sim\rangle$. The following lemma provides a crucial step for efficient listing of fuzzy tolerance classes.

LEMMA 3.64 (next extent)
The least extent A^+ of $\langle X, X, \sim\rangle$ which is greater (w.r.t. $<$) than a given fuzzy set $A \in L^X$ is given by

$$A^+ = A \oplus (i, j)$$

where (i, j) is the greatest one with $A <_{(i,j)} A \oplus (i, j)$.

PROOF. The assertion is an immediate consequence of [6, Theorem 8]. ∎

Our goal is to generate all classes of a fuzzy tolerance relation $\sim$. We know (Lemma 3.14) that classes of $\sim$ are particular extents of $\langle X, X, \sim\rangle$. Therefore, we may go through $\mathcal{B}(X, X, \sim)$ and for each $\langle A, B\rangle \in \langle X, X, \sim\rangle$ test by Theorem 3.61 whether A is a class (i.e. we have to check whether A is normal and whether $A = A^{\uparrow}$). Going through all $\langle A, B\rangle \in \langle X, X, \sim\rangle$ is possible due to Lemma 3.64: it is immediate that the least extent is $\emptyset^{\uparrow\sim\downarrow\sim}$. Furthermore, Lemma 3.64 gives a way to generate the immediate successor A^+ w.r.t. $<$ to any extent A. Since $<$ is a total ordering of all extents (Lemma 3.63), this procedure goes through all extents and ends up with X. The description of the algorithm follows.

INPUT: fuzzy tolerance $\sim$ on X
OUTPUT: a list of all $\sim$-classes

```
/* Generating fuzzy tolerance classes */
A := ∅↓↑
if A is normal and A = A↓ then store(A)
while A ≠ Y do
  A := A+
  if A is normal and A = A↓ then store(A)
```

It follows from the considerations above, that correctness of the present algorithm, i.e. the fact that the algorithm indeed outputs a list of all $\sim$-classes, follows from Lemma 3.64 (the algorithm goes through the extents of all formal fuzzy concepts) and from Theorem 3.61 (the algorithm outputs exactly those extents which are $\sim$-classes).

REMARK 3.65
In [11], the authors present an algorithm for generating all classes of a fuzzy tolerance relation with the set $[0, 1]$ of truth degrees equipped by $\min$ as conjunction (that is, they consider a particular case of ours). Their algorithm is, however, highly non-efficient. Namely, the authors show how to generate a fuzzy tolerance class given any permutation of the elements from X. In order to generate all fuzzy tolerance classes, one needs to go through all permutations of elements of X (and there are $|X|!$ premutations of X). Moreover, the fact that their procedure eventually generates all fuzzy tolerance classes is true only for $\min$-conjunction. There does not seem to be any obvious extension to other conjunctions.

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References

[1] M. Arbib. Tolerance automata. *Kybernetika* **3**, 223–233, 1967.

[2] R. Baeza-Yates and B. Ribiero-Neto. *Modern Information Retrieval.* Addison Wesley, New York, 1999.

[3] R. Bělohlávek. Fuzzy Galois connections. *Math. Logic Quarterly* **45**, 497–504, 1999.

[4] R. Bělohlávek. Similarity relations in concept lattices. *Journal of Logic and Computation*, **10**, 823–845, 2000.

[5] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles.* Kluwer Academic/Plenum Press, New York, 2002.

[6] R. Bělohlávek. Getting maximal rectangular submatrices from [0,1]-valued object-attribute tables: algorithms for fuzzy concept lattices (submitted). Preliminary version appeared in *Proceedings of the Fourth International Conference on Recent Advances in Soft Computing*. Nottingham, UK, 12-13 December, 2002, pp. 200–205.

[7] H. H. Bock. *Automatische Klassifikation.* Vandenhoeck & Ruprecht, Göttingen, 1974.

[8] S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra.* Springer-Verlag, New York/Heidelberg/Berlin, 1981.

[9] I. Chajda. *Algebraic Theory of Tolerance Relations*. Palacký University Press, Olomouc, 1991.

[10] V. V. Cross and T. A. Sudkamp. *Similarity and Compatibility in Fuzzy Set Theory: Assessment and Applications*. Physica-Verlag, Heidelberg, 2002.

[11] M. Das, M. K. Chakraborty and T. K. Ghoshal. Fuzzy tolerance relation, fuzzy tolerance space and basis. *Fuzzy Sets and Systems*, **97**, 361–369, 1998.

[12] B. S. Everitt. *Cluster Analysis*, 4th edn. Edward Arnold, 2001.

[13] B. Ganter and R. Wille. *Formal Concept Analysis. Mathematical Foundations.* Springer-Verlag, Berlin, 1999.

[14] G. Gerla. *Fuzzy Logic. Mathematical Tools for Approximate Reasoning.* Kluwer, Dordrecht, 2001.

[15] J. A. Goguen. L-fuzzy sets. *Journal of Mathematical Analysis and Applications*, **18**, 145–174, 1967.

[16] J. A. Goguen. The logic of inexact concepts. *Synthese*, **18**, 325–373, 1968–9.

[17] S. Gottwald. *A Treatise on Many-Valued Logics.* Research Studies Press, 2001.

[18] P. Hájek. *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.

[19] D. J. Hand, H. Mannila and P. Smyth. *Principles of Data Mining.* MIT Press, 2001.

[20] U. Höhle. On the fundamentals of fuzzy set theory. *Journal of Mathematical analysis and Applications*, **201**, 786–826, 1996.

[21] N. Jardine and R. Sibson. *Mathematical Taxonomy.* J. Wiley & Sons, London, 1971.

[22] F. Klawonn and J. L. Castro. Similarity in fuzzy reasoning. *Mathware & Soft Computing*, **2**, 197–228, 1995.

[23] E. P. Klement, R. Mesiar and E. Pap. *Triangular Norms.* Kluwer, Dordrecht, 2000.

[24] G. J. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic. Theory and Applications.* Prentice Hall, 1995.

[25] D. M. Krantz, R. D. Luce, P. Suppes and A. Tversky. *Foundations of Measurement, Vol. I, II, III.* Academic Press, New York, 1971, 1989, 1990.

[26] V. Novák, I. Perfilieva and J. Močkoř. *Mathematical Principles of Fuzzy Logic.* Kluwer, Boston, MA, 1999.

[27] J. Pogonowski. *Tolerance Spaces with Applications to Linguistics.* Adam Mickiewicz University Press, Poznan, Poland, 1981.

[28] S. A. Sloman and L. J. Rips. *Similarity and Symbols in Human Thinking.* A Bradford Book, MIT Press, Cambridge, MA, 1998.

[29] P. H. A. Sneath and R. R. Sokal. *Numerical Taxonomy. The Principles and Practice of Numerical Classification.* W. H. Freeman, San Francisco, 1973.

[30] H. Thiele. On the mutual definability of fuzzy tolerance relations and fuzzy tolerance coverings. *Proceedings of the Twenty-fifth International Symposium on Multiple-Valued Logic (ISVL'95)*, pp. 140–145, IEEE, 1995.

[31] A. Tversky. Features of similarity. *Psychological Review*, **84**, 327–352, 1977.

[32] L. A. Zadeh. Similarity relations and fuzzy orderings. *Information Sciences*, **3**, 159–176, 1971.

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