Journal of Computer and System Sciences 82 (2016) 357–365

Contents lists available at ScienceDirect

Journal of Computer and System Sciences

www.elsevier.com/locate/jcss

Bases of closure systems over residuated lattices ☆

Radim Belohlavek, Jan Konecny *

Dept. Computer Science, Palacky University Olomouc 17, listopadu 12, 771 49 Olomouc, Czech Republic

ARTICLE INFO

Article history:
Received 15 January 2015
Received in revised form 4 June 2015
Accepted 26 July 2015
Available online 7 August 2015

Keywords:
Closure operator
Base
Residuated lattice
Fuzzy logic

ABSTRACT

We present results on bases of closure systems over residuated lattices, which appear in applications of fuzzy logic. Unlike the Boolean case, the situation is not straightforward as there are two non-commuting generating operations involved. We present a decomposition theorem for a general closure operator and utilize it for computing generators and bases of the closure system. We show that bases are not unique and may in general have different sizes, and obtain a constructive description of the size of a largest base. We prove that if the underlying residuated lattice is a chain, all bases have the same size.

1. Introduction

Closure systems play a crucial role in various parts of mathematics and computer science, including algebra, logic, programming, databases, data analysis and management of data in general. It is well known that every closure system $\mathcal{S}$ in a set U, i.e. a system $\mathcal{S} \subseteq 2^U$ closed with respect to arbitrary intersections, has a unique base. This base, i.e. an inclusion-minimal subsystem of $\mathcal{S}$ that generates $\mathcal{S}$, consists of all the $\bigcap$-irreducible sets in $\mathcal{S}$—sets that cannot be obtained as intersections of other sets in $\mathcal{S}$. More generally, one may ask for a base of the closure system $[\mathcal{T}]$ generated by a given $\mathcal{T} \subseteq 2^U$. Again, such a base is unique and consists of the $\bigcap$-irreducible members of $\mathcal{T}$. Bases of closure systems, or dually, interior systems, are frequently encountered. In formal concept analysis, for instance, determining for a given input binary relation I between objects and attributes a minimal relation J such that the concept lattice of J, i.e. the lattice of all fixpoints of the Galois connection induced by J, is isomorphic to the concept lattice of I, is easily rephrased as the problem of determining bases of closure systems [6,8]. Another example, dual in that closure systems are replaced by interior systems, is the row/column-base of a Boolean matrix, which is just the base of the interior system generated by sets whose characteristic vectors are just the matrix rows/columns [14].

In this paper, we study bases of **L**-closure systems [2], which naturally appear when instead of bivalent (0–1, yes-or-no) data, the problems at hand involve data with grades from a partially ordered scale L (see e.g. [3–5,9,11,13]). In such a setting, sets or—more precisely—characteristic functions $A : U \to \{0, 1\}$ of sets are replaced by their generalizations $A : U \to L$, which are called L-sets, and the two-element Boolean algebra on $\{0, 1\}$ underlying the set calculus is replaced by its appropriate generalization **L**. In accordance with modern fuzzy logic [7,11–13], we take for **L** an arbitrary residuated lattice, leaving Boolean algebras, Heyting algebras, MV-algebras, BL-algebras, and other structures as special cases. In particular, if $L = \{0, 1\}$ then **L** becomes the two-element Boolean algebra and **L**-closure systems may be identified with ordinary closure systems. General **L**-closure systems represent a non-trivial generalization of ordinary ones. In addition to the fact that

☆ Preliminary results were presented at the NAFIPS 2013 conference.

* Corresponding author.
E-mail addresses: radim.belohlavek@acm.org (R. Belohlavek), jan.konecny@upol.cz (J. Konecny).

http://dx.doi.org/10.1016/j.jcss.2015.07.003

L involves intermediary values, i.e. those between 0 and 1, a substantial difference consists in the fact that while the only generating operation in ordinary closure systems is intersection, in **L**-closure systems there are two such operations: (infimum-based) intersection of L-sets and (residuum-based) multiplication of L-sets by constants in L, which is degenerate in the bivalent case $L = \{0, 1\}$. The properties of these operations, in particular the fact that the induced operators do not commute (multiplications of intersections is not the same as intersections of multiplications), make the problem of bases of **L**-closure systems non-trivial. For example, as we show in this paper, bases are not unique and in general, may have different sizes.

2. Preliminaries: residuated lattices, L-sets, L-closure systems

Recall that a (*complete*) *residuated lattice* [3,7,13,16] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist (the lattice order is denoted by $\leq$; 0 and 1 denote the least and greatest element, respectively);
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;
(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

Throughout the paper, **L** denotes an arbitrary complete residuated lattice. Common examples of complete residuated lattices include those defined on the real unit interval or on a finite chain $L = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$. For instance, for $L = [0, 1]$, we can use any left-continuous t-norm for $\otimes$, such as minimum, product, or Łukasewicz, and the corresponding residuum $\rightarrow$. Residuated lattices are commonly used in fuzzy logic [3,11,13], where the grades $a \in L$ are interpreted as degrees of truth and the operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of the (truth function of) conjunction and implication, respectively. The only residuated lattice with $L = \{0, 1\}$ coincides with the two-element Boolean algebra of classical logic in which case $\otimes$ and $\rightarrow$ are the truth functions of classical conjunction and implication. For more information on residuated lattices we refer to [3,7,13,16].

Given **L**, one may consider L-sets, i.e. generalizations of (characteristic functions of) ordinary sets, and their algebra. An L-*set* (or L-fuzzy set) in a universe set U is a mapping $A : U \rightarrow L$ assigning to every $u \in U$ an element $A(u) \in L$ interpreted as the truth degree to which u belongs to A. The set of all L-sets in U, denoted by L^U, is equipped with operations extending the classical set operations. For instance, the $\wedge$-based intersection is defined by $(A \wedge B)(u) = A(u) \wedge B(u)$. For $A, B \in L^U$, the degree of inclusion of A in B is defined by $S(A, B) = \bigwedge_{u \in U}(A(u) \rightarrow B(u))$. If $S(A, B) = 1$, which is equivalent to $A(u) \leq B(u)$ for each $u \in U$, one writes $A \subseteq B$ and says that A is included in B. For more details on L-sets, we refer to [3,11,13].

A system $\mathcal{S} \subseteq L^U$ is called an **L**-*closure system* in U [2] if

- $\mathcal{S}$ is closed under *left* $\rightarrow$*-multiplications*, i.e. $a \rightarrow A \in \mathcal{S}$ for each $a \in L$ and $A \in \mathcal{S}$;
- $\mathcal{S}$ is closed under $\bigwedge$*-intersections*, i.e. if $A_j \in \mathcal{S}$ ($j \in J$) then $\bigwedge_{j \in J} A_j \in \mathcal{S}$.

Here, $a \rightarrow A$ and $\bigwedge_{j \in J} A_j$ are defined by

$$(a \rightarrow A)(u) = a \rightarrow A(u) \quad \text{and} \quad (\textstyle\bigwedge_{j \in J} A_j)(u) = \bigwedge_{j \in J} A_j(u)$$

for any $u \in U$.

Remark 1. Since $0 \rightarrow a = 1$ and $1 \rightarrow a = a$ for any $a \in L$, we have $0 \rightarrow A = U$ and $1 \rightarrow A = A$ for each $A \in L^U$. Moreover, since U is the intersection of the empty system of L-sets, it belongs to any **L**-closure system. Therefore, if **L** is the two-element Boolean algebra, **L**-closure systems are just systems closed under $\bigwedge$-intersections and may be identified with ordinary closure systems. As a result, the notion of an **L**-closure system generalizes the ordinary notion of a closure system.

Let us also note that, analogously to the ordinary case, **L**-closure systems in U are in one-to-one correspondence with **L**-closure operators in U, i.e. mappings $C : L^U \rightarrow L^U$ satisfying

$$A \subseteq C(A), \quad S(A, B) \leq S(C(A), C(B)), \quad \text{and} \quad C(A) = C(C(A)),$$

for every $A, B \in L^U$. Namely, **L**-closure systems are just the sets of fixed points of **L**-closure operators [2].

3. Results

3.1. Bases of L-closure systems

Obviously, the set L^U is an **L**-closure system in U (the largest one) and one can easily see that an intersection of an arbitrary system of **L**-closure systems is an **L**-closure system. It hence follows from classical results [6] that for every system

$\mathcal{S} \subseteq L^U$ of L-sets there exists the least **L**-closure system $[\mathcal{S}]$ in U containing $\mathcal{S}$, namely the intersection of all **L**-closure systems in U that contain $\mathcal{S}$. Note also that the mapping sending $\mathcal{S}$ to $[\mathcal{S}]$ is an ordinary closure operator in L^U, i.e. it satisfies $\mathcal{S} \subseteq [\mathcal{S}]$; if $\mathcal{S} \subseteq \mathcal{T}$ then $[\mathcal{S}] \subseteq [\mathcal{T}]$; and $[[\mathcal{S}]] = [\mathcal{S}]$ for every $\mathcal{S}, \mathcal{T} \subseteq L^U$. These observations enable us to define the notion of a base.

Definition 1. A *base* of a set $\mathcal{T} \subseteq L^U$ is a set $\mathcal{S} \subseteq L^U$ such that

(a) $[\mathcal{S}] = \mathcal{T}$,
(b) $[\mathcal{P}] \neq \mathcal{T}$ for every $\mathcal{P} \subset \mathcal{S}$.

According to (a), a base of $\mathcal{T}$ *generates* $\mathcal{T}$ and we call $\mathcal{S}$ a set of *generators* of $\mathcal{T}$; according to (b), a base is *non-redundant*. More generally, we may consider bases with respect to a closure operator C in L^U which is different from [ ]. In such a case, we shall use the term C-*base*, C-*generators*, and the like. In this view, a base according to Definition 1 is a [ ]-base.

In what follows, we concentrate on the problem of describing and finding bases of **L**-closure systems, particularly of an **L**-closure system $[\mathcal{S}]$ generated by a given finite set $\mathcal{S} \subseteq L^U$ of L-sets in U. We start by describing a useful decomposition of [ ] into two other, simpler closure operators in L^U.

3.2. Decomposition of [ ]

For $\mathcal{S} \subseteq L^U$, let

$$[\mathcal{S}]_{\bigwedge} = \{\bigwedge \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S}\} \quad \text{and} \quad [\mathcal{S}]_{\rightarrow} = \{a \rightarrow A \mid a \in L,\ A \in \mathcal{S}\}. \tag{1}$$

That is, $[\mathcal{S}]_{\bigwedge}$ is the system of all $\bigwedge$-intersections of L-sets in $\mathcal{S}$ and $[\mathcal{S}]_{\rightarrow}$ is the system of all left $\rightarrow$-multiplications of L-sets in $\mathcal{S}$.

Lemma 1. *$[\ ]_{\bigwedge}$ and $[\ ]_{\rightarrow}$ are closure operators. That is, for any $\mathcal{S}, \mathcal{S}_1, \mathcal{S}_2 \subseteq L^U$,*

$$\mathcal{S} \subseteq [\mathcal{S}]_{\bigwedge};\ \mathcal{S}_1 \subseteq \mathcal{S}_2 \ \textit{implies}\ [\mathcal{S}_1]_{\bigwedge} \subseteq [\mathcal{S}_2]_{\bigwedge};\ [\mathcal{S}]_{\bigwedge} = [[\mathcal{S}]_{\bigwedge}]_{\bigwedge},$$
$$\mathcal{S} \subseteq [\mathcal{S}]_{\rightarrow};\ \mathcal{S}_1 \subseteq \mathcal{S}_2 \ \textit{implies}\ [\mathcal{S}_1]_{\rightarrow} \subseteq [\mathcal{S}_2]_{\rightarrow};\ [\mathcal{S}]_{\rightarrow} = [[\mathcal{S}]_{\rightarrow}]_{\rightarrow}.$$

Proof. Easy by checking the definitions. $\mathcal{S} \subseteq [\mathcal{S}]_{\rightarrow}$ follows from the fact that $1 \rightarrow A = A$ for every $A \in L^U$; $[\mathcal{S}]_{\rightarrow} = [[\mathcal{S}]_{\rightarrow}]_{\rightarrow}$ follows from the fact that $a \rightarrow (b \rightarrow A) = (a \otimes b) \rightarrow A$ for every $a, b \in L$ and $A \in L^U$. $\square$

The next lemma makes precise the intuition that $[\mathcal{S}]$ is obtained by iteratively computing the left $\rightarrow$-multiplications and $\bigwedge$-intersections of L-sets in $\mathcal{S}$.

Lemma 2. *Define $[\mathcal{S}]^i$ as follows*

$$[\mathcal{S}]^0 := \mathcal{S},$$
$$[\mathcal{S}]^{i+1} := [[\mathcal{S}]^i]_{\rightarrow} \cup [[\mathcal{S}]^i]_{\bigwedge} \ \textit{for any } i \geq 0.$$

Then $[\mathcal{S}] = \bigcup_{i=0}^{\infty} [\mathcal{S}]^i$.

Proof. Since $[\mathcal{T}]_{\rightarrow} \subseteq [\mathcal{T}]$ and $[\mathcal{T}]_{\wedge} \subseteq [\mathcal{T}]$ for any $\mathcal{T} \subseteq L^U$, one obtains $[\mathcal{S}]^i \subseteq [\mathcal{S}]$ for any i, hence also $\bigcup_{i=0}^{\infty} [\mathcal{S}]^i \subseteq [\mathcal{S}]$. Therefore, it is sufficient to prove that $\bigcup_{i=0}^{\infty} [\mathcal{S}]^i$ is an **L**-closure system. This is easy to see because if $A \in \bigcup_{i=0}^{\infty} [\mathcal{S}]^i$, i.e. $A \in [\mathcal{S}]^j$ for some j, then $a \rightarrow A \in [[\mathcal{S}]^j]_{\rightarrow} \subseteq [\mathcal{S}]^{j+1} \subseteq \bigcup_{i=0}^{\infty} [\mathcal{S}]^i$, verifying closedness under left $\rightarrow$-multiplications. Closedness under $\bigwedge$-intersections is verified similarly. $\square$

The next theorem shows that the description from Lemma 2 can be simplified considerably. Namely, [ ] is the composition of the closure operators $[\ ]_{\rightarrow}$ and $[\ ]_{\bigwedge}$ defined above.

Theorem 3. *For any $\mathcal{S} \subseteq L^U$, we have*

(a) $[\mathcal{S}] = [[\mathcal{S}]_{\rightarrow}]_{\bigwedge}$,
(b) $[[\mathcal{S}]_{\bigwedge}]_{\rightarrow} \subseteq [[\mathcal{S}]_{\rightarrow}]_{\bigwedge}$.

Proof. (a) We first check that $[[\mathcal{S}]_{\to}]_{\bigwedge}$ is an **L**-closure system. Let $A \in [[\mathcal{S}]_{\to}]_{\bigwedge}$. Then A is of the form $A = \bigwedge_{j\in J} a_j \to A_j$ for some J, $a_j \in L$, and $A_j \in \mathcal{S}$. Since every complete residuated lattice satisfies $a \to \bigwedge_k b_k = \bigwedge_k(a \to b_k)$ and $a \to (b \to c) = (a \otimes b) \to c$, we have

$$a \to A = a \to \bigwedge_{j\in J}(a_j \to A_j) = \bigwedge_{j\in J}(a \to (a_j \to A_j)) = \bigwedge_{j\in J}((a \otimes a_j) \to A_j),$$

which shows that $a \to A \in [[\mathcal{S}]_{\to}]_{\bigwedge}$. Let now $A_j \in [[\mathcal{S}]_{\to}]_{\bigwedge}$ for $j \in J$. Each A_j is of the form $A_j = \bigwedge_{k_j\in K_j} a_{k_j} \to A_{k_j}$ for some $a_{k_j} \in L$ and $A_{k_j} \in \mathcal{S}$, and we have

$$\bigwedge_{j\in J} A_j = \bigwedge_{j\in J} \bigwedge_{k_j\in K_j} a_{k_j} \to A_{k_j},$$

which is clearly an element of $[[\mathcal{S}]_{\to}]_{\bigwedge}$. Next, due to Lemma 1, $[[\mathcal{S}]_{\to}]_{\bigwedge}$ contains $\mathcal{S}$. Now, if $\mathcal{S} \subseteq \mathcal{T}$ for some **L**-closure system $\mathcal{T}$, then $[\mathcal{S}]_{\to} \subseteq [\mathcal{T}]_{\to} = \mathcal{T}$ and thus $[[\mathcal{S}]_{\to}]_{\bigwedge} \subseteq [\mathcal{T}]_{\bigwedge} = \mathcal{T}$, due to Lemma 1 and the fact that $\mathcal{T}$ is an **L**-closure system. Hence, $[[\mathcal{S}]_{\to}]_{\bigwedge}$ is the least **L**-closure system that contains $\mathcal{S}$.

(b): Any $A \in [[\mathcal{S}]_{\bigwedge}]_{\to}$ is of the form $A = a \to \bigwedge_{j\in J} A_j$ for some $A_j \in \mathcal{S}$. Since $a \to \bigwedge_{j\in J} A_j = \bigwedge_{j\in J}(a \to A_j)$, A is also of the form of an $\bigwedge$-intersection of left $\to$-multiplications of L-sets in $\mathcal{S}$ and hence belongs to $[[\mathcal{S}]_{\to}]_{\bigwedge}$. $\square$

As the next example shows, the inclusion converse to Theorem 3 (b) does not hold.

Remark 2. In the following example as well as in the examples below, we use the following convention. $A = \langle 0.5, 0.5, 1\rangle$ means that A is an L-set in a three-element universe U with a fixed order of elements to which the first, second, and the third element belong to A degrees 0.5, 0.5, and 1, respectively.

Example 1. Consider $L = [0, 1]$ equipped with the Łukasiewicz operations, i.e. $a \otimes b = \max(0, a + b - 1)$ and $a \to b = \min(1, 1 - a + b)$. Let

$$\mathcal{S} = \{A_1 = \langle 0.5, 0.5, 1\rangle, A_2 = \langle 1, 0.5, 0.5\rangle\}.$$

Then $[\mathcal{S}]_{\bigwedge} = \mathcal{S} \cup \{\langle 0.5, 0.5, 0.5\rangle, \langle 1, 1, 1\rangle\}$. Consider $B_1 = 0.6 \to A_1 = \langle 0.9, 0.9, 1\rangle$ and $B_2 = 0.7 \to A_2 = \langle 1, 0.8, 0.8\rangle$. Then $B_1 \wedge B_2 = \langle 0.9, 0.8, 0.8\rangle \notin [[\mathcal{S}]_{\bigwedge}]_{\to}$. On the other hand, we have $B_1 \wedge B_2 \in [\mathcal{S}] = [[\mathcal{S}]_{\to}]_{\bigwedge}$.

3.3. $[\,]_{\bigwedge}$-*bases and* $[\,]_{\to}$-*bases*

In this section, we describe $[\,]_{\bigwedge}$- and $[\,]_{\to}$-*bases* (see Definition 1 and the following remarks) of certain systems of L-sets.

Let for $\mathcal{V} \subseteq L^U$,

$$\mathrm{irr}_{\bigwedge}(\mathcal{V}) = \{A \in \mathcal{V} \mid A \notin [\mathcal{V} - \{A\}]_{\bigwedge}\}.$$

$\mathrm{irr}_{\bigwedge}(\mathcal{V})$ are elements in $\mathcal{V}$ that are not $\bigwedge$-intersections of other elements in $\mathcal{V}$, the $\bigwedge$-irreducibles. The following theorem is a folklore in lattice theory. Note also that the theorem follows from results on bases in domain theory on irreducibility [1,15,10].

Theorem 4. *For any finite set* $\mathcal{S} \subseteq L^U$, $\mathrm{irr}_{\bigwedge}(\mathcal{S})$ *is a unique* $[\,]_{\bigwedge}$*-base of* $[\mathcal{S}]_{\bigwedge}$.

In looking for $[\,]_{\to}$-bases of $[\mathcal{S}]_{\to}$, the basic idea is to use the following rule:

remove B from $\mathcal{S}$ if $B = a \to B'$ for some other $B' \in \mathcal{S}$ and $a \in L$,

i.e. to remove B if it can be obtained as a left $\to$-multiplication of some other $B' \in \mathcal{S}$. The following lemma shows an important observation, namely that such procedure will not cycle. For $\mathcal{S} \subseteq L^U$, let $\lhd$ denote the binary relation in L^U defined by

$B_1 \lhd B_2$ if and only if $B_2 = a \to B_1$ for some $a \in L$.

Lemma 5. *(a)* $\lhd$ *is reflexive, antisymmetric, and transitive.*
(b) $B_1 \lhd B_2$ *implies* $B_1 \subseteq B_2$.

Proof. (a): Reflexivity: Since $1 \to a = a$ for any $a \in L$, we have $B = 1 \to B$. Antisymmetry: suppose $B_2 = a_1 \to B_1$ and $B_1 = a_2 \to B_2$. Since $\to$ is antitone in the first argument, $a_1 \le 1$ yields $B_1 = 1 \to B_1 \subseteq a_1 \to B_1 = B_2$. Similarly, we get $B_2 \subseteq B_1$. Hence, $B_1 = B_2$, proving antisymmetry. Transitivity: If $B_1 \lhd B_2$ and $B_2 \lhd B_3$, i.e. $B_2 = a_1 \to B_1$ and $B_3 = a_2 \to B_2$

for some $a_1, a_2 \in L$, then since $\alpha \to (\beta \to \gamma) = (\alpha \otimes \beta) \to \gamma$, we obtain $B_3 = a_2 \to B_2 = a_2 \to (a_1 \to B_1) = (a_2 \otimes a_1) \to B_1$, proving $B_1 \lhd B_3$.

(b): If $B_1 \lhd B_2$, then for some $a_1 \in L$ we have $B_2 = a_1 \to B_1$. Due to antitony of $\to$ in the first argument, we get $B_1 = 1 \to B_1 \subseteq a_1 \to B_1 = B_2$. $\square$

Let $\mathrm{Min}_\lhd(\mathcal{S})$ denote the set of all elements in $\mathcal{S}$ minimal with respect to $\lhd$, i.e.

$$\mathrm{Min}_\lhd(\mathcal{S}) = \{B \in \mathcal{S} \mid B' \lhd B \text{ implies } B' = B \text{ for any } B' \in \mathcal{S}\}.$$

Theorem 6. *For every finite $\mathcal{S}$, $\mathrm{Min}_\lhd(\mathcal{S})$ is a unique $[\ ]_\to$-base of $[\mathcal{S}]_\to$.*

Proof. First we show $[\mathrm{Min}_\lhd(\mathcal{S})]_\to = [\mathcal{S}]_\to$. We have $[\mathrm{Min}_\lhd(\mathcal{S})]_\to = \{a \to B \mid a \in L, B \in \mathrm{Min}_\lhd(\mathcal{S})\}$ by definition. Since $\mathrm{Min}_\lhd(\mathcal{S}) \subseteq \mathcal{S}$, we have $[\mathrm{Min}_\lhd(\mathcal{S})]_\to \subseteq [\mathcal{S}]_\to$. On the other hand, let $B \in [\mathcal{S}]_\to$, i.e. $B = a_1 \to B_1$ for some $B_1 \in \mathcal{S}$. Take any $B_2 \in \mathrm{Min}_\lhd(\mathcal{S})$ such that $B_2 \lhd B_1$ (such B_2 exists since $\mathcal{S}$ is finite). Then $B_1 = a_2 \to B_2$ for some $a_2 \in L$, hence $B = a_1 \to B_1 = a_1 \to (a_2 \to B_2) = (a_1 \otimes a_2) \to B_2$, proving $B \in [\mathrm{Min}_\lhd(\mathcal{S})]_\to$. To sum up, $[\mathrm{Min}_\lhd(\mathcal{S})]_\to = [\mathcal{S}]_\to$.

If $\mathrm{Min}_\lhd(\mathcal{S})$ was redundant, there would be $B \in \mathrm{Min}_\lhd(\mathcal{S})$ such that $B \in [\mathrm{Min}_\lhd(\mathcal{S}) - \{B\}]_\to$. That is $B = a \to B'$ for some $a \in L$ and $B' \neq B$ minimal w.r.t. $\lhd$, which is a contradiction to the minimality of B w.r.t. $\lhd$.

Let $\mathcal{T}$ be another $[\ ]_\to$-base. Then since $\mathrm{Min}_\lhd(\mathcal{S}) \subseteq [\mathcal{S}]_\to = [\mathcal{T}]_\to$, for each $B \in \mathrm{Min}_\lhd(\mathcal{S})$ there exist $a_1 \in L$ and $B_1 \in \mathcal{T}$ such that $B = a_1 \to B_1$, i.e. $B_1 \lhd B$. As $B_1 \in \mathcal{T} \subseteq [\mathcal{T}]_\to = [\mathcal{S}]_\to$, there exist $a_2 \in L$ and $B_2 \in \mathrm{Min}_\lhd(\mathcal{S})$ such that $B_1 = a_2 \to B_2$, i.e. $B_2 \lhd B_1$. Due to transitivity of $\lhd$, we have $B_2 \lhd B$. Since $B, B_2 \in \mathrm{Min}_\lhd(\mathcal{S})$ and since B is minimal in $\mathcal{S}$, we obtain $B = B_2$. Observe that we have $B_2 \subseteq B_1 \subseteq B$, which is a direct consequence of Lemma 5 (b) and $B_2 \lhd B_1 \lhd B$. Therefore, $B = B_1 \in \mathcal{T}$ showing $\mathrm{Min}_\lhd(\mathcal{S}) \subseteq \mathcal{T}$. Since $\mathrm{Min}_\lhd(\mathcal{S})$ is a $[\ ]_\to$-base, we must have $\mathrm{Min}_\lhd(\mathcal{S}) = \mathcal{T}$ because otherwise $\mathcal{T}$ is redundant. $\square$

In view of Theorem 6, we denote $\mathrm{Min}_\lhd(\mathcal{S})$ also by $\mathrm{irr}_\to(\mathcal{S})$.

Remark 3. For the infinite case, neither of Theorem 4 and 6 holds. Consider $L = [0, 1]$ equipped with the Łukasiewicz operations (see Example 1) and $\mathcal{S}$ consisting of all singletons $\langle a \rangle$ with $a \in L - \{0\}$. Then $\mathrm{irr}_\bigwedge(\mathcal{S}) = \emptyset$ since for any $a \in L$, $\langle a \rangle = \bigwedge\{\langle b \rangle \mid b \in L, b > a\}$. Furthermore, $\mathrm{irr}_\to(\mathcal{S}) = \emptyset$ since for any $\langle b \rangle \in \mathcal{S}$, $\frac{b}{2} \neq b$ and yet $\langle \frac{b}{2} \rangle \lhd \langle b \rangle$ because $\langle b \rangle = (1 - \frac{b}{2}) \to \langle \frac{b}{2} \rangle$.

3.4. Two simple algorithms for computing sets of generators

The $[\ ]_\bigwedge$- and $[\ ]_\to$-bases described by Theorem 4 and Theorem 6 are easy to compute. We now show that they may be used to obtain simple algorithms for computing set of generators of $[\mathcal{S}]$ for a given finite $\mathcal{S}$. We need the following lemma.

Lemma 7. *If $[\mathcal{S}_1]_\to = [\mathcal{S}_2]_\to$ and $[\mathcal{S}_2]_\bigwedge = [\mathcal{S}_3]_\bigwedge$ then $[\mathcal{S}_1] = [\mathcal{S}_3]$.*

Proof. According to Theorem 3 (a), $[\mathcal{S}_1] = [[\mathcal{S}_1]_\to]_\bigwedge$ and $[\mathcal{S}_3] = [[\mathcal{S}_3]_\to]_\bigwedge$. Furthermore, Theorem 3 (b) yields $[[\mathcal{S}_3]_\bigwedge]_\to \subseteq [[\mathcal{S}_3]_\to]_\bigwedge$. Putting together with the assumptions and the extensivity and idempotency of $[\ ]_\bigwedge$, we obtain

$$\begin{aligned}[\mathcal{S}_1] &= [[\mathcal{S}_1]_\to]_\bigwedge = [[\mathcal{S}_2]_\to]_\bigwedge \subseteq [[[\mathcal{S}_2]_\bigwedge]_\to]_\bigwedge \\ &= [[[\mathcal{S}_3]_\bigwedge]_\to]_\bigwedge \subseteq [[[\mathcal{S}_3]_\to]_\bigwedge]_\bigwedge = [[\mathcal{S}_3]_\to]_\bigwedge = [\mathcal{S}_3].\end{aligned}$$

Conversely, we have

$$\begin{aligned}[\mathcal{S}_3] &= [[\mathcal{S}_3]_\to]_\bigwedge \subseteq [[[\mathcal{S}_3]_\bigwedge]_\to]_\bigwedge = [[[\mathcal{S}_2]_\bigwedge]_\to]_\bigwedge \\ &\subseteq [[[\mathcal{S}_2]_\to]_\bigwedge]_\bigwedge = [[[\mathcal{S}_1]_\to]_\bigwedge]_\bigwedge = [[\mathcal{S}_1]_\to]_\bigwedge = [\mathcal{S}_1]. \qquad \square\end{aligned}$$

One may look at Lemma 7 as follows. If $\mathcal{S}_1$ is the input system of L-sets, we obtain (a smaller) $\mathcal{S}_2$ with $[\mathcal{S}_1]_\to = [\mathcal{S}_2]_\to$, and then obtain from $\mathcal{S}_2$ some smaller $\mathcal{S}_3$ with $[\mathcal{S}_2]_\bigwedge = [\mathcal{S}_3]_\bigwedge$, then $\mathcal{S}_1$ and $\mathcal{S}_3$ generate the same **L**-closure systems. Symmetrically, one may consider $\mathcal{S}_3$ as the input and proceed analogously. In particular, in view of the preceding results, a good choice is to take $\mathrm{irr}_\bigwedge(\cdot)$ and $\mathrm{irr}_\to(\cdot)$ in this procedure. This consideration brings us to the following simple algorithms.

Algorithm 1.

input: finite $\mathcal{S} \subseteq L^U$
output: $\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))$

Algorithm 2.

input: finite $\mathcal{S} \subseteq L^U$
output: $\mathrm{irr}_\bigwedge(\mathrm{irr}_\to(\mathcal{S}))$

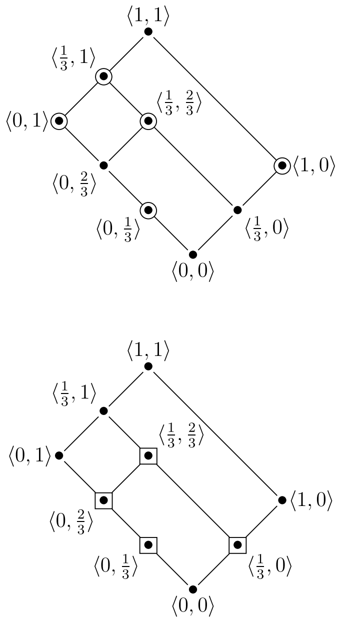


Fig. 1. The sets $\mathrm{irr}_{\bigwedge}(\mathcal{S})$ (top) and $\mathrm{irr}_{\rightarrow}(\mathcal{S})$ (bottom) of Example 2.

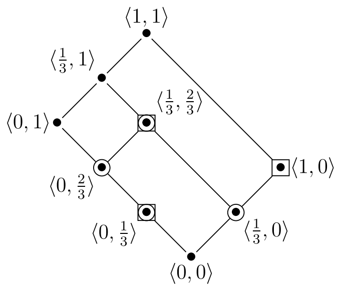


Fig. 2. Outputs of Algorithms 1 and 2 of Example 2. Squared nodes represent $\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))$; circled nodes represent $\mathrm{irr}_{\bigwedge}(\mathrm{irr}_{\rightarrow}(\mathcal{S}))$.

Theorem 8. *For a finite $\mathcal{S}$, both $\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))$ (output of Algorithm 1) and $\mathrm{irr}_{\bigwedge}(\mathrm{irr}_{\rightarrow}(\mathcal{S}))$ (output of Algorithm 2) are $[\ ]_{\bigwedge}$- and $[\ ]_{\rightarrow}$-non-redundant sets of generators of $[\mathcal{S}]$.*

Proof. Algorithm 1: Let $\mathcal{S}_3 = \mathcal{S}$, $\mathcal{S}_2 = \mathrm{irr}_{\bigwedge}(\mathcal{S}_3)$, and $\mathcal{S}_1 = \mathrm{irr}_{\rightarrow}(\mathcal{S}_2)$. Then Theorem 6 and Theorem 4 imply $[\mathcal{S}_1]_{\rightarrow} = [\mathcal{S}_2]_{\rightarrow}$ and $[\mathcal{S}_2]_{\bigwedge} = [\mathcal{S}_3]_{\bigwedge}$. Lemma 7 thus yields that $\mathcal{S}_1 = \mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))$, the output of Algorithm 1, is a set of generators of $[\mathcal{S}]$. Moreover, Theorem 6 implies that $\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))$ is $[\ ]_{\rightarrow}$-non-redundant. Since $\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S})) \subseteq \mathrm{irr}_{\bigwedge}(\mathcal{S})$, we clearly have $\mathrm{irr}_{\bigwedge}(\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))) = \mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))$, which means that $\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))$ is $[\ ]_{\bigwedge}$-non-redundant.

Symmetrically, putting $\mathcal{S}_1 = \mathcal{S}$, $\mathcal{S}_2 = \mathrm{irr}_{\rightarrow}(\mathcal{S}_1)$ and $\mathcal{S}_3 = \mathrm{irr}_{\bigwedge}(\mathcal{S}_2)$, we obtain that $\mathcal{S}_3 = \mathrm{irr}_{\bigwedge}(\mathrm{irr}_{\rightarrow}(\mathcal{S}))$, the output of Algorithm 2, is a $[\ ]_{\rightarrow}$-non-redundant and $[\ ]_{\bigwedge}$-non-redundant set of generators of $[\mathcal{S}]$. $\square$

As the next example shows, the outputs of Algorithms 1 and 2 may be different and not comparable with respect to set inclusion.

Example 2. Let $\mathbf{L}$ be the four-element Gödel chain, i.e. $L = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$,

$$a \otimes b = \min(a, b), \quad \text{and} \quad a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a > b. \end{cases} \tag{2}$$

One may check that $\mathcal{S} = \{\langle 1, 1\rangle, \langle 1, 0\rangle, \langle \frac{1}{3}, 1\rangle, \langle \frac{1}{3}, \frac{2}{3}\rangle, \langle \frac{1}{3}, 0\rangle, \langle 0, 1\rangle, \langle 0, \frac{2}{3}\rangle, \langle 0, \frac{1}{3}\rangle, \langle 0, 0\rangle\}$ is an $\mathbf{L}$-closure system. Its Hasse diagram with marked nodes corresponding to the elements in $\mathrm{irr}_{\bigwedge}(\mathcal{S})$ and $\mathrm{irr}_{\rightarrow}(\mathcal{S})$ are depicted in Fig. 1. One can see in Fig. 2 that the outputs of Algorithms 1 and 2 have common elements but are mutually incomparable and have different sizes.

The next example shows that although $[\ ]_{\rightarrow}$- and $[\ ]_{\bigwedge}$-non-redundant, the output of Algorithms 1 and 2 may be $[\ ]$-redundant.

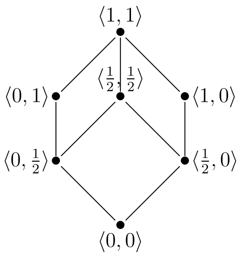


Fig. 3. **L**-closure system from Example 4.

Example 3. Consider again $L=[0,1]$ equipped with the Łukasiewicz operations. Let $\mathcal{S}=\{\langle 0.1,0.1\rangle,\langle 0.1,0.2\rangle,\langle 0.1,0.3\rangle\}$. As one easily checks, $\mathcal{S}$ is both $[\ ]_{\rightarrow}$- and $[\ ]_{\bigwedge}$-non-redundant, hence both Algorithms 1 and 2 produce $\mathcal{S}$. However, $\mathcal{S}$ is $[\ ]$-redundant, because $\langle 0.1,0.2\rangle\in[\mathcal{S}-\{\langle 0.1,0.2\rangle\}]$. Namely,

$$0.9\rightarrow\langle 0.1,0.1\rangle\wedge 1\rightarrow\langle 0.1,0.3\rangle=\langle 0.2,0.2\rangle\wedge\langle 0.1,0.3\rangle=\langle 0.1,0.2\rangle.$$

The next theorem shows an important condition for Algorithm 1 to produce a $[\ ]$-base.

Theorem 9. *If $\mathcal{S}$ is finite and closed under left $\rightarrow$-multiplications (in particular, if $\mathcal{S}$ is a finite* **L***-closure system), Algorithm 1 produces $[\ ]$-base of $[\mathcal{S}]$.*

Proof. Assume that $\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))$ is not a $[\ ]$-base. According to Theorem 8, $\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))$ generates $[\mathcal{S}]$, thus $\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))$ must be redundant. Hence, there exists $B\in\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))$ such that $B\in[\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))-\{B\}]$. According to Theorem 3, $B\in[\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))-\{B\}]_{\rightarrow}]_{\bigwedge}$, which means that

$$B=\bigwedge\nolimits_{B_i\in\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))-\{B\}}a_i\rightarrow B_i$$

for some a_is in L. Since $B_i\in\mathcal{S}$ and since $\mathcal{S}$ is closed under left $\rightarrow$-multiplications, we have $a_i\rightarrow B_i\in\mathcal{S}$. Furthermore, $B\in\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))\subseteq\mathrm{irr}_{\bigwedge}(\mathcal{S})$ implies $B=a_i\rightarrow B_i$ for some i. But then B obtains as a left $\rightarrow$-multiplication of $B_i\in\mathrm{irr}_{\bigwedge}(S)$, hence we cannot have $B\in\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))$, which contradicts the assumption. $\square$

Let us note that in most practical applications one works with a finite set L of degrees. In such case, one may compute a base of a given finite $\mathcal{S}$ according to Theorem 9 by transforming $\mathcal{S}$ to $\mathcal{S}'=[\mathcal{S}]_{\rightarrow}$, which is closed under left $\rightarrow$-multiplications due to Lemma 1, and then applying Algorithm 1 to $\mathcal{S}'$.

Remark 4. An analogous version of Theorem 9 does not hold for Algorithm 2. Namely, since $\mathcal{S}$ in Example 2 is an **L**-closure system, it is closed under $\bigwedge$-intersections as well as under left $\rightarrow$-multiplications. Yet the output of Algorithm 2, $\mathrm{irr}_{\bigwedge}(\mathrm{irr}_{\rightarrow}(\mathcal{S}))=\{\langle\frac{1}{3},\frac{2}{3}\rangle,\langle\frac{1}{3},0\rangle,\langle 0,\frac{2}{3}\rangle,\langle 0,\frac{1}{3}\rangle\}$, is not a $[\ ]$-base because

$$\langle 0,\tfrac{2}{3}\rangle=1\rightarrow\langle\tfrac{1}{3},\tfrac{2}{3}\rangle\wedge\tfrac{1}{3}\rightarrow\langle 0,\tfrac{1}{3}\rangle.$$

3.5. (Non-)uniqueness of bases and their sizes

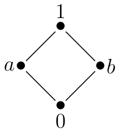
It is well known that an ordinary closure system has a unique base. Contrary to that, an **L**-closure system may have several bases.

Example 4. Let **L** be the three-element Gödel chain, i.e. $L=\{0,\frac{1}{2},1\}$ and the operations are given by (2) in Example 2. Consider the **L**-closure system $\mathcal{S}$ depicted in Fig. 3. One may check that

$$\{\langle 0,1\rangle,\langle 1,0\rangle,\langle\tfrac{1}{2},\tfrac{1}{2}\rangle\}\quad(\text{which equals }\mathrm{irr}_{\rightarrow}(\mathrm{irr}_{\bigwedge}(\mathcal{S}))),$$
$$\{\langle 0,\tfrac{1}{2}\rangle,\langle\tfrac{1}{2},0\rangle,\langle\tfrac{1}{2},\tfrac{1}{2}\rangle\}\quad(\text{which equals }\mathrm{irr}_{\bigwedge}(\mathrm{irr}_{\rightarrow}(\mathcal{S}))),$$
$$\{\langle 0,\tfrac{1}{2}\rangle,\langle 1,0\rangle,\langle\tfrac{1}{2},\tfrac{1}{2}\rangle\},\ \text{and}$$
$$\{\langle 0,1\rangle,\langle\tfrac{1}{2},0\rangle,\langle\tfrac{1}{2},\tfrac{1}{2}\rangle\}$$

are all bases of $\mathcal{S}$.

Furthermore, one may show that for non-linearly ordered residuated lattice, the bases may have different number of elements.



$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Fig. 4. Residuated lattice from Example 5.

Example 5. Let **L** be the residuated lattice whose lattice part is depicted in Fig. 4 (left), with $\otimes$ being $\wedge$ and $\to$ as shown in Fig. 4 (right). One can check, that $\{\langle a\rangle, \langle b\rangle\}$ and $\{\langle 0\rangle\}$ are bases of $\mathcal{S} = \{\langle 0\rangle, \langle a\rangle, \langle b\rangle, \langle 1\rangle\}$. This shows that bases of an **L**-closure system may have different sizes and be disjoint.

Notice that in Example 5, $\mathcal{S}$ is closed under left $\to$-multiplications. According to Theorem 9, $\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))$—the output of Algorithm 1—is a base of $\mathcal{S}$. As one checks, $\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S})) = \{\langle a\rangle, \langle b\rangle\}$ is the larger of the two bases. We are going to show that the base produced by Algorithm 1 is always the largest in size.

Lemma 10. *Let $\mathcal{S}$ be finite and let $\mathcal{C} \subseteq [\mathcal{S}]$ satisfy $\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S})) \subseteq [\mathcal{C}]_\to$. Then $[\mathcal{C}] = [\mathcal{S}]$.*

Proof. Clearly, $\mathcal{C} \subseteq [\mathcal{S}]$ implies $[\mathcal{C}] \subseteq [\mathcal{S}]$. Conversely, due to Theorem 8 and the assumption we get $[\mathcal{S}] = [\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))] \subseteq [[\mathcal{C}]_\to] = [\mathcal{C}]$. □

Lemma 11. *Let $\mathcal{S} = [[\mathcal{C}]_\to]_\wedge$. Then each $B \in \mathcal{S}$ is of the form*

$$B = \bigwedge\nolimits_{C\in\mathcal{C}}(a_C \to C) \tag{3}$$

for some $a_C \in L$.

Proof. Observe that since $B \in \mathcal{S} = [[\mathcal{C}]_\to]_\bigwedge$, B is an intersection of left $\to$-multiplications of elements in $\mathcal{C}$, i.e. is of the form $B = \bigwedge\mathcal{K}$ for some $\mathcal{K} = \bigcup_{C\in\mathcal{C}}\mathcal{K}_C$ where $\mathcal{K}_C = \{a \to C \mid a \in K_C\}$ for some $K_C \subseteq L$. Thus $B = \bigwedge_{C\in\mathcal{C}}\mathcal{K}_C$ and since every complete residuated lattice satisfies $\bigwedge_i(a_i \to b) = (\bigvee_i a_i) \to b$ and $0 \to a = 1$, B is actually in the form (3). □

Theorem 12. *Let $\mathcal{S}$ be finite and closed under left $\to$-multiplications. Then every base $\mathcal{B}$ of $[\mathcal{S}]$ has at most $|\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))|$ elements and has the property that for each $A \in \mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))$ there exists $B \in \mathcal{B}$ and $b \in L$ such that $A = b \to B$.*

Proof. Let $\mathcal{B}$ be a base of $[\mathcal{S}]$. Due to the assumptions and Theorem 3, $\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S})) \subseteq [\mathcal{S}] = [[\mathcal{B}]_\to]_\bigwedge$, hence each $A \in \mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))$ may be expressed as

$$A = \bigwedge\nolimits_{B\in\mathcal{B}}(a_B \to B) \tag{4}$$

for some $a_B \in L$ by Lemma 11. Observe that $\mathrm{irr}_\bigwedge(\mathcal{S}) = \mathrm{irr}_\bigwedge([\mathcal{S}]_\bigwedge)$, whence since $A \in \mathrm{irr}_\bigwedge(\mathcal{S})$, we also have $A \in \mathrm{irr}_\bigwedge([\mathcal{S}]_\bigwedge)$. As $\mathcal{S}$ is closed under left $\to$-multiplications, i.e. $\mathcal{S} = [\mathcal{S}]_\to$, we obtain $[\mathcal{S}]_\bigwedge = [[\mathcal{S}]_\to]_\bigwedge = [\mathcal{S}]$ on account of Theorem 3. Since $\mathcal{B} \subseteq [\mathcal{S}]$, every $a_B \to B$ in (4) is in $[\mathcal{S}]$ and thus in $[\mathcal{S}]_\bigwedge$. The above observation $A \in \mathrm{irr}_\bigwedge([\mathcal{S}]_\bigwedge)$ now implies that $A = a_B \to B$ for some $a_B \in L$ and $B \in \mathcal{B}$. Denote now for each $A \in \mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))$ by B_A some such B for which $A = a_B \to B$ and put $\mathcal{B}' = \{B_A \mid A \in \mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))\}$. Then $\mathcal{B}'$ clearly satisfies the conditions for $\mathcal{C}$ in Lemma 10 and thus generates $[\mathcal{S}]$. Since $\mathcal{B}' \subseteq \mathcal{B}$ and since $\mathcal{B}$ is a base, we get $\mathcal{B}' = \mathcal{B}$. Observing that $\mathcal{B}'$ has at most $|\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))|$ elements finishes the proof. □

Corollary 13. *Every base of a finite **L**-closure system has at most $|\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))|$ elements.*

All the four bases in Example 4, in which **L** is a chain, have equal size. We now show that this is no coincidence.

Theorem 14. *If **L** is a chain, then all bases of an arbitrary finite **L**-closure system have the same size.*

Proof. Assume by contradiction that $\mathcal{B}$ and $\mathcal{C}$ are two bases of a finite **L**-closure system $\mathcal{S}$ for which $|\mathcal{B}| < |\mathcal{C}|$. On account of Theorem 9 and Theorem 12, we may assume $\mathcal{C} = \mathrm{irr}_\to\mathrm{irr}_\bigwedge(\mathcal{S})$. Theorem 12 implies that for every $C \in \mathcal{C}$ there exists $b \in L$ and $B \in \mathcal{B}$ such that $C = b \to B$. As $|\mathcal{B}| < |\mathcal{C}|$, there must exist two mutually different $C_1, C_2 \in \mathcal{C}$ sharing the same element B, i.e. such that $C_1 = b_1 \to B$ and $C_2 = b_2 \to B$ for some $b_1, b_2 \in L$.

By Lemma 11 the element B is of the form

$$B = \bigwedge\nolimits_{C\in\mathcal{C}}(a_C \to C) \tag{5}$$

for some $a_C \in L$. Now observe that for the above elements $C_1, C_2 \in \mathcal{C}$ we have $a_{C_1} \to C_1 \subseteq a_{C_2} \to C_2$ or $a_{C_2} \to C_2 \subseteq a_{C_1} \to C_1$. Namely, we have $a_{C_i} \to C_i = a_{C_i} \to (b_i \to B) = (a_{C_i} \otimes b_i) \to B$. Since $\mathbf{L}$ is a chain, we have $a_{C_1} \otimes b_1 \leq a_{C_2} \otimes b_2$ or $a_{C_2} \otimes b_2 \leq a_{C_1} \otimes b_1$, hence $a_{C_2} \to C_2 \subseteq a_{C_1} \to C_1$ or $a_{C_1} \to C_1 \subseteq a_{C_2} \to C_2$, respectively, due to the antitony of $\to$ in its first argument.

Without loss of generality, assume $a_{C_1} \to C_1 \subseteq a_{C_2} \to C_2$. Then (5) may clearly be rewritten as

$$B = \bigwedge_{C \in \mathcal{C} - \{C_2\}} (a_C \to C).$$

As a result, we obtain

$$\begin{aligned} C_2 = b_2 \to B &= b_2 \to \bigwedge_{C \in \mathcal{C} - \{C_2\}} (a_C \to C) \\ &= \bigwedge_{C \in \mathcal{C} - \{C_2\}} (b_2 \to (a_C \to C)) = \bigwedge_{C \in \mathcal{C} - \{C_2\}} ((b_2 \otimes a_C) \to C), \end{aligned}$$

thus $C_2 \in \mathcal{C}$ may be obtained as an intersection of left $\to$-multiplications of the elements in $\mathcal{C} - \{C_2\}$, i.e. $C_2 \in [\mathcal{C} - \{C_2\}]$ which contradicts the assumption that $\mathcal{C}$ is a base and thus [ ]-non-redundant. □

4. Topics for future research

A natural continuation of the presented results is to examine problems connected to computation of bases of closure systems over residuated lattices. These include computing all bases, computing all bases of a given size, or estimating the number of bases. In addition, practically relevant seem some problems that are degenerate in case of ordinary closure operators such as describing and computing $\mathcal{B}$ which is an approximate base of $\mathcal{S}$ in that for every $A \in \mathcal{S}$ there exists $A' \in [\mathcal{B}]$ such that for every element u of the universe, $A'(u)$ is sufficiently close to $A(u)$.

Acknowledgments

The work was supported by grant No. GA15-17899S of the Czech Science Foundation.

References

[1] S. Abramsky, A. Jung, Domain theory, in: Handbook of Logic in Computer Science, Clarendon Press, Oxford, 1994, pp. 1–168.
[2] R. Belohlavek, Fuzzy closure operators, J. Math. Anal. Appl. 262 (2) (2001) 473–489.
[3] R. Belohlavek, Fuzzy Relational Systems: Foundations and Principles, Kluwer Academic Publishers, NY, 2002.
[4] R. Belohlavek, Concept lattices and order in fuzzy logic, Ann. Pure Appl. Logic 128 (1–3) (2004) 277–298.
[5] R. Belohlavek, J. Dvorak, J. Outrata, Fast factorization by similarity in formal concept analysis of data with fuzzy attributes, J. Comput. Syst. Sci. 73 (2007) 1012–1022.
[6] B. Davey, H. Priestley, Introduction to Lattices and Order, Cambridge University Press, 2002.
[7] Nikolaos Galatos, Peter Jipsen, T. Kowalski, H. Ono (Eds.), Residuated Lattices: An Algebraic Glimpse at Substructural Logics, Elsevier, 2007.
[8] B. Ganter, R. Wille, Formal Concept Analysis: Mathematical Foundations, Springer, 1999.
[9] G. Gerla, Fuzzy Logic: Mathematical Tools for Approximate Reasoning, Kluwer, 2001.
[10] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, Continuous Lattices and Domains, Encyclopedia of Mathematics and Its Applications, vol. 93, Cambridge University Press, 2003.
[11] S. Gottwald, A Treatise on Many-Valued Logics, Research Studies Press, Baldock, Hertfordshire, England, 2001.
[12] S. Gottwald, Mathematical fuzzy logics, Bull. Symb. Log. 14 (02) (2008) 210–239.
[13] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer, 1998.
[14] K.H. Kim, Boolean Matrix Theory and Applications, M. Dekker, 1982.
[15] L.-X. Lu, Q. Wang, Relative maximal filters in conditional semilattices, Fuzzy Systems Math. 2 (2011) 66–70.
[16] M. Ward, R.P. Dilworth, Residuated lattices, Trans. Am. Math. Soc. 45 (1939) 335–354.