On the Capability of Fuzzy Set Theory to Represent Concepts

RADIM BĚLOHLÁVEK,

Palacký University, Olomouc, Czech Republic

GEORGE J. KLIR,

HAROLD W. LEWIS

EILEEN WAY

Binghamton University - SUNY, USA

Abstract

The purpose of this paper is to examine the conclusions drawn by Osherson and Smith (1981) concerning the inadequacy of the apparatus of fuzzy set theory to represent concepts. Since Osherson and Smith derive their conclusions from specific examples, we show for each of these examples that the respective conclusion they arrive at is not warranted. That is, we demonstrate that fuzzy set theory is sufficiently expressive to represent the various strong intuitions and experimental evidence regarding the relation between simple and combined concepts that are described by Osherson and Smith. To pursue our arguments, we introduce a few relevant notions of fuzzy set theory.

Section 1. Introduction

Sometimes, a particular publication can have an extraordinary influence on an entire discipline with respect to a particular method. We feel that the frequently cited article by Osherson and Smith (1981) is such a publication, and that it has had an extraordinarily negative influence on the use of fuzzy logic and fuzzy set theory in psychology. We argue in this paper that all conclusions drawn by Osherson and Smith (abbreviated as Q&S in the following text) concerning the inadequacy of the apparatus of fuzzy set theory to represent concepts are refutable. Yet, their almost universal acceptance have caused the psychological research community to abandon this powerful representation tool.

The amount of influence that a critical publication by reputable scholars has on a field of research should not be underestimated. A paradigm case from a different field is that of Minsky and Papert's book *Perceptrons*, published in 1969 by the MIT Press. Minsky and Papert were interested in exploring the potential and limitations of early neural networks, principally known at that time through Frank Rosenblatt's work on the perceptron. Their book is a thorough mathematical analysis of the kinds of functions a two-layer perceptron is capable of computing. The well-known and devastating result of this analysis was that there are important functions that the perceptron could not perform.

Minsky and Papert's work resulted in the virtual demise of the research program in neural nets until the mid 1980's. In fairness to Minsky and Papert, the conclusion of their book does recommend that further research be done to determine the possibilities of multi-layer neural networks. Had this suggestion been taken seriously, the development of neural networks might be much farther along today. However, because of the excellent reputation of Minsky and Papert and the rigor of their analysis, most researchers relied on their 'intuitive judgment' that further work on multi-layer networks would prove to be 'sterile'. The artificial intelligence research community concluded for many years that Minsky and Papert's book had shown neural networks to be worthless as a research method for intelligent systems. As we all now know, this was an inaccurate assumption that delayed the development of what is now an important research area. Surely there is a lesson to be learned here. We feel that Osherson and Smith's seminal paper on prototype theory has incorrectly convinced the psychological research community that fuzzy set theory is a worthless tool for representing concepts.[1]

Since the publication of "The Adequacy of Prototype Theory as a Theory of Concepts," in 1981, in *Cognition*, there is scarcely any article or book on concepts that does not refer to O&S and does not summarize their criticism of fuzzy set theory. According to the Web of Science citation database, this article has been cited 134 times to date since its publication. This shows its extraordinary influence when compared with 38 citations for Osherson and Smith (1982) or 65 citations for Smith et al. (1982).

Although O&S's primary purpose in this article was to criticize prototype theory, they use fuzzy set theory to explicate the theory of prototypes. They state (p.265):

> Since, in addition, fuzzy-set theory is a natural complement to prototype theory…
> we shall evaluate prototype theory exclusively in the context of fuzzy-set theory.

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[1] It should be noted that Dominic Massaro and his research group provide an exception by using fuzzy set theory to represent human perception of speech; see, for example, Massaro (1993) and Massaro & Cohen (1993)

They reiterate the same a few pages later (p.271):

> Once again, it is fuzzy-set theory that saves prototype theory from inexplicitness.

Because of this tight connection between the two, O&S's negative conclusions reflect strongly on the value of fuzzy set theory in psychology. In fact, they make several explicit claims about the value of fuzzy set theory. For example, they state (p. 55):

> One thing is clear. Amalgamation of any number of current versions of prototype theory with Zadeh's (1965) rendition of fuzzy-set theory will not handle strong intuitions about the way concepts combine to form complex concepts and propositions.

The influence this article has had on the way in which the psychological community views fuzzy-set theory and fuzzy logic can be seen by the following sampling of quotes from the literature. J. Hampton (1993, pp. 81-82) states:

> Fuzzy logic is thus inappropriate for describing the case of conceptual combination of the kind exemplified in **pet fish**.

And again in 1997 he claims (Hampton, 1997, p. 140):

> Smith et al. (1988) took the striped apple example from Osherson and Smith (1981) and collected empirical evidence that it is indeed true that a picture of a brown apple (for example) is considered more typical of the conjunctive concept "brown apple" that it is of the simple concept "apple". The almost trivial nature of this demonstration highlights the failing of the fuzzy logic approach to cope with predictions of typicality in complex concepts.

Johnson-Laird (1983, p.199) says:

> All the obvious ways of specifying rules for *and* and *not* (in fuzzy logic) do violence to our intuitions about natural language. These arguments count decisively against the adoption of a fuzzy-set semantics for natural language. (See Osherson and Smith, 1981, for other, though related, shortcomings of fuzzy logic).

We claim that just as the artificial intelligence community was mistaken in accepting Minsky and Papert's intuition that perceptrons were a dead-end, so too is the psychological community mistaken to reject the power of fuzzy set theory as a representational tool. We feel that O&S's 1981 article has gone unanswered in specific terms too long and that the negative attitude it has generated in psychology towards fuzzy set theory has impeded possible progress in representing concepts.

In this paper, we address the supposed inadequacies and shortcomings of fuzzy set theory and show that in fact, it does have sufficient power and flexibility to capture O&S's or anyone else's 'strong intuitions' about concepts. In fact, fuzzy set theory has an expressive power that is far beyond that of classical set theory. For instance, there are no unique counterparts to the classical set-theoretic operations of complement, intersection and union in fuzzy set theory are not unique. There are, in fact, an infinite set of each of these operations from which to choose. Moreover, fuzzy set theory has aggregation operations and modifying operations that have no counterparts in classical set theory. Given this richness of representation, fuzzy sets can capture any relationships among natural categories revealed by empirical data or required by intuition.

Some may feel that this multitude of possible operations makes fuzzy set theory too flexible: it can fit any kind of data. However, this is precisely what we want in powerful representation formalism. There seems to be an unspoken expectation in O&S (and others) that somehow fuzzy set theory will automatically give the right answers to such difficult questions as conceptual combination. They state in another paper (Smith and Osherson, 1984, p. 339):

> Fuzzy-set theory (e.g., Zadeh, 1965) is of interest to cognitive scientists because it offers a calculus for combining prototype concepts.

There may be a hope that if fuzzy logic based upon fuzzy set theory truly represents human reasoning, that somehow the underlying logic of concepts might be revealed through its mechanisms. In this way, all we would need to do is to tap into the right logic and the problems of conceptual combination will be solved – the correct answers will fall out from fuzzy set theory automatically. And if this fails to occur, then fuzzy logic is just not the correct model for conceptual structure. However, we argue that it is a mistake to require that fuzzy set theory provide an underlying model of concepts and reasoning. Fuzzy set theory is a formal system, albeit one that has a great deal of expressive power, but how to use this expressive power must ultimately depend upon empirical data – it will not in itself answer the hard questions.

In Section 2, we introduce those notions of fuzzy set theory that are relevant to our discussion in this paper. We describe six different types of operations on fuzzy sets and show how they work. In Section 3, we respond in detail to each of O&S's arguments against the utility of fuzzy set theory for representing concepts. We show in each case that the conclusions stated by O&S are not warranted. Our overall conclusions are then presented in Section 4.

Section 2. Fuzzy Set Theory: Relevant Notions and Notation

Let X denote a conceptual domain in the sense used by O&S. To denote fuzzy sets defined on X, we adopt a notation that is currently predominant in the literature on fuzzy set theory. According to this notation, symbols of fuzzy sets, A, B, …, are not distinguished from symbols of their membership functions[2]. Since each fuzzy set is uniquely defined by one particular membership function, no ambiguity results from this double use of the same symbol. Following O&S, we consider only standard fuzzy sets, in which degrees of membership are expressed by real numbers in the unit interval [0, 1].

Contrary to the symbolic role of numbers 1 and 0 in characteristic functions of classical (crisp) sets, numbers assigned to objects in X by membership functions of fuzzy sets have clearly a numerical significance. This significance, which is preserved when crisp sets are viewed (from the standpoint of fuzzy set theory) as special fuzzy sets, allows us to manipulate fuzzy sets in numerous ways, some of which have no counterparts in classical set theory.

Since most of the arguments made by O&S in their paper involve operations on fuzzy sets, we need to introduce all relevant operations on fuzzy sets to facilitate our discussion. For our purpose in this paper, we present only a digest of the meaning and

[2] Observe that O&S use the term "characteristic function" (adopted from classical set theory) rather than the commonly used term "membership function."

basic properties of the various operations. The notation is adopted from (Klir and Yuan, 1995), where further details (including relevant proofs) are covered.

Each of the following six types of operations on fuzzy sets are relevant to our discussion:

- modifiers
- complements
- intersections
- unions
- implications
- averaging operations

Modifiers and complements are unary operations; intersections and unions are defined as binary operations, but their application can be extended to any number of fuzzy sets via their property of associativity; averaging operations, which are not associative, are defined, in general, as n-ary operations ($n \geq 2$).

All of these operations on fuzzy sets are induced by corresponding operations on [0,1] in the following way: An n-ary operation s on [0,1] extends to an n-ary operation on fuzzy sets (for simplicity, this extension of s is denoted also by s; moreover, we use a common name for both of the corresponding operations, i.e. the operation on [0,1] and the the operation on fuzzy sets) assigning to fuzzy sets A_1, $A_2 \ldots, A_n$ a fuzzy set $s(A_1, A_2 \ldots, A_n)$ whose membership function is given by

$$[s(A_1, A_2 \ldots, A_n)](x) = s(A_1(x), A_2(x), \ldots, A_n(x)).$$

The operation on fuzzy sets defined this way is said to be based on the corresponding operation on [0,1].

The purpose of *modifiers* is to modify fuzzy sets to account for linguistic hedges. Each modifier, m, is based on a one-to-one (and usually continuous) function of the form

$$m: [0, 1] \rightarrow [0, 1],$$

which is *order preserving*. Sometimes, it is also required that $m(0) = 0$ and $m(1) = 1$.

The most common modifiers either increase or decrease all values of a given membership function. A convenient class of functions, m_λ, that qualify as increasing or decreasing modifiers is defined by the simple formula

$$m_\lambda(a) = a^\lambda, \qquad (1)$$

where $\lambda \in (0, \infty)$ is a parameter whose value determines which way and how strongly m_λ modifies a given membership function. For all $a \in [0, 1]$, $m_\lambda(a) > a$ when $\lambda \in (0, 1)$, $m_\lambda(a) < a$ when $\lambda \in (1, \infty)$, and $m_\lambda(a) = a$ when $\lambda = 1$. The farther the value of λ from 1, the stronger the modifier m_λ.

In addition to the most common class of modifiers defined by (1), numerous other classes of modifiers were proposed and studied (Kerre and De Cock, 1999), some of which involve more than one parameter. Several modifiers may also be composed to form a more descriptive modifier. Which modifier to choose is basically an experimental question. Given any data regarding the meaning of a linguistic hedge in a given context, we need to choose a class of modifiers that qualitatively conforms to the data and, then, to select the modifier from the class (characterized by specific values of the parameters involved) that best fits the data.

Complements of fuzzy sets are based on functions, c, of the same form as modifiers, but they are *order reversing* and such that $c(0) = 1$ and $c(1) = 0$. Moreover, they are usually required to possess for all $a \in [0, 1]$ the property

$$c(c(a)) = a,$$

which is referred to as *involution*.

A practical class of involutive complements, c_λ, is defined for each $a \in [0,1]$ by the formula

$$c_\lambda(a) = (1 - a^\lambda)^{1/\lambda}, \tag{2}$$

where $\lambda \in (0, \infty)$ is a parameter whose values specify individual complements in the class. When $\lambda = 1$, the complement is usually referred to as the *standard complement*. Various other classes of complements are known and can be used when desirable. Some of them involve more than one parameter. As with modifiers, the choice of a particular complement in a given context is an experimental issue.

Intersections and unions of fuzzy sets, denoted in this paper by i and u, respectively, are generalizations of the operations of intersection and union in classical set theory. They are based on functions from $[0, 1]^2$ to $[0, 1]$ that are commutative, associative, and monotone nondecreasing. The only property in which they differ is a boundary condition: for all $a \in [0, 1]$, $i(a, 1) = a$ while $u(a, 0) = a$.

Contrary to their classical counterparts, intersections and unions of fuzzy sets are not unique. However, they are bounded for all a, b ∈ [0, 1] by the inequalities

$$i_{min}(a, b) \leq i(a, b) \leq \min(a, b),$$

$$\max(a, b) \leq u(a, b) \leq u_{max}(a, b),$$

where

$$i_{min}(a,b) = \begin{cases} \min(a,b) \text{ when } \max(a,b) = 1, \\ 0 \text{ otherwise,} \end{cases} \quad (3)$$

$$u_{max}(a,b) = \begin{cases} \max(a,b) \text{ when } \min(a,b) = 0, \\ 1 \text{ otherwise.} \end{cases} \quad (4)$$

Operations min and max are usually called *standard operations*, while operations i_{min} and u_{max} are often referred to as *drastic operations*. Numerous classes of functions are now available that fully cover these ranges of intersections and unions. Examples are classes i_λ and u_λ that are defined for all a, b ∈ [0, 1] by the formulas

$$i_\lambda(a, b) = 1 - \min\{1, [(1 - a)^\lambda + (1 - b)^\lambda]^{1/\lambda}\}, \quad (5)$$

$$u_\lambda(a, b) = \min[1, (a^\lambda + b^\lambda)^{1/\lambda}], \quad (6)$$

where $\lambda \in [0, \infty)$ is a parameter whose values specify individual intersections or unions in these classes. It is easy to show that the drastic operations are obtained for $\lambda = 0$, while the standard operations are obtained in the limit for $\lambda \to \infty$.

Functions i and u, which represent possible intersections and unions of fuzzy sets, respectively, are usually referred to in the literature as *triangular norms* (or *t-norms)* and *triangular conorms* (or *t-conorms*), respectively (Klement, et al., 2000; Klir and Yuan, 1995).

Another important class of operations are so-called fuzzy implications. Fuzzy implications are binary operations $\rightarrow$ on [0, 1] that are usually required to satisfy $0 \rightarrow a = 1$, $1 \rightarrow a = a$, $a \leq a'$ implies $a' \rightarrow b \leq a \rightarrow b$, $b \geq b'$ implies $a \rightarrow b \leq a \rightarrow b'$, and possibly also other conditions (see Klir and Yuan, 1995; Hájek, 1998). Fuzzy implications model the semantics of the implication connective in fuzzy setting and generalize in an obvious way the classical case. The most used example is the Lukasiewicz application defined by $a \rightarrow b = \min(1, 1 - a + b)$.

The last type of operations on fuzzy sets are *averaging operations*. These operations have no counterparts in classical set theory. Since they are not associative, averaging operations must be defined as functions of n arguments for any $n \geq 2$. That is, they are based on functions h of the form

$$h: [0, 1]^n \rightarrow [0, 1],$$

which are monotone nondecreasing and symmetric in all arguments, continuous, and idempotent. It is significant that

$$\min(a_1, a_2, \cdots, a_n) \leq h(a_1, a_2, \cdots, a_n) \leq \max(a_1, a_2, \cdots, a_n)$$

for any n-tuple $(a_1, a_2, \cdots, a_n) \in [0, 1]^n$. This means that the averaging operations fill the gap between intersections and unions.

One class of averaging operations, h_λ, which covers the entire interval between min and max operations, is defined for each n-tuple $(a_1, a_2, \cdots, a_n)$ in $[0, 1]^n$ by the formula

$$h_\lambda(a_1, a_2, \cdots, a_n) = \left(\frac{a_1^\lambda + a_2^\lambda + \cdots + a_n^\lambda}{n} \right)^{1/\lambda}, \quad (7)$$

where λ is a parameter whose range is the set of all real numbers except 0. For $\lambda = 0$, function h_λ is defined by the limit

$$\lim_{\lambda \to 0} h_\lambda(a_1, a_2, \cdots, a_n) = (a_1, a_2, \cdots, a_n)^{1/n},$$

which is the well-known geometric mean. Moreover,

$$\lim_{\lambda \to -\infty} h_\lambda(a_1, a_2, \cdots, a_n) = \min(a_1, a_2, \cdots, a_n),$$

$$\lim_{\lambda \to \infty} h_\lambda(a_1, a_2, \cdots, a_n) = \max(a_1, a_2, \cdots, a_n).$$

Other classes of averaging operations are now available, some of which use weighting factors to express relative importance of the individual fuzzy sets involved. For example, the function

$$h(a_i, w_i \mid i = 1,2,..., n) = \sum_{i=1}^{n} w_i a_i , \quad (8)$$

where the weighting factors w_i take usually values in the unit interval [0, 1] and

$$\sum_{i=1}^{n} w_i = 1,$$

expresses for each choice of values w_i the corresponding weighted average of values a_i (i = 1, 2, …, n). Again, the choice is an experimental issue.

Section 3. Response to Osherson and Smith

In this section, we respond in specific ways to all arguments regarding the utility of fuzzy set theory in representing and combining concepts that are pursued by Osherson and Smith (1981). These arguments led O&S to the conclusion that "fuzzy-set theory will not handle strong intuitions about the way concepts combine to form complex concepts and propositions" (p.55).

Our response is based on the following position. We do not contest the soundness of "strong intuitions" (the term used by O&S), presumably supported by outcomes of relevant experiments, regarding the relation between simple and combined concepts, as described by O&S. Our only aim is to answer the question: is the apparatus of fuzzy set theory capable of representing the various strong intuitions and/or experimental evidence described by O&S. Moreover, our answers are restricted to the specific examples employed by O&S. In each of the following subsections, we deal with one of the examples. For convenience, we use the simplified notation introduced in Sec. 2.

3.1. Striped Apples

In this example, the domain, F, is the set of all fruit, and three relevant concepts are considered: *apples*, *striped*, and *striped apple*. In their discussion, O&S use a drawing of a normal apple with stripes on it (p.44) and refer to this striped apple as a. They argue that the membership degree of a in the set of striped apples, SA, should be, from the psychological point of view, greater than the membership degree of a in the set of apples, A. That is, they argue that

$$SA(a) > A(a). \qquad (9)$$

So far so good. However, prior to stating this reasonable requirement, they assume, without any justification, that SA is an intersection of A and the set of all striped fruits, S. Using this assumption, they then argue that the intersection of A and S violates (9). Although this argument is correct, the assumption is wanting, as also recognized by

Lakoff [1987, p.142]: "The assumption that noun modifiers work by conjunction is grossly incorrect."

The degree SA(a) may be viewed in this case as an average of degrees A(a) and S(a), e.g., the weighted average

$$SA(a) = w\,A(a) + (1 - w)\,S(a),$$

where $w \in (0, 1)$.Then, (9) is satisfied whenever $S(a) > A(a)$. Considering O&S's drawing of the striped apple, a, it is reasonable to assume that a is psychologically more prototypical of a striped fruit than of an apple and, hence, $S(a) > A(a)$.

Another possibility is to view the concept *striped* as a noun modifier in which the magnitude of modifying A(a) depends on the degree of S(a). This view can be expressed, for example, by the function

$$SA(a) = [A(a)]^{1-\delta S(a)},$$

where $\delta \in (0, 1]$ is a parameter whose values is determined experimentally.

3.2. Logically Empty and Logically Universal Concepts

In Sec. 2.3.2 (pp.45, 46) of their paper, O&S argue that the concept *apple that is not an apple* is logically empty and, similarly, that the concept *fruit that either is or is not an apple* is logically universal. As a consequence, the sets representing these concepts are, respectively, the empty set and the universal set. These properties are of course correct in terms of classical logic and classical set theory, where they are called the *law of contradiction* and the *law of excluded middle*, respectively. However, O&S require that these laws hold in fuzzy set theory as well. They correctly show, that they, in fact, do not hold when standard operations of complement, intersection, and union are employed.

To require that the laws of contraction and excluded middle hold in fuzzy set theory to correctly represent concepts in natural language needs justification, which O&S do not provide. The requirement has already been disputed in numerous ways; see, e.g., Fuhrmann (1988, pp.323, 324), Lakoff (1987, p.141), and Zadeh (1982, p.588). Instead of presenting here the various arguments against the requirement, we prefer to show that the conclusions drawn by O&S regarding the two laws are erroneous regardless of whether we accept or reject the requirement.

Since the counterparts of the three classical set-theoretic operations (complements, intersections, unions) are not unique in fuzzy sets theory (as described in Sec.2), we can choose as needed different combination of these operations from the delimited classes of functions. Whatever combination of these functions we choose, some properties of the classical operations (properties of the underlying Boolean algebra) are inevitably violated. This is a consequence of imprecise boundaries of fuzzy sets. However, different combinations violate different properties, and this is crucial for our argument. The standard operations, for example, violate only the law of contradiction and the law of excluded middle, as correctly demonstrated by O&S. Some other combinations preserve these laws, but violate some other properties, usually distributivity and idempotence. For example, when the standard complement is combined with i_{min} and u_{max}, the laws of contradiction and excluded middle are satisfied, but they are also satisfied when the intersection and union operations are defined for all a, b $\in$ [0, 1] by the formulas

$$i(a, b) = \max(0, a + b - 1),$$

$$u(a, b) = \min(1, a + b),$$

respectively. Furthermore, a procedure is well established (Klir and Yuan, 1995, Sec.3.5) by which classes of operations can be constructed that satisfy the two laws. Consider, for example, the classes of complements, c_λ, and unions, u_λ, defined for each a, b $\in$ [0, 1] by the formulas

$$c_\lambda(a) = \frac{1-a}{1+\lambda a},$$

$$u_\lambda(a, b) = \min(1, a + b + \lambda ab),$$

where $\lambda \in (-1, \infty)$. Then,

$$u_\lambda(a, c_\lambda(a)) = 1$$

for all a $\in$ [0, 1] and each $\lambda \in (-1, \infty)$. This means that the law of excluded middle is satisfied for any pair (u_λ, c_λ) of operations from these classes.

It is clear that fuzzy set theory can represent cognitive situations in which the laws of contradiction and/or excluded middle should hold according to experimental evidence. Again, O&S are wrong in their conclusions.

3.3. Liquidity, Investment, and Wealth

In Sec. 2.4.3 (pp.46-48) of their paper, O&S discuss an example involving the concept *financial wealth* and its connection to concepts *liquidity* and *investment*. They consider three persons, A, B, C, whose assets are given in the following table:

Person	Liquidity	Investment
A	\$105,000	\$5,000
B	\$100,000	\$100,000
C	\$5,000	\$105,000

They describe the concepts *liquidity*, *investment*, and *wealth*, respectively, by fuzzy sets L, I, and W (our notation), and argue correctly that the following inequalities should be satisfied on intuitive grounds:

$$L(A) > L(B)$$
$$I(C) > I(B)$$
$$W(B) > W(A)$$
$$W(B) > W(C)$$

The basic issue now is how W(x) is determined for any $x \in X$ in terms of L(x) and I(x). That is, we want to find a function f such that

$$W(x) = f[L(x), I(x)]$$

is for any $x \in X$ sensible on intuitive and/or experimental grounds. O&S argue that "if fuzzy-set theory is to represent the conceptual connection among liquidity, investment, and wealth, it would seem that the only option is to employ fuzzy union," and what they mean is one particular fuzzy union, the one expressed by the max operation. This argument is clearly unwarranted, since f can be chosen from an infinite set (a continuum) of functions, as described in Sec. 2. Moreover, it is easy to see that the right function in this example should be an averaging function, which can be chosen, for example, from the class of functions characterized by (7). If there is no special experimental evidence to do otherwise, we may as well choose the arithmetic average (e.g., the function for $\lambda = 1$ in (7)), so that

$$W(x) = \frac{L(x) + I(x)}{2}$$

for all $x \in X$. Clearly, this function satisfies the required inequalities.

This example is particularly illuminating. It shows that fuzzy set theory has some capabilities that have no counterparts in classical set theory. Indeed, classical sets cannot be averaged!

3.4 Truth Conditions of Thoughts.

The last example discussed by Q & S concerns the so-called "truth conditions of thoughts". What Q&S mean by this term is basically the issue of determining the truth degree of a proposition of the form

All A's are B's. (1)

The authors claim that propositions of this form are normally assigned the truth condition

$$(\forall x \in D)(c_A(x) \leq c_B(x)), \quad (2)$$

where D is a universe of discourse and c_A and c_B are membership functions of fuzzy sets representing the terms A and B in (1), respectively. Then, they present a "counterintuitive result": Let D denote the universe of all animals and let A and B denote, respectively, the concepts *grizzly bear* and *inhabitant of North America*. That is, (1) becomes "All grizzly bears are inhabitants of North America". Then, if there is a squirrel (call it Sam) who lives on Mars and if c_A (Sam) = a > 0 and c_B (Sam) < a, it follows that c_A (Sam) $\leq c_B$ (Sam) is not the case, and thus the existence of a squirrel on Mars makes the truth value of "All grizzly bears are inhabitants of North America" 0 (false). The authors conclude that "...fuzzy set theory does not render prototype theory compatible with the truth conditions of inclusion."

The mistake O&S made is the failure to recognize that (2) is not the definition of the truth degree of (1). In fact, (2) defines a crisp (bivalent) relation S_{biv}, between fuzzy sets in D. S_{biv}, is verbally described as follows: "fuzzy sets c_A and c_B are in the relation S_{biv}, if and only if for all x from D it is true that the degree to which x belongs to c_A is less or equal to the degree to which x belongs to c_B." Then, the existence of Sam as described above makes the fact that S_{biv} holds between c_A and c_B false, which is completely in accord with the meaning of S_{biv}. To interpret (1) in the right way, let us rewrite (1) to:

For all x of the universe if x belongs to A then x belongs to B.

Using the basic principles of interpreting logical formulas in fuzzy logic, one obtains that the truth degree $S(c_A,c_B)$ of (1) is

$$S(c_A,c_B) = \inf_{x\in D}(c_A(x) \to c_B(x))$$

where $\to$ is a fuzzy implication, i.e. a binary operation in [0, 1] that corresponds to implication connective, and inf denotes the infimum (minimum if D is finite) in [0, 1].

As an illustration, if Lukasiewicz implication,

$$a \to b = \min(l, 1 - a + b),$$

is used, then one gets

$$S(c_A,c_B) = \inf_{x\in D} \min(1, 1 - c_A(x) + c_B(x).$$

Therefore, Sam's existence implies that the truth degree of "All grizzly bears are inhabitants of North America" is at most min(l, 1 − c_A (Sam) + c_B (Sam). If, for instance, c_A (Sam) = 0.1 and c_B (Sam) = 0.05 then the truth degree of (1) is at most 0.95 (the existence of other animals that count to a certain degree for grizzly bears and live outside North America can make $S(c_A,c_B)$ still smaller).

The meaning of (1), when interpreted correctly, therefore satisfies the basic intuitive requirements of O&S (note that the correct definition of the "subsethood degree" $S(c_A,c_B)$ can be found in Goguen's (1968-9), a paper to which Q & S refer).

We now briefly comment on further intuitive requirements and criticism formulated by O&S that are connected to propositions of the form (1) and to "truth conditions of thoughts." They give three reasons for their "distrust of a fuzzy-truth remedy to the problems cited with fuzzy inclusion." First, they claim that a calculus that allows one to assign partial truth degrees to (1) requires a non-trivial theory not yet undertaken. We have demonstrated that the right interpretation of (1) that was already available in 1968 (in a paper referred by Q&S) leads to results that are in accord with the intuition demonstrated by O&S's example. If one feels that the more squirrels on Mars, the smaller the truth degree of (1) should be the case, which is an issue mentioned by O&S, one actually has in mind a quantifier like "many", not "all", i.e. instead of (1) one deals with a proposition of the form "Most A's are B's." Quantifiers like "many", "most" etc. are traditionally studied by logic under the name generalized quantifiers. A good an up-to-date overview can be found in (Kryncki, et al, 1995); in the context of fuzzy logic, generalized quantifiers are studied in (Hájek, 1998, Chapter 8).

Second, O&S express a general distrust of a calculus that allows propositions intermediate truth degrees. They claim, without any serious justification, that "infinite valued logics violate strong intuitions about truth, validity, and consistency." Their only example is the following: O&S claim that in Lukasiewicz infinite valued logic, the sentence "If John is happy, and if John is happy only if business is good, then business is good" is not a tautology (i.e. has not always the truth degree 1), contrary to intuition.

However, this is not true: if p denotes "John is happy" and q denotes "business is good" then the above sentence corresponds to a propositional formula $\varphi = (p \,\&\, (p \Rightarrow q)) \Rightarrow q$. If $\nu(p)$ and $\nu(p)$ are truth degrees of p and q, then the truth degree assigned to φ in Lukasiewicz calculus is

$$\min(1, 1 - \max(0, \nu(p) + \min(1, 1 - \nu(p) + \nu(q)) - 1) + \nu(q))$$

which is always 1, i.e. φ is a tautology. Objections to graded approach to truth and to fuzzy logic can be confronted with literature; (Gottwald,2001; Hájek, 1998; Novák et al,1999) are examples of recent monographs where fuzzy logics are well-covered).

Third, Q&S "suspect" that the partial falsification proposal results from mistaking degrees of belief for degrees of truth." We strongly object to the claim that degrees of truth are in fact degrees of belief in disguise. Calculi for dealing with degrees of truth (fuzzy logic) and degrees of belief (Dempster-Shafer theory of evidence and related theories) are nowadays well developed (for Dempster-Shafer theory see e.g. (Shafer, 1976), and both epistemological and formal differences between truth and belief are well-recognized.

4. Conclusions

Our main points in this paper are that: (i) fuzzy set theory is a formalized language of much greater expressive power than classical set theory; (ii) to utilize this expressive power for representing and combining concepts, we need to construct relevant membership functions and operations by which they are combined or modified in the context of each particular application; and (iii) the construction may be guided by our strong intuitions (as understood by Q&S), but it must ultimately be based on experimental evidence.

The focus of this paper is intentionally very narrow: a demonstration that each of the examples of combining or modifying concepts that are discussed by O&S can adequately be handled by fuzzy set theory to satisfy the strong intuitions or experimental evidence. Although some criticism of Q&S's conclusions was expressed by, for example, Fuhrmann (1988, 1991), Lakoff (1987), and Zadeh (1982), this criticism is not sufficiently specific. We felt that a specific response to each of the specific conclusions made by O&S is long overdue.

To compensate for the intentional specificity of this article, we intend to discuss some broader issues of the role of fuzzy set theory in cognitive science in two companion papers. In one of them, we plan to present a comprehensive overview of current resources offered by fuzzy set theory to cognitive science. The other article will be more oriented to discussing the various conceptual, philosophical, and methodological issues regarding the use of fuzzy set theory for representing and manipulation concepts.

In writing these three articles, our intent is solely constructive. We want to show that the frequent negative statements about fuzzy set theory in cognitive science literature, usually based on the O&S's paper, are ill founded, and that shying away from fuzzy set theory is counterproductive to healthy development of the field. It is our hope that our articles will help to renew interest in fuzzy set theory by at least some cognitive scientists in the years ahead.

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