

Reductions of formal fuzzy contexts

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ABSTRACT

We present results on reductions of formal fuzzy contexts. First, we explain why this problem is more difficult than the straightforward reduction of classical formal contexts. To this end, we use illustrative examples together with an analysis based on the underlying closure structures and the notion of a base of a fuzzy closure system, which provides insight into the nature of the problem and its difficulty. We then present the notion of a reduction and theorems that lead to two reduction algorithms. The first one is a straightforward iterative reduction algorithm. The second one utilizes bases of certain systems of fuzzy sets. We include examples demonstrating both approaches. Finally, we discuss a previously proposed approach and outline directions for future research.

1. Introduction

Reduction of formal contexts is a classic topic in formal concept analysis (FCA). Its solution consists in removing all the rows that may be obtained as intersections of other rows, and the same for the columns. An easy reasoning yields that if applied to a formal context that is clarified (i.e., has no duplicate rows and columns), such a procedure results in the smallest formal context that is equivalent to the original formal context in that their concept lattices are isomorphic. For details, see chapter 1.2 of [1].

In FCA of formal fuzzy contexts (i.e., contexts with fuzzy, or graded, attributes), the problem of reduction is not properly understood yet. In fact, in spite of numerous publications on FCA of fuzzy contexts, only a few address this problem or contain results directly relevant to it. The first work directly motivated by this problem is [2] and its expanded version [3], both presenting results mainly in terms of fuzzy closure operators. Worth noting is also [4] which explores row and column closure spaces of matrices over residuated lattices. Another stream of research is represented by [5–7] where the authors develop notions and present an algorithm for reducing fuzzy contexts in the framework of multi-adjoint concept lattices. Let us also mention that in several papers, the authors explore what they term reduction of formal contexts (both ordinary and fuzzy) with a different aim, namely modifying a formal context so that the resulting concept lattice is smaller but still bearing reasonable information about the input data. Since the aim of such works is different from ours, we do not consider them in our paper.

In our pursuit to enhance understanding of reduction of fuzzy contexts, our goal in this paper is to

- define the problem of reduction of formal fuzzy contexts;
- explain why in a fuzzy setting the problem is considerably more involved than in the classical setting;
- analyze the problem in terms of basic mathematical structures behind FCA of fuzzy contexts to obtain a mathematically clean and conceptually simple view;
- present new theoretical results on reduction and use examples to illustrate the issues being addressed;

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- propose two reduction algorithms.

Since we also aim for simplicity, we intend to address the problem in an elementary way, e.g., omitting usage of more advanced structures such as a properly understood notion of a complete lattice in a fuzzy setting [8]. Even though such usage would make some of our results more elegant and more similar in their form to the classical results, it would make our results less accessible for readers not acquainted with those structures.

Our results may basically be presented in two ways: in terms of sets and relations, i.e., in the language of FCA, and in terms of matrices over residuated lattices. In this regard, we choose the first option due to the intended readership.

We consider the framework of FCA of formal fuzzy contexts, introduced in the late 1990s (see, e.g., [9,10]), in which elements of complete residuated lattices represent the degrees to which attributes apply to objects. This setting is chosen because it is well developed, relatively simple, and sufficiently general to demonstrate the issues under consideration.

2. Preliminaries

2.1. Fuzzy logic and fuzzy sets

As degrees of truth—which include the boundary cases 0 and 1 implicitly employed by ordinary FCA, but may also include intermediate ones such as $3/4$ —we use elements of complete residuated lattices [11–13], i.e., algebras $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ such that

- (i) $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice (with \leq denoting its partial order);
- (ii) \otimes is a commutative and associative operation for which $a \otimes 1 = a$;
- (iii) \otimes and \rightarrow satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

Residuated lattices are commonly used in fuzzy logic [11,12]: The elements $a \in L$ are interpreted as degrees of truth and the operations \otimes (multiplication) and \rightarrow (residuum) play the role of the (truth functions of) conjunction and implication, respectively. Common examples include residuated lattices defined on the real unit interval $L = [0, 1]$ or on a finite chain $L = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$. In both these cases, one may use, e.g., the Łukasiewicz operations, i.e., $a \otimes b = \max(0, a + b - 1)$ and $a \rightarrow b = \min(1, 1 - a + b)$, or the Gödel operations, i.e., $a \otimes b = \min(a, b)$ and $a \rightarrow b = 1$ if $a \leq b$ and $a \rightarrow b = b$ if $a > b$.

Given an arbitrary \mathbf{L} , classical, bivalent notions naturally obtain their generalizations. In particular, a fuzzy set (or \mathbf{L} -set if one needs to be explicit) in a universe set U is a mapping $A : U \rightarrow L$ assigning to every $u \in U$ a truth degree $A(u) \in L$ (degree to which u belongs to A). The set of all L -sets in U , denoted by L^U , inherits the operations of \mathbf{L} and itself becomes a complete residuated lattice. For instance, the \wedge -intersection of fuzzy sets $A, B \in L^U$ is defined by $(A \wedge B)(u) = A(u) \wedge B(u)$ and generalizes the intersection of classical sets.

Two further notions are of particular importance in what follows. A singleton given by $u \in U$ and $a \in L$ is a fuzzy set $\{^a/u\}$ defined by $\{^a/u\}(u) = a$ and $\{^a/u\}(v) = 0$ for $v \neq u$. For $A, B \in L^U$, the degree of inclusion of A in B is defined by

$$S(A, B) = \bigwedge_{u \in U} (A(u) \rightarrow B(u)); \tag{1}$$

clearly, $S(A, B)$ is the truth degree of the formula representing “for each $u \in U$, if u is in A then u is in B .” One writes $A \leq B$ if $S(A, B) = 1$, i.e., if A is (fully) included in B ; due to the properties of residuated lattices, this “crisp” inclusion of fuzzy sets satisfies

$$A \leq B \quad \text{iff} \quad \text{for each } u \in U : A(u) \leq B(u). \tag{2}$$

Importantly, the only residuated lattice \mathbf{L} with $L = \{0, 1\}$ coincides with the two-element Boolean algebra of classical logic in which case \otimes and \rightarrow are the truth functions of classical conjunction and implication. This provides a bridge to classical notions: Taking the two-element Boolean algebra for \mathbf{L} , fuzzy sets in U become (characteristic functions of) subsets of U , the degree of inclusion (1) represents inclusion of classical sets, etc.

2.2. Formal fuzzy contexts and their concept lattices

Take an arbitrary complete residuated lattice \mathbf{L} . A formal fuzzy context is a triplet $\langle X, Y, I \rangle$ in which X and Y are non-empty sets (of objects and attributes) and $I \in L^{X \times Y}$ is a fuzzy relation, for which the degree $I(x, y)$ is interpreted as the degree to which the object $x \in X$ has the attribute $y \in Y$. Below, we mostly assume $X = \{1, \dots, n\}$ and $Y = \{1, \dots, m\}$; we also write L^k to denote $L^{\{1, \dots, k\}}$, thus also L^n for L^X , etc., for convenience in some situations.

The concept-forming operators $\uparrow : L^X \rightarrow L^Y$ and $\downarrow : L^Y \rightarrow L^X$ are defined by

$$C^\uparrow(y) = \bigwedge_{x=1}^n (C(x) \rightarrow I(x, y)) \quad \text{and} \quad D^\downarrow(x) = \bigwedge_{y=1}^m (D(y) \rightarrow I(x, y)), \tag{3}$$

for each $C \in L^X$, $D \in L^Y$, $y \in \{1, \dots, m\}$, and $x \in \{1, \dots, n\}$. A formal (fuzzy) concept of $\langle X, Y, I \rangle$ is any pair $\langle C, D \rangle$ of fuzzy sets $C \in L^X$ of objects (extent) and $D \in L^Y$ of attributes (intent) for which $C^\uparrow = D$ and $D^\downarrow = C$.

Notice that using basic semantic rules of fuzzy logic [12], $C^\uparrow(y)$ is the truth degree of the formula saying “for each object x , if x is in C then x has y ,” i.e., “ y is shared by all objects in C .” Hence the above definition of a formal concept is just a many-valued-logic interpretation of the core idea “the intent consists of all attributes shared by all objects of the extent, and dually for the extents.”

Notice also that if L is the two-element Boolean algebra, the above “fuzzy notions” become the classical notions of formal contexts, concept-forming operators, etc.¹ This is how basic notions of ordinary FCA [1] obtain their generalizations in the presented fuzzy logic setting.

The set

$$B(X, Y, I) = \{ \langle C, D \rangle \mid C \in L^X, D \in L^Y, C^\dagger = D, D^\dagger = C \}$$

of all formal concepts of a given fuzzy context $\langle X, Y, I \rangle$ equipped with the partial order

$$\langle C_1, D_1 \rangle \leq \langle C_2, D_2 \rangle \text{ iff } C_1 \leq C_2 \text{ (iff } D_2 \leq D_1),$$

representing a subconcept-superconcept hierarchy, is called the (fuzzy) concept lattice of $\langle X, Y, I \rangle$. Each $B(X, Y, I)$ equipped with \leq is indeed a complete lattice and its structure is described by a corresponding basic theorem of fuzzy concept lattices. Note also that a proper structural notion of a fuzzy concept lattice generalizing the classical one in broader mathematical considerations is yet even more general and is based on the notion of a fuzzy order. See [8,11,14,15] for both notions, the corresponding basic theorems, the underlying mathematical structures, and further issues.

Analogously to the classical case, a formal fuzzy context $\langle X, Y, I \rangle$ may be represented by a table (matrix) containing $I(x, y)$ at the intersection of the row and column representing $x \in X$ and $y \in Y$, respectively. Since

$$\{^1/x\}^\dagger(y) = I(x, y) \quad \text{and} \quad \{^1/y\}^\dagger(x) = I(x, y),$$

for the singletons $\{^1/x\}$ and $\{^1/y\}$, the fuzzy set of attributes represented by the table row for x is just the intent of the object concept $\langle \{^1/x\}^\dagger, \{^1/x\}^\dagger \rangle$, and dually for y .

3. Why is the problem more difficult than classical reduction?

Before answering this question, let us note that even though some considerations and results below are valid for arbitrary formal fuzzy contexts, in most cases pertaining to the reduction problem we assume that

- X and Y are finite; the set L of truth degrees of L is finite;
- $\langle X, Y, I \rangle$ is clarified, i.e., $x_1 \neq x_2$ implies $\{^1/x_1\}^\dagger \neq \{^1/x_2\}^\dagger$ (there are no identical rows) and $y_1 \neq y_2$ implies $\{^1/y_1\}^\dagger \neq \{^1/y_2\}^\dagger$ (no identical columns).²

The reasons are similar to those in the ordinary case [1, chap. 1.2].³

3.1. Recalling classical reduction

Consider a clarified classical formal context represented by

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

where the rows represent the objects x_1, x_2 , and x_3 , respectively. Then row 1 is reducible because

$$\{x_1\}^\dagger = \{x_2\}^\dagger \cap \{x_3\}^\dagger,$$

i.e., row 1 is obtained as the intersection of rows 2 and 3. No other row is reducible, hence the reduction of rows leads to

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

A dual procedure of column reduction would then lead to the reduced context $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; column 1 was removed because a “full column”—as the intersection of an empty set of columns—is always reducible. The concept lattices of the original formal context and the reduced one are isomorphic.

¹ Strictly speaking, this is true when one identifies sets and relations with their characteristic functions.

² As in the ordinary case, if $\langle X_2, Y_2, I_2 \rangle$ is obtained from $\langle X_1, Y_1, I_1 \rangle$ by clarification (i.e., removing duplicate rows and columns) then $B(X_2, Y_2, I_2)$ is isomorphic to $B(X_1, Y_1, I_1)$.

³ For the finiteness of L , notice that, e.g., for $L = [0, 1]$ equipped with the Łukasiewicz operations, $X = Y = \{*\}$, $I(*, *) = 0.5$, the fuzzy concept lattice $B(X, Y, I)$ is isomorphic to $\langle [0.5, 1], \leq \rangle$, hence infinite, implying technical inconveniences. Most importantly, an infinite L would prevent most algorithmic considerations.

3.2. Example demonstrating the difficulty in a fuzzy setting

In the classical setting, an object is reducible if its intent (row) is obtained by the intersection—the fundamental operation defining the concept-forming operators—applied to other objects’ intents (rows). In a fuzzy setting,

the concept-forming operators[†] and[‡] involve two operations: \wedge and \rightarrow .

That is to say, in addition to the intersection (i.e., \wedge -intersection; Section 2.1) as in the classical case, they involve the residuum (many-valued implication) \rightarrow . Hence,

the meaning of “a row is obtained from other rows” is richer in a fuzzy setting.

Consider the residuated lattice given by the Gödel operations on $L = \{0, 1/2, 1\}$ and a clarified formal fuzzy context represented by

$$\begin{pmatrix} 1/2 & 0 \\ 1 & 0 \\ 1/2 & 1 \end{pmatrix}.$$

On the one hand,

$$\{1/x_2\}^\dagger = 1/2 \rightarrow \{1/x_1\}^\dagger,$$

i.e., row 2 is obtained by \rightarrow applied to (1/2 and) row 1. On the other hand,

$$\{1/x_1\}^\dagger = \{1/x_2\}^\dagger \wedge \{1/x_3\}^\dagger,$$

i.e., row 1 is obtained as the \wedge -intersection of rows 2 and 3.

While all the involved sets are finite, row 2 is reducible using row 1, and row 1 is reducible using row 2 and row 3. Such a circular dependence cannot occur in the classical case because if—in a clarified classical context—row i is reducible using some rows including row j then row j cannot be reducible using some rows including row i . This indicates the difficulty encountered in a fuzzy setting.

3.3. Mathematical explanation⁴

We now provide a view of the reduction problem for fuzzy contexts in terms of basic mathematical structures underlying FCA of fuzzy contexts.

Note first that, trivially as in the classical case, the mapping $\pi : \langle A, B \rangle \mapsto A$ is an \leq -isomorphism between the fuzzy concept lattice $B(X, Y, I)$ and the system

$$\text{Ext}(X, Y, I) = \{C \in L^X \mid C = C^{\dagger\dagger}\}$$

of all extents of $\langle X, Y, I \rangle$ ordered by the inclusion \leq of fuzzy sets defined by (2). In the classical case, $\text{Ext}(X, Y, I)$ forms a closure system on X [1]. In a fuzzy setting, $\text{Ext}(X, Y, I)$ forms a fuzzy closure system on X (for this notion and its properties utilized below, see [15]):

A system $S \subseteq L^U$ is a fuzzy closure system on U if

- $a \rightarrow A \in S$ for any $a \in L$ and $A \in S$,
- $\bigwedge_{j \in J} A_j \in S$ for any $A_j \in S$ ($j \in J$),

where $a \rightarrow A \in L^U$ (\rightarrow -multiplication or \rightarrow -shift) and $\bigwedge_{j \in J} A_j \in L^U$ (\wedge -intersection) are defined by

$$(a \rightarrow A)(u) = a \rightarrow A(u) \quad \text{and} \quad \left(\bigwedge_{j \in J} A_j\right)(u) = \bigwedge_{j \in J} A_j(u).$$

As in the ordinary case, fuzzy closure systems on U are in one-to-one correspondence with fuzzy closure operators on U [15], i.e., mappings $C : L^U \rightarrow L^U$ satisfying

$$A \leq C(A), \quad S(A, B) \leq S(C(A), C(B)), \quad \text{and} \quad C(A) = C(C(A)),$$

for every $A, B \in L^U$.

Note that if \mathbf{L} is the two-element Boolean algebra, fuzzy closure systems on U essentially become classical closure systems on U because the operation $a \rightarrow A$ is trivial in this case, and may thus be disregarded. Namely, for each (characteristic function of an) ordinary set A , $0 \rightarrow A = U$ is the full set which is contained in every closure system and $1 \rightarrow A = A$ leaves A unchanged. Likewise, fuzzy closure operators become classical closure operators.

Clearly, it follows from classical results [16] that for each system $S \subseteq L^U$ of fuzzy sets there exists the least fuzzy closure system $[S]$ on U containing S , and that the mapping $S \mapsto [S]$ is an (ordinary) closure operator on L^U : $S \subseteq [S]$; if $S \subseteq \mathcal{T}$ then $[S] \subseteq [\mathcal{T}]$; and $[[S]] = [S]$ for every $S, \mathcal{T} \subseteq L^U$.

In this perspective, the composition $\dagger\dagger$ is a fuzzy closure operator on X and $\text{Ext}(X, Y, I)$ is just the system of its fixpoints. The relevance of the closure operator $[\]$ with $U = X$ is apparent from the following theorem.

⁴ For our purpose here, we consider attribute reduction, as it allows for a little simpler presentation: It leads to considerations on the system of extents which is isomorphic to the fuzzy concept lattice of a given fuzzy context (rather than dually isomorphic as with the system of intents).

Theorem 1. Consider the system $S = \{\{^1/y\}^\downarrow \mid y \in Y\}$ of all columns of a fuzzy context $\langle X, Y, I \rangle$. Then

$$\text{Ext}(X, Y, I) = [S],$$

i.e., $\text{Ext}(X, Y, I)$ is the fuzzy closure system on X generated by the fuzzy sets $\{^1/y\}^\downarrow \in L^X$ (columns).

Proof. The claim follows from the following established facts [11]: First, due to its definition, $\text{Ext}(X, Y, I)$ is just the set of all fixpoints of the fuzzy closure operator ${}^{\uparrow\downarrow} : L^X \rightarrow L^X$, and thus forms a fuzzy closure system on X . Second, each $\{^1/y\}^\downarrow$ is an extent, whence $S \subseteq \text{Ext}(X, Y, I)$. Third, if a fuzzy closure system on X includes S , then—due to the above defining property of a fuzzy closure system— \mathcal{T} contains each \bigwedge -intersection of \rightarrow -shifts of the elements of S , and since each extent is an \bigwedge -intersection of some \rightarrow -shifts of the elements of S , \mathcal{T} includes $\text{Ext}(X, Y, I)$. (Note that in terms of row and columns spaces of matrices over residuated lattices, the claim has been established in [4, Theorem 3.1].) \square

The following notion, which is essential in our considerations, was studied in [3]:⁵ Consider the above closure operator $[] : 2^{L^U} \rightarrow 2^{L^U}$. A base of a fuzzy closure system $S \subseteq L^U$ is a set $\mathcal{T} \subseteq L^U$ such that

- $[\mathcal{T}] = S$ (\mathcal{T} generates the fuzzy closure system),
- $[\mathcal{P}] \subset S$ for every $\mathcal{P} \subset \mathcal{T}$ (\mathcal{T} is non-redundant).

In view of Theorem 1, it is apparent that the problem of reduction of attributes may be rephrased as the problem of finding a base of $\text{Ext}(X, Y, I)$. Now, while the classical case is straightforward, the situation in a fuzzy setting is more difficult:

- (a) In the classical setting, i.e., with \mathbf{L} being the two-element Boolean algebra, the above implies that reducing attributes of an ordinary formal context $\langle X, Y, I \rangle$ is equivalent to finding a base of the ordinary closure system generated by $S = \{\{y\}^\downarrow \mid y \in Y\}$ (columns of the context). Chapter 1.2 in [1] implies that such a base is unique and consists of all the columns that cannot be obtained as intersections of other columns.⁶
- (b) In a fuzzy setting, the situation is different. It has been shown [3] that even when U and L are finite, the following may happen for a fuzzy closure system $[S]$ generated by a (finite) system $S \subseteq L^U$ of fuzzy sets:
 - (i) $[S]$ may have several bases;
 - (ii) $[S]$ may have bases of different sizes;
 - (iii) a base of $[S]$ may contain fuzzy sets that are not present in S ; it may even happen that no element of the base belongs to S .

The reason is the involvement of two operations, \wedge and \rightarrow , in the concept-forming operators (rather than just \wedge as in the classical case). As we show below, this manifests itself in that $[]$ results from two simpler operators, $[]_\wedge$ and $[]_\rightarrow$, which interact in a non-trivial way. Namely, we shall prove that

$$[] = []_\wedge \vee []_\rightarrow,$$

i.e., $[]$ is the supremum of $[]_\wedge$ and $[]_\rightarrow$ in the lattice of closure operators on L^U .

Now, consider (b) and the fuzzy context $\langle X, Y, I \rangle$ with $X = U$ and the columns equal to the fuzzy sets in S . If reduction is conceived as finding a base, (b) implies that reduction of fuzzy contexts behaves differently compared to the classical case: A given fuzzy context may have several different reductions, these may be of different sizes, and may contain columns (and rows) that were not present in the context. Although condition (iii) in particular may suggest that the term “reduction” is not appropriate in this case, an analogy with linear algebra shows that there is nothing wrong with conditions (i)–(iii). In particular, (i)–(iii) imply that one may explore various kinds of reductions, such as the rows and columns of the reduced context indeed appear in the original context.

Given this basic outlook, we now proceed by presenting theorems that provide further insight into our problem and two reduction algorithms.

4. Reductions and reduction algorithms

4.1. The concept of reduction

In terms of reducing formal fuzzy contexts,⁷ a direct consequence of the situation described above is that a new formal context, whose fuzzy concept lattice is isomorphic to the one of the input fuzzy context, may result by removing certain rows of the input context and replacing them by rows that are not present in the input context. This leads to the following general notion of reduction of a formal fuzzy context.

We start by a straightforward notion of reducibility of objects (rows) and attributes (columns) of a clarified fuzzy context $\langle X, Y, I \rangle$: An object $x \in X$ is *reducible* (in $\langle X, Y, I \rangle$) if there exists $A \in L^X$ such that $A(x) = 0$ and $A^\uparrow = \{^1/x\}^\uparrow$. Dually, an attribute $y \in Y$ is reducible if for some $B \in L^Y$ with $B(y) = 0$ one has $B^\downarrow = \{^1/y\}^\downarrow$. Finally, $\langle X, Y, I \rangle$ is *object-reduced* (row-reduced) if no $x \in X$ is reducible, and dually for the attributes; $\langle X, Y, I \rangle$ is *reduced* if it is both object- and attribute-reduced.

⁵ Below, we shall use a slightly more general setting compared to [3].

⁶ The uniqueness of a base of classical closure systems also appears, e.g., in [17, Theorem 1. 1. 1] (in a dual form for interior spaces associated to Boolean matrices).

⁷ As noted at the start of Section 3, we assume that the considered contexts are clarified.

Remark 1. (a) Reducibility of x says that the row $\{^1/x\}^\uparrow$ may be obtained using other rows (rows of those x' for which $A(x') \neq 0$) using the operator $^\uparrow$. It is easy to see that in the classical setting, the above notion of reducibility of x means that the row of x may be obtained as an intersection of rows of x' of a certain (classical) set $A \subseteq X$ not containing x , i.e., means that row x is reducible in the classical sense. Note that this interpretation of reducibility applies in the fuzzy setting too because for a fuzzy set $A \in L^X$ we have $A^\uparrow(\cdot) = \bigwedge_{x \in X} A(x) \rightarrow I(x, \cdot)$, whence the fuzzy set $A^\uparrow \in L^Y$ may be interpreted as a (generalized) intersection of rows $I(x, \cdot) = \{^1/x\}^\uparrow$ of x in A , in which the rows of x are adjusted by the membership degrees $A(x)$ of x ; see [11, p. 81]. Our concept thus generalizes classical reducibility.

(b) It thus is immediate to see that when one removes a reducible object x from a given fuzzy context $\langle X, Y, I \rangle$, the resulting fuzzy context with the set $X - \{x\}$ of objects has the same intents as the original context. Therefore, when $\langle Z, Y, J \rangle$ results from $\langle X, Y, I \rangle$ by a successive removal of reducible rows (i.e., removing some x_1 reducible in $\langle X, Y, I \rangle$, followed by removing some x_2 that is reducible in the resulting smaller context with the object set $X - \{x_1\}$, etc.), we have

$$\text{Int}(X, Y, I) = \text{Int}(Z, Y, J).$$

The same applies dually to the attributes. \square

To define the concept of a reduction of a given fuzzy context, we first need a notion of equivalence of contexts. The intended meaning is that the fuzzy concept lattices of equivalent fuzzy contexts be isomorphic under an appropriate notion of isomorphism. Such a notion was first explored in [8], has further been studied in [4], and is utilized in what follows. We call two formal fuzzy contexts $\langle X_1, Y_1, I_1 \rangle$ and $\langle X_2, Y_2, I_2 \rangle$ *equivalent* if there is a bijective mapping $h : \text{Ext}(X_1, Y_1, I_1) \rightarrow \text{Ext}(X_2, Y_2, I_2)$ such that both h and h^{-1} are c-morphisms between the fuzzy closure spaces $\text{Ext}(X_1, Y_1, I_1)$ and $\text{Ext}(X_2, Y_2, I_2)$. Recall from [4] that a c-morphism between fuzzy closure spaces $S_1 \subseteq L^U$ and $S_2 \subseteq L^V$ is a mapping $h : S_1 \rightarrow S_2$ such that $h(a \rightarrow A) = a \rightarrow h(A)$ and $h(\bigwedge_{k \in K} A_k) = \bigwedge_{k \in K} h(A_k)$ for every $a \in L$ and $A, A_k \in S_1$ ($k \in K$).

Remark 2. Due to an alternative characterization of c-morphisms, the equivalence of $\langle X_1, Y_1, I_1 \rangle$ and $\langle X_2, Y_2, I_2 \rangle$ means that there is a bijective mapping $h : \text{Ext}(X_1, Y_1, I_1) \rightarrow \text{Ext}(X_2, Y_2, I_2)$ satisfying

$$S(A_1, A_2) = S(h(A_1), h(A_2)) \tag{4}$$

for each $A_1, A_2 \in \text{Ext}(X_1, Y_1, I_1)$. Since $S(A_1, A_2) = S(B_2, B_1)$ for any pair of fuzzy concepts $\langle A_1, B_1 \rangle$ and $\langle A_2, B_2 \rangle$ of a given fuzzy context [11], one may replace extents by intents in the above definition of equivalence of contexts. Moreover, (4) means that $\mathcal{B}(X_1, Y_1, I_1)$ and $\mathcal{B}(X_2, Y_2, I_2)$ are isomorphic as fuzzy ordered sets in that there exists a bijection $h : \mathcal{B}(X_1, Y_1, I_1) \rightarrow \mathcal{B}(X_2, Y_2, I_2)$ such that for every $\langle A_1, B_1 \rangle, \langle C_1, D_1 \rangle \in \mathcal{B}(X_1, Y_1, I_1)$ one has

$$\langle A_1, B_1 \rangle \preceq_1 \langle C_1, D_1 \rangle = h(A_1, B_1) \preceq_2 h(C_1, D_1),$$

with the fuzzy orders on $\mathcal{B}(X_i, Y_i, I_i)$ defined by $\langle A_1, B_1 \rangle \preceq_1 \langle C_1, D_1 \rangle = S(A_1, C_1)$ or, equivalently, $\langle A_1, B_1 \rangle \preceq_1 \langle C_1, D_1 \rangle = S(D_1, B_1)$ for $i = 1, 2$.⁸ The existence of an isomorphism between $\mathcal{B}(X_1, Y_1, I_1)$ and $\mathcal{B}(X_2, Y_2, I_2)$ provides an intuitive alternative definition of equivalence of $\langle X_1, Y_1, I_1 \rangle$ and $\langle X_2, Y_2, I_2 \rangle$. \square

A formal fuzzy context $\langle X_2, Y_2, I_2 \rangle$ is a *reduction* of $\langle X_1, Y_1, I_1 \rangle$ if the two contexts are equivalent and $\langle X_2, Y_2, I_2 \rangle$ has no reducible objects or attributes.⁹ It will be apparent that both our algorithms presented in this section produce reductions of the input fuzzy contexts.

In the following sections, we present two algorithms to compute row-reduced formal fuzzy contexts. Obviously, dual approaches result in column-reduced contexts.¹⁰ A reduced context is thus obtained by row-reducing the input context followed by column-reducing the row-reduced context.

4.2. Straightforward iterative method

We shall utilize the following characterization of reducibility. Here, $I(x, \cdot)$ denotes $\{^1/x\}^\uparrow$, i.e., the row of the object x , and dually $I(\cdot, y)$ denotes $\{^1/y\}^\downarrow$. In addition, for $A \in L^U$, we denote by $A - u$ the fuzzy set in L^U given by

$$(A - u)(v) = \begin{cases} 0 & \text{if } u = v, \\ A(v) & \text{otherwise,} \end{cases}$$

for each $v \in U$.

Theorem 2. An object $x \in X$ is reducible if and only if

$$(I(x, \cdot)^\downarrow - x)^\uparrow = I(x, \cdot).$$

⁸ Note that this implies, in particular, that $\mathcal{B}(X_1, Y_1, I_1)$ and $\mathcal{B}(X_2, Y_2, I_2)$ are isomorphic as partially ordered sets for the partial orders \preceq_i defined as $\langle A_1, B_1 \rangle \preceq_i \langle C_1, D_1 \rangle$ iff $(\langle A_1, B_1 \rangle \preceq_i \langle C_1, D_1 \rangle) = 1$.

⁹ We admit that one may wonder whether “reduction” is a proper term given that it may contain rows and columns not present in the input context. We nevertheless choose this term to remain consistent with the terminology of ordinary FCA.

¹⁰ A column-reduced context of a given $\langle X, Y, I \rangle$ is obtained by applying a row-reducing algorithm to the transposed $\langle X, Y, I \rangle$, i.e., to $\langle Y, X, I^{-1} \rangle$, followed by transposing the result.

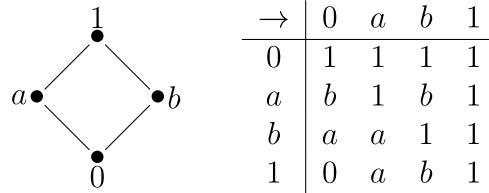


Fig. 1. Residuated lattice used in the introductory example of Section 4.2.

An attribute $y \in Y$ is reducible if and only if

$$(I(\cdot, y)^\uparrow - y)^\downarrow = I(\cdot, y).$$

Proof. Due to duality, we only prove the first part. If $(I(x, \cdot)^\downarrow - x)^\uparrow = I(x, \cdot)$ then, because $I(x, \cdot) = \{^1/x\}^\uparrow$, putting $A = I(x, \cdot)^\downarrow - x$ we obviously have $A(x) = 0$ and $A^\uparrow = \{^1/x\}^\uparrow$, whence x is reducible.

If x is reducible, i.e., there exists $A \in L^X$ with $A(x) = 0$ for which $A^\uparrow = \{^1/x\}^\uparrow$, then $A^\downarrow = \{^1/x\}^\downarrow = I(x, \cdot)^\downarrow$, whence $(I(x, \cdot)^\downarrow - x)^\uparrow = (A^\downarrow - x)^\uparrow$. We therefore need to check

$$(A^\downarrow - x)^\uparrow = \{^1/x\}^\uparrow.$$

Since $A(x) = 0$ and $A \subseteq A^\uparrow$, we have $A = A - x \subseteq A^\uparrow - x$, and so

$$(A^\downarrow - x)^\uparrow \subseteq A^\uparrow = \{^1/x\}^\uparrow$$

due to the antitony of $^\uparrow$. Conversely, since $A^\uparrow - x \subseteq A^\uparrow$, the antitony of $^\uparrow$ and the chain rule for the fuzzy Galois connections $^\uparrow$ and $^\downarrow$ yields

$$(A^\uparrow - x)^\downarrow \supseteq A^{\uparrow\downarrow} = A^\uparrow = \{^1/x\}^\uparrow,$$

finishing the proof. \square

Our setting does not permit us, even when all involved sets are finite, to remove all reducible rows simultaneously, as the following example shows. Consider the four-element residuated lattice whose lattice part is depicted in Fig. 1 (left), with \otimes being \wedge and \rightarrow as shown in Fig. 1 (right). In the context given by

$$I = \begin{pmatrix} a & 1 \\ a & b \\ 1 & b \end{pmatrix},$$

all rows are reducible, because

$$\begin{aligned} (a \ 1) &= b \rightarrow (a \ b), \\ (a \ b) &= (a \ 1) \wedge (1 \ b), \text{ and} \\ (1 \ b) &= a \rightarrow (a \ b). \end{aligned}$$

This leads to Algorithm 1—a straightforward iterative algorithm based on the reducibility criterion of Theorem 2. Note that this algorithm indeed computes a row-reduced context $\langle Z, Y, J \rangle$ equivalent to the input context $\langle X, Y, I \rangle$. Namely, $\langle Z, Y, J \rangle$ is clearly row-reduced. Furthermore, by the definition of a reducible object, removing such an object does not change the system of intents, hence the rows of $\langle Z, Y, J \rangle$ generate $\text{Int}(X, Y, I)$. Since, trivially, the rows of $\langle Z, Y, J \rangle$ generate $\text{Int}(Z, Y, J)$, one has $\text{Int}(X, Y, I) = \text{Int}(Z, Y, J)$, which yields that $B(X, Y, I)$ and $B(Z, Y, J)$ are isomorphic, i.e., $\langle X, Y, I \rangle$ and $\langle Z, Y, J \rangle$ are equivalent.

The following example illustrates this algorithm.

Example 1. Consider the five-element Gödel chain L (see Section 2.1) and a formal fuzzy context given by

$$I = \begin{pmatrix} 0.5 & 0.5 & 1 & 0.25 \\ 0.5 & 0 & 0.5 & 0.25 \\ 0.5 & 0.5 & 0.5 & 0.25 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 1 & 1 & 1 & 0.25 \end{pmatrix}.$$

Initialize by $J = I$ and consider the first row, i.e., $I(1, \cdot) = (0.5 \ 0.5 \ 1 \ 0.25)$. We check its reducibility using the characterization in Theorem 2. Since

$$\begin{aligned} I(1, \cdot)^{\downarrow J} &= (1 \ 0.5 \ 0.5 \ 0.5 \ 0.5), \\ I(1, \cdot)^{\downarrow J} - 1 &= (0 \ 0.5 \ 0.5 \ 0.5 \ 0.5), \end{aligned}$$

Algorithm 1: Computing a row-reduced fuzzy context.

Input: formal fuzzy context $\langle X, Y, I \rangle$
Output: row-reduced $\langle Z, Y, J \rangle$ equivalent to $\langle X, Y, I \rangle$
 $\langle Z, Y, J \rangle \leftarrow \langle X, Y, I \rangle$
for each $x \in X$ **do**
 if x is reducible in $\langle Z, Y, J \rangle$ **then**
 remove x from Z
 remove the row of x from J
 end
end
return $\langle Z, Y, J \rangle$

$$(I(1, \cdot)^{\downarrow J} - 1)^{\uparrow J} = (0.5 \quad 0.5 \quad 0.5 \quad 0.25) \neq I(1, \cdot),$$

the first row is not reducible. Next, we check the reducibility of the second row, $I(2, \cdot) = (0.5 \quad 0 \quad 0.5 \quad 0.25)$: As

$$\begin{aligned} I(2, \cdot)^{\downarrow J} &= (1 \quad 1 \quad 1 \quad 0.5 \quad 1), \\ I(2, \cdot)^{\downarrow J} - 2 &= (1 \quad 0 \quad 1 \quad 0.5 \quad 1), \\ (I(2, \cdot)^{\downarrow J} - 2)^{\uparrow J} &= (0.5 \quad 0.5 \quad 0.5 \quad 0.25) \neq I(2, \cdot), \end{aligned}$$

the second row is not reducible either. For the third row, $I(3, \cdot) = (0.5 \quad 0.5 \quad 0.5 \quad 0.25)$, we obtain

$$\begin{aligned} I(3, \cdot)^{\downarrow J} &= (1 \quad 1 \quad 1 \quad 1 \quad 1), \\ I(3, \cdot)^{\downarrow J} - 3 &= (1 \quad 1 \quad 0 \quad 1 \quad 1), \\ (I(3, \cdot)^{\downarrow J} - 3)^{\uparrow J} &= (0.5 \quad 0.5 \quad 0.5 \quad 0.25) = I(3, \cdot), \end{aligned}$$

hence the third row of I is reducible in J and we remove it from J . Now we have

$$J = \begin{pmatrix} 0.5 & 0.5 & 1 & 0.25 \\ 0.5 & 0 & 0.5 & 0.25 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 1 & 1 & 1 & 0.25 \end{pmatrix}.$$

Proceeding the same way, one finds that the fourth row of I is not reducible in J and that the fifth row of I is reducible in J . We thus remove the fifth row of I from J and obtain

$$J = \begin{pmatrix} 0.5 & 0.5 & 1 & 0.25 \\ 0.5 & 0 & 0.5 & 0.25 \\ 0.5 & 0.5 & 0.5 & 0.5 \end{pmatrix}.$$

Since we have processed all rows of I , the fuzzy relation J represents the result of the algorithm, i.e., a row-reduced fuzzy context.

4.3. Method based on bases of fuzzy closure systems

Our second method is based on a further development of the insight presented in Section 3.3. Since we consider row-reduction in this section (cf. the last paragraph of Section 4.1), we are interested in the dual form of Theorem 1, according to which

$$\text{Int}(X, Y, I) = [S]$$

for the system $S = \{\{\uparrow/x\} \mid x \in X\}$ of rows of a given fuzzy context $\langle X, Y, I \rangle$ and the (ordinary) closure operator $[\] : 2^{L^Y} \rightarrow 2^{L^Y}$ returning the fuzzy closure system $[S]$ generated by S .

In the considerations below, we shall need a generalization of the notion of a base introduced in Section 3.3: For a closure operator $C : 2^U \rightarrow 2^U$, a C -base of a fixpoint S of C ¹¹ is a set $\mathcal{T} \subseteq U$ such that

- $C(\mathcal{T}) = S$ (\mathcal{T} generates S),
- $C(\mathcal{P}) \subset S$ for every $\mathcal{P} \subset \mathcal{T}$ (\mathcal{T} is non-redundant).

Observe that since being a fixpoint for S means that $S = C(S)$, the first condition is equivalent to $C(\mathcal{T}) = C(S)$. Clearly, for $U = L^Y$ and for $C = [\]$, we obtain the notion of a base used in Section 3.3.

The following characterization shall be used repeatedly:

¹¹ Note that even in the above setting with $U = L^Y$, the set S need not be a fuzzy closure system in our considerations below.

Lemma 1. Let $C(\mathcal{T}) = S$. Then \mathcal{T} is a C -base of S iff $A \notin C(\mathcal{T} - \{A\})$ for each $A \in \mathcal{T}$.

Proof. Let \mathcal{T} be a C -base of S . If $A \in C(\mathcal{T} - \{A\})$, then the properties of closure yield $C(\mathcal{T} - \{A\}) = C(C(\mathcal{T} - \{A\})) \supseteq C(C(\mathcal{T} - \{A\}) \cup \{A\}) = C((\mathcal{T} - \{A\}) \cup \{A\}) = C(\mathcal{T})$, contradicting the non-redundancy of a C -base.

Conversely, let $A \notin C(\mathcal{T} - \{A\})$ for each $A \in \mathcal{T}$ and let $\mathcal{P} \subset \mathcal{T}$. Take any $A \in \mathcal{T} - \mathcal{P}$. Since, clearly, $C(\mathcal{T} - \{A\}) \subset C(\mathcal{T})$, we obtain $C(\mathcal{P}) \subseteq C(\mathcal{T} - \{A\}) \subset C(\mathcal{T})$, i.e., \mathcal{T} is non-redundant, and thus forms a C -base of S . \square

As mentioned in Section 3.3, the closure operator $[\]$ is obtained from two simpler operators, $[\]_{\wedge}$ and $[\]_{\rightarrow}$, that correspond to the operations \wedge and \rightarrow involved in the concept-forming mappings \uparrow and \downarrow . These operators are defined by

$$[S]_{\wedge} = \{ \bigwedge \mathcal{T} \mid \mathcal{T} \subseteq S \} \quad \text{and} \quad [S]_{\rightarrow} = \{ a \rightarrow A \mid a \in L, A \in S \}. \tag{5}$$

That is, $[S]_{\wedge}$ is the system of all \wedge -intersections of the fuzzy sets in S and $[S]_{\rightarrow}$ is the system of all \rightarrow -shifts of the fuzzy sets in S . These were identified and studied in [3], where it was observed that both $[\]_{\wedge}$ and $[\]_{\rightarrow}$ are ordinary closure operators. While some further results on $[\]$, $[\]_{\wedge}$, and $[\]_{\rightarrow}$ have been established in [3], we present additional results and a new perspective on the previous findings. Unlike [3], which directly derives the results from the particular forms of $[\]$, $[\]_{\wedge}$, and $[\]_{\rightarrow}$, we consider these three operators as elements of the lattice

$$\langle C\mathcal{O}_U, \leq \rangle$$

of all (ordinary) closure operators on U [18,19] for $U = L^Y$, with the lattice order \leq defined componentwise, i.e., $C \leq D$ iff $C(S) \subseteq D(S)$ for each $S \subseteq U$, and \wedge and \vee denoting the infimum and supremum, respectively. In this view, we attempt to derive results by using as much as possible the general properties of $\langle C\mathcal{O}_U, \leq \rangle$, and resort to the particular forms of $[\]$, $[\]_{\wedge}$, and $[\]_{\rightarrow}$ only when necessary.

The main result used in the following is this:

Theorem 3. For any complete residuated lattice L and set Y (both possibly infinite):

$$[\] = [\]_{\wedge} \vee [\]_{\rightarrow}, \tag{6}$$

i.e., $[\]$ is the supremum of $[\]_{\wedge}$ and $[\]_{\rightarrow}$ in the lattice $\langle C\mathcal{O}_{L^Y}, \leq \rangle$.

$$[\] = [\]_{\wedge} \circ [\]_{\rightarrow}, \tag{7}$$

i.e., $[S] = [[S]_{\rightarrow}]_{\wedge}$ for each system $S \subseteq L^Y$ of fuzzy sets in Y .

Proof. (6): Let $C \in C\mathcal{O}_{L^Y}$ be a closure operator on L^Y such that $[\]_{\wedge} \leq C$ and $[\]_{\rightarrow} \leq C$. This means that for each $S \subseteq L^Y$ one has (a) $[S]_{\wedge} \subseteq C(S)$ and (b) $[S]_{\rightarrow} \subseteq C(S)$. Now, (a) and (b) mean that $C(S)$ contains all \wedge -intersections and all \rightarrow -shifts of the fuzzy sets in S . Since $[S]$ (as a fuzzy closure system generated by S) is the smallest system containing all these \wedge -intersections and \rightarrow -shifts, we get $[S] \subseteq C(S)$. In view of $[S]_{\wedge} \subseteq [S]$ and $[S]_{\rightarrow} \subseteq [S]$ this means that $[\]$ is the claimed supremum.

(7): Let us start with an observation on general closure operators: If $D \circ C \leq C \circ D$ in $\langle C\mathcal{O}_U, \leq \rangle$, then $C \circ D$ is the supremum of C and D .¹² Indeed, from $D \circ C \leq C \circ D$ one easily obtains that $C \circ D$ is idempotent, and since $C \circ D$ is obviously extensive and monotone, it is a closure operator; if $C, D \leq K$ for some $K \in C\mathcal{O}_U$, then $C \circ D(A) = C(D(A)) \subseteq C(K(A)) \subseteq K(K(A)) = K(A)$, hence $C \circ D \leq K$ proving that $C \circ D = C \vee D$. Now, put $C = [\]_{\wedge}$ and $D = [\]_{\rightarrow}$, and observe that $D \circ C \leq C \circ D$, i.e., $[[S]_{\wedge}]_{\rightarrow} \subseteq [[S]_{\rightarrow}]_{\wedge}$ for each S . Indeed, $A \in [[S]_{\wedge}]_{\rightarrow}$ means that $A = a \rightarrow \bigwedge_{j \in J} A_j$ for some $a \in L$ and $A_j \in S$. Using the distributivity of \rightarrow and \wedge , e.g., [11,13], one obtains $a \rightarrow \bigwedge_{j \in J} A_j = \bigwedge_{j \in J} a \rightarrow A_j$, which means $A \in [[S]_{\rightarrow}]_{\wedge}$. Now, the general observation yields $C \circ D = C \vee D$, i.e., $[\]_{\wedge} \circ [\]_{\rightarrow} = [\]_{\wedge} \vee [\]_{\rightarrow}$, whence (7) follows from (6). \square

Remark 3. Note that [3] also contains the formula $[S] = \bigcup_{i=0}^{\infty} [S]^i$ with $[S]^0 = S$ and $[S]^{i+1} = [[S]^i]_{\rightarrow} \cup [[S]^i]_{\wedge}$, which is based on the idea that one has to repeatedly apply the generating operations. In view of the inclusion $[[S]_{\wedge}]_{\rightarrow} \subseteq [[S]_{\rightarrow}]_{\wedge}$ used in the proof of Theorem 3, this formula collapses to (7). Note also that the converse inclusion, i.e., $[[S]_{\wedge}]_{\rightarrow} \supseteq [[S]_{\rightarrow}]_{\wedge}$, fails to hold [3]. \square

Formula (7) suggests the following idea: Since the closure $[S]$ represents successive “wrapping”, i.e., taking $[S]_{\rightarrow}$ followed by taking $[[S]_{\rightarrow}]_{\wedge}$, it seems reasonable to obtain a base of $[S]$ by computing a $[\]_{\wedge}$ -base from S followed by computing a $[\]_{\rightarrow}$ -base from this $[\]_{\wedge}$ -base. We shall show that by a slight adjustment, this idea indeed leads to a feasible algorithm to compute $[\]$ -bases, i.e., a row-reducing algorithm.

For $S \subseteq L^Y$, let \triangleleft denote the binary relation on L^Y defined by

$$B_1 \triangleleft B_2 \text{ if and only if } B_2 = a \rightarrow B_1 \text{ for some } a \in L.$$

The following theorem was obtained in [3]:¹³

Theorem 4. For every finite $S \subseteq L^Y$,

¹² This claim may also be obtained using simple reasoning and the results in [19], but we present a more direct argument.

¹³ Part (b) appears to be folklore in lattice theory and was not proved in [3]. Since we were not able to find a complete proof (apart from proof sketches hinting at the finiteness of S), we provide in the appendix an analysis of this claim (Lemma 4, Lemma 5, and Theorem 7) and prove it for a more general assumption allowing possibly infinite S . In particular, (b) follows from Theorem 7.

(a) \triangleleft is a partial order on L^Y and the set $\text{irr}_{\rightarrow}(S)$ of its minimal elements, i.e.,

$$\text{irr}_{\rightarrow}(S) = \{B \in S \mid B' \triangleleft B \text{ implies } B' = B \text{ for any } B' \in S\},$$

is the only $[\]_{\rightarrow}$ -base of $[S]_{\rightarrow}$.

(b) the set

$$\text{irr}_{\wedge}(S) = \{A \in S \mid A \notin [S - \{A\}]_{\wedge}\}.$$

is the only $[\]_{\wedge}$ -base of $[S]_{\wedge}$.

We now obtain a theorem justifying our second reduction algorithm below.

Theorem 5. If $S \subseteq L^U$ is finite and closed under \rightarrow -shifts (i.e., $[S]_{\rightarrow} \subseteq S$), then $\text{irr}_{\rightarrow}(\text{irr}_{\wedge}(S))$ is a $[\]$ -base of $[S]$.

This theorem was obtained in [3] using the specific forms of $[\]$, $[\]_{\wedge}$, $[\]_{\rightarrow}$. We now obtain it as a particular instance of a claim (Lemma 2 below) regarding general closure operators so that we resort to the specific properties of $[\]$, $[\]_{\wedge}$, $[\]_{\rightarrow}$ only when needed. Even though parts of the general claim are valid for closure operators on arbitrary sets U , we restrict to the finite case, which is relevant for computing a base.

Thus, consider closure operators $E : 2^U \rightarrow 2^U$ such that every fixpoint $S \subseteq U$ of E has an E -base. For an arbitrary $\mathcal{T} \subseteq U$ (not necessarily a base) denote by $b_E(\mathcal{T})$ some E -base of $E(\mathcal{T})$. One may consider b_E as a function whose output $b_E(\mathcal{T})$ is one of the possibly several existing E -bases of the fixpoint $E(\mathcal{T})$ generated by \mathcal{T} . For instance, $b_E(\mathcal{T})$ may be an E -base of $E(\mathcal{T})$ which is easily computable from \mathcal{T} —an interpretation which corresponds to our situation.

Lemma 2. Let $C, D \in \mathcal{C}\mathcal{O}_U$ for a finite U and assume

- (i) $D \circ C \leq C \circ D$,
- (ii) $b_C(\mathcal{T}) \subseteq \mathcal{T}$ and $b_D(\mathcal{T}) \subseteq \mathcal{T}$ for any $\mathcal{T} \subseteq U$,
- (iii) $b_C(\mathcal{T})$ is the only C -base of $C(\mathcal{T})$ for any $\mathcal{T} \subseteq U$.

Then $C \circ D \in \mathcal{C}\mathcal{O}_U$ and if S is D -closed then $b_D(b_C(S))$ is a $C \circ D$ -base of $C \circ D(S)$.

Proof. The fact that $C \circ D$ is a closure operator follows by an easy argument [19]. It remains to check that $b_D(b_C(S))$ (a) generates $C \circ D(S)$ and (b) is non-redundant.

(a): To verify $C \circ D(b_D(b_C(S))) = C \circ D(S)$, we prove both inclusions: Using the assumptions for b_D and b_C , the extensivity of C and D , and the idempotency of $C \circ D$,

$$\begin{aligned} C \circ D(b_D(b_C(S))) &= C(D(b_D(b_C(S)))) = C(D(b_C(S))) \\ &\subseteq C(D(C(b_C(S)))) = C(D(C(S))) \\ &\subseteq C(D(C(D(S)))) = C(D(S)) = C \circ D(S). \end{aligned}$$

On the other hand, taking into account b_D and b_C , and using $D \circ C \leq C \circ D$ and the extensivity of C ,

$$\begin{aligned} C \circ D(b_D(b_C(S))) &= C(D(b_D(b_C(S)))) = C(D(b_C(S))) \\ &= C(C(D(b_C(S)))) \supseteq C(D(C(b_C(S)))) \\ &= C(D(C(S))) \supseteq C(D(S)) = C \circ D(S). \end{aligned}$$

(b) If $b_D(b_C(S))$ is not a $C \circ D$ -base then Lemma 1 yields some $B \in b_D(b_C(S))$ for which

$$B \in D \circ C(b_D(b_C(S)) - \{B\}). \tag{8}$$

We show that (8) leads to a contradiction.

Let us start by observing two facts. First, due to assumption (ii), $b_D(b_C(S)) - \{B\} \subseteq S$, and since S is D -closed, we have

$$D(b_D(b_C(S)) - \{B\}) \subseteq D(b_D(b_C(S))) \subseteq D(S) \subseteq S. \tag{9}$$

Second, since $B \in b_D(b_C(S))$, (ii) leads to $B \in b_C(S)$. Next, we obtain that

$$B \in D(b_D(b_C(S)) - \{B\}). \tag{10}$$

Namely, (8) means that B is in the C -closure of the set $D(b_D(b_C(S)) - \{B\})$. Using $D(b_D(b_C(S)) - \{B\}) \subseteq S$ of (9), the fact that $b_C(S)$ is a unique C -base of $C(S)$ due to (iii), and $B \in b_C(S)$, it follows that (10) must be the case. Indeed, otherwise the finiteness of U enables us to obtain from $\mathcal{T} := D(b_D(b_C(S)) - \{B\})$ a C -base of $C(S)$ not containing B , i.e., different from $b_C(S)$: Remove elements from \mathcal{T} to obtain a C -non-redundant set with the same generating power, then keep adding to this set elements from $b_C(S)$ until a C -base is obtained.

Now, in view of Lemma 1 and $B \in b_D(b_C(S))$, condition (10) presents a contradiction to the fact that $b_D(b_C(S))$ is a D -base (of $D(b_D(b_C(S)))$), finishing the proof. \square

Proof of Theorem 5. Putting $U = L^Y$, $C = [\]_{\wedge}$, and $D = [\]_{\rightarrow}$, we verify that the assumptions of Lemma 2 are met. Indeed, (i) translates to $[[S]_{\wedge}]_{\rightarrow} \subseteq [[S]_{\rightarrow}]_{\wedge}$ which is established in the proof of (7) in Theorem 3; (ii) translates to $\text{irr}_{\wedge}(S) \subseteq S$ and $\text{irr}_{\rightarrow}(S) \subseteq S$, which is clear from the description of irr_{\wedge} and irr_{\rightarrow} in Theorem 4; (iii) follows from Theorem 7 in the appendix. The assertion of Lemma 2 now yields the theorem. \square

With regard to the above idea of obtaining a base of $[S]$ from S by successive “unwrapping,” i.e., by computing a $[]_{\wedge}$ -base from S followed by computing from it a $[]_{\rightarrow}$ -base, [Theorem 5](#) yields the following: The adjustments needed for this idea to work consist of making the input S closed under \rightarrow -shifts, which is easily done by computing $[S]_{\rightarrow}$, and applying the idea to $[S]_{\rightarrow}$. Such procedure then ensures that a $[]$ -base of $[S]$ is obtained for any input S .

It is now clear from the above discussions that if such procedure is applied to a system $S = \{\{^1/x\}^\uparrow \mid x \in X\}$ of rows of a given formal context $\langle X, Y, I \rangle$, it returns a system \mathcal{T} of rows of a row-reduced context that is equivalent to the given $\langle X, Y, I \rangle$, i.e., a row-reduced context whose fuzzy concept lattice is isomorphic to the concept lattice $\mathcal{B}(X, Y, I)$ of the input context. In detail, consider the new context $\langle Z, Y, J \rangle$ with $Z = \mathcal{T}$ for which

$$J(z, y) = z(y)$$

for each $z \in Z$ and $y \in Y$ (the rows of the new objects $z \in Z$ are the fuzzy sets of \mathcal{T}). The above results imply that the fuzzy closure system generated by the rows of $\langle X, Y, I \rangle$ equals the one generated by the rows of $\langle Z, Y, J \rangle$. Since these two fuzzy closure systems are just $\text{Int}(X, Y, I)$ and $\text{Int}(Z, Y, J)$, respectively, we get

$$\text{Int}(X, Y, I) = \text{Int}(Z, Y, J),$$

which implies that $\mathcal{B}(X, Y, I)$ and $\mathcal{B}(Z, Y, J)$ are isomorphic, i.e., $\langle X, Y, I \rangle$ and $\langle Z, Y, J \rangle$ are equivalent. The procedure is summarized as [Algorithm 2](#).

Algorithm 2: Computing a row-reduced fuzzy context.

Input: formal fuzzy context $\langle X, Y, I \rangle$
Output: row-reduced $\langle Z, Y, J \rangle$ equivalent to $\langle X, Y, I \rangle$
 $S \leftarrow \{\{^1/x\}^\uparrow \mid x \in X\}$
 $Z \leftarrow \text{irr}_{\rightarrow}(\text{irr}_{\wedge}([S]_{\rightarrow}))$
 $J(z, y) \leftarrow z(y)$ for each $z \in Z, y \in Y$
 return $\langle Z, Y, J \rangle$

Example 2. Consider the same formal fuzzy context as in the previous example:

$$I = \begin{pmatrix} 0.5 & 0.5 & 1 & 0.25 \\ 0.5 & 0 & 0.5 & 0.25 \\ 0.5 & 0.5 & 0.5 & 0.25 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ 1 & 1 & 1 & 0.25 \end{pmatrix}.$$

Then

$$S = \{(0.5 \ 0.5 \ 1 \ 0.25), (0.5 \ 0 \ 0.5 \ 0.25), (0.5 \ 0.5 \ 0.5 \ 0.25), (0.5 \ 0.5 \ 0.5 \ 0.5), (1 \ 1 \ 1 \ 0.25)\}.$$

The following schema shows intermediate results of [Algorithm 2](#). First, we compute all the shifts $a \rightarrow A$ for $a \in L$ and $A \in S$ (column $[S]_{\rightarrow}$). Second, we eliminate the \wedge -reducible elements (column $\text{irr}_{\wedge}([S]_{\rightarrow})$) and then eliminate the \rightarrow -reducible elements (column $\text{irr}_{\rightarrow}(\text{irr}_{\wedge}([S]_{\rightarrow}))$).

$[S]_{\rightarrow}$	$\text{irr}_{\wedge}([S]_{\rightarrow})$	$\text{irr}_{\rightarrow}(\text{irr}_{\wedge}([S]_{\rightarrow}))$
(0.5 0 0.5 0.25)	\wedge -reducible	—
(0.5 0.5 0.5 0.25)	\wedge -reducible	—
(0.5 0.5 0.5 0.5)	OK	OK
(0.5 0.5 1 0.25)	OK	OK
(1 0 1 0.25)	OK	OK
(1 1 1 0.25)	OK	\rightarrow -reducible
(1 1 1 1)	\wedge -reducible	—

This leads to the resulting fuzzy context whose rows are the fuzzy sets that survived the elimination:

$$J = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 1 & 0.25 \\ 1 & 0 & 1 & 0.25 \end{pmatrix}.$$

Notice that J contains a row (the last one) that is not present in the input matrix I .

Remark 4. As is clear from the above description, [Algorithm 2](#) is based on computing a base of the system $\text{Int}(X, Y, I)$ of intents, regarded as a fuzzy closure system generated by the system $S = \{\{^1/x\}^\uparrow \mid x \in X\}$ of rows of a given fuzzy context $\langle X, Y, I \rangle$. In particular, [Algorithm 2](#) utilizes the base described in [Theorem 5](#). But note that the simple iterative [Algorithm 1](#), which consists in a successive removal of reducible objects, i.e., on the idea inspired by the classical reduction [\[1\]](#), may also be thought of as being based on computing a base of $\text{Int}(X, Y, I)$. Namely, the rows of the row-reduced context produced by [Algorithm 1](#) form a base: according to the considerations in [Section 4.1](#), they generate $\langle X, Y, I \rangle$; according to [Lemma 1](#), they are non-redundant. In this perspective, the two algorithms differ in the way they compute a base of $\text{Int}(X, Y, I)$.

4.4. Reconstruction of formal concepts of the original context

In this section, we establish how the formal concepts of the original fuzzy context can be reconstructed from the formal concepts of its reduction, and vice versa, for the classes of reductions presented in [Sections 4.2](#) and [4.3](#). Since a complete reduction consists of a row reduction followed by a column reduction (or in the reverse order), it is sufficient to analyze this relationship for object reduction (cf. the end of [Section 4.1](#)). Dual considerations naturally apply to attribute reduction.

Let $\langle X, Y, I \rangle$ be a formal fuzzy context and let $\langle Z, Y, J \rangle$ be an equivalent row-reduced context. For the methods in [Sections 4.2](#) and [4.3](#), the equivalence of $\langle X, Y, I \rangle$ and $\langle Z, Y, J \rangle$ means that their systems of intents coincide, i.e.,¹⁴

$$\text{Int}(X, Y, I) = \text{Int}(Z, Y, J). \tag{11}$$

Consequently, for every formal concept $\langle C, D \rangle \in \mathcal{B}(X, Y, I)$, there is a unique corresponding formal concept $\langle C', D \rangle \in \mathcal{B}(Z, Y, J)$, for which $C' = D^{\downarrow J}$.

For the relationship between the extents $C \in \text{Ext}(X, Y, I)$ and $C' \in \text{Ext}(Z, Y, J)$ we need the following lemma.

Lemma 3 ([\[20\]](#)). For arbitrary formal fuzzy contexts $\langle X_1, Y, I_1 \rangle$ and $\langle X_2, Y, I_2 \rangle$, we have $\text{Int}(X_1, Y, I_1) \subseteq \text{Int}(X_2, Y, I_2)$ if and only if there exists a fuzzy relation $R \in L^{X_1 \times X_2}$ such that $I_1 = R \triangleleft I_2$, where \triangleleft denotes the composition operator defined by

$$(R \triangleleft I_2)(x_1, y) = \bigwedge_{x_2 \in X_2} (R(x_1, x_2) \rightarrow I_2(x_2, y)).$$

Therefore, due to [\(11\)](#), [Lemma 3](#) applied to $\text{Int}(X, Y, I) \subseteq \text{Int}(Z, Y, J)$ and $\text{Int}(Z, Y, J) \subseteq \text{Int}(X, Y, I)$ yields the existence of fuzzy relations $T \in L^{X \times Z}$ and $R \in L^{Z \times X}$, respectively, for which

$$I = T \triangleleft J \quad \text{and} \quad J = R \triangleleft I. \tag{12}$$

According to the criteria for solving fuzzy relational equations [\[11, chap. 6\]](#), solutions T and R of the two equations in [\(12\)](#) can be obtained as¹⁵

$$T = I \triangleleft J^{-1} \quad \text{and} \quad R = J \triangleleft I^{-1},$$

with J^{-1} and I^{-1} denoting the inverse relations, i.e.,

$$T(x, z) = \bigwedge_{y \in Y} (I(x, y) \rightarrow J(z, y)) \quad \text{and} \quad R(z, x) = \bigwedge_{y \in Y} (J(z, y) \rightarrow I(y, x)). \tag{13}$$

Now, the fuzzy relations T and R represent translations between the extents of the concept lattice $\mathcal{B}(X, Y, I)$ of the original formal context and the extents of the concept lattice $\mathcal{B}(Z, Y, J)$ of the reduced formal context, as shown by the following theorem.

Theorem 6. Let $\langle X, Y, I \rangle$ and $\langle Z, Y, J \rangle$ be fuzzy contexts satisfying [\(12\)](#), and T and R be the fuzzy relations defined in [\(13\)](#).

(a) Let $\langle C', D \rangle \in \mathcal{B}(Z, Y, J)$. The extent C of the original context corresponding to C' , i.e., the fuzzy set C for which $\langle C, D \rangle \in \mathcal{B}(X, Y, I)$, is reconstructed by¹⁶

$$C = T \triangleleft C'. \tag{14}$$

(b) Let $\langle C, D \rangle \in \mathcal{B}(X, Y, I)$. The extent C' of the reduced context corresponding to C , i.e., the fuzzy set C' for which $\langle C', D \rangle \in \mathcal{B}(Z, Y, J)$, is reconstructed by

$$C' = R \triangleleft C. \tag{15}$$

Proof. (a) Let $\langle C', D \rangle \in \mathcal{B}(Z, Y, J)$, i.e., $C' = D^{\downarrow J}$ and $C'^{\uparrow J} = D$. We need to show that $C = T \triangleleft C'$ satisfies $C = D^{\downarrow I}$. Using [\(3\)](#), [\(12\)](#), and the standard properties of residuated lattices regarding the distributivity of \rightarrow and \bigwedge [\[11\]](#), we obtain

$$D^{\downarrow I}(x) = \bigwedge_{y \in Y} (D(y) \rightarrow I(x, y))$$

¹⁴ The question of possible other row-reductions remains an open problem.

¹⁵ In fact, $T = I \triangleleft J^{-1}$ and $R = J \triangleleft I^{-1}$ are the greatest solutions, but this fact does not play a significant role in our treatment.

¹⁶ Using a commonly employed abuse of notation identifying L^U with $L^{U \times \{*\}}$ as well as with $L^{U \times \{*\}}$. Thus, $(T \triangleleft C')(x) = \bigwedge_{z \in Z} (T(x, z) \rightarrow C(z))$. The same applies to formula [\(15\)](#), i.e., $(R \triangleleft C)(z) = \bigwedge_{x \in X} (R(z, x) \rightarrow C(x))$.

$$\begin{aligned}
 &= \bigwedge_{y \in Y} \left(D(y) \rightarrow \bigwedge_{z \in Z} (T(x, z) \rightarrow J(z, y)) \right) \\
 &= \bigwedge_{z \in Z} \bigwedge_{y \in Y} (T(x, z) \rightarrow (D(y) \rightarrow J(z, y))) \\
 &= \bigwedge_{z \in Z} \left(T(x, z) \rightarrow \bigwedge_{y \in Y} (D(y) \rightarrow J(z, y)) \right) \\
 &= \bigwedge_{z \in Z} (T(x, z) \rightarrow D^{\downarrow J}(z)) = \bigwedge_{z \in Z} (T(x, z) \rightarrow C'(z)) \\
 &= (T \triangleleft C')(x),
 \end{aligned}$$

proving $C = D^{\downarrow I}$.

(b) Follows analogously by swapping the roles of I and J and using $J = R \triangleleft I$. \square

We illustrate the reconstruction process described by Theorem 6 using the fuzzy contexts from Examples 1 and 2.

Example 3. Consider the five-element Gödel chain \mathbf{L} , the fuzzy context $\langle X, Y, I \rangle$ and the row-reduced fuzzy context $\langle Z, Y, J \rangle$ computed using Algorithm 1 in Example 1, with $Z = \{1, 2, 4\}$ representing the first, the second, and the fourth rows of I that survived the elimination.

The translation fuzzy relations $T \in L^{X \times Z}$ and $R \in L^{Z \times X}$ computed using (13) are as follows:

$$T = \begin{pmatrix} 1 & 0 & 0.5 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0.25 & 0 & 1 \\ 0.5 & 0 & 0.5 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 1 & 0 & 0.5 & 0.5 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0.25 & 0 & 0.25 & 1 & 0.25 \end{pmatrix}.$$

Consider the formal fuzzy concept $\langle C', D \rangle$ of the reduced context $\langle Z, Y, J \rangle$ determined by the intent (one of the rows of J) $D = (0.5 \ 0.5 \ 0.5 \ 0.5)$. We have $C' = D^{\downarrow J} = (0.25 \ 0 \ 1)$.

Using Theorem 6, the corresponding extent $C \in \text{Ext}(X, Y, I)$ of the original context $\langle X, Y, I \rangle$ is obtained by (14), i.e., via $C = T \triangleleft C'$. Thus, computing for each object $x \in X$ the membership degree $C(x) = \bigwedge_{z \in Z} (T(x, z) \rightarrow C'(z))$ leads to

$$\begin{aligned}
 C(1) &= (1 \rightarrow 0.25) \wedge (0 \rightarrow 0) \wedge (0.5 \rightarrow 1) = 0.25 \wedge 1 \wedge 1 = 0.25, \\
 C(2) &= (1 \rightarrow 0.25) \wedge (1 \rightarrow 0) \wedge (1 \rightarrow 1) = 0.25 \wedge 0 \wedge 1 = 0, \\
 C(3) &= (1 \rightarrow 0.25) \wedge (0 \rightarrow 0) \wedge (1 \rightarrow 1) = 0.25 \wedge 1 \wedge 1 = 0.25, \\
 C(4) &= (0.25 \rightarrow 0.25) \wedge (0 \rightarrow 0) \wedge (1 \rightarrow 1) = 1 \wedge 1 \wedge 1 = 1, \\
 C(5) &= (0.5 \rightarrow 0.25) \wedge (0 \rightarrow 0) \wedge (0.5 \rightarrow 1) = 0.25 \wedge 1 \wedge 1 = 0.25.
 \end{aligned}$$

This gives us the reconstructed extent $C = (0.25 \ 0 \ 0.25 \ 1 \ 0.25)$. One can easily verify directly using $\langle X, Y, I \rangle$ that, indeed, $D^{\downarrow I} = C$.

Conversely, one may obtain C' from C using (15).

Remark 5. With regard to the considerations preceding Theorem 6, observe that in Example 3, one may use a fuzzy relation R different from $R = J \triangleleft I^{-1}$ that is used in Theorem 6 and in Example 3. For example, since all rows of $\langle Z, Y, J \rangle$ are also rows of $\langle X, Y, I \rangle$ in this case, we can let R be the identity relation restricted to $Z \times X$, i.e.,

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

That clearly gives the same result for the reconstructed C' because $C' = R \triangleleft C = C \cap Z$. \square

Example 4. Similarly, one can translate the extents for the row-reduced context $\langle Z, Y, J \rangle$ obtained using Algorithm 2 in Example 2. The fuzzy relations $T \in L^{X \times Z}$ and $R \in L^{Z \times X}$ are computed again using (13):

$$T = \begin{pmatrix} 0.5 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0.25 & 0 \\ 0.5 & 0.5 & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0.25 & 0 & 0.25 & 1 & 0.25 \\ 1 & 0 & 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 0.5 & 0.5 & 1 \end{pmatrix}.$$

Take again the intent $D = (0.5 \ 0.5 \ 0.5 \ 0.5)$ (the same as in Example 3) and consider its extent C' in the context $\langle Z, Y, J \rangle$, i.e., $C' = D^{\downarrow J} = (1 \ 0.25 \ 0)$. This way, we obtained a formal concept $\langle C', D \rangle \in \mathcal{B}(Z, Y, J)$.

To reconstruct the corresponding $\langle C, D \rangle \in \mathcal{B}(X, Y, I)$, we again use (14),

$$C(1) = (0.5 \rightarrow 1) \wedge (1 \rightarrow 0.25) \wedge (0 \rightarrow 0) = 1 \wedge 0.25 \wedge 1 = 0.25,$$

$$\begin{aligned}
 C(2) &= (1 \rightarrow 1) \wedge (1 \rightarrow 0.25) \wedge (1 \rightarrow 0) = 1 \wedge 0.25 \wedge 0 = 0, \\
 C(3) &= (1 \rightarrow 1) \wedge (1 \rightarrow 0.25) \wedge (0 \rightarrow 0) = 1 \wedge 0.25 \wedge 1 = 0.25, \\
 C(4) &= (1 \rightarrow 1) \wedge (0.25 \rightarrow 0.25) \wedge (0 \rightarrow 0) = 1 \wedge 1 \wedge 1 = 1, \\
 C(5) &= (0.5 \rightarrow 1) \wedge (0.5 \rightarrow 0.25) \wedge (0 \rightarrow 0) = 1 \wedge 0.25 \wedge 1 = 0.25.
 \end{aligned}$$

This results in the same extent $C = (0.25 \quad 0 \quad 0.25 \quad 1 \quad 0.25)$ as in [Example 3](#). In the other direction, the fuzzy relation R may be used to obtain C' from C using [\(15\)](#). \square

Remark 6. Consider two formal concepts $\langle A, B \rangle, \langle C, D \rangle \in \mathcal{B}(X, Y, I)$ and their corresponding concepts $\langle A', B \rangle, \langle C', D \rangle \in \mathcal{B}(Z, Y, J)$ in the row-reduced context. Since $S(C, A) = S(B, D)$ generally holds for formal fuzzy concepts [\[11\]](#), applying [\(15\)](#) yields

$$S(C, A) = S(B, D) = S(C', A') = S(R \triangleleft C, R \triangleleft A)$$

By setting $C = \{^1/x\}^{\uparrow I \downarrow I}$, we obtain

$$S(\{x\}^{\uparrow I \downarrow I}, A) = S(R \triangleleft \{^1/x\}^{\uparrow I \downarrow I}, R \triangleleft A).$$

Since the left-hand side reduces to $A(x)$, we have

$$A(x) = S(R \triangleleft \{^1/x\}^{\uparrow I \downarrow I}, R \triangleleft A). \tag{16}$$

If the reduced context $\langle Z, Y, J \rangle$ contains only rows from $\langle X, Y, I \rangle$ we can further simplify [\(16\)](#) using $R \triangleleft C = C \cap Z$. Thus, we have

$$A(x) = S(\{^1/x\}^{\uparrow I \downarrow I} \cap Z, A \cap Z). \tag{17}$$

The resulting expression [\(17\)](#) is a direct generalization of the formula

$$x \in A \iff \{x\}^{\uparrow I \downarrow I} \cap Z \subseteq A \cap Z \tag{18}$$

known from ordinary FCA [\[1, p. 33\]](#). Note also that the ordinary formula [\(18\)](#) represents all that is needed for the reconstruction of the extent A of the original context $\langle X, Y, I \rangle$ from the corresponding extent A' of the reduced context $\langle Z, Y, J \rangle$.¹⁷ Namely, $\langle Z, Y, J \rangle$ contains only rows from $\langle X, Y, I \rangle$ in the ordinary case, whence the above situation resulting in [\(18\)](#)—which describes just a particular option in a fuzzy setting—represents a general situation in ordinary FCA. \square

5. Concluding remarks

In this paper, we examined the problem of reduction of formal fuzzy contexts. We first explained the problem and its non-triviality when compared to the classical case. The view of the problem in terms of closure structures underlying FCA in a fuzzy setting, which we provided, enables a clear understanding of the problem. We also presented further results obtained from the provided view along with two simple reduction algorithms. The present contributions are meant to initiate further research on the practically important problem of reduction: The research topics include:

- A significant contribution to the problem of reduction of fuzzy contexts was recently provided by Antoni et al. [\[5\]](#). The authors consider the problem within the so-called multi-adjoint framework. The core contribution of their work is a procedure for attribute reduction, which is based on their previous theoretical results by Cornejo, Medina, and Ramírez-Poussa [\[6,7\]](#). The approach involves several stages. First, generating a list of attributes and all of their \rightarrow -shifts [\[5, Algorithm 1\]](#), followed by identifying the extents of \wedge -irreducible concepts [\[5, Algorithm 2\]](#). The attributes are categorized as absolutely necessary, relatively necessary, and absolutely unnecessary ([\[5, Algorithms 3–7\]](#)). Finally, the algorithm utilizes a heuristic inspired by the linear QUICKREDUCT II algorithm to select a small subset of attributes [\[5, Algorithm 9\]](#), which results in an attribute-reduced context. Since the multi-adjoint framework is more general than our framework, the discussed algorithm is applicable to the fuzzy contexts we investigate. A detailed comparison of [\[5\]](#) to our approach, which we performed, is beyond the scope of our paper, and we plan to include it in our future writings. Compared to both of our algorithms, the one in [\[5\]](#) is considerably more complex. Note also that the algorithm in [\[5\]](#) always returns some columns of the input context, which is the same as with our [Algorithm 1](#), but different from our [Algorithm 2](#) which may replace some of the removed columns by new ones. For a linear residuated lattice, all three algorithms return a reduced context of the same size, because as we proved in [\[3\]](#), all bases of fuzzy closure systems have the same size in this case. For non-linear residuated lattices, while the algorithm [\[5\]](#) does not always return a smallest base, we expect it to deliver a smaller base on average compared to our [Algorithm 1](#). Overall, even though a clean, elementary formulation of the reduction problem is missing in [\[5–7\]](#) in our view, the authors present several interesting ideas that are worth further exploration and are expected to improve our first algorithm.
- Further work on algorithms is needed. Due to our results, two basic classes of algorithms need to be explored. The first one includes those returning rows and columns of the original formal fuzzy context (our first algorithm and the one in [\[5\]](#) belong to this class). The second one consists of those possibly returning different rows and columns, which may result in a smaller formal context. A computational complexity analysis of the problem and the algorithms also provides an interesting topic.

¹⁷ Bearing in mind that $A' = A \cap Z$, formula [\(18\)](#) yields a constructive description of A : A consists of those objects x for which $\{x\}^{\uparrow I \downarrow I} \cap Z$ is contained in A' .

- We obtained some further results on questions directly related to the reduction problem, which employ finer notions of reduction and equivalence of formal fuzzy contexts with respect to reduction. These may be regarded as a continuation of [2]. Further questions include a characterization of formal fuzzy contexts that are equivalent in that their concept lattices are isomorphic with a notion of isomorphism properly reflecting the structure of fuzzy concept lattices [8]; furthermore, the two kinds of reductions proposed in our paper suggest to examine further classes of reductions and their properties. An abstract view of the involved structures, such as lattice orders, in the present setting of a many-valued logic may enable one to reformulate some of the results presented in this paper in a more general, yet cleaner mathematical language, but may also provide valuable insights.
- Worth pursuing are also particular instances of the reduction problem arising when the residuated lattice of truth degrees, which determines the properties of the concept-forming operators, is chosen from a certain class of algebras such as the class of all MV-algebras, all Gödel algebras, or—more particularly—all finite chains equipped with the Łukasiewicz operations. We already know that if the residuated lattice is linear, all bases, which we use to obtain reductions, have the same size.

CRedit authorship contribution statement

Radim Belohlavek: Writing – review & editing, Writing – original draft, Investigation, Formal analysis, Conceptualization; **Jan Konecny:** Writing – review & editing, Writing – original draft, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A.

Consider a complete lattice $\langle V, \leq \rangle$ with infima \bigwedge and assume that it satisfies the ascending chain condition (ACC): There is no infinite sequence $v_1 < v_2 < v_3 < \dots$ [16]. The finiteness of V clearly implies the ACC, but there are infinite V satisfying the ACC (wide, but short). The ACC means (is equivalent to) that every non-empty subset of V has a maximal element, which property we shall use. Note that in our paper we use the following results for the particular case $V = L^U$ (set of all fuzzy sets in U) with finite L and U , in which case the ACC is clearly satisfied.

Consider any $\mathcal{V} \subseteq V$ (and below also $S, \mathcal{T} \subseteq V$) and put

$$\text{irr}_{\bigwedge}(\mathcal{V}) = \{A \in \mathcal{V} \mid A \notin [\mathcal{V} - \{A\}]_{\bigwedge}\}.$$

$\text{irr}_{\bigwedge}(\mathcal{V})$ are elements in \mathcal{V} that are not \bigwedge -intersections of other elements in \mathcal{V} , the \bigwedge -irreducibles.

Lemma 4. $[\mathcal{V}]_{\bigwedge} = [\text{irr}_{\bigwedge}(\mathcal{V})]_{\bigwedge}$.

Proof. $[\mathcal{V}]_{\bigwedge} \supseteq [\text{irr}_{\bigwedge}(\mathcal{V})]_{\bigwedge}$ follows from $\mathcal{V} \supseteq \text{irr}_{\bigwedge}(\mathcal{V})$. Conversely, if $A \in [\mathcal{V}]_{\bigwedge}$, then $A = \bigwedge \mathcal{U}$ for some $\mathcal{U} \subseteq \mathcal{V}$. To prove the claim, it is therefore sufficient to prove that every $B \in \mathcal{V}$ is the \bigwedge -intersection of some elements in $\text{irr}_{\bigwedge}(\mathcal{V})$. By contradiction, suppose $B \notin [\text{irr}_{\bigwedge}(\mathcal{V})]_{\bigwedge}$ for some $B \in \mathcal{V}$. By virtue of the ACC, assume that B is maximal with this property. Then $B < \bigwedge \mathcal{P}$ where $\mathcal{P} = \{C \in \text{irr}_{\bigwedge}(\mathcal{V}) \mid B < C\}$. Since $B \notin [\text{irr}_{\bigwedge}(\mathcal{V})]_{\bigwedge}$, we have $B \notin \text{irr}_{\bigwedge}(\mathcal{V})$, i.e., B is \bigwedge -reducible, hence, $B = \bigwedge \mathcal{Q}$ where $\mathcal{Q} = \{C \in \mathcal{V} \mid B < C\}$. Observe now that $\mathcal{Q} - \mathcal{P}$ is non-empty (otherwise $\mathcal{Q} \subseteq \mathcal{P}$, whence $B < \bigwedge \mathcal{P} \leq \bigwedge \mathcal{Q}$) and that every $C \in \mathcal{Q} - \mathcal{P}$ is \bigwedge -reducible.

If $C \in [\text{irr}_{\bigwedge}(\mathcal{V})]_{\bigwedge}$ for each $C \in \mathcal{Q} - \mathcal{P}$ then each such C would be an \bigwedge -intersection of some elements in $\text{irr}_{\bigwedge}(\mathcal{V})$, and since $B < C$, each $C \in \mathcal{Q} - \mathcal{P}$ would be an \bigwedge -intersection of some elements in \mathcal{P} . Consequently, $\bigwedge \mathcal{Q} - \mathcal{P}$ would be an \bigwedge -intersection of some elements in \mathcal{P} . Now,

$$B = \bigwedge \mathcal{Q} = \bigwedge (\mathcal{P} \cup (\mathcal{Q} - \mathcal{P})) = \bigwedge \mathcal{P} \wedge \bigwedge \mathcal{Q} - \mathcal{P}$$

would imply that $B = \bigwedge \mathcal{P}$ which is not the case.

Therefore, there exists $C \in \mathcal{Q} - \mathcal{P}$ with $C \notin [\text{irr}_{\bigwedge}(\mathcal{V})]_{\bigwedge}$. This contradicts the maximality of B because $B < C$ for each $C \in \mathcal{Q} - \mathcal{P}$. \square

Lemma 5. $[S]_{\bigwedge} = [\mathcal{T}]_{\bigwedge}$ if and only if $\text{irr}_{\bigwedge}(S) = \text{irr}_{\bigwedge}(\mathcal{T})$.

Proof. If $\text{irr}_{\bigwedge}(S) = \text{irr}_{\bigwedge}(\mathcal{T})$, then Lemma 4 yields $[S]_{\bigwedge} = [\text{irr}_{\bigwedge}(S)]_{\bigwedge} = [\text{irr}_{\bigwedge}(\mathcal{T})]_{\bigwedge} = [\mathcal{T}]_{\bigwedge}$.

Conversely, let $[S]_{\bigwedge} = [\mathcal{T}]_{\bigwedge}$. Due to symmetry, we only prove $\text{irr}_{\bigwedge}(S) \subseteq \text{irr}_{\bigwedge}(\mathcal{T})$. Let $B \in \text{irr}_{\bigwedge}(S)$. Then $B \in S \subseteq [S]_{\bigwedge}$ and the assumption and Lemma 4 imply $B \in [\mathcal{T}]_{\bigwedge} = [\text{irr}_{\bigwedge}(\mathcal{T})]_{\bigwedge}$, from which it follows that $B = \bigwedge \mathcal{U}$ for some $\mathcal{U} \subseteq \text{irr}_{\bigwedge}(\mathcal{T})$. Now, since $\text{irr}_{\bigwedge}(\mathcal{T}) \subseteq \mathcal{T} \subseteq [S]_{\bigwedge}$, the assumption $[S]_{\bigwedge} = [\mathcal{T}]_{\bigwedge}$ yields that every $C \in \mathcal{U}$ is an \bigwedge -intersection of some elements $B' \in S$. Therefore, if $B < C$ for every $C \in \mathcal{U}$, B would be an \bigwedge -intersection of elements of S , namely the elements B' , that are greater than B , which is a contradiction to the assumption $B \in \text{irr}_{\bigwedge}(S)$. It follows that there must be $C \in \mathcal{U}$ such that $B = C$, and since $\mathcal{U} \subseteq \text{irr}_{\bigwedge}(\mathcal{T})$, we proved $B \in \text{irr}_{\bigwedge}(\mathcal{T})$. \square

We obtain the following claim:

Theorem 7. $\text{irr}_{\wedge}(S)$ is the only $[\]_{\wedge}$ -base of $[S]_{\wedge}$.

Proof. According to Lemma 4, $\text{irr}_{\wedge}(S)$ generates $[S]_{\wedge}$. Furthermore, $\text{irr}_{\wedge}(S)$ is non-redundant. Namely, if $[\mathcal{R}]_{\wedge} = [\text{irr}_{\wedge}(S)]_{\wedge}$ for some $\mathcal{R} \subset \text{irr}_{\wedge}(S)$, then for any $B \in \text{irr}_{\wedge}(S) - \mathcal{R}$ we have $[\text{irr}_{\wedge}(S)]_{\wedge} \supseteq [\text{irr}_{\wedge}(S) - \{B\}]_{\wedge} \supseteq [\mathcal{R}]_{\wedge} = [\text{irr}_{\wedge}(S)]_{\wedge}$, hence $[\text{irr}_{\wedge}(S)]_{\wedge} = [\text{irr}_{\wedge}(S) - \{B\}]_{\wedge}$. Since $\text{irr}_{\wedge}(S) - \{B\} \subseteq S - \{B\}$, we obtain $B \in [S]_{\wedge} = [\text{irr}_{\wedge}(S)]_{\wedge} = [\text{irr}_{\wedge}(S) - \{B\}]_{\wedge} \subseteq [S - \{B\}]_{\wedge}$, a contradiction to $B \in \text{irr}_{\wedge}(S)$. If \mathcal{T} is another $[\]_{\wedge}$ -base of $[S]_{\wedge}$ then clearly $\mathcal{T} = \text{irr}_{\wedge}(\mathcal{T})$ (otherwise \mathcal{T} is redundant). Moreover, by definition, $[\mathcal{T}]_{\wedge} = [S]_{\wedge}$ and hence $\text{irr}_{\wedge}(\mathcal{T}) = \text{irr}_{\wedge}(S)$ by Lemma 5. As a result, $\mathcal{T} = \text{irr}_{\wedge}(S)$, finishing the proof. \square

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