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Concept lattices of isotone vs. antitone Galois connections in graded setting: Mutual reducibility revisited

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ABSTRACT

It is well known that concept lattices of isotone and antitone Galois connections induced by an ordinary binary relation and its complement are isomorphic, via a natural isomorphism mapping extents to themselves and intents to their complements. It is also known that in a fuzzy setting, this and similar kinds of reduction fail to hold. In this note, we show that when the usual notion of a complement, based on a residuum w.r.t. 0, is replaced by a new one, based on residua w.r.t. arbitrary truth degrees, the above-mentioned reduction remains valid. For ordinary relations, the new and the usual complement coincide. The result we present reveals a new, deeper root of the reduction: It is not the availability of the law of double negation but rather the fact that negations are implicitly present in the construction of concept lattices of isotone Galois connections.

1. Problem setting

As is well-known, a given ordinary binary relation $I \in \{0,1\}^{X\times Y}$ (representing, e.g. a yes/no relationship between objects $x \in X$ and attributes $y \in Y$) induces two important pairs of operators between $\{0,1\}^X$ and $\{0,1\}^Y$. Namely, a pair $\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ defined by

$$\begin{aligned} A^{\uparrow_I} &= \{y \in Y \mid \text{ for each } x \in A : \langle x,y\rangle \in I\},\\ B^{\downarrow_I} &= \{x \in X \mid \text{ for each } y \in B : \langle x,y\rangle \in I\}, \end{aligned} \tag{1}$$

and a pair $\langle {}^{\cap_I}, {}^{\cup_I}\rangle$ defined by

$$\begin{aligned} A^{\cap_I} &= \{y \in Y \mid \text{there exists } x \in A \text{ such that } \langle x,y\rangle \in I\},\\ B^{\cup_I} &= \{x \in X \mid \text{for each } y \in Y : \langle x,y\rangle \in I \text{ implies } y \in B\}, \end{aligned} \tag{2}$$

for all subsets A of X and B of Y. These operators are employed in several areas including data analysis, such as formal concept analysis in particular [8] or association rules, logic and reasoning about data [7], or ordered sets and their applications [6]. It is well known that the two pairs of operators are mutually definable [7]. An important consequence is that with $\neg$ denoting the set complement, the sets of fixpoints, i.e. the concept lattices

$$\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I) \quad \text{and} \quad \mathcal{B}(X^{\uparrow_{\neg I}}, Y^{\downarrow_{\neg I}}, \neg I) \text{ are isomorphic as lattices,} \tag{3}$$

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(or, equivalently, $\mathcal{B}(X^{\cap_{\neg I}}, Y^{\cup_{\neg I}}, \neg I)$ and $\mathcal{B}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$ are isomorphic), with $\langle A, B\rangle \mapsto \langle A, \neg B\rangle$ being an isomorphism. Hence, in particular,

$$\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \mathrm{Ext}(X^{\uparrow_{\neg I}}, Y^{\downarrow_{\neg I}}, \neg I), \tag{4}$$

i.e. the corresponding sets of extents are equal. Here, the concept lattices and the sets of extents of a binary relation $I \in \{0,1\}^{X\times Y}$ are defined by

$$\mathcal{B}(X^{\uparrow_I}, Y^{\downarrow_I}, I) = \{\langle C, D\rangle \in \{0,1\}^X \times \{0,1\}^Y | C^{\uparrow_I} = D, D^{\downarrow_I} = C\}, \tag{5}$$

$$\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I) = \{\langle C, D\rangle \in \{0,1\}^X \times \{0,1\}^Y | C^{\cap_I} = D, D^{\cup_I} = C\}, \tag{6}$$

$$\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I) = \{C \in \{0,1\}^X | \langle C, D\rangle \in \mathcal{B}(X^{\uparrow_I}, Y^{\downarrow_I}, I) \text{ for some } D\}, \tag{7}$$

$$\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \{C \in \{0,1\}^X | \langle C, D\rangle \in \mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I) \text{ for some } D\}. \tag{8}$$

The above reducibility results mean that, in a sense, one need not investigate the properties of the concept lattices of $\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ and $\langle {}^{\cap_I}, {}^{\cup_I}\rangle$ separately because the properties of one are derivable from those of the other.

However, as shown in [9], when fuzzy relations instead of ordinary relations I are considered (i.e. graded attributes rather than yes/no attributes are considered), the above mutual reducibility results are no longer true. In this note, we show that when the notion of a complement of a fuzzy relation is defined in a new way, (3) and (4) remain valid even in the setting of fuzzy relations. We also show that in the other direction, the reducibility results cannot be saved even with the new notion of complement. Since in the case of ordinary relations the new notion of complement coincides with the usual one, our result puts the known reducibility results in a different perspective that we discuss.

2. Result and remarks

We assume that the set L of truth degrees along with the truth functions $\otimes$ of conjunction and $\to$ of implication forms a complete residuated lattice, i.e. a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ satisfying: $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice; $\langle L, \otimes, 1\rangle$ is a commutative monoid; $\otimes$ and $\to$ satisfy adjointness, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \to c$. We assume that the reader is familiar with examples and properties of residuated lattices [2,10,11,13].

A fuzzy relation $I \in L^{X\times Y}$ induces two pairs of operators between L^X and L^Y, i.e. the sets of all fuzzy sets in X and Y, defined by

$$A^{\uparrow_I}(y) = \bigwedge_{x\in X}(A(x) \to I(x,y)), \quad B^{\downarrow_I}(x) = \bigwedge_{y\in Y}(B(y) \to I(x,y)), \tag{9}$$

$$A^{\cap_I}(y) = \bigvee_{x\in X}(A(x) \otimes I(x,y)), \quad B^{\cup_I}(x) = \bigwedge_{y\in Y}(I(x,y) \to B(y)), \tag{10}$$

for all fuzzy sets $A \in L^X$ and $B \in L^Y$. Clearly, (9) and (10) generalize the above operators defined by (1) and (2) (just put $L = \{0,1\}$). $\mathcal{B}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$, $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$, $\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$, and $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ are defined by the same formulas as in (5)–(8) with $\{0,1\}$ replaced by L. For more information we refer, e.g. to [2–4,9].

As was mentioned above, when I is a fuzzy relation (3) and (4) fail to hold. This fact was for the first time observed in [9] and is well known. In this observation, however, it is crucial that the complement $\neg I$ of a fuzzy relation I between X and Y is conceived as a fuzzy relation between X and Y defined by

$$\neg I(x,y) = I(x,y) \to 0. \tag{11}$$

That is, one uses the truth function $\neg$ of negation defined by

$$\neg a = a \to 0, \tag{12}$$

for each $a \in L$ and the standard way of defining a complement of a fuzzy set by means of $\neg$.

As we show in what follows, there is another notion of complement of I. Both $\neg I$, as defined above, and the new notion of complement coincide with the ordinary notion of complement in the ordinary case, i.e. when $L = \{0,1\}$. However, the new notion of complement has the advantage that a part of the reducibility results, namely (3) and (4), remain true even when I is a fuzzy relation (see Remark 2b for a reducibility result that does not hold with any notion of complement).

The classical notion of complement $\neg I$ of a fuzzy relation may be looked at the following way. Each attribute $y \in Y$ in the data table representing I is replaced by its complement. That it, each fuzzy set $I_y \in L^X$, representing attribute y, defined by $I_y(x) = I(x,y)$ is replaced in the table by its complement $\neg I_y$ defined by

$$(\neg I_y)(x) = \neg(I_y(x)), \quad \text{i.e.} \quad (\neg I_y)(x) = I_y(x) \to 0.$$

The complement (12) is in fact the residuum of a w.r.t. 0. However, one may also consider a residuum of $a \in L$ w.r.t. to an arbitrary element $b \in L$, i.e. one may consider

$$\neg_b a = a \to b, \tag{13}$$

of which $\neg a$ is a particular case because $\neg a = \neg_0 a$. In addition to $\neg I_y$, the "negation relative to 0" one may therefore also consider $\neg_b I_y$, the "negation relative to b", for other degrees b, defined by

$$(\neg_b I_y)(x) = \neg_b(I_y(x)), \quad \text{i.e.} \quad (\neg_b I_y)(x) = I_y(x) \rightarrow b.$$

For every original attribute y, I_y may therefore be replaced not just by the complement $\neg_0 I_y$ w.r.t. 0 but by several complements $\neg_b I_y$ w.r.t. $b \in K$ with $K \subseteq L$ being a set of selected values, bringing us the following definition.

Definition 1. For a set $K \subseteq L$, the K-complement of a fuzzy relation I between X and Y is a fuzzy relation $\neg_K I$ between X and $Y \times K$ defined by

$$(\neg_K I)(x, \langle y, b\rangle) = \neg_b I(x, y), \tag{14}$$

for every $x \in X$, $y \in Y$, and $b \in K$.

Remark 1.

(a) Going from I to $\neg_K I$ may be seen as replacing every attribute $y \in Y$, represented by I_y in I, by a collection of new attributes $\langle y, b\rangle \in Y \times K$, represented by $\neg_b I_y$ in $\neg_K I$ for $b \in K$.
(b) Clearly, for $K = \{0\}$, $\neg_K I$ may be identified with $\neg I$, because $Y \times \{0\}$ may be identified with Y and $\neg_K I(x, \langle y, \{0\}\rangle) = \neg I(x, y)$.
(c) Observe that for $L = \{0, 1\}$ (the ordinary case), $\neg_{L-\{1\}} I = \neg_{\{0\}} I$, i.e. in view of (b) of this Remark, $\neg_{L-\{1\}} I$ may be identified with the classical complement $\neg I$ of I.

In view of Remark 1c, there are two ways to generalize the notion of a complement of an ordinary relation I between X and Y to a fuzzy setting:

(i) First, a complement of I may be defined as a fuzzy relation between X and Y by (11).
(ii) Second, a complement of I may be defined as a fuzzy relation between X and $Y \times K$ by (14) with $K = L - \{1\}$.

While (3) and (4) fail to hold in a fuzzy setting for (i), they do hold in a fuzzy setting with the complement understood according to (ii):

Theorem 1. *For a fuzzy relation I between X and Y, let $\lrcorner I$ denote $\neg_{L-\{1\}} I$. Then $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ are isomorphic as lattices, with the mappings $\langle A, B\rangle \mapsto \langle A, D\rangle$, where*

$$D(y, b) = \neg_b B(y), \tag{15}$$

for $y \in Y$, $b \in L - \{1\}$, and $\langle A, D\rangle \mapsto \langle A, B\rangle$, where

$$B(y) = \bigwedge_{b \in L-\{1\}} \neg_b D(y, b), \tag{16}$$

for $y \in Y$, being the isomorphism and its inverse. Hence, in particular,

$$\text{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \text{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I). \tag{17}$$

Proof. We first prove (17). Since ${}^{\uparrow_{\lrcorner I}\downarrow_{\lrcorner I}}$ is an **L**-closure operator in X [2], it follows that $\text{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ is an **L**-closure system in X, i.e. it is closed under arbitrary $\bigwedge$-intersections and left $\rightarrow$-multiplications. This means that for all $A_j \in \text{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$, $j \in J$, we have $\bigwedge_{j \in J} A_j \in \text{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ and for each $a \in L$ and $A \in \text{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ we have $a \rightarrow A \in \text{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ with $a \rightarrow A \in L^X$ defined by $(a \rightarrow A)(x) = a \rightarrow A(x)$ for each $x \in X$. Moreover, [4, Theorem 2 (10)] implies that $\text{Ext}(X^{\uparrow_{\lrcorner I}}, Y \times (L - \{1\})^{\downarrow_{\lrcorner I}}, \lrcorner I)$ is the least **L**-closure system in X containing every column of $\lrcorner I$, i.e. every $\neg_b I_y$ for each $b \in L - \{1\}$.

To prove (17), it is therefore sufficient to show that $\text{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ is the least **L**-closure system in X containing every column of $\lrcorner I$. This assertion follows from the fact that $\text{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ is always an **L**-closure system and from the following claim. □

Claim 1. $\text{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ *consists of all possible $\bigwedge$-intersections of fuzzy sets $\neg_b I_y (y \in Y, b \in L - \{1\})$.*

*Namely, if $\mathcal{S}$ is an **L**-closure system that contains every column of $\lrcorner I$, it contains all intersections of the columns of $\lrcorner I$ and, due to Claim, it contains* $\text{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$*. Therefore, to prove (17), it remains to prove Claim.*

Proof of Claim 1. Since ${}^{\cap_I}$ and ${}^{\cup_I}$ form an isotone Galois connection, we have

$$\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I) = \{B^{\cup_I} | B \in L^Y\}. \tag{18}$$

On one hand, every $B^{\cup_I}$ is an intersection of fuzzy sets of the form $\neg_b I_y$ because

$$B^{\cup_I}(x) = \bigwedge_{y \in Y}(I(x,y) \to B(y)) = \bigwedge_{y \in Y} \neg_{B(y)} I_y. \tag{19}$$

On the other hand, consider an arbitrary intersection A of $\neg_b I_y$s, i.e. $A = \bigwedge_{\langle y,b\rangle \in P} \neg_b I_y$ for some $P \subseteq Y \times (L - \{1\})$. Define $B(y) = \bigwedge_{\langle y,b\rangle \in P} b$. Then

$$A(x) = \bigwedge_{y \in Y} \bigwedge_{\langle y,b\rangle \in P} (I(x,y) \to b) = \bigwedge_{y \in Y} I(x,y) \to \bigwedge_{\langle y,b\rangle \in P} b = \bigwedge_{y \in Y} I(x,y) \to B(y) = B^{\cup_I}(x),$$

hence $A \in \mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$, finishing the proof of Claim and hence also the proof of (17).

Now, since $\mathrm{Ext}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathrm{Ext}(X^{\uparrow_\lrcorner}, Y \times (L - \{1\})^{\downarrow_\lrcorner}, \lrcorner I)$ are isomorphic as lattices to $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X^{\uparrow_\lrcorner}, Y \times (L - \{1\})^{\downarrow_\lrcorner}, \lrcorner I)$, respectively, it follows that $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X^{\uparrow_\lrcorner}, Y \times (L - \{1\})^{\downarrow_\lrcorner}, \lrcorner I)$ are isomorphic as lattices.

Take any $\langle A, B\rangle \in \mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and the corresponding $\langle A, D\rangle \in \mathcal{B}(X^{\uparrow_\lrcorner}, Y \times (L - \{1\})^{\downarrow_\lrcorner}, \lrcorner I)$. Then

$$\begin{aligned} D(y,b) = A^{\uparrow_\lrcorner}(y,b) &= \bigwedge_{x \in X} A(x) \to \lrcorner I(x, \langle y,b\rangle) = \bigwedge_{x \in X} A(x) \to (I(x,y) \to b) = \bigwedge_{x \in X} ((A(x) \otimes I(x,y)) \to b) \\ &= \left[\bigvee_{x \in X} (A(x) \otimes I(x,y))\right] \to b = A^{\cap_I}(y) \to b = B(y) \to b = \neg_b B(y), \end{aligned}$$

verifying (15). To check (16), consider any $A \in L^X$ and the corresponding $B = A^{\cap_I}$ and $D = A^{\uparrow_\lrcorner}$. Observe first that

$$B(y) \leqslant \neg_b D(y,b), \tag{20}$$

for each $b \in L - \{1\}$. Indeed, taking into account $a \leqslant (a \to b) \to b = \neg_b \neg_b a$ for any $a \in L$ and (15), we have $B(y) \leqslant \neg_b \neg_b B(y) = \neg_b D(y,b)$. This verifies the "$\leqslant$" part of (16). Let now $c = B(y)$. If $c < 1$, then c is one of the degrees from $L - \{1\}$ over which the infimum in (16) is taken and since $\neg_c D(y,c) = \neg_c \neg_c B(y) = \neg_c \neg_c c = c = B(y)$ in this case, the infimum in (16) is indeed equal to $B(y)$. If $c = 1$ then due to (20), $\neg_b D(y,b) = 1$ for each $b \in L - \{1\}$, hence also the infimum in (16) is equal to 1, i.e. equal to $B(y)$. □

Remark 2

(a) One easily checks that since $\neg_1 I_y(x) = 1$ for each $x \in X$, one may replace $L - \{1\}$ by L in Theorem 1.
(b) A converse statement to Theorem 1 does not hold. That is, there is no notion of a complement $\sim$ such that for any fuzzy relation I, $\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$ is equal to $\mathrm{Ext}(X^{\cap_{\sim I}}, Z^{\cup_{\sim I}}, \sim I)$ for any suitable Z. This is because for some fuzzy relations I, $\mathrm{Ext}(X^{\uparrow_I}, Y^{\downarrow_I}, I)$ is not a system of extents of any fuzzy relation J w.r.t. the operators ${}^{\cap_J}$ and ${}^{\cup_J}$ [5].
(c) In view of Remark 1c, Theorem 1 generalizes (3) and (4) and its proof does not use the law of double negation.

3. Conclusions

We proposed a new notion of complement of a fuzzy relation. We showed that this notion helps to save certain results that are known not to hold with the ordinary notion of a complement. A further exploration of the new notion of complement remains a subject for future research.

It is an interesting question to explore to what extent the new notion may be used in various other areas of fuzzy set theory to replace the usual notion of complement in such a way that the resulting concepts behave as in the classical, bivalent case. In particular, in the context of closure structures associated to fuzzy relations, it seems reasonable to use the new notion of complement to define a new semantics of failure dependencies in knowledge spaces with graded knowledge states [1]. Another topic worth further investigation is provided by [12]. One of the main results in [12] is a description of a scaling of a fuzzy relation $I \in L^{X \times Y}$ to an ordinary relation $I_c \subseteq (X \times L) \times (Y \times L)$ such that $\mathcal{B}(X^{\cap_I}, Y^{\cup_I}, I)$ and $\mathcal{B}(X \times L^{\uparrow_{I_c}}, Y \times L^{\downarrow_{I_c}}, I_c)$ are isomorphic as lattices. This result is a consequence of Theorem 1. Furthermore, [12] considers general isotone Galois connections that employ linguistic hedges to parameterize the concept of an isotone Galois connection and to reduce the size of the resulting concept lattice. An analogous reduction may be obtained by using $\neg_K I$ with $K \subseteq L - \{1\}$. These issues will be subject to a future work.

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