ARTICLE IN PRESS
FSS:7435

Available online at www.sciencedirect.com

ScienceDirect

Fuzzy Sets and Systems ••• (••••) •••–•••

FUZZY sets and systems

www.elsevier.com/locate/fss

A reduction theorem to compute fixpoints of fuzzy closure operators

Radim Belohlavek, Jan Konecny *

Palacký University, Olomouc, Czech Republic

Received 9 October 2017; received in revised form 9 May 2018; accepted 27 May 2018

Abstract

We present a reduction theorem which relates sets of fixpoints of fuzzy closure operators to sets of fixpoints of ordinary closure operators. As a result we obtain a method to compute sets of fixpoints of fuzzy closure operators by algorithms available for ordinary operators. We also provide explicit descriptions of selected algorithms which result from the presented approach.

Keywords: Formal concept analysis; Fuzzy closure operator; Fixed point; Algorithm

1. Introduction

In fuzzy logic, closure structures appear in several contexts and have been examined since the early explorations, including those on Pavelka-style logic and fuzzy topology. A number of recent investigations studied computation of systems of fixpoints of fuzzy closure operators. Except for other reasons, a particular practical reason is that these systems appear in a number of problems pertaining to relational data. A natural way to compute sets of fixpoints of fuzzy closure operators to generalize the available algorithms for the ordinary operators (see e.g. [7] for an overview). These particular generalizations are, nevertheless, not straightforward since in the fuzzy setting, there are two generating operations involved, namely the minimum and residuum, instead of a single operation, namely the minimum, which is involved in the ordinary case. Our main concern in this paper is an alternative prospect which derives from the general question of relationships between ordinary and fuzzy notions. We examine whether and to what extent fuzzy closure operators are reducible to ordinary closure operators. Some general relationships were obtained in [4,25], see also [5], but in this paper we are particularly interested in reduction results convenient for computing the fixpoints of fuzzy closure operators.

2. Preliminaries

We employ complete residuated lattices as the structures of truth degrees and assume familiarity with their properties; see e.g. [5,16]: a complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$

* Corresponding author.
E-mail address: jan.konecny@upol.cz (J. Konecny).

https://doi.org/10.1016/j.fss.2018.05.015
0165-0114/

Please cite this article in press as: R. Belohlavek, J. Konecny, A reduction theorem to compute fixpoints of fuzzy closure operators, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.05.015

is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid; and $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

for each $a, b, c \in L$. Recall that commonly employed residuated lattices are induced by left-continuous t-norms $\otimes$, in which case $L = [0, 1]$, $\wedge = \min$, $\vee = \max$, and $a \rightarrow b = \max\{c \mid a \otimes c \leq b\}$; this, in particular, includes the three basic t-norms: the Łukasiewicz, Gödel (minimum), and Goguen (product) t-norms. Another common choice is a finite chain $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1]$ $(a_0 < \cdots < a_n)$ with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain; a finite Gödel chain consists of L and restrictions of the Gödel operations on $[0, 1]$ to L.

By L^U we denote the collection of all L-sets in U, i.e. all fuzzy sets $A : U \rightarrow L$. The operations with L-sets are defined componentwise as usual. Given $A, B \in L^U$, the subsethood degree (or, degree of inclusion of A in B) is defined by

$$S(A, B) = \bigwedge_{u \in U} (A(u) \rightarrow B(u)), \tag{2}$$

which generalizes the classical subsethood $\subseteq$. Put verbally, $S(A, B)$ represents a degree to which A is a subset of B. We write $A \subseteq B$ iff $S(A, B) = 1$; as a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$.

For our purpose, $\mathbf{L}$ needs to be equipped with operations representing intensifying linguistic hedges such as "very" or "rather," for which aim we use the concept close to the one used in [16,17]. By a truth-stressing hedge (or just a hedge) on $\mathbf{L}$, we mean a unary mapping * on L for which

$$1^* = 1, \tag{3}$$

$$a^* \leq a, \tag{4}$$

$$(a \rightarrow b)^* \leq a^* \rightarrow b^*, \tag{5}$$

$$a^{**} = a^*, \tag{6}$$

for each $a, b \in L$. Conditions (3) and (5) imply monotony: if $a \leq b$ then $a^* \leq b^*$. Two boundary cases are: (i) identity, i.e. $a^* = a$ $(a \in L)$; (ii) globalization, i.e. $1^* = 1$ and $a^* = 0$ for $a < 1$ [27]. Fig. 1 presents examples of truth-stressing hedges.

We shall need the following property of hedges:

Lemma 1. *Any truth-stressing hedge satisfies*

$$(a^* \otimes b^*)^* = a^* \otimes b^* \tag{7}$$

for all $a, b \in L$.

Proof. Adjointness implies that $a^* \otimes b^* \leq (a^* \otimes b^*)^*$ iff $a^* \leq b^* \rightarrow (a^* \otimes b^*)^*$. The latter condition holds true because as $a^* \leq b^* \rightarrow (a^* \otimes b^*)$, monotony of *, (5) and (6) imply $a^* = a^{**} \leq (b^* \rightarrow (a^* \otimes b^*))^* \leq b^{**} \rightarrow (a^* \otimes b^*)^* = b^* \rightarrow (a^* \otimes b^*)^*$.

The converse inequality follows directly from (4). □

Remark 1. Lemma 1 was pointed out to us by an anonymous referee, who also observed that $\langle L, \otimes, 1\rangle$ forms a quantale and that the hedge * forms a quantic conucleus on $\langle L, \otimes, 1\rangle$ [26, Definition 3. 1. 2].

An important special case of a complete residuated lattice with hedge is the two-element Boolean algebra $\langle\{0, 1\}, \wedge, \vee, \otimes, \rightarrow, ^*, 0, 1\rangle$, denoted by $\mathbf{2}$, of classical logic, in which $0^* = 0$, $1^* = 1$.

3. Main results

Throughout this section, $\mathbf{L}$ denotes a complete residuated lattice $\mathbf{L}$ with a truth-stressing hedge $*$. Our results concern the following, general concept of a fuzzy closure operator [3,8] (as we shall see, this concept covers the particular cases we need). An $\mathbf{L}^*$-closure operator in a set U is a mapping $C : L^U \rightarrow L^U$ satisfying

Please cite this article in press as: R. Belohlavek, J. Konecny, A reduction theorem to compute fixpoints of fuzzy closure operators, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.05.015

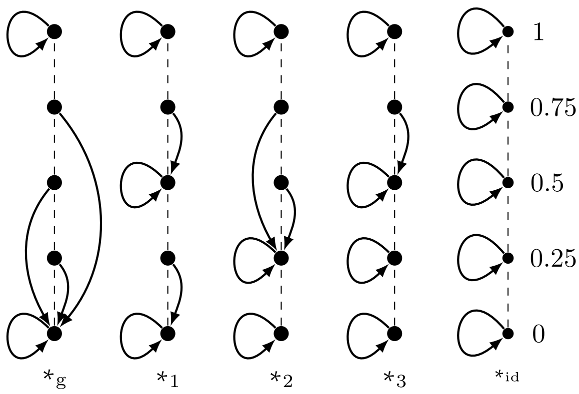

Fig. 1. Truth-stressing hedges on 5-element chain with Łukasiewicz operations $\mathbf{L} = \langle\{0, 0.25, 0.5, 0.75, 1\}, \min, \max, \otimes, \to, 0, 1\rangle$. The leftmost hedge denoted by ^{*g} is the globalization, the rightmost hedge denoted by ^{*id} is the identity.

$$A \subseteq C(A), \quad S(A, B)^* \leq S(C(A), C(B)), \quad C(A) = C(C(A)),$$

for every $A, B \in L^U$. Notice in particular the second condition which expresses a strong version of monotony involving graded inclusion. There exists a one-to-one correspondence between $\mathbf{L}^*$-closure operators and so-called $\mathbf{L}^*$-closure systems, which are defined as follows. Let $b \to A$ (called also a left $\to$-multiplication of A by b or a b-shift of A) and $\bigwedge_{j\in J} A_j$ be defined for $b \in L$ and fuzzy sets $A, A_j \in L^U$ $(j \in J)$ by $(b \to A)(u) = b \to A(u)$ and $(\bigwedge_{j\in J} A_j)(u) = \bigwedge_{j\in J} A_j(u)$. An $\mathbf{L}^*$-closure system in U is a system $\mathcal{S} \subseteq L^U$ of fuzzy sets in U satisfying:

$$a^* \to A \in \mathcal{S} \quad \text{whenever } a \in L \text{ and } A \in \mathcal{S},$$
$$\bigwedge_{j\in J} A_j \in \mathcal{S} \quad \text{whenever } A_j \in \mathcal{S}\ (j \in J),$$

i.e. $\mathcal{S}$ is closed under a^*-shifts and arbitrary intersections.

$\mathbf{L}^*$-closure operators and systems arise in formal concept analysis of data with fuzzy attributes [5,25] the following way. For a formal fuzzy context $\langle X, Y, I\rangle$ (i.e. X, Y, and I are a set of objects, a set of attributes, and a fuzzy relation between these sets, respectively), let operators $^\uparrow : L^X \to L^Y$ and $^\downarrow : L^Y \to L^X$ be defined as follows:

$$A^\uparrow(y) = \bigwedge_{x\in X}(A(x) \to I(x, y)) \quad \text{and} \quad B^\downarrow(x) = \bigwedge_{y\in Y}(B(y)^* \to I(x, y)). \tag{8}$$

A fuzzy concept lattice of $\langle X, Y, I\rangle$ is the set $\mathcal{B}(X, Y^*, I)$ of all fixpoints $\langle^\uparrow, ^\downarrow\rangle$, i.e.

$$\mathcal{B}(X, Y^*, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^\uparrow = B,\ B^\downarrow = A\}.$$

Note that this is a particularly important special case of so-called concept lattices with hedges; see [6,10]. The two boundary instances of hedges, namely $*$ being identity and globalization, are particularly important: With identity, one obtains the fuzzy concept lattices of [6,25], while for globalization one obtains the one-sided, or crisply generated, fuzzy concept lattices [10,28,20].

For our purpose, we shall need the following natural partial order $\leq$ of truth-stressing hedges: For hedges $*_1$ and $*_2$ on $\mathbf{L}$, put

$$*_1 \leq *_2 \quad \text{iff} \quad a^{*_1} \leq a^{*_2} \quad \text{for every } a \in L.$$

It is immediate that for the globalization $*_g$, the identity $*_{id}$, and an arbitrary hedge $*$ one has

$$*_g \leq * \leq *_{id}.$$

Fig. 2 illustrates the partial order $\leq$ on the set of hedges from Fig. 1. One easily checks the following claim:

Observation 1. Let $\text{fix}(*) = \{a \in L \mid a^* = a\}$ (fixpoints of $*$). Then $*_1 \leq *_2$ iff $\text{fix}(*_1) \subseteq \text{fix}(*_2)$.

We now present some easy observations connecting the above notions.

Please cite this article in press as: R. Belohlavek, J. Konecny, A reduction theorem to compute fixpoints of fuzzy closure operators, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.05.015

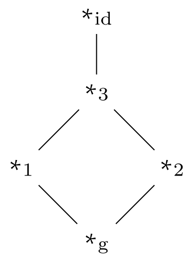

Fig. 2. Hasse diagram of the partial order $\leq$ on hedges from Fig. 1.

Observation 2.

(i) The set

$$\mathrm{Ext}(X, Y^*, I) = \{A \in L^X \mid \langle A, B\rangle \in \mathcal{B}(X, Y^*, I) \text{ for some } B\}$$

of all extents of $\langle X, Y, I\rangle$ forms an $\mathbf{L}^*$-closure system.

(ii) The compound mapping $C : L^X \to L^X$ defined by

$$C(A) = A^{\uparrow\downarrow}$$

for each $A \in L^X$ is an $\mathbf{L}^*$-closure operator; $\mathrm{Ext}(X, Y^*, I)$ is the set of all fixpoints of C.

(iii) If $*_1 \leq *_2$ then $\mathcal{B}(X, Y^{*_1}, I) \subseteq \mathcal{B}(X, Y^{*_2}, I)$.

Proof. Claims (i) and (ii) are well known and may be proved by adaptation of the proofs from the Boolean case using the properties of residuated lattices and hedges.

(iii): Let $\langle A_1, B_1\rangle \in \mathcal{B}(X, Y^{*_1}, I)$. By definition, we have $A_1 = B_1^{\downarrow}$, i.e. $A_1(x) = \bigwedge_{y\in Y}(B_1(y)^{*_1} \to I(x, y))$. Since $*_1 \leq *_2$, Observation 1 yields $\mathrm{fix}(*_1) \subseteq \mathrm{fix}(*_2)$. Since $B_1(y)^{*_1} \in \mathrm{fix}(*_1)$, we get $B_1(y)^{*_1} \in \mathrm{fix}(*_2)$, whence $(B_1(y)^{*_1})^{*_2} = B_1(y)^{*_1}$. Putting now $D = B_1^{*_1}$ we obtain $\bigwedge_{y\in Y}(D(y)^{*_2} \to I(x, y)) = \bigwedge_{y\in Y}(B_1(y)^{*_1} \to I(x, y)) = A_1(x)$ and since $A_1^{\uparrow} = B_1$, we conclude that $\langle A_1, B_1\rangle$ is a formal concept in $\mathcal{B}(X, Y^{*_2}, I)$ generated by the fuzzy set D. □

As was mentioned above, we assume that the $\mathbf{L}^*$-closure operator C, for which we intend to compute the set of its fixpoints, is given by a system $\mathcal{S} \subseteq L^U$ of fuzzy sets in U in that C is the operator generated by $\mathcal{S}$. That is, the set of all fixpoints of C is just the $\mathbf{L}^*$-closure system generated by $\mathcal{S}$, i.e. the least $\mathbf{L}^*$-closure system that contains $\mathcal{S}$. The assumption that a closure operator C is given by a system $\mathcal{S}$ is common when dealing constructively with closure operators; in addition, it fits well the situation in formal concept analysis in which the concept lattice is given by formal context (a formal context can be regarded as a collection of its columns, i.e. a collection $\mathcal{S}$ of the type mentioned above).

To proceed, we need the following notions. For any system $\mathcal{S} \subseteq L^U$ of fuzzy sets in U, denote by

- $[\mathcal{S}]^*$ the least $\mathbf{L}^*$-closure system containing $\mathcal{S}$; that is: (a) $[\mathcal{S}]^*$ is an $\mathbf{L}^*$-closure system, (b) $[\mathcal{S}]^*$ contains $\mathcal{S}$ (in that $[\mathcal{S}]^* \supseteq \mathcal{S}$), and (c) $[\mathcal{S}]^*$ is contained in any other $\mathbf{L}^*$-closure system containing $\mathcal{S}$;
- $[\mathcal{S}]_{\bigwedge}$ the least system containing $\mathcal{S}$ that is closed under $\bigwedge$-intersections,
- $[\mathcal{S}]^*_{\to}$ the least system containing $\mathcal{S}$ that is closed under a^*-shifts for every $a \in L$.

In view of the above considerations, we shall consider a formal context $\langle X, Y, I\rangle$ as represented by the system $\mathcal{S} = \{I_y \in L^X \mid y \in Y\}$ of fuzzy sets I_y (columns of the context) defined by

$$I_y(x) = I(x, y)$$

for each $x \in X$. The following theorem says that the $\mathbf{L}^*$-closure system generated by $\mathcal{S}$ is just the set of all extents of the fuzzy concept lattice $\mathcal{B}(X, Y^*, I)$ of $\langle X, Y, I\rangle$:

Please cite this article in press as: R. Belohlavek, J. Konecny, A reduction theorem to compute fixpoints of fuzzy closure operators, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.05.015

Theorem 3. $[\mathcal{S}]^* = \mathrm{Ext}(X, Y^*, I)$.

Proof. The fact that $\mathrm{Ext}(X, Y^*, I)$ is an $\mathbf{L}^*$-closure system is due to Observation 2. The fact that $\mathrm{Ext}(X, Y^*, I)$ contains $\mathcal{S}$ follows from the property that any column $I_y \in \mathcal{S}$ is an extent. Namely, it is the extent of the formal concept $\langle \{^1/y\}^{\downarrow}, \{^1/y\}^{\downarrow\uparrow}\rangle$, since $\{^1/y\}^{\downarrow}(x) = \{^1/y\}(y)^* \to I(x, y) = I(x, y) = I_y(x)$. Now, if an arbitrary $\mathbf{L}^*$-closure system $\mathcal{T}$ contains $\mathcal{S}$, it also contains any extent in $\mathrm{Ext}(X, Y^*, I)$ because each such extent A is by definition in the form $A = \bigwedge_{y\in Y}(b_y^* \to I_y)$ for some fuzzy set B with $B(y) = b_y$ and, clearly, $\bigwedge_{y\in Y}(b_y^* \to I_y)$ is an intersection of a^*-shifts of elements in $\mathcal{S}$. □

A significant insight into our problem is contained in the following claim.

Theorem 4. *For every system $\mathcal{S} = \{I_y \in L^X \mid y \in Y\}$ we have*

$$[\mathcal{S}]_{\bigwedge} = \{\bigwedge \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S}\}, \tag{9}$$

$$[\mathcal{S}]^*_{\to} = \{a^* \to I_y \mid a \in L,\ y \in Y\}, \tag{10}$$

$$[\mathcal{S}]^* = [[\mathcal{S}]^*_{\to}]_{\bigwedge}. \tag{11}$$

Proof. (9) is easy to check (it actually follows from general results concerning lattices).

(10): Clearly, S $1^* \to I_y = 1 \to I_y = I_y$, whence $\mathcal{S} \subseteq [\mathcal{S}]^*_{\to}$. Since every system closed under a^*-shifts obviously contains $[\mathcal{S}]^*_{\to}$, it is now sufficient to show that $[\mathcal{S}]^*_{\to}$ itself is closed under a^*-shifts. Take thus any element in $[\mathcal{S}]^*_{\to}$, i.e. an element of the form $a^* \to I_y$. Due to (7) and the fact that $\alpha \to (\beta \to \gamma) = (\alpha \otimes \beta) \to \gamma$, we obtain

$$b^* \to (a^* \to I_y) = (b^* \otimes a^*) \to I_y = (b^* \otimes a^*)^* \to I_y.$$

Since $(b^* \otimes a^*)^* \to I_y$ itself is a member of $[\mathcal{S}]^*_{\to}$, the proof is finished. (11): Verifying that $[[\mathcal{S}]^*_{\to}]_{\bigwedge}$ contains $\mathcal{S}$ and that $[[\mathcal{S}]^*_{\to}]_{\bigwedge}$ is contained in every system closed under a^*-shifts and $\bigwedge$-intersections is routine. It thus remains to show that $[[\mathcal{S}]^*_{\to}]_{\bigwedge}$ is closed under $\bigwedge$-intersections and a^*-shifts. Closedness under $\bigwedge$-intersections is easy to see. Let thus $a \in L$ and consider an arbitrary element of $[[\mathcal{S}]^*_{\to}]_{\bigwedge}$, i.e. an element of the form $\bigwedge_{y\in Z}(a_y^* \to I_y)$ for $Z \subseteq Y$. Then

$$a^* \to \bigwedge_{y\in Z}(a_y^* \to I_y) = \bigwedge_{y\in Z}(a^* \to (a_y^* \to I_y)) =$$
$$= \bigwedge_{y\in Z}(a^* \otimes a_y^*) \to I_y) = \bigwedge_{y\in Z}((a^* \otimes a_y^*)^* \to I_y) \in [[\mathcal{S}]^*_{\to}]_{\bigwedge},$$

verifying closedness of $[[\mathcal{S}]^*_{\to}]_{\bigwedge}$ under a^*-shifts. □

Remark 2. Observe that (11) presents an important decomposition result: In order to compute the desired $\mathbf{L}^*$-closure system generated by $\mathcal{S}$, i.e. to compute $[\mathcal{S}]^*$, it suffices to perform two steps: (i) compute the $[\ ]^*_{\to}$-closure of $\mathcal{S}$, i.e. $[\mathcal{S}]^*_{\to}$; apply to the result of (i) the operator of $[\ ]_{\bigwedge}$-closure. Due to (10), to obtain $[\mathcal{S}]^*_{\to}$, it suffices to generate all the a^*-shifts of elements in $\mathcal{S}$. Due to (9), computing the $[\ ]_{\bigwedge}$-closure is analogous to the problem of computing closure systems in the ordinary case.

Another result we shall employ is the following:

Theorem 5. *If $*_1 \le *_2$ then*

$$[\mathcal{S}]^{*_2} = [[\mathcal{S}]^{*_2}_{\to}]^{*_1}. \tag{12}$$

Proof. $[[\mathcal{S}]^{*_2}_{\to}]^{*_1} \subseteq [\mathcal{S}]^{*_2}$: Due to Theorem 4, each element in $[[\mathcal{S}]^{*_2}_{\to}]^{*_1}$ is of the form

$$\bigwedge_{y\in Z}(a_y^{*_1} \to (b_y^{*_2} \to I_y)), \tag{13}$$

for some $Z \subseteq Y$ and $a_y, b_y \in L$. The assumption $*_1 \le *_2$ and Observation 1 leads to $\mathrm{fix}(*_1) \subseteq \mathrm{fix}(*_2)$. As $a_y^{*_1} \in \mathrm{fix}(*_1)$, we have $a_y^{*_1} \in \mathrm{fix}(*_2)$, whence $a_y^{*_1} = (a_y^{*_1})^{*_2}$. We thus obtain

Please cite this article in press as: R. Belohlavek, J. Konecny, A reduction theorem to compute fixpoints of fuzzy closure operators, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.05.015

$$\bigwedge_{y\in Z}(a_y^{*_1} \to (b_y^{*_2} \to I_y)) = \bigwedge_{y\in Z}(a_y^{*_1} \otimes b_y^{*_2}) \to I_y) =$$
$$= \bigwedge_{y\in Z}(((a_y^{*_1})^{*_2} \otimes b_y^{*_2}) \to I_y) =$$
$$= \bigwedge_{y\in Z}(((a_y^{*_1})^{*_2} \otimes b_y^{*_2})^{*_2} \to I_y)),$$

which is an element of $[\mathcal{S}]^{*_2}$ on account of Theorem 4.

$[\mathcal{S}]^{*_2} \subseteq [[\mathcal{S}]^{*_2}_{\to}]^{*_1}$: Putting $a_y = 1$ in (13), which is the form of an arbitrary element in $[[\mathcal{S}]^{*_2}_{\to}]^{*_1}$, we obtain $\bigwedge_{y\in Z}(b_y^{*_2} \to I_y)$, which is the form of a general element in $[\mathcal{S}]^{*_2}$, proving the claim. □

The significance of Theorem 5 appears in the following viewpoint: Suppose we have a method to compute fixpoints of $\mathbf{L}^{*_1}$-closure operators, which are induced by a collection of fuzzy sets, and need to compute fixpoints of an $\mathbf{L}^{*_2}$-closure operator induced by $\mathcal{S}$ with $*_1 \leq *_2$. To do so, it is sufficient to run the method on $[\mathcal{S}]^{*_2}_{\to}$, i.e. the system of all a^{*_2}-shifts of the L-sets in $\mathcal{S}$.

In particular, Theorem 5 is of considerable interest if $*_2$ is the given hedge $*$ and $*_1$ is the globalization $*_g$. Then

$$\mathrm{Ext}(X, Y^*, I) = [\mathcal{S}]^* = [[\mathcal{S}]^*_{\to}]^{*_g} = [\{a^* \to I_y \mid y \in Y\}]^{*_g}.$$

The thus outlined reduction now leads to a feasible end because, as the next theorem shows, $[\ ]^{*_g}$ may be computed using classical algorithms that compute fixpoints of closure operators.

Theorem 6. *Given a system $\mathcal{T} = \{J_z \in L^X \mid z \in Z\}$ of fuzzy sets, consider the following operator $cl_{\mathcal{T}} : 2^Z \to 2^Z$:*

$$cl_{\mathcal{T}}(D) = \{z \in Z \mid J_z(x) \geq \textstyle\bigwedge_{z'\in D} J_{z'}(x) \text{ for all } x \in X\}.$$

$cl_{\mathcal{T}}$ is a (classical) closure operator in Z for which

$$\mathrm{Ext}(X, Z^{*_g}, J) = [\mathcal{T}]^{*_g} = \{\textstyle\bigwedge_{z\in D} J_z \mid D \in \mathrm{fix}(cl_{\mathcal{T}})\}.$$

Proof. For globalization, the concept lattice $\mathcal{B}(X, Z^{*_g}, J)$ coincides with the so-called crisply generated fuzzy concept lattice [10]. Now, according to [9], every concept $\langle A, B\rangle \in \mathcal{B}(X, Z^{*_g}, J)$ is uniquely determined by the 1-cut ${}^1B = \{y \in Y \mid B(y) = 1\}$ of its intent B, in that $A = ({}^1B)^{\downarrow}$ and $B = ({}^1B)^{\downarrow\uparrow}$. According to [10], the infima in $\mathrm{Int}(X, Z^{*_g}, J)$ are given by

$$\inf_{k\in K} B_k = (\textstyle\bigwedge_{k\in K} B_k)^{\downarrow\uparrow}.$$

Furthermore, for any $*$,

$$((\textstyle\bigwedge_{k\in K} B)^{\downarrow\uparrow})^* = (\textstyle\bigwedge_{k\in K} B_k)^*.$$

Indeed, one always has $(\bigwedge_{k\in K} B_k)^{\downarrow\uparrow} \supseteq (\bigwedge_{k\in K} B_k)^*$; furthermore, idempotency and isotony of $*$ imply $((\bigwedge_{k\in K} B_k)^{\downarrow\uparrow})^* \supseteq (\bigwedge_{k\in K} B_k)^*$. Conversely, if B_k are intents, we have $B_k = B_k^{\downarrow\uparrow}$, hence $\bigwedge_{k\in K} B_k \subseteq B_k$ and isotony of ${}^{\downarrow\uparrow}$ imply $(\bigwedge_{k\in K} B_k)^{\downarrow\uparrow} \subseteq B_k^{\downarrow\uparrow} = B_k$, hence $(\bigwedge_{k\in K} B_k)^{\downarrow\uparrow} \subseteq \bigwedge_{k\in K} B_k$ from which we get $((\bigwedge_{k\in K} B_k)^{\downarrow\uparrow})^* \subseteq (\bigwedge_{k\in K} B_k)^*$.

Now, one easily checks that if $*$ is the globalization $*_g$, we have

$$(\textstyle\bigwedge_{k\in K} B_k)^{*_g} = \textstyle\bigwedge_{k\in K} B_k^{*_g}.$$

Since $B^{*_g} = {}^1B$, it follows that the system $\mathcal{U} = \{{}^1B \mid B \in \mathrm{Int}(X, Z^{*_g}, J)\}$ equipped with $\subseteq$ is isomorphic to $\mathrm{Int}(X, Z^{*_g}, J)$, and hence a closure system. For the ordinary closure operator $cl_{\mathcal{U}}$ corresponding to $\mathcal{U}$ and arbitrary $D \subseteq Z$,

$$cl_{\mathcal{U}}(D) = \bigcap\{D' \in \mathcal{U} \mid D \subseteq D'\} =$$
$$= \bigcap\{{}^1B \mid B \in \mathrm{Int}(X, Z^{*_g}, J), D \subseteq B\} =$$
$$= {}^1\bigcap\{B \in \mathrm{Int}(X, Z^{*_g}, J) \mid D \subseteq B\} = {}^1(D^{\downarrow\uparrow}).$$

One easily checks that ${}^1(D^{\downarrow\uparrow}) = cl_{\mathcal{T}}(D)$, whence $cl_{\mathcal{U}}(D) = cl_{\mathcal{T}}(D)$. Since for every intent $B \in \mathrm{Int}(X, Z^{*_g}, J)$, the corresponding extent is $B^{\downarrow} = ({}^1B)^{\downarrow} = \bigwedge_{z\in {}^1B} J_z$; the proofs is finished. □

Please cite this article in press as: R. Belohlavek, J. Konecny, A reduction theorem to compute fixpoints of fuzzy closure operators, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.05.015

From the previous results in this section, the following theorem, which is the main claim in this section, directly follows:

Theorem 7. *For $\mathcal{S} = \{I_y \in L^X \mid y \in Y\}$, put $Z = Y \times \mathrm{fix}(*)$, $\mathcal{T} = \{a \to I_y \mid a \in \mathrm{fix}(*), I_y \in \mathcal{S}\}$, and consider the ordinary closure operator $cl_{\mathcal{T}}$ from Theorem 6. Then*

$$\mathrm{Ext}(X, Y^*, I) = [\mathcal{S}]^* = \{\textstyle\bigwedge_{\langle y,a\rangle \in D}(a \to I_y) \mid D \in \mathrm{fix}(cl_{\mathcal{T}})\}.$$

4. Application: a principle to obtain algorithms computing fuzzy concept lattices

As an example of possible applications of the reduction results obtained in the previous section, in particular Theorem 7, we now present a principle which makes it possible to obtain algorithms for computing fuzzy concept lattices $\mathcal{B}(X, Y^*, I)$, almost for free, from algorithms for computing ordinary concept lattices [15,22]. The key ingredient is a kind of wrapping of the given algorithm for ordinary concept lattices. Correctness of the procedure follows from the theorems presented above.

For the sake of simplicity, we do not distinguish between the intents in $\mathrm{Int}(X, Z^{*_g}, J)$ and their 1-cuts, and thus identify them. In particular, instead of fuzzy concepts $\langle A, B\rangle \in \mathcal{B}(X, Z^{*_g}, J)$ we actually work with pairs $\langle A, {}^1B\rangle$. Note that here, ${}^1B = \{z \in Z \mid B(z) = 1\}$ is the 1-cut of B. This simplification is possible due to the results in [9,10]: the fuzzy concepts $\langle A, B\rangle$ are the crisply generated fuzzy concepts of [9], while the corresponding pairs $\langle A, {}^1B\rangle$ are the one-sided fuzzy concepts of [20].

The principle, i.e. a method to obtain fuzzy concept lattices as described above, is as follows:

Step 1 (scaling): Define for an input fuzzy context $\langle X, Y, I\rangle$ a new fuzzy context $\langle X, Z, J\rangle$ where $Z = Y \times \mathrm{fix}(*)$ and the fuzzy relation $J \in L^{X\times Z}$ is given by

$$J(x, \langle y, a\rangle) = a \to I(x, y).$$

Transform $\langle X, Y, I\rangle$ to $\langle X, Z, J\rangle$.

Interestingly, one needs not explicitly form the possibly large new fuzzy context $\langle X, Z, J\rangle$. For the given purpose, it is sufficient to obtain the new operators induced by J, which are defined as follows:

$$A^{\Uparrow} = \{\langle y, a\rangle \mid A \subseteq a \to I_y\},$$
$$B^{\Downarrow} = \bigwedge_{\langle y,a\rangle \in B} a \to I_y. \tag{14}$$

The operators $\langle \Uparrow, \Downarrow\rangle$ are indeed the operators induced by $\langle X, Z, J\rangle$, provided one identifies the intents B with their 1-cuts ${}^1B$ as described above.

Step 2 (ordinary computation): Compute $\mathcal{B}(X, Z^{*_g}, J)$ using an available algorithm for computing fixpoints of ordinary closure operators. The available algorithm is to be applied to the intents of $\mathcal{B}(X, Z^{*_g}, J)$ which are ordinary sets due to our identification of intents B with their 1-cuts ${}^1B$.

Step 3 (transformation of results): Transform the computed concept lattice $\mathcal{B}(X, Z^{*_g}, J)$ to the fuzzy concept lattice $\mathcal{B}(X, Y^*, I)$.

Due to the above results, the two concept lattices are isomorphic and have the same extents. The transformation is therefore straightforward: Each $\langle A, B\rangle \in \mathcal{B}(X, Z^{*_g}, J)$ gets transformed to its corresponding concept $\langle A, A^{\uparrow}\rangle \in \mathcal{B}(X, Y^*, I)$. Moreover, one easily observes [10] that computing $A^{\uparrow}$ can be simply done as follows:

$$A^{\uparrow} = \lceil B \rceil \quad \text{where} \quad \lceil B \rceil(y) = \bigvee\{a \mid \langle y, a\rangle \in B\}. \tag{15}$$

An example of the wrapping procedure is presented in Fig. 3.

We now present two algorithms to compute a fuzzy concept lattice $\mathcal{B}(X, Y^*, I)$, which result by the application of the above wrapping procedure. Throughout the rest of this section, we assume that the elements of Z are represented by integers in $\{1, \ldots, |Z|\}$.

Please cite this article in press as: R. Belohlavek, J. Konecny, A reduction theorem to compute fixpoints of fuzzy closure operators, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.05.015

	α	β	γ
A	0.5	0	1
B	1	0.5	1
C	0	0.5	0.5
D	0.5	0.5	1

Step 1: (scaling)
Not performed explicitly,
but computed by columns on demand.

	$\langle 0,\alpha\rangle$	$\langle 0.5,\alpha\rangle$	$\langle 1,\alpha\rangle$	$\langle 0,\beta\rangle$	$\langle 0.5,\beta\rangle$	$\langle 1,\beta\rangle$	$\langle 0,\gamma\rangle$	$\langle 0.5,\gamma\rangle$	$\langle 1,\gamma\rangle$
A	1	1	0.5	1	0.5	0	1	1	1
B	1	1	1	1	1	0.5	1	1	1
C	1	0.5	0	1	1	0.5	1	1	0.5
D	1	1	0.5	1	1	0.5	1	1	1

Step 2: (ordinary computation)
Performed by any ordinary algorithm.

	extent	intent
⓪	$\{^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{D}\}$	$\{\langle 0,\alpha\rangle,\langle 0.5,\alpha\rangle,\langle 1,\alpha\rangle,\langle 0,\beta\rangle,\langle 0.5,\beta\rangle,\langle 1,\beta\rangle,\langle 0,\gamma\rangle,\langle 0.5,\gamma\rangle,\langle 1,\gamma\rangle\}$
①	$\{^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{\langle 0,\alpha\rangle,\langle 0.5,\alpha\rangle,\langle 1,\alpha\rangle\ \langle 0,\beta\rangle,\langle 0.5,\beta\rangle,\ \langle 0,\gamma\rangle,\langle 0.5,\gamma\rangle,\langle 1,\gamma\rangle\}$
②	$\{^{0.5}/\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{D}\}$	$\{\langle 0,\alpha\rangle,\langle 0.5,\alpha\rangle,\ \langle 0,\beta\rangle,\langle 0.5,\beta\rangle,\langle 1,\beta\rangle,\langle 0,\gamma\rangle,\langle 0.5,\gamma\rangle,\langle 1,\gamma\rangle\}$
③	$\{^{0.5}/\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{C}, \mathrm{D}\}$	$\{\langle 0,\alpha\rangle,\langle 0.5,\alpha\rangle,\ \langle 0,\beta\rangle,\langle 0.5,\beta\rangle,\ \langle 0,\gamma\rangle,\langle 0.5,\gamma\rangle,\langle 1,\gamma\rangle\}$
④	$\{^{0.5}/\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{\langle 0,\alpha\rangle,\ \langle 0,\beta\rangle,\langle 0.5,\beta\rangle,\ \langle 0,\gamma\rangle,\langle 0.5,\gamma\rangle\}$
⑤	$\{\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{C}, \mathrm{D}\}$	$\{\langle 0,\alpha\rangle,\langle 0.5,\alpha\rangle,\ \langle 0,\beta\rangle,\langle 0.5,\beta\rangle,\ \langle 0,\gamma\rangle,\langle 0.5,\gamma\rangle,\langle 1,\gamma\rangle\}$
⑥	$\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{\langle 0,\alpha\rangle,\ \langle 0,\beta\rangle,\ \langle 0,\gamma\rangle,\langle 0.5,\gamma\rangle\}$

Step 3: (transformation of results)
Apply $\lceil\cdot\rceil$, see (15), to the intents.

	extent	intent
⓪	$\{^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{D}\}$	$\{\alpha,\beta,\gamma\}$
①	$\{^{0.5}/\mathrm{B}, {}^{0.5}/\mathrm{C}, {}^{0.5}/\mathrm{D}\}$	$\{\alpha, {}^{0.5}/\beta, \gamma\}$
②	$\{^{0.5}/\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{D}\}$	$\{^{0.5}/\alpha, \beta, \gamma\}$
③	$\{^{0.5}/\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{C}, \mathrm{D}\}$	$\{^{0.5}/\alpha, {}^{0.5}/\beta, \gamma\}$
④	$\{^{0.5}/\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{^{0.5}/\beta, {}^{0.5}/\gamma\}$
⑤	$\{\mathrm{A}, \mathrm{B}, {}^{0.5}/\mathrm{C}, \mathrm{D}\}$	$\{^{0.5}/\alpha, \gamma\}$
⑥	$\{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}$	$\{^{0.5}/\gamma\}$

Fig. 3. Example of the wrapping procedure.

4.1. Generalized Close-by-One

The first algorithm is based on the well-known CbO algorithm [21]. For brevity, we employ the simplified algorithm [19] instead of the original CbO [21]. The generalization to fuzzy attributes we present below can be applied the same way to the successors of CbO, such as fCbO [24] and InClose family [1].

The core of CbO is the function GENERATEFROM (see Algorithm 1), which has two input parameters: $\langle A, B\rangle$ (initial fuzzy concept of $\mathcal{B}(X, Z^*, J)$) and $z \in Z$ (the first attribute to be processed). Starting with $\langle A, B\rangle$, the procedure then descends in a recursive manner through a part of the fuzzy concept lattice. With its inputs $\langle A, B\rangle$ and $y \in Y$, GENERATEFROM first produces $\langle A, \lceil B\rceil\rangle$ (line 1) and checks its stopping condition (lines 2–4). The procedure ends when $\langle A, B\rangle$ is equal to $\langle Z^{\Downarrow}, Z\rangle$ or if $y > n$. If it does not end, it browses through all attributes $j \in Z$, $j \geq z$, that do not belong to the intent B (lines 5 and 6). For every such attribute, a new concept

Algorithm 1: GENERATEFROM($\langle A, B\rangle, z$).

```
1  output ⟨A, ⌈B⌉⟩
2  if B = Z or z > n then
3  |  return
4  end
5  for j from z up to n do
6  |  if j ∉ B then
7  |  |  C := A ∩ {j}^⇓
8  |  |  D := C^⇑
9  |  |  if B ∩ Z_j = D ∩ Z_j then
10 |  |  |  GENERATEFROM(⟨C, D⟩, j + 1)
11 |  |  end
12 |  end
13 end
14 return
```

$$\langle C, D\rangle = \langle A \cap \{j\}^{\Downarrow}, (A \cap \{j\}^{\Downarrow})^{\Uparrow}\rangle \tag{16}$$

is created (lines 7 and 8). A canonicity test on $\langle C, D\rangle$ (line 9) is then performed to verify whether a continuation with $\langle C, D\rangle$ and $j+1$ by recursively calling GENERATEFROM is needed or not. This test is based on comparing $B \cap Z_j = D \cap Z_j$ where $Z_j \subseteq Z$ is given by

$$Z_j = \{z \in Z \mid z < j\}.$$

The loop in lines 5–13 then goes on with the next value of j.

Note that when comparing to the ordinary version, the only changes in the function GENERATEFROM are as follows:

- The operators (14) (lines 7 an 8) are used to form the concepts (16) of $\mathcal{B}(X, Z^{*g}, J)$. This corresponds to Step 1 above.
- The concept $\langle A, B\rangle \in \mathcal{B}(X, Z^{*g}, J)$ is transformed to the concept $\langle A, \lceil B\rceil\rangle \in \mathcal{B}(X, Y^*, I)$ in line 1 before it is presented as the output. This corresponds to Step 3 above.

4.2. Generalized Lindig's algorithm

The Lindig's algorithm was proposed in [23]. Compared to the original version, which computes the concepts in a bottom–up manner, we exchange the roles of objects and attributes as it is more convenient for our purpose. We therefore compute the concepts in a top-down manner.

The main part of Lindig's algorithm is the function NEIGHBORS (Algorithm 2), which computes for a given fuzzy concept $\langle A, B\rangle \in \mathcal{B}(X, Z^{*g}, J)$ a set N of concepts that are the lower neighbors of $\langle A, B\rangle$. In line 2, N is first initialized to the empty set. NEIGHBORS then browses all attributes that are not in B (the loop in lines 3–11). In lines 7 and 8, it forms a new concept $\langle C, D\rangle = \langle (B \cup \{z\})^{\Downarrow}, (B \cup \{z\})^{\Downarrow\Uparrow}\rangle$. In order to avoid duplicities and false lower neighbors the function employs a test, which involves a set *min* of minimal possible generators. This set is initialized in line 1 to the set of all attributes not present in B. If a new intent D includes a possible generator that is different from z (line 6) then z is deleted from *min* (line 9). If not, the newly formed concept $\langle C, D\rangle$ is the lower neighbor of $\langle A, B\rangle$ and is included in N (line 7).

The procedure LINDIG (Algorithm 3) first generates the largest concept. Subsequently, for every concept generated for the first time, it computes all its lower neighbors. The procedure LINDIG makes use of a tree of concepts which makes it possible to check whether a given concept has already been computed. This property is used to generate the concepts of $\mathcal{B}(X, Z^{*g}, J)$. Then, the procedure executes an additional loop (lines 12–13) to transform their intents by using $\lceil\,\rceil$.

Note that compared to the original LINDIG, the changes are as follows:

- In lines 4 and 5 of NEIGHBORS (Algorithm 2), the operators (14) to form concepts (16) in $\mathcal{B}(X, Z^{*g}, J)$ are used. This corresponds to Step 1 above.

Please cite this article in press as: R. Belohlavek, J. Konecny, A reduction theorem to compute fixpoints of fuzzy closure operators, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.05.015

Algorithm 2: NEIGHBORS($\langle A, B\rangle$).

Data: concept $\langle A, B\rangle$
Result: Set N of lower neighbors of $\langle A, B\rangle$

$min := Z - B$
$N := \emptyset$
for $z \in Z - B$ **do**
  $C := (B \cup \{z\})^{\Downarrow}$
  $D := C^{\Uparrow}$
  if $min \cap ((D - B) - \{z\}) = \emptyset$ **then**
    $N := N \cup \{\langle C, D\rangle\}$
  else
    $min := min - \{z\}$
  end
end
return N

Algorithm 3: LINDIG($\langle X, Y, I\rangle$).

Data: **L**-context $\langle X, Y, I\rangle$
Result: $\mathcal{B}(X, Y^*, I)$

1 $c := \langle X, X^{\Uparrow}\rangle$
2 $\mathcal{T}$ is a concept tree consisting of the root node c
3 **while** $c \neq$ `null` **do**
4   **for** $n \in$ NEIGHBORS(c) **do**
5   **if** $n \notin \mathcal{T}$ **then**
6   insert n to $\mathcal{T}$
7   **end**
8   set n to be a lower neighbor of c
9   **end**
10   $c :=$ Next($c, \mathcal{T}$)
11 **end**
12 **for** $\langle A, B\rangle \in \mathcal{T}$ **do**
13   change intent of $\langle A, B\rangle$ to $\lceil B\rceil$
14 **end**
15 **return** $\mathcal{T}$

– The intents of fuzzy concepts stored in $\mathcal{T}$ are transformed in the loop on lines 12–14 of LINDIG. This corresponds to Step 3 above.

5. Extension to interior structures

As is well known, interior structures are fully dual in the classical, Boolean setting to closure structures. That is, once we have results for closure structures, the corresponding results for interior structures follow by duality. Due to a lack of the law of double negation in fuzzy logic, this kind of duality is not available in a fuzzy setting to full extent. This is why fuzzy interior structures as well as fuzzy concept lattices involving interior operators have been studied separately in the literature. The aim of this section is to present the main results of the previous sections in their setting for interior structures. We omit proofs because these may be obtained fairly routinely given the proofs for closure structures in the previous sections.

Recall that an $\mathbf{L}^*$-interior operator is a mapping $K : L^U \to L^U$ satisfying

$$K(A) \subseteq A \qquad S(A, B)^* \leq S(K(A), K(B)) \qquad K(A) = K(K(A)) \tag{17}$$

for all $A, B \in L^U$. The basic notions regarding fuzzy concept lattices that involve interior structures are defined as follows [18,2]: Given a formal fuzzy context $\langle X, Y, I\rangle$, consider the mappings ${}^{\wedge} : L^X \to L^Y$ and ${}^{\vee} : L^Y \to L^X$ defined by

$$A^{\wedge}(y) = \textstyle\bigwedge_{x\in X}(I(x, y) \to A(x)) \text{ and } B^{\vee}(x) = \textstyle\bigvee_{y\in Y}(B(y)^* \otimes I(x, y)). \tag{18}$$

Please cite this article in press as: R. Belohlavek, J. Konecny, A reduction theorem to compute fixpoints of fuzzy closure operators, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.05.015

The set

$$\mathcal{B}^{\wedge\vee}(X, Y^*, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^\wedge = B,\ B^\vee = A\}$$

of all fixpoints of $\langle {}^\wedge, {}^\vee\rangle$ is a kind of fuzzy concept lattice of I derived from the operators $^\wedge$ and $^\vee$ and we shall call it just the fuzzy concept lattice of I in this section. The collections of its extents and the collection of its intents are denoted by $\mathrm{Ext}^{\wedge\vee}(X, Y, I)$ and $\mathrm{Int}^{\wedge\vee}(X, Y, I)$. For any system $\mathcal{S} \subseteq L^U$ of fuzzy sets in U, one naturally defines notions analogous to those for closure structures as follows:

- $\langle\mathcal{S}\rangle^*$ the least $\mathbf{L}^*$-interior system containing $\mathcal{S}$,
- $[\mathcal{S}]_\bigvee$ the least system containing $\mathcal{S}$ that is closed under $\bigvee$-unions,
- $[\mathcal{S}]^*_\otimes$ the least system containing $\mathcal{S}$ that is closed under a^*-multiplications for every $a \in L$.

Here, closedness under $\bigvee$-unions means that $\bigvee_{j\in J} A_j \in \mathcal{S}$ whenever $A_j \in \mathcal{S}$ for all $j \in J$; closedness under a^*-multiplications means that $a^* \otimes A \in \mathcal{S}$ whenever $A \in \mathcal{S}$. The main claims for closure structures generalize to the setting of interior structures as follows:

Theorem 8. $\langle\mathcal{S}\rangle^* = \mathrm{Ext}^{\wedge\vee}(X, Y^*, I)$.

Theorem 9. *For every system $\mathcal{S} = \{I_y \in L^X \mid y \in Y\}$ we have*

$$[\mathcal{S}]_\bigvee = \{\bigvee \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S}\}, \tag{19}$$
$$[\mathcal{S}]^*_\otimes = \{a^* \otimes I_y \mid a \in L,\ y \in Y\}, \tag{20}$$
$$\langle\mathcal{S}\rangle^* = [[\mathcal{S}]^*_\otimes]_\bigvee. \tag{21}$$

Theorem 10. *If $*_1 \leq *_2$ then*

$$\langle\mathcal{S}\rangle^{*_2} = [[\mathcal{S}]^{*_2}_\otimes]^{*_1}. \tag{22}$$

Theorem 11. *For a system $\mathcal{T} = \{J_z \in L^X \mid z \in Z\}$, the operator $cl_\mathcal{T} : 2^Z \to 2^Z$ defined by*

$$cl_\mathcal{T}(D) = \{z \in Z \mid J_z(x) \leq \textstyle\bigvee_{z'\in D} J_{z'}(x) \text{ for all } x \in X\}$$

is an ordinary closure in Z such that

$$\mathrm{Ext}^{\wedge\vee}(X, Z^{*_g}, J) = [\mathcal{T}]^{*_g} = \{\textstyle\bigvee_{z\in D} J_z \mid D \in \mathrm{fix}(cl_\mathcal{T})\}.$$

Theorem 12. *For $\mathcal{S} = \{I_y \in L^X \mid y \in Y\}$, put $Z = Y \times \mathrm{fix}(*)$, $\mathcal{T} = \{a \otimes I_y \mid a \in \mathrm{fix}(*), y \in Y\}$, and consider the ordinary closure operator $cl_\mathcal{T}$ from Theorem 11. Then*

$$\mathrm{Ext}^{\wedge\vee}(X, Y^*, I) = \langle\mathcal{S}\rangle^* = \{\textstyle\bigvee_{\langle y,a\rangle\in D}(a \otimes I_y) \mid D \in \mathrm{fix}(cl_\mathcal{T})\}.$$

Based on Theorem 12 the wrapping procedure for the interior setting has the following form:

Step 1 (scaling): Transform the input fuzzy context $\langle X, Y, I\rangle$ to the fuzzy context $\langle X, Z, J\rangle$ where $Z = Y \times \mathrm{fix}(*)$ and $J \in L^{X\times Z}$ is given by

$$J(x, \langle y, a\rangle) = a \otimes I(x, y).$$

Analogously as in the case of closure structures, one need not form $\langle X, Z, J\rangle$. Instead the following operators can be used:

$$A^{\curlywedge} = \{\langle y, a\rangle \mid a \otimes I_y \subseteq A\},$$
$$B^{\curlyvee} = \bigvee_{\langle y,a\rangle\in A} a \otimes I_y.$$

Step 2 (ordinary computation) Compute $\mathcal{B}^{\wedge\vee}(X, Z^{*_g}, J)$ using an algorithm for computing fixpoints of an ordinary closure operator (apply this algorithm to the intents of $\mathcal{B}^{\wedge\vee}(X, Z^{*_g}, J)$, which are ordinary sets due to our identification of intents with their 1-cuts).

Step 3 (transformation of results): Transform $\mathcal{B}^{\wedge\vee}(X, Z^{*_g}, J)$ to the required fuzzy concept lattice $\mathcal{B}^{\wedge\vee}(X, Y^*, I)$. Again, it is sufficient to transform the intents using the mapping $\lceil\cdot\rceil$ (15).

Remark 3. Notice that some results involve closure operators in spite of the fact that the framework in this section is based on interior structures. Notice also that the framework still involves the same notion of a truth-stressing hedge as the framework with closure structures, rather than a dual notion of hedge. In view of this non-symmetry w.r.t. results in the previous sections, the fact that the above results hold true for the interior-based framework and have a form very similar to that for the closure-based framework is interesting and worth mentioning.

Remark 4. There is also a dual way to approach interior-based concept lattices [18]. Given a formal fuzzy context $\langle X, Y, I\rangle$, the concept forming operators $^{\cap} : L^X \to L^Y$ and $^{\cup} : L^Y \to L^X$ are defined by

$$A^{\cap}(y) = \bigvee_{x\in X}(B(y) \otimes I(x, y)) \text{ and } B^{\cup}(x) = \bigwedge_{y\in Y}(I(x, y) \to A(x)^{\bullet}), \tag{23}$$

where $\bullet : L \to L$ is a truth-depressing hedge (a counterpart of a truth-stressing hedges representing linguistic hedges "slightly" or "more or less"). The extents in this case form certain fuzzy closure systems, which are not yet well described in the literature. Their study is a subject of our future research.

6. Conclusions and further research

Whether and to what extent it is possible to apply results regarding ordinary closure operators to general fuzzy closure operators is a question important not only for its theoretical appeal but also because fuzzy closure operators appear in several practically significant areas. In this paper, we obtained certain reduction results allowing one to apply results from the ordinary setting to a fuzzy setting. These results complement other existing results on the relationship of ordinary and fuzzy closure structures, see e.g. [5]. The following topics seem worth future investigation. First, as far as the wrapping procedure obtained in this paper is concerned, a comprehensive experimental evaluation of efficiency of various ways to compute fixpoints of fuzzy closure operators remains an open research topic. The present results might become a relevant component in this endeavor. Second, directly related to fuzzy concept lattices are attribute dependencies and other structures that appear when working with relational data, e.g. [11–14]. Extending the methods worked out in this paper to these structures represents another broad research topic. Third, a challenging topic for future research is to examine reduction results for general relational structures.

Acknowledgements

Supported by Grant No. GA15-17899S of the Czech Science Foundation. Preliminary results in this paper appeared in the Proceedings of FUZZ-IEEE 2017, pp. 1–6.

References

[1] S. Andrews, A 'best-of-breed' approach for designing a fast algorithm for computing fixpoints of Galois connections, Inf. Sci. 295 (2015) 633–649.
[2] E. Bartl, R. Belohlavek, J. Konecny, V. Vychodil, Isotone Galois connections and concept lattices with hedges, in: Int. IEEE Conference on Intelligent Systems, IEEE IS 2008, Varna, Bulgaria, 2008, pp. 15-24–15-28.
[3] R. Belohlavek, Fuzzy closure operators, J. Math. Anal. Appl. 262 (2) (October 2001) 473–489.
[4] R. Belohlavek, Reduction and a simple proof of characterization of fuzzy concept lattices, Fundam. Inform. 46 (4) (December 2001) 277–285.
[5] R. Belohlavek, Fuzzy Relational Systems: Foundations and Principles, Kluwer Academic Publishers, Norwell, USA, 2002.
[6] R. Belohlavek, Concept lattices and order in fuzzy logic, Ann. Pure Appl. Log. 128 (1–3) (2004) 277–298.
[7] R. Belohlavek, B. De Baets, J. Outrata, V. Vychodil, Computing the lattice of all fixpoints of a fuzzy closure operator, IEEE Trans. Fuzzy Syst. 18 (3) (2010) 546–557.
[8] R. Belohlavek, T. Funioková, V. Vychodil, Fuzzy closure operators with truth stressers, Log. J. IGPL 13 (5) (2005) 503–513.

[9] R. Belohlavek, V. Sklenář, J. Zacpal, Crisply generated fuzzy concepts, in: Formal Concept Analysis, in: Lecture Notes in Computer Science, vol. 3403, 2005, pp. 268–283.
[10] R. Belohlavek, V. Vychodil, Formal concept analysis and linguistic hedges, Int. J. Gen. Syst. 41 (5) (2012) 503–532.
[11] R. Belohlavek, V. Vychodil, Attribute dependencies for data with grades I, Int. J. Gen. Syst. 45 (7–8) (2016) 864–888.
[12] R. Belohlavek, V. Vychodil, Attribute dependencies for data with grades II, Int. J. Gen. Syst. 46 (1) (2017) 66–92.
[13] R. Belohlavek, V. Vychodil, Relational similarity-based model of data I: Foundations and query systems, Int. J. Gen. Syst. 46 (7) (2017) 671–751.
[14] R. Belohlavek, V. Vychodil, Relational similarity-based model of data II: Dependencies in data, Int. J. Gen. Syst. 47 (1) (2018) 1–50.
[15] B. Ganter, R. Wille, Formal Concept Analysis – Mathematical Foundations, Springer, 1999.
[16] P. Hájek, Metamathematics of Fuzzy Logic, Trends in Logic, Springer, November 2001.
[17] P. Hájek, On very true, Fuzzy Sets Syst. 124 (3) (2001) 329–333.
[18] J. Konecny, Isotone fuzzy Galois connections with hedges, Inf. Sci. 181 (10) (2011) 1804–1817, Special Issue on Information Engineering Applications Based on Lattices.
[19] P. Krajca, J. Outrata, V. Vychodil, Parallel algorithm for computing fixpoints of Galois connections, Ann. Math. Artif. Intell. 59 (2) (2010) 257–272.
[20] S. Krajci, A generalized concept lattice, Log. J. IGPL 13 (5) (2005) 543–550.
[21] S.O. Kuznetsov, A fast algorithm for computing all intersections of objects from an arbitrary semilattice, Nauchn. Tek. Inf., Ser. 2, Inf. Processy Sistemy (1) (1993) 17–20.
[22] S.O. Kuznetsov, S. Obiedkov, Comparing performance of algorithms for generating concept lattices, J. Exp. Theor. Artif. Intell. 14 (2002) 189–216.
[23] C. Lindig, Fast concept analysis, in: Working with Conceptual Structures – Contributions to ICCS 2000, 2000, pp. 152–161.
[24] J. Outrata, V. Vychodil, Fast algorithm for computing fixpoints of Galois connections induced by object-attribute relational data, Inf. Sci. 185 (1) (2012) 114–127.
[25] S. Pollandt, Fuzzy Begriffe: Formale Begriffsanalyse von unscharfen Daten, Springer-Verlag, Berlin–Heidelberg, 1997.
[26] K.I. Rosenthal, Quantales and Their Applications, Pitman Research Notes in Mathematics Series, vol. 234, Longman Scientific & Technical, Harlow, UK, 1990.
[27] G. Takeuti, S. Titani, Globalization of intuitionistic set theory, Ann. Pure Appl. Log. 33 (1987) 195–211.
[28] S.B. Yahia, A. Jaoua, Discovering knowledge from fuzzy concept lattice, in: Data Mining and Computational Intelligence, in: Studies in Fuzziness and Soft Computing, vol. 68, Physica-Verlag GmbH, Heidelberg, Germany, 2001, pp. 167–190.

Please cite this article in press as: R. Belohlavek, J. Konecny, A reduction theorem to compute fixpoints of fuzzy closure operators, Fuzzy Sets Syst. (2018), https://doi.org/10.1016/j.fss.2018.05.015