Soft Computing (2018) 22:6299–6310
https://doi.org/10.1007/s00500-017-2972-1

FOUNDATIONS

A calculus for containment of fuzzy attributes

Radim Belohlavek[1] · Jan Konecny[1]

Published online: 11 December 2017

Abstract

Dependencies in data describing objects and their attributes represent a key topic in understanding relational data. In this paper, we examine certain dependencies of data described by fuzzy attributes such as *green* or *high performance*, i.e. attributes which apply to objects to certain degrees. Such attributes subsume Boolean attributes as a particular case. We utilize the framework of residuated structures of truth degrees as developed in modern fuzzy logic and examine several fundamental problems for our dependencies. These include connections to existing dependencies for fuzzy as well as Boolean attributes, connections to interior- and closure-like structures, definition and properties of semantic entailment including an efficient check of entailment, various model-theoretical properties, a logical calculus of the dependencies inspired by the well-known Armstrong rules with its ordinary-style as well as graded-style syntactico-semantical completeness, fully informative sets of all dependencies that are valid in given data including a constructive description of minimal such sets, as well as various other problems.

Keywords Fuzzy logic · Dependencies of fuzzy attributes · Fuzzy closure structures · Formal concept analysis

1 Introduction

1.1 Problem setting

We assume that the dependencies we study pertain to data in the form of a table with rows and columns corresponding to objects x in a set X and their logical attributes y in a set Y, respectively. While in the classic, Boolean case, every attribute y either applies or does not apply to any given object x, we assume a more general setting in which the attributes are fuzzy. That is, with every attribute y and object x, there is associated a degree to which y applies to x. We furthermore assume that these truth degrees form a partially ordered set L bounded by 0 and 1 (representing falsity and truth, respectively) which is, moreover, equipped with truth functions of logical connectives such as conjunction and implications, as detailed below. The classical case then becomes a particular case in which the only members of L are 0 and 1 and in which the truth functions are the truth functions of Boolean logical connectives.

Communicated by A. Di Nola.

✉ Jan Konecny
jan.konecny@upol.cz

[1] Department of Computer Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic

In our paper, we examine certain dependencies which concern containment of attributes. In particular, we introduce basic syntactic and semantic notions which are inspired by two basic meanings of containment of fuzzy attributes, namely binary and graded containment (Sect. 2), explore connections to interior-like structures and outline ramifications of these connections (Sect. 3), develop a logic for our dependencies with two kinds of completeness (Sect. 4), and provide results regarding minimal fully informative sets of if–then rules (Sect. 5).

1.2 Preliminaries

As the above-mentioned scales of truth degrees, we use complete residuated lattices. Since these are well known (Belohlavek 2012; Goguen 1969; Hájek 1998), we restrict to recalling basic facts. A *complete residuated lattice* with a truth-stressing hedge (shortly, a hedge) is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid; $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

for each $a, b, c \in L$; hedge * satisfies

$$1^* = 1, \quad a^* \leq a, \quad (a \to b)^* \leq a^* \to b^*, \quad a^{**} = a^*, \tag{2}$$

for each $a, b \in L$, $a_i \in L$ ($i \in I$). Elements a of L are called truth degrees. $\otimes$ and $\to$ are (truth functions of) many-valued conjunction and implication. Hedge * may be seen as a (truth function of) logical connective "very true." Properties (2) have natural interpretations, e.g. second one can be read: "if a is very true, then a is true," the third one as: "if $a \to b$ is very true and if a is very true, then b is very true." Note that other properties of hedges are sometimes imposed, see, e.g. (Hájek 1998).

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\to$; or with L being a finite chain with appropriate operations. Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^* = a$ ($a \in L$), and (ii) globalization: $a^* = 1$ for $a = 1$ and $a^* = 0$ for $a < 1$. An important special case of a complete residuated lattice with hedge is the two-element Boolean algebra $\langle\{0, 1\}, \wedge, \vee, \otimes, \to, {}^*, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge, \vee, \otimes, \to$ of $\mathbf{2}$ are the truth functions (interpretations) of the corresponding logical connectives of the classical logic and $0^* = 0$, $1^* = 1$.

We exploit the usual notions and notation: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A : U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A." The collection of all $\mathbf{L}$-sets in U is denoted by $\mathbf{L}^U$. The operations with $\mathbf{L}$-sets are defined componentwise. Binary $\mathbf{L}$-relations between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$. That is, a binary $\mathbf{L}$-relation $I \in \mathbf{L}^{X\times Y}$ is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I). An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is called empty (denoted by $\emptyset$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in \mathbf{L}^X$, $a \otimes A \in \mathbf{L}^X$ and $a \to A \in \mathbf{L}^X$ are defined by

$$(a \otimes A)(x) = a \otimes A(x) \text{ and } (a \to A)(x) = a \to A(x).$$

Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A, B) = \textstyle\bigwedge_{u\in U}\big(A(u) \to B(u)\big), \tag{3}$$

which generalizes the ordinary subsethood relation $\subseteq$. $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. Throughout the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice with a hedge.

In what follows, we utilize the following inequality which is easy to prove using the definitions.

Lemma 1 *For $A, B \in \mathbf{L}^U$, we have*

$$S(A, B)^* \leq S(A^*, B^*). \tag{4}$$

1.3 Previous work

Explorations of various kinds of dependencies among Boolean attributes, developments of various logical calculi describing such dependencies as well as various computational problems related to these dependencies represent fundamental issues in data management and have been thoroughly studied in the past.

Most important among such dependencies are various kinds of if–then rules. These rules, which basically describe that when certain attributes are present then certain other attributes are present as well, are thoroughly examined along with their connections to functional dependencies in the classic book (Maier 1983). They have further been explored for data analytic purposes, and their algorithmic properties are examined in the influential paper (Guigues and Duquenne 1986) and in (Ganter and Wille 1999), which works pay a particular attention to extraction of a smallest fully informative set of dependencies from Boolean data. Taking into account almost valid if–then rules leads to association rules and considerations regarding confidence and support of such rules. Related explorations represent a major research stream in data mining, and we refer to (Zhang and Zhang 2002) for an overview and to (Rauch 2005) for related logical calculi of these rules as well as calculi of much more general dependencies in Boolean data, namely those of (Hájek and Havránek 2012) which subsume association rules as a very particular case.

Directly connected to the topic of our paper are recent explorations of dependencies in data with fuzzy rather than Boolean attributes, i.e. attributes such as for graded attributes, such as *green* or *high performance*. These have been codeveloped by one of the present authors in a series of papers, see (Belohlavek and Vychodil 2016, 2017) for a comprehensive treatment and (Belohlavek and Vychodil 2006) for early explorations. In this paper, we present a logic of if–then rules $A \Rightarrow B$ for graded attributes whose basic meaning is: if all attributes of an object are contained A then they are contained in B. These rules have the same syntactic form as those in (Belohlavek and Vychodil 2016, 2017), but have a different semantics: they represent restrictions on what attributes may be possessed by objects. The technical difference from the rules in (Belohlavek and Vychodil 2016, 2017) consists in the fact that the new rules are closely related to interior-like structures, while the former rules are related to closure-like structures. In the Boolean case, the two kinds of dependencies are mutually reducible: The reducibility derives from the fact that interior- and closure-like structures are mutually reducible in the Boolean case. In the setting of fuzzy attributes, such reducibility is not available, as is well known, due to the lack of the law of double negation. Consequently, our new kind of dependencies needs to be carefully explored anew. In a broader perspective, the new rules manifest the variety of structures naturally associated with object-attribute data, which has also been examined, e.g. in (Belohlavek and Konecny 2012; Ciucci et al. 2014; Georgescu and Popescu 2004; Konecny 2011), and further contribute to understanding these structures.

2 Syntax and semantics

Suppose $\mathbf{L}$ is a complete residuated lattice with a hedge (i.e. a scale of grades equipped with logical operations) and Y be a set of (symbols of) fuzzy attributes. Each expression of the form

$$A \Rightarrow B,$$

in which A and B are fuzzy sets of attributes (i.e. $A, B \in \mathbf{L}^Y$) is called a *fuzzy attribute implication* (FAI) over Y; FAIs are our basic formulas. While they are identical with the formulas in (Belohlavek and Vychodil 2016, 2017) as far syntax is concerned, their semantics is different. Put verbally, the intended meaning is:

> If all attributes of an object are contained in A then they are contained in B.

Since in a fuzzy setting, whether an object has an attribute is a matter of degree, validity of our formulas is a matter of degree as well. In the semantics described, one needs to be careful about the meaning of containment since there are two natural options possible—taking containment as bivalent or graded. We provide a general semantics which covers both these options as particular cases.

Let x denote an object and $M \in \mathbf{L}^Y$ a fuzzy set representing the attributes of x, i.e. for each $y \in Y$ the degree to which object x has attribute y is M. Our aim is to define the truth degree, denoted $||A \Rightarrow B||_M$, of $A \Rightarrow B$ in M, i.e. the truth degree to which $A \Rightarrow B$ is true for object x. As mentioned above, we provide a general definition which subsumes two particular cases, one for bivalent and one for graded containment. For bivalent containment, the fact that $A \Rightarrow B$ is fully true in M (in symbols $||A \Rightarrow B||_M = 1$) means:

$$\text{if} \quad M \subseteq A \quad \text{then} \quad M \subseteq B, \tag{5}$$

where $M \subseteq A$ denotes full containment, i.e. $M(y) \leq A(y)$ for all $y \in Y$. For a graded containment, the fact that $A \Rightarrow B$ is fully true in M means:

$$S(M, A) \leq S(M, B), \tag{6}$$

i.e. a degree of inclusion of M in A is less than or equal to the degree of inclusion of M in B, cf. (3). Now, both approaches can be obtained as particular cases of the following definition, in which the hedge * acts as a parameter (see below):

Definition 1 For a fuzzy attribute implication $A \Rightarrow B$ and a fuzzy set M of attributes (of some object x), we define the *degree* $||A \Rightarrow B||_M \in L$ *to which* $A \Rightarrow B$ *is valid in* M as follows:

$$||A \Rightarrow B||_M = S(M, A)^* \to S(M, B). \tag{7}$$

One easily verifies that if * is globalization and identity, respectively, (7) meets the above cases corresponding to bivalent and graded inclusion, (5) and (6), respectively. Let us emphasize, however, that the degree of validity $||A \Rightarrow B||_M$ is a general truth degree in L, i.e. it need not be equal to 0 or 1.

Clearly, from the perspective of the current literature on structures related to fuzzy attributes, our formulas $A \Rightarrow B$ may be interpreted in tables with fuzzy attributes. Note that each such table may be identified with a triplet $\langle X, Y, I\rangle$, in which X and Y are sets of objects (table rows) and attributes (table columns), and I is a fuzzy relation between X and Y for which $I(x, y)$ is interpreted as the degree to which the object x has the attribute y. The corresponding definitions needed are as follows

Definition 2 For a collection $\mathcal{M}$ of fuzzy sets M of attributes in Y, we define the degree to which $A \Rightarrow B$ is valid in $\mathcal{M}$ as follows:

$$||A \Rightarrow B||_{\mathcal{M}} = \textstyle\bigwedge_{M \in \mathcal{M}} ||A \Rightarrow B||_M. \tag{8}$$

For a table $\langle X, Y, I\rangle$ with fuzzy attributes, we define the degree to which $A \Rightarrow B$ is valid in $\langle X, Y, I\rangle$ by

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\{I_x | x \in X\}}, \tag{9}$$

where I_x denotes the fuzzy set representing the row corresponding to object x, i.e. $I_x(y) = I(x, y)$ for each $y \in Y$.

Remark 1 (*alternative to failure dependencies*) The dependencies just introduced represent a natural alternative to failure dependencies in the theory of knowledge spaces (Doignon and Falmagne 2012) in the setting in which mastering pieces of knowledge is a matter of degree (Bartl and Belohlavek 2011). Recall that in ordinary knowledge spaces, one deals with subsets of items in a set Q (representing questions in some area, for instance). If M is a subset of Q representing a given person's state of knowledge (a set of questions the person is capable of answering correctly), a failure dependency $A \Rightarrow B$, where $A, B \subseteq Q$, holds for M if

$$A \cap M = \emptyset \text{ implies } B \cap M = \emptyset,$$

i.e. failure in answering all questions in A implies failure in answering all questions in B. This notion has been extended in (Bartl and Belohlavek 2011) to a graded setting in which M, A, and B are allowed to be fuzzy sets in Q in which the membership degrees model situations in which mastering of or failure on a particular question is a matter of degree. In particular, the degree to which $A \Rightarrow B$ is true in M has been defined in (Bartl and Belohlavek 2011) by

$$S(A \otimes M, \emptyset)^* \to S(B \otimes M, \emptyset);$$

here, $(A \otimes M)(q) = A(q) \otimes M(q)$ is the $\otimes$-based intersection of fuzzy sets A and M. This definition corresponds directly to the above definition from the ordinary case, but it has the disadvantage that it involves the many-valued negation $\neg a = a \to 0$ associated with the underlying structure $\mathbf{L}$ of truth degrees. This property is disadvantageous because such negation lacks certain properties of ordinary negation (e.g. the law of double negation) due to which fact only certain properties from the ordinary setting carry over to the fuzzy setting.

The dependencies we study in the present paper, nevertheless, offer another way to capture failure dependencies. Namely, it is easily seen that in the ordinary setting a failure dependency $C \Rightarrow D$ is true in a state $M \subseteq Q$ iff the following holds true: if $M \subseteq \overline{C}$ then $M \subseteq \overline{D}$. Denoting $A = \overline{C}$ and $B = \overline{D}$, this may be rewritten as: if $M \subseteq A$ then $M \subseteq B$. While C and D represent failures on questions in failure dependencies, A and B in the new kind of dependencies (which is obviously the kind studied in this paper) represent mastering of questions. Namely, the meaning of this new kind of dependency is described as: if all questions the individual has mastered are in A, then all questions he has mastered are in B as well. Hence, the new type of dependencies may aptly be called *mastering dependencies.*

A direct generalization of mastering dependencies to a fuzzy setting clearly yields the dependencies whose semantics is defined by Definition 1. While failure and mastering dependencies are equivalent in the ordinary setting (due to the law of double negation, a mastering dependency $A \Rightarrow B$ is equivalent to the failure dependency $\overline{A} \Rightarrow \overline{B}$), they are no longer equivalent in a fuzzy setting (clearly, they are equivalent if the fuzzy logic connective of negation involved satisfies the law of double negation, such as the Łukasiewicz negation for instance). In a fuzzy setting, they both describe the same type of dependency, but technically, mastering dependencies, as formalized by Definition 1, are more convenient because they do not involve the possibly problematic logical connective of negation. □

Our semantics of fuzzy attribute implications is closely connected to particular Galois-like connections and their fixpoints. These Galois-like connections have been introduced in (Georgescu and Popescu 2004), see also (Konecny 2011): For $\langle X, Y, I\rangle$ as above, consider the operators ${}^{\cap} : \mathbf{L}^X \to \mathbf{L}^Y$ and ${}^{\cup} : \mathbf{L}^Y \to \mathbf{L}^X$ defined by

$$A^{\cap}(y) = \bigvee_{x \in X}(A(x)^* \otimes I(x, y)), \tag{10}$$

$$B^{\cup}(x) = \bigwedge_{y \in Y}(I(x, y) \to B(y)), \tag{11}$$

for any fuzzy set A in X and B in Y. Let furthermore $\mathcal{B}(X^{\cap}, Y^{\cup}, I) = \{\langle A, B\rangle \mid A^{\cap} = B, B^{\cup} = A\}$ denote the lattice of the fixpoints of ${}^{\cap}$ and ${}^{\cup}$ and $\mathrm{Int}(X^{\cap}, Y^{\cup}, I) = \{B \mid \langle A, B\rangle \in \mathcal{B}(X^{\cap}, Y^{\cup}, I)$ for some $A\}$ the corresponding system of intents (i.e. of the second components of the fixpoints). We also use just $\mathrm{Int}(I)$ instead of $\mathrm{Int}(X^{\cap}, Y^{\cup}, I)$. The following theorem reveals an important relationship: the validity of our attribute dependencies $A \Rightarrow B$ in a table $\langle X, Y, I\rangle$ coincides with the validity in the intents of $\langle X, Y, I\rangle$ and also with the degree to which $A^{\cup\cap}$ is contained in B. This theorem is utilized below in a characterization of complete sets of fuzzy attribute implications.

Theorem 1 *Given a table $\langle X, Y, I\rangle$ with fuzzy attributes and a fuzzy attribute implication $A \Rightarrow B$ over Y, we have*

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathrm{Int}(I)} = S(A^{\cup\cap}, B). \tag{12}$$

Proof First, we check $||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathrm{Int}(I)}$. Observe, that $||A \Rightarrow B||_{\langle X,Y,I\rangle} \le ||A \Rightarrow B||_{\mathrm{Int}(I)}$ iff for each $M \in \mathrm{Int}(I)$ we have

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} \le ||A \Rightarrow B||_M,$$

i.e.

$$\bigwedge_{x \in X}(S(I_x, A)^* \to S(I_x, B)) \le S(M, A)^* \to S(M, B).$$

As $I_x(y) = I(x, y)$, we have

$$S(I_x, A) = \bigwedge_{y \in Y} I_x(y) \to A(y) = A^{\cup}(x).$$

Therefore, the last inequality is equivalent to

$$\bigwedge_{x \in X}(A^{\cup}(x)^* \to B^{\cup}(x)) \le S(M, A)^* \to S(M, B),$$

i.e. to

$$S(A^{\cup}, B^{\cup}) = \bigwedge_{x \in X}(A^{\cup}(x)^* \to B^{\cup}(x)) \le S(M, A)^* \to S(M, B),$$

which is equivalent to

$$S(M, A)^* \otimes S(A^{\Cup}, B^{\Cup}) \leq S(M, B) \tag{13}$$

due to adjointness of $\otimes$ and $\rightarrow$. Thus, it suffices to prove (13) for each $M \in \mathrm{Int}(I)$. For this purpose, consider the operator $^{\Cap}$, the "unhedged" version of $^{\cap}$ defined by

$$A^{\Cap}(y) = \bigvee_{x \in X}(A(x) \otimes I(x, y)).$$

The pair $\langle^{\Cap}, ^{\Cup}\rangle$ forms an isotone **L**-Galois connection and hence satisfies $S(C_1, C_2) \leq S(C_1^{\Cap}, C_2^{\Cap})$, $S(D_1, D_2) \leq S(D_1^{\Cup}, D_2^{\Cup})$, and $D^{\Cup\Cap} \subseteq D$. Due to the fact that $M = M^{\Cup\cap}$ and since $S(C, D) \otimes S(D, E) \leq S(C, E)$, we obtain

$$\begin{aligned}
S(M, A)^* \otimes S(A^{\Cup}, B^{\Cup}) &\leq S(M^{\Cup}, A^{\Cup})^* \otimes S(A^{\Cup}, B^{\Cup}) \\
&\leq S(M^{\Cup *}, A^{\Cup *}) \otimes S(A^{\Cup}, B^{\Cup}) \\
&\leq S(M^{\Cup *}, B^{\Cup}) \\
&\leq S(M^{\Cup * \Cap}, B^{\Cup\Cap}) \\
&= S(M^{\Cup\cap}, B^{\Cup\Cap}) = S(M, B^{\Cup\Cap}) \leq S(M, B),
\end{aligned}$$

verifying (13) and thus $||A \Rightarrow B||_{\langle X,Y,I\rangle} \leq ||A \Rightarrow B||_{\mathrm{Int}(I)}$. To check $||A \Rightarrow B||_{\langle X,Y,I\rangle} \geq ||A \Rightarrow B||_{\mathrm{Int}(I)}$, it is sufficient to observe that for each $x \in X$, $I_x \in \mathrm{Int}(I)$.

Second, we check $||A \Rightarrow B||_{\langle X,Y,I\rangle} = S(A^{\Cup\cap}, B)$. We have

$$\begin{aligned}
||A \Rightarrow B||_{\langle X,Y,I\rangle} &= ||A \Rightarrow B||_{\{I_x | x \in X\}} \\
&= \bigwedge_{x \in X} (S(I_x, A)^* \rightarrow S(I_x, B)) \\
&= \bigwedge_{x \in X} (A^{\Cup}(x)^* \rightarrow B^{\Cup}(x)) \\
&= \bigwedge_{x \in X} (A^{\Cup *}(x) \rightarrow B^{\Cup}(x)) \\
&= S(A^{\Cup *}, B^{\Cup}) \\
&= S(A^{\Cup * \cap}, B) \\
&= S(A^{\Cup\cap}, B),
\end{aligned}$$

proving the claim. □

Having defined validity, we now consider theories of our fuzzy attribute implications and models of these theories. Recall that according to a seminal work of Pavelka (Pavelka 1979a, b, c), a theory in a fuzzy setting naturally consists of formulas to which degrees of truth are attached, i.e. a theory is a fuzzy set of formulas; see also (Gerla 2001; Hájek 1998). Therefore, we define a *theory* to be a fuzzy set T of fuzzy attribute implications. We furthermore say that a theory is *crisp* if T is crisp as a fuzzy set, in which case we write $A \Rightarrow B \in T$ if $T(A \Rightarrow B) = 1$ and $A \Rightarrow B \notin T$ if $T(A \Rightarrow B) = 0$.

Note that the degree to which an implication $A \Rightarrow B$ is a member of T, i.e. the degree $T(A \Rightarrow B)$, may naturally be interpreted as the degree to which the validity of $A \Rightarrow B$ is assumed. In addition, T may alternatively be regarded as a fuzzy set of dependencies extracted from data, in which case $T(A \Rightarrow B)$ is interpreted as the degree to which $A \Rightarrow B$ is valid in the data.

The set $\mathrm{Mod}(T)$ of all *models* of a given theory T is then defined as

$$\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid \text{for each } A, B \in \mathbf{L}^Y : T(A \Rightarrow B) \leq ||A \Rightarrow B||_M\}.$$

Observe that according to this definition, M is a model of T if for every implication $A \Rightarrow B$ it holds that the degree to which $A \Rightarrow B$ is valid in M is greater than or equal than the degree $T(A \Rightarrow B)$ that the theory "prescribes" for $A \Rightarrow B$. Clearly, if T is crisp then $\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid \text{ for each } A \Rightarrow B \in T : ||A \Rightarrow B||_M = 1\}$.

The degree to which a given implication $A \Rightarrow B$ *semantically follows* from a theory T of implications is then naturally defined by

$$||A \Rightarrow B||_T = \bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M.$$

Interestingly, the general concept of degree of validity of fuzzy attribute implications may be reduced to the seemingly narrower, particular concept of full validity (i.e. validity to degree 1):

Lemma 2 *For a fuzzy attribute implication $A \Rightarrow B$, a fuzzy set M of attributes, and a truth degree $c \in L$ we have*

$$c \leq ||A \Rightarrow B||_M \textit{ iff } ||A \Rightarrow c \rightarrow B||_M = 1.$$

Proof Using $\alpha \rightarrow (\beta \rightarrow \gamma) = \beta \rightarrow (\alpha \rightarrow \gamma)$ and $\alpha \rightarrow \bigwedge_k \beta_k = \bigwedge_k(\alpha \rightarrow \beta_k)$, one easily obtains $||A \Rightarrow (c \rightarrow B)||_M = c \rightarrow ||A \Rightarrow B||_M$. The claim then follows from the fact that $\alpha \rightarrow \beta = 1$ iff $\alpha \leq \beta$. □

We may now, in a sense, reduce the concepts of model and entailment for general theories (i.e. theories which involve truth degrees) to the concepts of model and entailment for crisp theories:

Lemma 3 *For a theory T of fuzzy attribute implications, denote by* $\mathrm{cr}(T)$ *a crisp theory as follows:*

$$\mathrm{cr}(T) = \{A \Rightarrow T(A \Rightarrow B) \rightarrow B \mid A, B \in \mathbf{L}^Y \textit{ and } T(A \Rightarrow B) \rightarrow B \neq Y\}. \tag{14}$$

Then,

$$\mathrm{Mod}(T) = \mathrm{Mod}(\mathrm{cr}(T)), \textit{ and} \tag{15}$$

$$||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{cr}(T)} \tag{16}$$

for any fuzzy attribute implication $A \Rightarrow B$.

Proof The equality in (15) is a direct consequence of Lemma 2. The equality in (16) follows by definition from (15). □

Interestingly, one may now reduce the concept of general degree of entailment from a theory to that of bivalent (i.e. to degree 1) entailment from a crisp theory:

Lemma 4 *For a fuzzy attribute implication* $A \Rightarrow B$ *and a theory* T *of implications, we have*

$$||A \Rightarrow B||_T = \bigvee\{c \in L \mid ||A \Rightarrow c \rightarrow B||_T = 1\},$$
$$||A \Rightarrow B||_T = \bigvee\{c \in L \mid ||A \Rightarrow c \rightarrow B||_{\mathrm{cr}(T)} = 1\}.$$

Proof Using Lemma 2, we have

$$\begin{aligned} ||A \Rightarrow B||_T &= \textstyle\bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M \\ &= \bigvee\{c \in L \mid c \leq ||A \Rightarrow B||_M \text{ for each } M \in \mathrm{Mod}(T)\} \\ &= \bigvee\{c \in L \mid ||A \Rightarrow c \rightarrow B||_T = 1\}, \end{aligned}$$

establishing the first equality. The second one is a direct consequence of the first and of (16). □

Lemma 4 is conveniently used when later proving graded completeness theorem for our logic.

3 Models and their connection to interior-like structures

We now explore the structure of models of theories and establish important connections to interior-like structures. Let us recall from (Belohlavek et al. 2005) that an $\mathbf{L}^*$-interior operator on a set Y is a mapping $I : \mathbf{L}^Y \rightarrow \mathbf{L}^Y$ which satisfies

$$I(A) \subseteq A, \tag{17}$$
$$S(A_1, A_2)^* \leq S(I(A_1), I(A_2)), \tag{18}$$
$$I(A) = I(I(A)), \tag{19}$$

for every $A, A_1, A_2 \in \mathbf{L}^Y$. Note that when $L = \{0, 1\}$ (two-valued Boolean case) then the concept of an $\mathbf{L}^*$-interior operator coincides (modulo identifying crisp fuzzy sets with ordinary sets) with the ordinary concept of interior operator.

Let us further recall (Belohlavek et al. 2005) that an $\mathbf{L}^*$-interior system on Y is a set $\mathcal{S} \subseteq \mathbf{L}^Y$ of fuzzy sets in Y which is closed under unions of fuzzy sets and so-called a^*-multiplications of fuzzy sets; that is to say, $\mathcal{S}$ is required to satisfy:

$$\text{if } A_j \in \mathcal{S} \text{ for every } j \in J \text{ then } \textstyle\bigcup_{j \in J} A_j \in \mathcal{S}, \tag{20}$$
$$\text{if } A \in \mathcal{S} \text{ and } a \in L \text{ then } a^* \otimes A \in \mathcal{S}. \tag{21}$$

Now again, for $L = \{0, 1\}$, $\mathbf{L}^*$-interior systems coincide with ordinary interior systems.

Let now for an $\mathbf{L}^*$-interior system $\mathcal{S}$ on Y and an $\mathbf{L}^*$-interior operator I on Y define the mapping $I_{\mathcal{S}}$ and the system $\mathcal{S}_I$ by

$$I_{\mathcal{S}}(B) = \bigcup_{i \in I} (S(A_i, B)^* \otimes A_i)$$

and

$$\mathcal{S}_I = \{A \in \mathbf{L}^Y \mid A = I(A)\}.$$

Then, as proved in (Belohlavek et al. 2005), $I_{\mathcal{S}}$ is an $\mathbf{L}^*$-interior operator on Y and $\mathcal{S}_I$ is an $\mathbf{L}^*$-interior system on Y; furthermore, the mappings sending $\mathcal{S}$ to $I_{\mathcal{S}}$ and I to $\mathcal{S}_I$ are bijective and are mutually inverse.

Interestingly, as the following two theorems show, models of theories of our formulas are just the $\mathbf{L}^*$-interior systems on Y.

Theorem 2 *Let* T *be a theory of fuzzy attribute implications over* Y. *Then, the system* $\mathrm{Mod}(T)$ *of all models of* T *is an* $\mathbf{L}^*$*-interior system on* Y.

Proof According to the definition, we need to verify that $\mathrm{Mod}(T)$ satisfies conditions (20) and (21). By virtue of Lemma 3, we may safely suppose that T is crisp.

For (20): If $M_j \in \mathrm{Mod}(T)$ for $j \in J$, then $||A \Rightarrow B||_{M_j} = 1$, i.e. $S(M_j, A)^* \leq S(M_j, B)$, for any $A \Rightarrow B \in T$. Now, since $(\bigwedge_{j \in J} a_j)^* \leq \bigwedge_{j \in J} a_j^*$, we get

$$\begin{aligned} S(\textstyle\bigcup_{j \in J} M_j, A)^* &= (\textstyle\bigwedge_{j \in J} S(M_j, A))^* \\ &\leq \textstyle\bigwedge_{j \in J} S(M_j, A)^* \\ &\leq \textstyle\bigwedge_{j \in J} S(M_j, B) = S(\textstyle\bigcup_{j \in J} M_j, B), \end{aligned}$$

proving $||A \Rightarrow B||_{\bigcup_{j \in J} M_j} = 1$, hence $\bigcup_{j \in J} M_j \in \mathrm{Mod}(T)$.

For (21): If $M \in \mathrm{Mod}(T)$, then for each $A \Rightarrow B \in T$ we have $||A \Rightarrow B||_M = 1$, i.e. $S(M_j, A)^* \leq S(M_j, B)$. For each $a \in L$, we thus have

$$\begin{aligned} S(a^* \otimes M, A)^* &= (a^* \rightarrow S(M, A))^* \leq a^{**} \rightarrow S(M, A)^* \\ &= a^* \rightarrow S(M, A)^* \leq a^* \rightarrow S(M, B) = S(a^* \otimes M, B), \end{aligned}$$

establishing $||A \Rightarrow B||_{a^* \otimes M} = 1$, whence $a^* \otimes M \in \mathrm{Mod}(T)$. □

Theorem 3 *For every* $\mathbf{L}^*$*-interior system* $\mathcal{S}$ *on* Y, *there is a theory* T *of fuzzy attribute implications over* Y *whose models are just the elements of* $\mathcal{S}$, *i.e. for which* $\mathcal{S} = \mathrm{Mod}(T)$.

Proof We verify that the theory $T = \{A \Rightarrow I_{\mathcal{S}}(A) \mid A \in \mathbf{L}^Y\}$ has the required property. Take any $M \in \mathcal{S}$. Then, since $I_{\mathcal{S}}$ is the corresponding operator, $M = I_{\mathcal{S}}(M)$. According to (18), we obtain

$$S(M, A)^* \leq S(I_{\mathcal{S}}(M), I_{\mathcal{S}}(A)) = S(M, I_{\mathcal{S}}(A)).$$

This way, we proved that $||A \Rightarrow I_{\mathcal{S}}(A)||_M = 1$, hence $M \in \mathrm{Mod}(T)$. This first inclusion, $\mathcal{S} \subseteq \mathrm{Mod}(T)$, is therefore established.

We establish the second inclusion by showing that if $M \notin \mathcal{S}$ then $M \notin \mathrm{Mod}(T)$. Take any $M \notin \mathcal{S}$. Then, clearly, $M \neq I_{\mathcal{S}}(M)$. Since $I_{\mathcal{S}}$ is an interior operator, we have $I_{\mathcal{S}}(M) \subset M$, cf. (17). Consequently, $S(M, I_{\mathcal{S}}(M)) \neq 1$ due to the definition of graded inclusion S. Since

$$||M \Rightarrow I_{\mathcal{S}}(M)||_M = S(M, M)^* \rightarrow S(M, I_{\mathcal{S}}(M))$$

and since $S(M, M)^* = 1^* = 1$, we obtain

$$\begin{aligned} ||M \Rightarrow I_{\mathcal{S}}(M)||_M &= 1 \rightarrow S(M, I_{\mathcal{S}}(M)) \\ &= S(M, I_{\mathcal{S}}(M)) \neq 1. \end{aligned}$$

This means that M is not a model of T, i.e. $M \notin \mathrm{Mod}(T)$, finishing the proof. □

Since $\mathrm{Mod}(T)$ is an $\mathbf{L}^*$-interior system on Y, we may consider for every $A \in \mathbf{L}^Y$ the largest model in $\mathrm{Mod}(T)$ covered by A. This largest model, which is clearly $I_{\mathrm{Mod}(T)}(A)$, has a very important property. Namely, as shown in the next theorem, the degree $||A \Rightarrow B||_T$ of entailment of any $A \Rightarrow B$ from T equals the degree to which $A \Rightarrow B$ is valid in this single model $I_{\mathrm{Mod}(T)}(A)$, as well as to the degree to which this model is included in B:

Theorem 4 *Let $A \Rightarrow B$ be an arbitrary fuzzy attribute implication, and let T be any theory of implications. Then,*

the degree of entailment $||A \Rightarrow B||_T$ equals
the degree of validity $||A \Rightarrow B||_{I_{\mathrm{Mod}(T)}(A)}$ equals
the degree of inclusion $S(I_{\mathrm{Mod}(T)}(A), B)$.

Proof Since $I_{\mathrm{Mod}(T)}(A)$ is a model of T, the definition of semantic entailment yields $||A \Rightarrow B||_T \leq ||A \Rightarrow B||_{I_{\mathrm{Mod}(T)}(A)}$. Due to property (17), which is obeyed by $I_{\mathrm{Mod}(T)}$, and since $1^* = 1$, we get $S(I_{\mathrm{Mod}(T)}(A), A)^* = 1$. Now, since by definition,

$$\begin{aligned} &||A \Rightarrow B||_{I_{\mathrm{Mod}(T)}(A)} \\ &\quad = S(I_{\mathrm{Mod}(T)}(A), A)^* \rightarrow S(I_{\mathrm{Mod}(T)}(A), B), \end{aligned}$$

and since $1 \rightarrow a = a$, we easily obtain

$$||A \Rightarrow B||_{I_{\mathrm{Mod}(T)}(A)} = S(I_{\mathrm{Mod}(T)}(A), B).$$

Consider now any model M of T. As $I_{\mathrm{Mod}(T)}$ is the interior operator corresponding to T, we have $M = I_{\mathrm{Mod}(T)}(M)$. Applying (18) and $M = I_{\mathrm{Mod}(T)}(M)$, we obtain

$$\begin{aligned} &S(M, A)^* \otimes S(I_{\mathrm{Mod}(T)}(A), B) \\ &\quad \leq S(I_{\mathrm{Mod}(T)}(M), I_{\mathrm{Mod}(T)}(A)) \otimes S(I_{\mathrm{Mod}(T)}(A), B) \\ &\quad \leq S(I_{\mathrm{Mod}(T)}(M), B) = S(M, B). \end{aligned}$$

The adjointness property applied to the previous inequality finally yields

$$S(I_{\mathrm{Mod}(T)}(A), B) \leq S(M, A)^* \rightarrow S(M, B),$$

for each $M \in \mathrm{Mod}(T)$, which is the required inequality because $S(M, A)^* \rightarrow S(M, B) = ||A \Rightarrow B||_M$. We proved $S(I_{\mathrm{Mod}(T)}(A), B) \leq ||A \Rightarrow B||_T$. □

4 Syntactico-semantical completeness

In this section, we introduce an axiomatic system for our logic, which is inspired by the classic Armstrong system (Armstrong 1974). We then proceed to establish two kinds of completeness theorem for our system. First is the ordinary-style completeness according to which an arbitrary implication $A \Rightarrow B$ is provable from a crisp theory T of implications if and only if $A \Rightarrow B$ semantically follows from T to degree 1. Second is the graded-style completeness according to which it holds that the degree of provability from a theory T of an arbitrary implication $A \Rightarrow B$ equals the degree to which $A \Rightarrow B$ semantically follows from T.

We start by presenting our basic deduction rules:

(Ax) (from any premises) infer conclusion $A \Rightarrow A \cup B$,
(DCut) from premises $A \Rightarrow B$ and $B \cap C \Rightarrow D$ infer conclusion $A \cap C \Rightarrow D$,
(Sh) from premise $A \Rightarrow B$ infer conclusion $c^* \rightarrow A \Rightarrow c^* \rightarrow B$

for each $A, B, C, D \in \mathbf{L}^Y$, and $c \in L$. Note that the fuzzy set $c^* \rightarrow A$ is defined by $(c^* \rightarrow A)(y) = c^* \rightarrow A(y)$. Note furthermore that the rule (Ax) is essentially an axiom, and that according to this rule, any formula of the form $A \Rightarrow A \cup B$ can be derived in a single inference step.

While (Ax) and (DCut) are inspired by the ordinary rules of axiom and cut (in fact, (DCut) is dual to the usual rule of cut), rule (Sh) is a new rule in our setting. It is easy to see that if the hedge * is the globalization, rule (Sh) may be dropped. This is because if c equals 1 (Sh) clearly says "from $A \Rightarrow B$ infer $A \Rightarrow B$" and is thus a trivial rule. If $c < 1$, then because $c^* = 0$, rule (Sh) allows us to infer from $A \Rightarrow B$ the trivial conclusion $Y \Rightarrow Y$, can be inferred by (Ax) and thus can be omitted.

For ordinary provability, we use the usual notions. Thus, for an ordinary (i.e. crisp) theory T we denote by $T \vdash_{\mathcal{R}} A \Rightarrow B$ the fact that $A \Rightarrow B$ is provable from T, which means that it may be derived from the implications in T using a set $\mathcal{R}$ of deduction rules. Furthermore, we call a deduction rule of the form "from $\varphi_1, \ldots, \varphi_n$ infer φ" derivable from a set $\mathcal{R}$ of other rules if $\{\varphi_1, \ldots, \varphi_n\} \vdash_{\mathcal{R}} \varphi$. If the subscript $\mathcal{R}$ is omitted, it means that $\mathcal{R}$ consists of (Ax)–(Sh).

It is a matter of routine verification that the following rules are derivable from (Ax) and (DCut):

(Ref) (from any premises) infer conclusion $A \Rightarrow A$,
(Wea) from premise $A \Rightarrow B$ infer conclusion $A \cap C \Rightarrow B$,
(Add) from premises $A \Rightarrow B$ and $A \Rightarrow C$ infer conclusion $A \Rightarrow B \cap C$,
(Pro) from premise $A \Rightarrow B \cap C$ infer conclusion $A \Rightarrow B$,
(Tra) from premises $A \Rightarrow B$ and $B \Rightarrow C$ infer conclusion $A \Rightarrow C$,

for each $A, B, C, D \in \mathbf{L}^Y$.

4.1 Ordinary-style completeness

Before proving the completeness theorem, we need some auxiliary results. We call a deduction rule *sound* if every model of its premises is also a model of its conclusions, which for a rule "from $\varphi_1, \ldots, \varphi_n$ infer φ" means that if $M \in \mathrm{Mod}(\{\varphi_1, \ldots, \varphi_n\})$ then $M \in \mathrm{Mod}(\{\varphi\})$. Soundness of a rule thus means that whenever the premises are true, the conclusion is true as well. Our rules are sound:

Lemma 5 *The deduction rules* (Ax)*,* (DCut)*, and* (Sh) *are sound.*

Proof (Ax): Clearly, for any M we have $S(M, A)^* \leq S(M, A \cup B)$, i.e. $||A \Rightarrow A \cup B||_M = 1$, proving $M \in \mathrm{Mod}(\{A \Rightarrow A \cup B\})$.

(DCut): We need to check that $M \in \mathrm{Mod}(\{A \Rightarrow B, B \cap C \Rightarrow D\})$ implies $M \in \mathrm{Mod}(\{A \cap C \Rightarrow D\})$. We prove a stronger claim, namely

$$(||A \Rightarrow B||_M)^* \otimes (||B \cap C \Rightarrow D||_M)^* \leq ||A \cap C \Rightarrow D||_M.$$

As one easily observes, this claim is equivalent to

$$S(M, A \cap C)^* \otimes \left[S(M, A)^* \to S(M, B)\right]^* \otimes \left[S(M, B \cap C)^* \to S(M, D)\right]^* \leq S(M, D).$$

The latter inequality holds true since

$$\begin{aligned} &S(M, A \cap C)^* \otimes (S(M, A)^* \to S(M, B))^* \otimes \left[S(M, B \cap C)^* \to S(M, D)\right]^* \\ &\leq (S(M, A)^* \wedge S(M, C)^*) \otimes (S(M, A)^* \to S(M, B))^* \otimes \left[S(M, B \cap C)^* \to S(M, D)\right]^* \\ &\leq S(M, A)^* \otimes \left[S(M, A)^* \to S(M, B)\right]^* \otimes S(M, C)^* \otimes \left[S(M, B \cap C)^* \to S(M, D)\right] \\ &\leq S(M, B)^* \otimes S(M, C)^* \otimes \left[S(M, B \cap C)^* \to S(M, D)\right] \\ &\leq S(M, D). \end{aligned}$$

(Sh): Let $M \in \mathrm{Mod}(\{A \Rightarrow B\})$. We have to show that $M \in \mathrm{Mod}(\{c^* \to A \Rightarrow c^* \to B\})$. Observe first that $M \in \mathrm{Mod}(\{A \Rightarrow B\})$ iff $||A \Rightarrow B||_M = 1$ iff $S(M, A)^* \leq S(M, B)$ iff

$$\text{for each } y \in Y: \ M(y) \otimes S(M, A)^* \leq B(y). \qquad (22)$$

Now, $M \in \mathrm{Mod}(\{c^* \to A \Rightarrow c^* \to B\})$ iff $S(M, c^* \to A)^* \leq S(M, c^* \to B)$ iff for each $y \in Y$ we have $M(y) \otimes S(M, c^* \to A)^* \leq c^* \to B(y)$, i.e. $M(y) \otimes c^* \otimes S(M, c^* \to A)^* \leq B(y)$, which is true:

$$\begin{aligned} &M(y) \otimes c^* \otimes S(M, c^* \to A)^* \\ &\leq M(y) \otimes c^* \otimes (c^{**} \to S(M, A)^*) \\ &= M(y) \otimes c^* \otimes (c^* \to S(M, A)^*) \\ &\leq M(y) \otimes S(M, A)^* \leq B(y), \end{aligned}$$

the last inequality being true due to (22), proving the soundness of (Sh). □

Call an ordinary theory T of fuzzy attribute implications *semantically closed* if all its semantics consequences are already contained in T, i.e. if for every $A \Rightarrow B$ we have $A \Rightarrow B \in T$ if and only if $||A \Rightarrow B||_T = 1$. Analogously, T is called *syntactically closed* if the same is true for its syntactic consequences (formulas provable from T), i.e. if for every $A \Rightarrow B$ we have $A \Rightarrow B \in T$ if and only if $T \vdash A \Rightarrow B$.

Lemma 6 *If an ordinary theory T of fuzzy attribute implications is semantically closed, it is also syntactically closed.*

Proof One may easily observe that an ordinary theory T of FAIs is syntactically closed iff we have:

$A \Rightarrow A \cup B \in T$,
if $A \Rightarrow B \in T$ and $B \cap C \Rightarrow D \in T$ then $A \cap C \Rightarrow D \in T$,
if $A \Rightarrow B \in T$ then $c^* \to A \Rightarrow c^* \to B \in T$

for each $A, B, C, D \in \mathbf{L}^Y$, and $c \in L$. Therefore, we have to show that for each deduction rule "from $\varphi_1, \ldots, \varphi_n$ infer φ," i.e. one of (Ax)–(Sh), $\varphi_1, \ldots \varphi_n \in T$ implies $\varphi \in T$. Let thus $\varphi_1, \ldots \varphi_n \in T$. Since $\{\varphi_1, \ldots \varphi_n\} \subseteq T$, for any model $M \in \mathrm{Mod}(T)$ we have

$$M \in \mathrm{Mod}(\{\varphi_1, \ldots \varphi_n\}).$$

Since each of the rules (Ax)–(Sh) is sound by Lemma 5, we conclude $M \in \text{Mod}(\{\varphi\})$. Since M is an arbitrary model of T, this shows that φ semantically follows from T. Since T is semantically closed, we get $\varphi \in T$. ◻

The following lemma is crucial for our proof of completeness. Note that similarly as in case of Armstrong rules, the assumption of finiteness may be dropped if we employ infinitary rules (we omit this technical issue).

Lemma 7 *If an ordinary theory T of fuzzy attribute implications over a finite set Y and a finite set L of truth degrees is syntactically closed, it is also semantically closed.*

Proof To verify that a syntactically closed theory T is semantically closed, it is sufficient to verify that $\{A \Rightarrow B \mid ||A \Rightarrow B||_T = 1\} \subseteq T$. To check this inclusion, we show that $A \Rightarrow B \notin T$ implies $A \Rightarrow B \notin \{A \Rightarrow B \mid ||A \Rightarrow B||_T = 1\}$. Notice for this purpose that since T is closed syntactically, it is also closed under any of the derived rules (Ref)–(Tra) listed above.

Assume $A \Rightarrow B \notin T$. We demonstrate that a model $M \in \text{Mod}(T)$ exists that is not a model of $A \Rightarrow B$, and this clearly implies the required inclusion $A \Rightarrow B \notin \{A \Rightarrow B \mid ||A \Rightarrow B||_T = 1\}$. Consider the fuzzy set $M = A^-$ of attributes defined as follows: A^- is the smallest fuzzy set for which $A \Rightarrow A^-$ is in T.

Observe first that such A^- exists: Since $A \Rightarrow A \in T$ due to (Ref), the set $S = \{C \mid A \Rightarrow C \in T\}$ is non-empty; moreover, since Y and L are finite, S is finite; finally, S is closed under intersections because if $A \Rightarrow C_1, \ldots, A \Rightarrow C_n \in T$ then $A \Rightarrow \bigcap_{i=1}^n C_i \in T$ by a repeated use of (Add).

Next, we show that (a) A^- is not a model of $A \Rightarrow B$, i.e. $||A \Rightarrow B||_{A^-} \neq 1$, and that (b) A^- is a model of T, i.e. $||C \Rightarrow D||_{A^-} = 1$ for every $C \Rightarrow D \in T$.

(a): Assume, by way of contradiction, that $||A \Rightarrow B||_{A^-} = 1$. From $A^- \subseteq A$ it follows $A^- \subseteq B$ because $1 = ||A \Rightarrow B||_{A^-} = S(A^-, A)^* \to S(A^-, B) = 1 \to S(A^-, B) = S(A^-, B)$. An application of rule (Pro) to $A \Rightarrow A^- \in T$ now yields $A \Rightarrow B \in T$, which is a contradiction with our assumption.

(b): For any $C \Rightarrow D \in T$, we need to check that $||C \Rightarrow D||_{A^-} = 1$. That is, to check $S(A^-, C)^* \to S(A^-, D) = 1$. The latter equality holds iff

$$S(A^-, C)^* \otimes A^- \subseteq D, \text{ i.e. iff } A^- \subseteq S(A^-, C)^* \to D.$$

To verify the last inclusion, we prove

$$A \Rightarrow S(A^-, C)^* \to D \in T.$$

Observe that this is indeed sufficient, as A^- is the smallest fuzzy set with $A \Rightarrow A^- \in T$. The following three assertions are furthermore available:

(i) $A \Rightarrow A^- \in T$ (directly from the definition of A^-),
(ii) $A^- \Rightarrow S(A^-, C)^* \to C \in T$
(namely, $A^- \Rightarrow S(A^-, C)^* \to C$ is an instance of (Ax) because $A^- \subseteq S(A^-, C)^* \to C$),
(iii) $S(A^-, C)^* \to C \Rightarrow S(A^-, C)^* \to D \in T$ (just apply (Sh) to $C \Rightarrow D \in T$).

Applying now (Tra) twice to (i), (ii), and (iii), $A \Rightarrow S(A^-, C)^* \to D \in T$ is readily obtained. ◻

We thus obtain the ordinary-style completeness for our logic:

Theorem 5 *For finite Y and L, and an ordinary theory T of fuzzy attribute implications, $T \vdash A \Rightarrow B$ if and only if $||A \Rightarrow B||_T = 1$, i.e. $A \Rightarrow B$ is provable from T iff $A \Rightarrow B$ semantically follows from T.*

Proof Denote by $\text{syn}(T)$ and $\text{sem}(T)$ the least syntactically and semantically closed ordinary theory of FAIs that contains T, respectively. It is easily shown that both syn and sem are closure operators and that $\text{syn}(T) = \{A \Rightarrow B \mid T \vdash A \Rightarrow B\}$ and $\text{sem}(T) = \{A \Rightarrow B \mid ||A \Rightarrow B||_T = 1\}$. To prove the claim, it is thus sufficient to establish $\text{syn}(T) = \text{sem}(T)$. As $\text{syn}(T)$ is syntactically closed, it is also semantically closed by Lemma 7, which means $\text{sem}(\text{syn}(T)) \subseteq \text{syn}(T)$. Therefore, since $T \subseteq \text{syn}(T)$, monotony of sem yields

$$\text{sem}(T) \subseteq \text{sem}(\text{syn}(T)) \subseteq \text{syn}(T).$$

In a similar manner, we get $\text{syn}(T) \subseteq \text{sem}(T)$, showing $\text{syn}(T) = \text{sem}(T)$, completing the proof. ◻

4.2 Graded-style completeness

Even though Theorem 5 connects provability and entailment, one may naturally ask if general degrees of entailment—different from 1 to which Theorem 5 restricts—may be characterized by a kind of generalized provability concept. For this purpose, we employ a concept of degree of provability of $A \Rightarrow B$ from T, denoted by $|A \Rightarrow B|_T$. With this concept in hand, we establish that $|A \Rightarrow B|_T = ||A \Rightarrow B||_T$, which equality may be looked at as expressing a completeness-to-degrees of our logic. This concept is inspired by the framework of Pavelka-style logic (Pavelka 1979a, b, c), see also, e.g. (Gerla 2001; Hájek 1998).

Let therefore T be a theory of fuzzy attribute implications, $A \Rightarrow B$ be an implication, and define the *degree to which $A \Rightarrow B$ is provable from T* by

$$|A \Rightarrow B|_T = \bigvee \{c \in L \mid \text{cr}(T) \vdash A \Rightarrow c \otimes B\}. \quad (23)$$

Note that in this definition, $\text{cr}(T)$ is defined as in Lemma 3. Observe that we made use in this definition of Lemma 4 which reduces entailment to an arbitrary degree to entailment to degree 1. One then obtains our graded-style completeness theorem:

Theorem 6 *For a theory T over finite Y and finite $\mathbf{L}$, it holds*

$$|A \Rightarrow B|_T = ||A \Rightarrow B||_T.$$

Proof Directly by using Lemma 4 and Theorem 5 and the above considerations. □

5 Non-redundant bases

In this section, we describe certain non-redundant fully informative sets of implications true in a given table $\langle X, Y, I\rangle$.

Definition 3 We call an ordinary theory (i.e. a set) T of fuzzy attribute implications *complete in* a table $\langle X, Y, I\rangle$ if for each implication $A \Rightarrow B$ we have $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\langle X,Y,I\rangle}$. If, moreover, no proper subset of T is complete in $\langle X, Y, I\rangle$, we call T a *base of* $\langle X, Y, I\rangle$.

Clearly, if T is complete then each $A \Rightarrow B \in T$ is valid in $\langle X, Y, I\rangle$ to degree 1 and, furthermore, for any other implication $C \Rightarrow D$, the degree $||C \Rightarrow D||_{\langle X,Y,I\rangle}$ to which $C \Rightarrow D$ is valid in $\langle X, Y, I\rangle$ is equal to the degree $||C \Rightarrow D||_T$ to which $C \Rightarrow D$ semantically follows from T. In this sense, bases are non-redundant ordinary theories with complete information about validity in data. The following theorem characterizes complete ordinary theories:

Theorem 7 *An ordinary theory T is complete if and only if* $\mathrm{Mod}(T) = \mathrm{Int}(I)$.

Proof Take a complete T and assume that $M \in \mathrm{Mod}(T)$, i.e. M is a model of T. Due to (12), we have $||M \Rightarrow M^{\cup\cap}||_{\mathrm{Int}(I)} = S(M^{\cup\cap}, M^{\cup\cap}) = 1$. Hence, $||M \Rightarrow M^{\cup\cap}||_T = 1$ because T is complete and because we have (12). From $M \in \mathrm{Mod}(T)$, we get $||M \Rightarrow M^{\cup\cap}||_M = 1$, hence $1 = S(M, M)^* \leq S(M, M^{\cup\cap})$ and, therefore, $M \subseteq M^{\cup\cap}$. Now, one always has $M^{\cup\cap} \subseteq M$, whence $M \in \mathrm{Int}(I)$, finishing the proof of $\mathrm{Mod}(T) \subseteq \mathrm{Int}(I)$.

Let furthermore $M \in \mathrm{Int}(I)$. Due to (12), $||A \Rightarrow B||_M \geq ||A \Rightarrow B||_{\mathrm{Int}(I)} = ||A \Rightarrow B||_{\mathrm{Mod}(T)} = 1$ for every $A \Rightarrow B \in T$. This means that M is a model of T, establishing $\mathrm{Int}(I) \subseteq \mathrm{Mod}(T)$.

On the other hand, from $\mathrm{Mod}(T) = \mathrm{Int}(I)$ we obtain using (12) that $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{Int}(I)} = ||A \Rightarrow B||_{\langle X,Y,I\rangle}$, i.e. T is complete. □

We now describe bases of $\langle X, Y, I\rangle$ that may be obtained from systems of so-called pseudo-intents of $\langle X, Y, I\rangle$. In case of attribute implications over Boolean attributes, the notion of a pseudo-intent goes back to (Guigues and Duquenne 1986) [see also (Ganter and Wille 1999)]; our notion for graded attributes is inspired by (Belohlavek and Vychodil 2016, 2017).

Definition 4 We call a system $\mathcal{P}$ of fuzzy sets in Y a *system of pseudo-intents* of a table $\langle X, Y, I\rangle$ if for every fuzzy set $P \in \mathbf{L}^Y$ the following holds: $P \in \mathcal{P}$ if and only if $P \neq P^{\cup\cap}$ and for each $Q \in \mathcal{P}$ with $Q \neq P$ we have $||Q \Rightarrow Q^{\cup\cap}||_P = 1$.

From now on, let $\mathcal{P}$ denote a system of pseudo-intents of $\langle X, Y, I\rangle$. We need the following auxiliary results.

Lemma 8 *For a system of pseudo-intents $\mathcal{P}$ of $\langle X, Y, I\rangle$, consider the set $T = \{P \Rightarrow P^{\cup\cap} \mid P \in \mathcal{P}\}$ of implications. Then,* $\mathrm{Mod}(T) \subseteq \mathrm{Int}(I)$*, i.e. every model of T in an intent of I.*

Proof We proceed by way of contradiction. To show that $\mathrm{Mod}(T) \subseteq \mathrm{Int}(I)$, we assume $M \notin \mathrm{Int}(I)$ which means $M \neq M^{\cup\cap}$. As M is a model of T, it follows that for each $Q \in \mathcal{P}$ one has $||Q \Rightarrow Q^{\cup\cap}||_M = 1$. By definition of a system of pseudo-intents, $M \in \mathcal{P}$ which implies that $M \Rightarrow M^{\cup\cap} \in T$. Now,

$$\begin{aligned} ||M \Rightarrow M^{\cup\cap}||_M &= S(M, M)^* \to S(M, M^{\cup\cap}) \\ &= S(M, M^{\cup\cap}) \neq 1, \end{aligned}$$

a contradiction to $M \in \mathrm{Mod}(T)$. □

Lemma 9 *For any $A, M \in \mathbf{L}^Y$, we have $||A \Rightarrow A^{\cup\cap}||_M = 1$ for every $A \in \mathbf{L}^Y$ and $M \in \mathrm{Int}(I)$.*

Proof Let $M \in \mathrm{Int}(I)$, i.e. $M = M^{\cup\cap}$. We have

$$\begin{aligned} S(M, A)^* &\leq S(M^{\cup}, A^{\cup})^* \\ &\leq S(M^{\cup *}, A^{\cup *}) \\ &\leq S(M^{\cup\cap}, A^{\cup\cap}) \\ &= S(M, A^{\cup\cap}). \end{aligned}$$

Thus, $S(M, A)^* \to S(M, A^{\cup\cap}) = 1$, i.e. $||A \Rightarrow A^{\cup\cap}||_M = 1$. □

Lemma 10 *For any system $\mathcal{P}$ of pseudo-intents of a table $\langle X, Y, I\rangle$, the set $T = \{P \Rightarrow P^{\cup\cap} \mid P \in \mathcal{P}\}$ is complete in $\langle X, Y, I\rangle$.*

Proof Due to (12), it suffices to verify that for any implication $A \Rightarrow B$ we have

$$||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{Int}(I)}$$

On the one hand, $||A \Rightarrow B||_T \leq ||A \Rightarrow B||_{\mathrm{Int}(I)}$ follows from the fact that every intent in $\mathrm{Int}(I)$ is a model of T, which was established in Lemma 9. Conversely, $||A \Rightarrow B||_T \geq ||A \Rightarrow B||_{\mathrm{Int}(I)}$ follows from Lemma 8. □

We thus obtain the main result in the present section.

Theorem 8 $T = \{P \Rightarrow P^{\cup\cap} \mid P \in \mathcal{P}\}$ *is a base of* $\langle X, Y, I\rangle$.

Proof By Lemma 10, T is complete. It remains to check minimality of T. Let $T' \subset T$. Take some $P \in \mathcal{P}$ such that $P \Rightarrow P^{\cup\cap}$ does not belong to T'. Definition 4 implies that $||Q \Rightarrow Q^{\cup\cap}||_P = 1$ for every $Q \in \mathcal{P}$ and $Q \neq P$, which means $P \in \text{Mod}(T')$. Since $||P \Rightarrow P^{\cup\cap}||_P = S(P, P^{\cup\cap}) \neq 1$, we obtain $||P \Rightarrow P^{\cup\cap}||_{T'} \neq 1$. On the other hand, Lemma 9 and (12) yield $||P \Rightarrow P^{\cup\cap}||_{\langle X,Y,I\rangle} = 1$, hence T' is not complete in $\langle X, Y, I\rangle$. □

In the remainder, we show that if L is finite and * is the globalization, the base T in Theorem 8 is in fact the smallest one. For this purpose, we need the following auxiliary result.

Lemma 11 *Suppose that fuzzy sets P and Q in Y are pseudo-intents or intents of* $\langle X, Y, I\rangle$, *i.e.* $P, Q \in \mathcal{P} \cup \text{Int}(I)$, *satisfying*

$$S(Q, P)^* \leq S(P \cup Q, P^{\cup\cap})$$

and

$$S(P, Q)^* \leq S(P \cup Q, Q^{\cup\cap}).$$

Then, $P \cup Q$ *is an intent of* $\langle X, Y, I\rangle$, *i.e.* $P \cup Q \in \text{Int}(I)$.

Proof Let T be the set of fuzzy attribute implications in Theorem 8 and consider its subset $T' = T - \{P \Rightarrow P^{\cup\cap}, Q \Rightarrow Q^{\cup\cap}\}$. On account of Definition 4 and Lemma 9, we obtain than both P and Q are models of T'. It follows that for every implication $A \Rightarrow B$ in T' one has $S(P, A)^* \leq S(P, B)$ and $S(Q, A)^* \leq S(Q, B)$. Therefore, $S(P \cup Q, A)^* = (S(P, A) \wedge S(Q, A))^* \leq S(P, A)^* \wedge S(Q, A)^* \leq S(P, B) \wedge S(Q, B) = S(P \cup Q, B)$. This inequality implies that $P \cup Q \in \text{Mod}(T')$. On account of Lemma 8, it remains to prove that $P \cup Q \in \text{Mod}(\{P \Rightarrow P^{\cup\cap}, Q \Rightarrow Q^{\cup\cap}\})$. Due to the two inequalities assumed in the present lemma, $S(P \cup Q, P)^* = S(Q, P)^* \leq S(P \cup Q, P^{\cup\cap})$ and $S(P \cup Q, Q)^* = S(P, Q)^* \leq S(P \cup Q, Q^{\cup\cap})$, which means by definition that both the required conditions, $||P \Rightarrow P^{\cup\cap}||_{P\cup Q} = 1$ and $||Q \Rightarrow Q^{\cup\cap}||_{P\cup Q} = 1$, are met and hence $P \cup Q \in \text{Mod}(\{P \Rightarrow P^{\cup\cap}, Q \Rightarrow Q^{\cup\cap}\})$ is indeed the case. □

Now we obtain:

Theorem 9 *Let* $\mathbf{L}$ *be a finite residuated lattice with* * *being the globalization, let* Y *be finite, let* $T = \{P \Rightarrow P^{\cup\cap} \mid P \in \mathcal{P}\}$. *If* T' *is complete in* $\langle X, Y, I\rangle$ *then* $|T| \leq |T'|$.

Proof We prove the claim by showing that for each $P \in \mathcal{P}$, T' contains an implication $A \Rightarrow B$ with $P \subseteq A$ and $A^{\cup\cap} = P^{\cup\cap}$ and that for mutually different $P_1, P_2 \in \mathcal{P}$, their corresponding implications in T' are also different.

Consider any pseudo-intent $P \in \mathcal{P}$. Because, by definition of a pseudo-intent, $P \neq P^{\cup\cap}$, and because T' is complete, we obtain by virtue of Theorem 7 that an implication $A \Rightarrow B$ exists in T' for which $||A \Rightarrow B||_P \neq 1$. As * is the globalization, we have $P \subseteq A$ and $P \nsubseteq B$. Thus, $P^{\cup\cap} \subseteq A^{\cup\cap}$. Now, since T' is complete and due to (12), we conclude $S(A^{\cup\cap}, B) = 1$, whence $A^{\cup\cap} \subseteq B$. As $P \nsubseteq B$, we finally get $P \nsubseteq A^{\cup\cap}$.

We now easily obtain that $A^{\cup\cap} \cup P \notin \text{Int}(I)$: $P \nsubseteq A^{\cup\cap}$ implies $A^{\cup\cap} \subset A^{\cup\cap} \cup P$; $P \subseteq A$ and $A^{\cup\cap} \subseteq A$ yield $A^{\cup\cap} \cup P \subseteq A$, hence $(A^{\cup\cap} \cup P)^{\cup\cap} \subseteq A^{\cup\cap}$ by monotony of $^{\cup\cap}$. To sum up, $(A^{\cup\cap} \cup P)^{\cup\cap} \subset A^{\cup\cap} \cup P$, i.e. $A^{\cup\cap} \cup P$ is not an intent.

We now verify $A^{\cup\cap} = P^{\cup\cap}$. First, the above-observed fact $P \subseteq A$ implies $P^{\cup\cap} \subseteq A^{\cup\cap}$. It remains to prove $A^{\cup\cap} \subseteq P^{\cup\cap}$. This inclusion readily follows from $A^{\cup\cap} \subseteq P$ which we now verify. By contradiction, if $A^{\cup\cap} \nsubseteq P$ then the fact $P \nsubseteq A^{\cup\cap}$ observed above and Lemma 11 yield $A^{\cup\cap} \cup P \in \text{Int}(I)$, contradicting the above observation.

It remains to check that if $P_1, P_2 \in \mathcal{P}$ are different, then no single $A \Rightarrow B \in T'$ can satisfy $P_1, P_2 \subseteq A$ and $P_1^{\cup\cap} = A^{\cup\cap} = P_2^{\cup\cap}$. By contradiction, assume that $A \Rightarrow B \in T'$ has this property. Observe first that $P_1 \subset P_2$ cannot be the case: Due to the definition of a pseudo-intent, $P_1 \subset P_2$ implies $P_1 \subseteq P_2^{\cup\cap}$, hence $A^{\cup\cap} = P_1^{\cup\cap} \subset P_1 \subseteq P_2^{\cup\cap} = A^{\cup\cap}$, a contradiction. Similarly one observes that $P_2 \subset P_1$ cannot be the case, hence we have $S(P_1, P_2) < 1$ and $S(P_2, P_1) < 1$, and thus, $S(P_1, P_2)^* = 0$ and $S(P_2, P_1)^* = 0$. Lemma 11 now yields $P_1 \cup P_2 \in \text{Int}(I)$. As $P_1, P_2 \subseteq A$, we have $P_1 \cup P_2 \subseteq A$ and thus also $(P_1 \cup P_2)^{\cup\cap} \subseteq A^{\cup\cap}$. Since P_1 is a pseudo-intent, we have $P_1^{\cup\cap} \subset P_1$, hence also $P_1^{\cup\cap} \subset P_1 \cup P_2 = (P_1 \cup P_2)^{\cup\cap} \subseteq A^{\cup\cap}$, a contradiction to the assumption $P_1^{\cup\cap} = A^{\cup\cap}$. □

6 Conclusions and further issues

We examined a logic for dependencies describing containment of fuzzy attributes and established several results for this logic. The grades are assumed to be members of a complete residuated lattice and our semantics is based on the operations in this lattice. The dependencies involved may be seen as dual to those of (Belohlavek and Vychodil 2016, 2017). However, the lack of certain laws in residuated structures of truth degrees, such as the law of double negation, prevents a reduction of the present dependencies to those from (Belohlavek and Vychodil 2016, 2017) and requires a separate inquiry.

Among the main results established in the paper are: results regarding validity of dependencies, their models, and entailment; connections to related structures, particularly to isotone Galois connections and the lattices of their fixpoints; an axiomatic system for reasoning with the dependencies including two versions of completeness theorem;

basic results on bases, i.e. minimal fully informative sets of dependencies that are true in a given data. To keep the paper concise, we did not examine related computational problems. These problems include the problem of computing a degree of entailment of a given dependency from a given set of dependencies, the problem of computing various kinds of bases, including the significant base we described, and various other problems ordinarily studied for data dependencies. Such problems remain a future research topic.

Acknowledgements The paper is a substantially extended version of a paper presented by the authors at the International Symposium on Knowledge Acquisition and Modeling. This study was funded by Grant No. GA15-17899S of the Czech Science Foundation.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants performed by any of the authors.

References

Armstrong WW (1974) Dependency structures of data base relationships. In: IFIP congress, vol 74. Geneva, pp 580–583
Bartl E, Belohlavek R (2011) Knowledge spaces with graded knowledge states. Inf Sci 181(8):1426–1439
Belohlavek R (2012) Fuzzy relational systems: foundations and principles, vol 20. Kluwer, New York
Belohlavek R, Konecny J (2012) Concept lattices of isotone vs. antitone Galois connections in graded setting: mutual reducibility revisited. Inf Sci 199:133–137
Belohlavek R, Vychodil V (2006) Attribute implications in a fuzzy setting. In: ICFCA 2006, international conference on formal concept analysis, LNAI, vol 3874. Springer, pp 45–60
Belohlavek R, Vychodil V (2016) Attribute dependencies for data with grades I. Int J Gen Syst 45:66–92
Belohlavek R, Vychodil V (2017) Attribute dependencies for data with grades II. Int J Gen Syst 46:66–92
Belohlavek R, Funioková T, Vychodil V (2005) Fuzzy closure operators with truth stressers. Log J IGPL 13(5):503–513
Ciucci D, Dubois D, Prade H (2014) The structure of oppositions in rough set theory and formal concept analysis-toward a new bridge between the two settings. In: International symposium on foundations of information and knowledge systems. Springer, pp 154–173
Doignon JP, Falmagne JC (2012) Knowledge spaces. Springer, Berlin
Ganter B, Wille R (1999) Formal concept analysis: mathematical foundations. Springer, Berlin
Georgescu G, Popescu A (2004) Non-dual fuzzy connections. Arch Math Log 43(8):1009–1039
Gerla G (2001) Fuzzy logic: mathematical tools for approximate reasoning. Springer, Dordrecht
Goguen JA (1969) The logic of inexact concepts. Synthese 19(3–4):325–373
Guigues JL, Duquenne V (1986) Familles minimales d'implications informatives résultant d'un tableau de données binaires. Math Sci Hum 95:5–18
Hájek P (1998) Metamathematics of fuzzy logic, vol 4. Kluwer, Dordrecht
Hájek P, Havránek T (2012) Mechanizing hypothesis formation: mathematical foundations for a general theory. Springer, Berlin
Konecny J (2011) Isotone fuzzy Galois connections with hedges. Inf Sci 181(10):1804–1817
Maier D (1983) Theory of relational databases. Computer Science Press, Rockville
Pavelka J (1979a) On fuzzy logic i. Z Math Log Grundl Math 22:45–52
Pavelka J (1979b) On fuzzy logic ii. Z Math Log Grundl Math 22:119–134
Pavelka J (1979c) On fuzzy logic iii. Z Math Log Grundl Math 22:447–464
Rauch J (2005) Logic of association rules. Appl Intell 22(1):9–28
Zhang C, Zhang S (2002) Association rule mining: models and algorithms. Springer, Berlin