On Elkan's Theorems: Clarifying Their Meaning via Simple Proofs

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This article deals with the claims that "a standard version of fuzzy logic collapses mathematically to two-valued logic" made by Charles Elkan in two papers [Proc 11th National Conf on AI, Menlo Park, CA: AAAI Press, 1993, pp 698–703; IEEE Expert 1994;9:3–8]. Although Elkan's effort to trivialize fuzzy logic has been questioned by numerous authors, our aim is to examine in detail his formal arguments and make some new observations. We present alternative, considerably simpler proofs of Elkan's theorems and use these proofs to argue that Elkan's claims are unwarranted.

1. INTRODUCTION

In 1993, Charles Elkan presented a paper at the 11th National Conference on Artificial Intelligence, which was also published in the conference proceedings.[1] In the paper, Elkan attempted to show that "as a formal system, a standard version of fuzzy logic collapses mathematically to two-valued logic." For this purpose, he employed one theorem, to which we refer in this article as *Elkan's first theorem.* One year later, a debate regarding a revised version of Elkan's paper[2] was organized in *IEEE Expert* (August 1994), in which 15 invited responses to the paper (written by 22 authors) were published. In these responses, various misconceptions and other shortcomings of the paper were identified. Later, some of these shortcomings were examined more thoroughly in additional papers.[3–8] The theorem in Elkan's revised paper, to which we refer as *Elkan's second theorem,* is different from the theorem in his original paper. In both cases, proofs of the theorems are quite long. Because it is common to take the length of a proof as a measure of profundity of the proven theorem, Elkan's theorems may look on the surface to be quite profound. The purpose of this article is to demonstrate the contrary. This is

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INTERNATIONAL JOURNAL OF INTELLIGENT SYSTEMS, VOL. 22, 203–207 (2007)
 Published online in Wiley InterScience (www.interscience.wiley.com). • DOI 10.1002/int.20194

accomplished by (1) presenting short proofs of both theorems, and (2) by using these proofs to show that assumptions upon which the theorems are based define formal systems that are not capable of dealing with fuzziness.

The article is organized as follows. Elkan's theorems are presented in Section 2. This is followed in Section 3 by presenting simple proofs of both theorems. Based on the insight from these proofs, we then argue that the crucial assumptions employed in these theorems assert in each case that the formal system of concern is, in fact, a system of "nonfuzzy logic."

2. ELKAN'S THEOREMS

In his first theorem, Elkan considers a logical system with connectives of conjunction ($\wedge$), disjunction ($\vee$), and negation ($\neg$). A truth degree from $[0,1]$ is assigned to each formula A (Elkan uses "assertion" instead of "formula"); this truth degree is denoted by $t(A)$. The properties of the system that Elkan requires are summarized in the following definition[1]:

Definition 1. *Let A and B be arbitrary assertions. Then*

- $t(A \wedge B) = \min\{t(A), t(B)\}$
- $t(A \vee B) = \max\{t(A), t(B)\}$
- $t(\neg A) = 1 - t(A)$
- $t(A) = t(B)$ *if A and B are logically equivalent*

In addition to this, Elkan says,[1] "In the last case of this definition, let 'logically equivalent' mean equivalent according to the rules of classical two-valued propositional calculus." Then, Elkan asserts and proves for this system the following theorem.

Theorem 1 (Elkan[1]). *For any two formulas A and B, either* $t(B) = t(A)$ *or* $t(B) = 1 - t(A)$.

In his second theorem, Elkan considers again a system with properties given in Definition 1, but he explains the meaning of the term "logically equivalent" in Definition 1 very differently. He says,[2] "Depending on how the phrase 'logically equivalent' is understood, Definition 1 yields different formal systems. A fuzzy logic system is intended to allow an indefinite variety of numerical truth values. However, for many notions of logical equivalence, only two different truth values are possible given the postulates of Definition 1." Then, Elkan asserts and proves the following theorem.

Theorem 2 (Elkan[2]). *Given the formal system of Definition 1, if* $\neg(A \wedge \neg B)$ *and* $B \vee (\neg A \wedge \neg B)$ *are logically equivalent, then for any two formulas A and B, either* $t(B) = t(A)$ *or* $t(B) = 1 - t(A)$.

3. SIMPLE PROOFS OF ELKAN'S THEOREMS AND ASSOCIATED REMARKS

In this section, we present simple proofs of both Elkan's theorems and make remarks regarding the meaning of assumptions upon which the theorems are based.

Simple proof of Theorem 1. Observe that $A \wedge \neg A$ is logically equivalent to $B \wedge \neg B$ (in classical two-valued calculus). By direct application of the assumptions, $\min(t(A), 1 - t(A)) = t(A \wedge \neg A) = t(B \wedge \neg B) = \min(t(B), 1 - t(B))$ for A and B. Because $\min(t(\theta), 1 - t(\theta))$ equals $t(\theta)$ or $1 - t(\theta)$, for any θ, we conclude $t(B) = t(A)$ or $t(B) = 1 - t(A)$. ■

In the following remark we argue that the last assumption of Definition 1 is unnatural for fuzzy logic and, in fact, makes Elkan's logical system a "nonfuzzy logic."

Remark 1 (on assumptions of Theorem 1). Note first that assumption $t(A \vee B) = \max\{t(A), t(B)\}$ of Definition 1 was not used in our proof, and, therefore, it can be omitted from the first version of Elkan's theorem. As to the rest of the assumptions, both $t(A \wedge B) = \min\{t(A), t(B)\}$ and $t(\neg A) = 1 - t(A)$ are quite reasonable and, in fact, are often used in applications of fuzzy logic. Let us now concentrate on the last assumption, that is, on

$$t(A) = t(B) \text{ if } A \text{ and } B \text{ are logically equivalent}$$

In our proof, we used a particular instance of this assumption. Namely, we took $A \wedge \neg A$ and $B \wedge \neg B$ as a pair of logically equivalent formulas in our proof. It is interesting to use these formulas to show that the last assumption makes Elkan's system defined by Definition 1, in a sense, a system of "nonfuzzy logic."

We argue as follows: $t(\theta \wedge \neg\theta)$ gives us in fuzzy logic nontrivial and useful information about formula θ. To see this, assume that θ denotes the assertion "x is a red apple." Observe that $t(\theta \wedge \neg\theta)$ ranges between 0 and 0.5. If $t(\theta \wedge \neg\theta) = 0$, then we have $t(\theta) = 0$ or $t(\neg\theta) = 0$, that is, $t(\theta) = 1$. That is, if $t(\theta \wedge \neg\theta) = 0$, then either θ is completely false or θ is completely true. On the other hand, if $t(\theta \wedge \neg\theta) = 0.5$, then we have $t(\theta) = 0.5$, that is, θ is completely a borderline proposition. In general, the closer $t(\theta \wedge \neg\theta)$ is to 0.5, the more borderline a case of being a red apple θ describes; the closer $t(\theta \wedge \neg\theta)$ is to 0, the more clear-cut a case θ describes. That is, $t(\theta \wedge \neg\theta)$ can be taken as a measure to which θ describes a borderline case. Now, the assumption that $t(A \wedge \neg A) = t(B \wedge \neg B)$ for every pair of propositions A and B says that the degree to which A describes a borderline case is the same as the degree to which B describes a borderline case. Needless to say, such an assumption is absurd *because* the aim of fuzzy logic is just the opposite. Namely, the principal aim of fuzzy logic is to enable us to describe a whole spectrum of cases, ranging from completely borderline cases to completely clear-cut ones. Using an analogy with probability theory, the last assumption is analogous to assuming that a probability assignment is restricted to 0 and 1 only. This would be saying right in the beginning that a probability calculus is, in fact, not capable of dealing with randomness. In the same way, to accept the last assumption of Definition 1 is to say right in the beginning that our fuzzy logic is, in fact, not capable of dealing with fuzziness.

Simple proof of Theorem 2. Denote $a = t(A)$ and $b = t(B)$. Then we have

$$\max(b, 1 - a) \leq \max(b, 1 - b) \tag{1}$$

Indeed, we have $\max(b,1 - a) = 1 - \min(a,1 - b) = t(\neg(A \wedge \neg B)) = t(B \vee (\neg A \wedge \neg B)) = \max(b, \min(1 - a, 1 - b)) = \min(\max(b, 1 - a), \max(b, 1 - b))$, which is clearly equivalent to Inequality (1) because $x \leq y$ iff $x = \min(x, y)$. Now, Inequality (1) is equivalent to $1 - a \leq \max(b, 1 - b)$. Without loss of generality, we can assume that $a \leq 1 - a$ (otherwise take $\neg A$ instead of A). Then we have $\max(a, 1 - a) = 1 - a \leq \max(b, 1 - b)$ and, by symmetry, also $\max(b, 1 - b) \leq \max(a, 1 - a)$. Then $b = a$ or $b = 1 - a$ immediately follows. ■

As in the case of Theorem 1, we argue in the following remark that the crucial assumption of Theorem 2 is unnatural from the point of view of fuzzy logic.

Remark 2 (on assumptions of Theorem 2). Unlike Theorem 1, Theorem 2 uses a particular scheme of equivalent formulas as one of its assumptions. Nevertheless, it can be shown in a manner similar to that in Remark 1 that this assumption is unnatural and makes Elkan's system a "nonfuzzy logic." Namely, following our proof of Theorem 2, we arrived at $\max(a, 1 - a) = \max(b, 1 - b)$ for $a = t(A)$ and $b = t(B)$. Because $\max(a, 1 - a) = \max(b, 1 - b)$ is equivalent to $\min(a, 1 - a) = \min(b, 1 - b)$, the assumption of equivalency of $\neg(A \wedge \neg B)$ and $B \vee (\neg A \wedge \neg B)$ yields $t(A \wedge \neg A) = t(B \wedge \neg B)$. Now, the argument in Remark 1 showing that the logical system in Elkan's first theorem is, in fact, a system of "nonfuzzy logic" applies to the second Elkan's theorem as well.

4. CONCLUSIONS

The motivation for introducing and developing fuzzy logic has been its capability to capture borderline cases, in which propositions are not required to be true or false, but are allowed to have intermediate truth degrees. Although both Elkan's theorems purportedly deal with a system of fuzzy logic (Definition 1), they are, in fact, based on assumptions that exclude all borderline cases. This fact was already recognized by some authors who responded to Elkan's papers. The intent of the simple proofs of Elkan's theorems and their analysis presented here is to make this fact more transparent.

Acknowledgments

R. Bělohlávek acknowledges support by grant MSM 6198959214.

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