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Row and Column Spaces of Matrices over Residuated Lattices

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Abstract. We present results regarding row and column spaces of matrices whose entries are elements of residuated lattices. In particular, we define the notions of a row and column space for matrices over residuated lattices, provide connections to concept lattices and other structures associated to such matrices, and show several properties of the row and column spaces, including properties that relate the row and column spaces to Schein ranks of matrices over residuated lattices. Among the properties is a characterization of matrices whose row (column) spaces are isomorphic. In addition, we present observations on the relationships between results established in Boolean matrix theory on one hand and formal concept analysis on the other hand.

Keywords: graded matrix, formal concept analysis, morphism

1. Introduction

The results presented in this paper are motivated by recent results on decompositions of matrices over residuated lattices and factor analysis of data described by such matrices, see e.g. [5, 6, 9]. The results reveal a fundamental role of closure and interior structures, most importantly concept lattices, for the decompositions. In particular, the results motivate us to investigate the calculus of matrices over residuated lattices. Such matrices include Boolean matrices as a particular case. Therefore, we investigate in the setting of matrices over residuated lattices the notions known from Boolean matrices that are relevant to matrix decompositions. The most important among them are the notions of a row and column space. These notions are the main subject of the present paper. In addition to obtain appropriate generalizations of these notions for matrices over residuated lattices and the results regarding these notions, our goal is to establish links between the matrix-like setting and the setting of interior/closure structures of formal concept analysis. Note that most of the notions and results we establish for matrices remain true when rephrased in terms of relations between possibly infinite sets; for this to be true, however, the residuated lattices need to be complete.

2. Preliminaries: Matrices, Decompositions, Concept Lattices

Matrices We deal with matrices whose degrees are elements of residuated lattices. Recall that a (complete) residuated lattice [3, 13, 20] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist (the lattice order is denoted by $\leq$; 0 and 1 denote the least and greatest element, respectively);

(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation that is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;

(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

Throughout the paper, $\mathbf{L}$ denotes an arbitrary (complete) residuated lattice. Common examples of complete residuated lattices include those defined on the real unit interval, i.e. $L = [0, 1]$, or on a finite chain in a unit interval, e.g. $L = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$. For instance, for $L = [0, 1]$, we can use any left-continuous t-norm for $\otimes$, such as minimum, product, or Łukasiewicz, and the corresponding residuum $\rightarrow$ [3, 13, 20]. Residuated lattices are commonly used in fuzzy logic [3, 12, 13]. Elements $a \in L$ are called grades (degrees of truth). Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of a (truth function of) conjunction and implication, respectively.

We deal with (de)compositions $I = A * B$ which involve an $n \times m$ matrix I, an $n \times k$ matrix A, and a $k \times m$ matrix B. We assume that $I_{ij}, A_{il}, B_{lj} \in L$. That is, all the matrix entries are elements of a given residuated lattice $\mathbf{L}$. Therefore, examples of matrices I which are subject to the decomposition are

$$\begin{pmatrix} 1.0 & 1.0 & 0.0 & 0.0 & 0.6 & 0.4 \\ 1.0 & 0.9 & 0.0 & 0.0 & 1.0 & 0.8 \\ 1.0 & 1.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 1.0 & 0.5 & 0.0 & 0.7 & 1.0 & 0.4 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The second matrix emphasizes that binary matrices are a particular case for $L = \{0, 1\}$. The i-th row and the j-th column of I are denoted by $I_{i\_}$ and $I_{\_j}$, respectively.

Composition Operators We use three matrix composition operators, $\circ$, $\triangleleft$, and $\triangleright$, and consider the corresponding decompositions $I = A \circ B$, $I = A \triangleleft B$, and $I = A \triangleright B$. In the decompositions, I_{ij} is interpreted as the degree to which the object i has the attribute j; A_{il} as the degree to which the factor

l applies to the object i; B_{lj} as the degree to which the attribute j is a manifestation (one of possibly several manifestations) of the factor l. The composition operators are defined by

$$(A \circ B)_{ij} = \bigvee_{l=1}^{k} A_{il} \otimes B_{lj}, \quad (1)$$
$$(A \triangleleft B)_{ij} = \bigwedge_{l=1}^{k} A_{il} \rightarrow B_{lj}, \quad (2)$$
$$(A \triangleright B)_{ij} = \bigwedge_{l=1}^{k} B_{lj} \rightarrow A_{il}. \quad (3)$$

Note that these operators were extensively studied by Bandler and Kohout, see e.g. [16]. They have natural verbal descriptions. For instance, $(A \circ B)_{ij}$ is the truth degree of the proposition "there is factor l such that l applies to object i and attribute j is a manifestation of l"; $(A \triangleleft B)_{ij}$ is the truth degree of "for every factor l, if l applies to object i then attribute j is a manifestation of l". Note also that for $L = \{0, 1\}$, $A \circ B$ coincides with the well-known Boolean product of matrices [15].

Decomposition Problem Given an $n \times m$ matrix I and a composition operator $*$ (i.e., $\circ$, $\triangleleft$, or $\triangleright$), the decomposition problem consists in finding a decomposition $I = A * B$ of I into an $n \times k$ matrix A and a $k \times m$ matrix B with the number k (number of factors) as small as possible. The smallest k is called the *Schein rank* of I and is denoted by $\rho_{\mathrm{s}}(I)$ (to make the type of product explicit, also by $\rho_{\mathrm{s}\circ}(I)$, $\rho_{\mathrm{s}\triangleleft}(I)$, and $\rho_{\mathrm{s}\triangleright}(I)$). Looking for decompositions $I = A * B$ can be seen as looking for factors in data described by I. That is, decomposing I can be regarded as factor analysis in which the data as well as the operations used are different from the ordinary factor analysis [14].

Concept Lattices Associated to I Let $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, m\}$. Recall that L^U denotes the set of all L-sets in U, i.e. all mappings from U to L Consider the following pairs of operators induced by matrix I. First, the pair $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ of operators ${}^{\uparrow} : L^X \rightarrow L^Y$ and ${}^{\downarrow} : L^Y \rightarrow L^X$ is defined by

$$C^{\uparrow}(j) = \bigwedge_{i=1}^{n}(C(i) \rightarrow I_{ij}), \quad D^{\downarrow}(i) = \bigwedge_{j=1}^{m}(D(j) \rightarrow I_{ij}), \quad (4)$$

for $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, n\}$. Second, the pair $\langle {}^{\cap}, {}^{\cup} \rangle$ of operators ${}^{\cap} : L^X \rightarrow L^Y$ and ${}^{\cup} : L^Y \rightarrow L^X$ is defined by

$$C^{\cap}(j) = \bigvee_{i=1}^{n}(C(i) \otimes I_{ij}), \quad D^{\cup}(i) = \bigwedge_{j=1}^{m}(I_{ij} \rightarrow D(j)), \quad (5)$$

for $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, n\}$. Third, the pair $\langle {}^{\wedge}, {}^{\vee} \rangle$ of operators ${}^{\wedge} : L^X \rightarrow L^Y$ and ${}^{\vee} : L^Y \rightarrow L^X$ is defined by

$$C^{\wedge}(j) = \bigwedge_{i=1}^{n}(I_{ij} \rightarrow C(i)), \quad D^{\vee}(i) = \bigvee_{j=1}^{m}(D(j) \otimes I_{ij}), \quad (6)$$

for $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, n\}$. $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ forms an antitone Galois connection [1], $\langle {}^{\cap}, {}^{\cup} \rangle$ and $\langle {}^{\wedge}, {}^{\vee} \rangle$ each form an isotone Galois connection [11]. To emphasize that the operators are induced by I, we also denote the operators by $\langle {}^{\uparrow_I}, {}^{\downarrow_I} \rangle$, $\langle {}^{\cap_I}, {}^{\cup_I} \rangle$, and $\langle {}^{\wedge_I}, {}^{\vee_I} \rangle$. Furthermore, denote the corresponding sets of fixpoints by $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$, $\mathcal{B}(X^{\cap}, Y^{\cup}, I)$, and $\mathcal{B}(X^{\wedge}, Y^{\vee}, I)$, i.e.

$$\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I) = \{\langle C, D \rangle \mid C^{\uparrow} = D,\ D^{\downarrow} = C\},$$
$$\mathcal{B}(X^{\cap}, Y^{\cup}, I) = \{\langle C, D \rangle \mid C^{\cap} = D,\ D^{\cup} = C\},$$
$$\mathcal{B}(X^{\wedge}, Y^{\vee}, I) = \{\langle C, D \rangle \mid C^{\wedge} = D,\ D^{\vee} = C\}.$$

The sets of fixpoints are complete lattices, called concept lattices associated to I, and their elements are called formal concepts. Note that these operators and their sets of fixpoints have extensively been studied, see e.g. [1, 2, 4, 11, 18]. Clearly, $\langle C, D\rangle \in \mathcal{B}(X^{\cap}, Y^{\cup}, I)$ iff $\langle D, C\rangle \in \mathcal{B}(Y^{\wedge}, X^{\vee}, I^{\mathrm{T}})$, where I^{T} denotes the transpose of I; so one could consider only one pair, $\langle{}^{\cap_I}, {}^{\cup_I}\rangle$ or $\langle{}^{\wedge_I}, {}^{\vee_I}\rangle$, and obtain the properties of the other pair by a simple translation. Note also that if $L = \{0, 1\}$, $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ coincides with the ordinary concept lattice of the formal context consisting of X, Y, and the binary relation (represented by) I.

It is well known that for $L = \{0, 1\}$, each of the three operators is definable by any of the remaining two and that, as a consequence, $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ is isomorphic to $\mathcal{B}(X^{\cap}, Y^{\cup}, \overline{I})$ with $\langle A, B\rangle \mapsto \langle A, \overline{B}\rangle$ being an isomorphism ($\overline{U}$ denotes the complement of U).

The mutual definability fails for general L because it is based on the law of double negation which does not hold for general residuated lattices. A simple framework that enables us to consider all the three operators as particular types of a more general operator is provided in [6], cf. also [11] for another possibility. For simplicity, we do not work with the general approach and use the three operators because they are well known.

The concept lattices associated to I play a fundamental role for decompositions of I. Namely, given a set $\mathcal{F} = \{\langle C_1, D_1\rangle, \dots, \langle C_k, D_k\rangle\}$ of L-sets C_l and D_l in $\{1, \dots, n\}$ and $\{1, \dots, m\}$, respectively, define $n \times k$ and $k \times m$ matrices $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ (we assume a fixed indexing of the elements of $\mathcal{F}$) by

$$(A_{\mathcal{F}})_{il} = (C_l)(i) \quad \text{and} \quad (B_{\mathcal{F}})_{lj} = (D_l)(j).$$

This says: the l-th column of $A_{\mathcal{F}}$ is the transpose of the vector corresponding to C_l and the l-th row of $B_{\mathcal{F}}$ is the vector corresponding to D_l. Then, we have:

Theorem 2.1. ([6])
For every $n \times m$ matrix I over a residuated lattice, $\rho_{\mathrm{s}\circ}(I)$, $\rho_{\mathrm{s}\triangleleft}(I)$, $\rho_{\mathrm{s}\triangleright}(I) \le \min(m, n)$. In addition,
($\circ$) there exists $\mathcal{F} \subseteq \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ with $|\mathcal{F}| = \rho_{\mathrm{s}\circ}(I)$ such that $I = A_{\mathcal{F}} \circ B_{\mathcal{F}}$;
($\triangleleft$) there exists $\mathcal{F} \subseteq \mathcal{B}(X^{\cap}, Y^{\cup}, I)$ with $|\mathcal{F}| = \rho_{\mathrm{s}\triangleleft}(I)$ such that $I = A_{\mathcal{F}} \triangleleft B_{\mathcal{F}}$;
($\triangleright$) there exists $\mathcal{F} \subseteq \mathcal{B}(X^{\wedge}, Y^{\vee}, I)$ with $|\mathcal{F}| = \rho_{\mathrm{s}\triangleright}(I)$ such that $I = A_{\mathcal{F}} \triangleright B_{\mathcal{F}}$.

Note that Theorem 2.1 says that if $I = A \circ B$ for $n \times k$ and $k \times m$ matrices A and B, then there exists $\mathcal{F} \subseteq \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ with $|\mathcal{F}| \le k$ such that for the $n \times |\mathcal{F}|$ and $|\mathcal{F}| \times m$ matrices $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ we have $I = A_{\mathcal{F}} \circ B_{\mathcal{F}}$; the same for $\triangleleft$ and $\triangleright$ (this is the way the theorem is phrased in [6]).

3. Row and Column Spaces

In this section, we define the notions of row and column spaces for matrices over residuated lattices and establish their properties and connections to concept lattices and other structures known from formal concept analysis.

In what follows, we denote

$$X = \{1, \dots, n\}, \quad Y = \{1, \dots, m\}, \quad F = \{1, \dots, k\}.$$

We assume that $E^{\wedge_A \wedge_B}$ stands for $(E^{\wedge_A})^{\wedge_B}$ and the like. For convenience and since there is no danger of misunderstanding, we take the advantage of identifying $n \times m$ matrices over residuated lattices (the set of all such matrices is denoted by $L^{n\times m}$) with binary fuzzy relations between X and Y (the set of all such relations is denoted by $L^{X\times Y}$). Also, we identify vectors with n components over residuated lattices (the set of all such vectors is denoted by L^n) with fuzzy sets in X (the set of all such fuzzy sets is denoted by L^X). As usual, we identify vectors with n components with $1 \times n$ matrices.

Using the terminology known from Boolean matrices [15], we define the following notions.

Definition 3.1. $V \subseteq L^n$ is called an *i-subspace* if

- V is closed under $\otimes$-multiplication, i.e. for every $a \in L$ and $C \in V$ we have $a \otimes C \in V$ (here, $a \otimes C$ is defined by $(a \otimes C)(i) = a \otimes C(i)$ for $i = 1, \ldots, n$);
- V is closed under $\bigvee$-union, i.e. for $C_j \in V$ $(j \in J)$ we have $\bigvee_{j\in J} C_j \in V$ (here, $\bigvee_{j\in J} C_j$ is defined by $(\bigvee_{j\in J} C_j)(i) = \bigvee_{j\in J} C_j(i)$).

$V \subseteq L^n$ is called a *c-subspace* if

- V is closed under left $\rightarrow$-multiplication (or $\rightarrow$-shift), i.e. for every $a \in L$ and $C \in V$ we have $a \rightarrow C \in V$ (here, $a \rightarrow C$ is defined by $(a \rightarrow C)(i) = a \rightarrow C(i)$ for $i = 1, \ldots, n$);
- V is closed under $\bigwedge$-intersection, i.e. for $C_j \in V$ $(j \in J)$ we have $\bigwedge_{j\in J} C_j \in V$ (here, $\bigwedge_{j\in J} C_j$ is defined by $(\bigwedge_{j\in J} C_j)(i) = \bigwedge_{j\in J} C_j(i)$).

Remark 3.1. (1) If elements of V are regarded as fuzzy sets, the concepts of an i-subspace and a c-subspace coincide with the concept of a fuzzy interior system and a fuzzy closure system as defined in [2, 7].

(2) For $L = \{0, 1\}$ the concept of an i-subspace coincides with the concept of a subspace from the theory of Boolean matrices [15]. In fact, closedness under $\otimes$-multiplication is satisfied for free in the case of Boolean matrices. Note also that for Boolean matrices, V forms a c-subspace iff $\overline{V} = \{\overline{C} \mid C \in V\}$ forms an i-subspace (with $\overline{C}$ defined by $\overline{C}(i) = \overline{C(i)}$ where $\overline{a} = a \rightarrow 0$, i.e. $\overline{0} = 1$ and $\overline{1} = 0$), and vice versa. However, such a reducibility among the concepts of i-subspace and c-subspace is not available in general because in residuated lattices, the law of double negation (saying that $(a \rightarrow 0) \rightarrow 0 = a$) does not hold.

Definition 3.2. The *i-span* (*c-span*) of $V \subseteq L^n$ is the intersection of all i-subspaces (c-subspaces) of L^n that contain V, hence itself an i-subspace (c-subspace) of L^n.

The *row i-space* (*row c-space*) of matrix $I \in L^{n\times m}$ is the i-span (c-span) of the set of all rows of I (considered as vectors from L^n). The *column i-space* (*column c-space*) is defined analogously as the i-span (c-span) of the set of columns of I. The row i-space, row c-space, column i-space, and column c-space of matrix I is denoted by $R_i(I)$, $R_c(I)$, $C_i(I)$, $C_c(I)$.

For a concept lattice $\mathcal{B}(X^\vartriangle, Y^\triangledown, I)$, where $\langle{}^\vartriangle, {}^\triangledown\rangle$ is either of $\langle{}^\uparrow, {}^\downarrow\rangle$, $\langle{}^\cap, {}^\cup\rangle$, and $\langle{}^\wedge, {}^\vee\rangle$, denote the corresponding sets of extents and intents by $\mathrm{Ext}(X^\vartriangle, Y^\triangledown, I)$ and $\mathrm{Int}(X^\vartriangle, Y^\triangledown, I)$. That is,

$$
\begin{aligned}
\mathrm{Ext}(X^\vartriangle, Y^\triangledown, I) &= \{C \in L^X \mid \langle C, D\rangle \in \mathcal{B}(X^\vartriangle, Y^\triangledown, I) \text{ for some } D\},\\
\mathrm{Int}(X^\vartriangle, Y^\triangledown, I) &= \{D \in L^Y \mid \langle C, D\rangle \in \mathcal{B}(X^\vartriangle, Y^\triangledown, I) \text{ for some } C\},
\end{aligned}
$$

A fundamental connection between the row and column spaces on one hand, and the concept lattices on the other hand, is described in the following theorem (I^{T} denotes the transpose of I).

Theorem 3.1. For a matrix $I \in L^{n\times m}$, $X = \{1,\dots,n\}$, $Y = \{1,\dots,m\}$, we have

$$\begin{aligned} R_i(I) &= \mathrm{Int}(X^{\cap}, Y^{\cup}, I) = \mathrm{Ext}(Y^{\wedge}, X^{\vee}, I^{\mathrm{T}}), & (7)\\ R_c(I) &= \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I) = \mathrm{Ext}(Y^{\uparrow}, X^{\downarrow}, I^{\mathrm{T}}), & (8)\\ C_i(I) &= \mathrm{Ext}(X^{\wedge}, Y^{\vee}, I) = \mathrm{Int}(Y^{\cap}, X^{\cup}, I^{\mathrm{T}}), & (9)\\ C_c(I) &= \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I) = \mathrm{Int}(Y^{\uparrow}, X^{\downarrow}, I^{\mathrm{T}}). & (10) \end{aligned}$$

Proof:
(7): To establish $R_i(I) = \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$, notice that $\mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ is just the set of all fixpoints of the fuzzy interior operator ${}^{\cup\cap}$ (see e.g. [7, 11]), i.e. a fuzzy interior system. To see that this fuzzy interior system is the least one that contains all rows of I, it is sufficient to observe that every intent $D \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ is a $\bigvee$-union of $\otimes$-multiplications of rows of I and that $\mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ contains every row of I. To observe this fact, consider the corresponding formal concept $\langle C, D\rangle \in \mathcal{B}(X^{\cap}, Y^{\cup}, I)$. It follows from the description of suprema in $\mathcal{B}(X^{\cap}, Y^{\cup}, I)$ that

$$\begin{aligned} \langle C, D\rangle &= \textstyle\bigvee_{x\in X}\langle \{{}^{C(x)}\!/x\}^{\cap\cup}, \{{}^{C(x)}\!/x\}^{\cap}\rangle = \\ & \quad \langle(\textstyle\bigvee_{x\in X}\{{}^{C(x)}\!/x\})^{\cap\cup}, \textstyle\bigvee_{x\in X}\{{}^{C(x)}\!/x\}^{\cap}\rangle, \end{aligned}$$

(note that $\{{}^{a}\!/x\}$ denotes a singleton fuzzy set A defined by $A(u) = a$ for $u = x$ and $A(u) = 0$ for $u \neq x$) and hence

$$\begin{aligned} D &= \textstyle\bigvee_{x\in X}\{{}^{C(x)}\!/x\}^{\cap} = \\ &= \textstyle\bigvee_{x\in X} C(x) \otimes \{{}^{1}\!/x\}^{\cap}. \end{aligned}$$

In addition, $\langle\{{}^{1}\!/x\}^{\cap\cup}, \{{}^{1}\!/x\}^{\cap}\rangle$ is a particular formal concept from $\mathcal{B}(X^{\cap}, Y^{\cup}, I)$. It is now sufficient to realize that $\{{}^{1}\!/x\}^{\cap}$ is just the x-th row of I.

The second equality of (7) is immediate. (9) is a consequence of (7) when taking a transpose of I. Namely, in such case extents and intents switch their roles.

(8): Similarly, to establish $R_c(I) = \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$, notice that $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ is just the set of all fixpoints of the fuzzy closure operator ${}^{\downarrow\uparrow}$ (see e.g. [1, 2]), i.e. a fuzzy closure system. To see that $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ is the least fuzzy closure system which contains all rows of I, it is sufficient to observe that every intent $D \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ is an $\bigwedge$-intersection of $\rightarrow$-shifts of rows of I and that $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ contains every row of I. To observe this fact, consider the corresponding formal concept $\langle C, D\rangle \in \mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$. Then it follows from the description of suprema in $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ that

$$\begin{aligned} \langle C, D\rangle &= \textstyle\bigvee_{x\in X}\langle \{{}^{C(x)}\!/x\}^{\uparrow\downarrow}, \{{}^{C(x)}\!/x\}^{\uparrow}\rangle = \\ & \quad \langle(\textstyle\bigvee_{x\in X}\{{}^{C(x)}\!/x\})^{\uparrow\downarrow}, \textstyle\bigwedge_{x\in X}\{{}^{C(x)}\!/x\}^{\uparrow}\rangle, \end{aligned}$$

and hence

$$\begin{aligned} D &= \textstyle\bigwedge_{x\in X}\{{}^{C(x)}\!/x\}^{\uparrow} = \\ &= \textstyle\bigwedge_{x\in X} C(x) \rightarrow \{{}^{1}\!/x\}^{\uparrow}. \end{aligned}$$

In addition, $\langle\{{}^{1}\!/x\}^{\uparrow\downarrow}, \{{}^{1}\!/x\}^{\uparrow}\rangle$ is a particular formal concept from $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$. It is now sufficient to realize that $\{{}^{1}\!/x\}^{\uparrow}$ is just the x-th row of I.

Again, (10) is a consequence of (8) when taking the transpose of I. $\square$

The following lemma provides us with the relationships between the row and column spaces of matrices and their compositions. Recall that $X = \{1, \dots, n\}$, $Y = \{1, \dots, m\}$, and $F = \{1, \dots, k\}$.

Lemma 3.1. For matrices $A \in L^{n\times k}$ and $B \in L^{k\times m}$,

$$\begin{aligned} R_i(A \circ B) &\subseteq R_i(B), & (11)\\ C_i(A \circ B) &\subseteq C_i(A), & (12)\\ R_c(A \triangleleft B) &\subseteq R_c(B), & (13)\\ C_c(A \triangleright B) &\subseteq C_c(A). & (14)\end{aligned}$$

In addition,

$$\begin{aligned} C_c(A \triangleleft B) &\subseteq \mathrm{Ext}(X^{\cap}, F^{\cup}, A), & (15)\\ R_c(A \triangleright B) &\subseteq \mathrm{Int}(F^{\wedge}, Y^{\vee}, B). & (16)\end{aligned}$$

Proof:
(11): According to [8, Theorem 4],

$$\mathrm{Int}(X^{\cap_{A\circ B}}, Y^{\cup_{A\circ B}}, A \circ B) \subseteq \mathrm{Int}(F^{\cap_B}, Y^{\cup_B}, B).$$

Due to Theorem 3.1, $\mathrm{Int}(X^{\cap_{A\circ B}}, Y^{\cup_{A\circ B}}, A\circ B) = R_i(A\circ B)$ and $\mathrm{Int}(F^{\cap_B}, Y^{\cup_B}, B) = R_i(B)$, whence the claim.

The other inclusions follow analogously from

$$\begin{aligned} \mathrm{Ext}(X^{\wedge_{A\circ B}}, Y^{\vee_{A\circ B}}, A \circ B) &\subseteq \mathrm{Ext}(X^{\wedge_A}, F^{\vee_A}, A),\\ \mathrm{Int}(X^{\uparrow_{A\triangleleft B}}, Y^{\downarrow_{A\triangleleft B}}, A \triangleleft B) &\subseteq \mathrm{Int}(F^{\uparrow_B}, Y^{\downarrow_B}, B),\\ \mathrm{Ext}(X^{\uparrow_{A\triangleright B}}, Y^{\downarrow_{A\triangleright B}}, A \triangleright B) &\subseteq \mathrm{Ext}(X^{\uparrow_A}, F^{\downarrow_A}, A),\\ \mathrm{Ext}(X^{\uparrow_{A\triangleleft B}}, Y^{\downarrow_{A\triangleleft B}}, A \triangleleft B) &\subseteq \mathrm{Ext}(X^{\cap_A}, F^{\cup_A}, A),\\ \mathrm{Int}(X^{\uparrow_{A\triangleright B}}, Y^{\downarrow_{A\triangleright B}}, A \triangleright B) &\subseteq \mathrm{Int}(F^{\wedge_B}, Y^{\vee_B}, B),\end{aligned}$$

proved in [8, Theorem 4], and Theorem 3.1. □

The necessary and sufficient conditions for inclusions of row and column spaces of two matrices are the subject of the following theorem.

Theorem 3.2. Consider matrices $I \in L^{n\times m}$, $A \in L^{n\times k}$, and $B \in L^{k\times m}$.

$$R_i(I) \subseteq R_i(B) \text{ iff there exists a matrix } A' \in L^{n\times k} \text{ such that } I = A' \circ B, \quad (17)$$

$$C_i(I) \subseteq C_i(A) \text{ iff there exists a matrix } B' \in L^{k\times m} \text{ such that } I = A \circ B', \quad (18)$$

$$R_c(I) \subseteq R_c(B) \text{ iff there exists a matrix } A' \in L^{n\times k} \text{ such that } I = A' \triangleleft B, \quad (19)$$

$$C_c(I) \subseteq C_c(A) \text{ iff there exists a matrix } B' \in L^{k\times m} \text{ such that } I = A \triangleright B'. \quad (20)$$

In addition,

$$C_c(I) \subseteq \mathrm{Ext}(X^{\cap}, F^{\cup}, A) \quad (21)$$
$$\text{iff there exists a matrix } B' \in L^{k\times m} \text{ such that } I = A \triangleleft B',$$

$$R_c(I) \subseteq \mathrm{Int}(F^{\wedge}, Y^{\vee}, B) \quad (22)$$
$$\text{iff there exists a matrix } A' \in L^{n\times k} \text{ such that } I = A' \triangleright B.$$

Proof:
(17): "⇒": Let $R_i(I) \subseteq R_i(B)$, i.e. by Theorem 3.1, $\mathrm{Int}(X^{\cap}, Y^{\cup}, I) \subseteq \mathrm{Int}(F^{\cap}, Y^{\cup}, B)$. Every $H \in \mathrm{Int}(F^{\cap}, Y^{\cup}, B)$ can be written as

$$H = \bigvee_{1 \leq l \leq k} c_l \otimes B_{l\_}.$$

Thus every $H \in \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ can be written as $\bigvee_{1 \leq l \leq k} c_l \otimes B_{l\_}$. Therefore, since every row $I_{i\_}$ of I belongs to $\mathrm{Int}(X^{\cap}, Y^{\cup}, I)$, $I_{i\_}$ can be written as

$$I_{i\_} = \bigvee_{1 \leq l \leq k} c_{li} \otimes B_{l\_}.$$

Now, we get the required matrix A' by putting $A'_{il} = c_{li}$. "⇐" is established in Lemma 3.1.

(18) follows from (17), (9) and the fact that $(C \circ D)^{\mathrm{T}} = D^{\mathrm{T}} \circ C^{\mathrm{T}}$.

(19): "⇒": Let $R_c(I) \subseteq R_c(B)$, i.e. by Theorem 3.1, $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I) \subseteq \mathrm{Int}(F^{\uparrow}, Y^{\downarrow}, B)$. Every $H \in \mathrm{Int}(F^{\uparrow}, Y^{\downarrow}, B)$ can be written as

$$H = \bigwedge_{1 \leq l \leq k} c_l \to B_{l\_}.$$

Thus every $H \in \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ can be written in this form as well. Therefore, since every row $I_{i\_}$ of I belongs to $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$, $I_{i\_}$ can be written as

$$I_{i\_} = \bigwedge_{1 \leq l \leq k} c_{li} \to B_{l\_}.$$

Now, we get the required matrix A' by putting $A'_{il} = c_{li}$. "⇐" is established in Lemma 3.1.

(20) follows from (19), (10) and the fact that $(C \triangleleft D)^{\mathrm{T}} = D^{\mathrm{T}} \triangleright C^{\mathrm{T}}$.

(21): "⇒": Let $C_c(I) \subseteq \mathrm{Ext}(X^{\cap}, F^{\cup}, A)$, i.e. by Theorem 3.1, $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I) \subseteq \mathrm{Ext}(X^{\cap}, F^{\cup}, A)$. Every $H \in \mathrm{Ext}(F^{\cap}, Y^{\cup}, B)$ and thus in particular every $H \in \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$ can be written as $\bigwedge_{1 \leq l \leq k} A_{\_l} \to c_l$. Therefore, since every column $I_{\_i}$ of I belongs to $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$, $I_{\_i}$ can be written as

$$I_{\_i} = \bigwedge_{1 \leq l \leq k} A_{\_l} \to c_{li}$$

Now, we get the required matrix B' by putting $B'_{li} = c_{li}$. Again, "⇐" is established in Lemma 3.1.

(22): follows from (21), (8) and the fact that $(C \triangleleft D)^{\mathrm{T}} = D^{\mathrm{T}} \triangleright C^{\mathrm{T}}$.

□

As a corollary, we obtain the following theorem.

Theorem 3.3. Let I and J be $n \times m$ matrices.

(a) If $R_i(I) = R_i(J)$ and $I = A \circ B$ for some $A \in L^{n \times k}, B \in L^{k \times m}$ then there exists $A' \in L^{n \times k}$ such that $J = A' \circ B$.

(b) If $C_i(I) = C_i(J)$ and $I = A \circ B$ for some $A \in L^{n \times k}, B \in L^{k \times m}$ then there exists $B' \in L^{k \times m}$ such that $J = A \circ B'$.

(c) If $R_c(I) = R_c(J)$ and $I = A \triangleleft B$ for some $A \in L^{n \times k}, B \in L^{k \times m}$ then there exists $A' \in L^{n \times k}$ such that $J = A' \triangleleft B$.

(d) If $C_c(I) = C_c(J)$ and $I = A \triangleright B$ for some $A \in L^{n\times k}, B \in L^{k\times m}$ then there exists $B' \in L^{k\times m}$ such that $J = A \triangleright B'$.

In addition,

(e) If $C_c(I) = C_c(J)$ and $I = A \triangleleft B$ for some $A \in L^{n\times k}, B \in L^{k\times m}$ then there exists $B' \in L^{k\times m}$ such that $J = A \triangleleft B'$.

(f) If $R_c(I) = R_c(J)$ and $I = A \triangleright B$ for some $A \in L^{n\times k}, B \in L^{k\times m}$ then there exists $A' \in L^{n\times k}$ such that $J = A' \triangleright B$.

Proof:
(a): If $I = A \circ B$ then, according to (17), $R_i(I) \subseteq R_i(B)$. Since $R_i(J) = R_i(I)$, we also have $R_i(J) \subseteq R_i(B)$. Another application of (17) yields A' for which $J = A' \circ B$. The proof for (b)–(f) is similar. □

We now show two theorems, well known from the Boolean matrix theory [15], as corollaries of the above results. As is mentioned above, for $L = \{0, 1\}$, the row i-space $R_i(I)$ of I coincides with the row space of the Boolean matrix I as defined in the Boolean matrix theory; likewise, $C_i(I)$ coincides with the column space of I as defined in the Boolean matrix theory.

Remark 3.2. From the point of view of concept lattices, as developed within formal concept analysis [10], the row space of a Boolean matrix I, i.e. $R_i(I)$, is dually isomorphic as a lattice to the lattice of all intents of the ordinary concept lattice of the complement of I, i.e. to $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, \overline{I})$. Namely, according to Theorem 3.1, $R_i(I) = \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ and it is well known that for $L = \{0, 1\}$, $\mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ is dually isomorphic to $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, \overline{I})$ with $D \mapsto \overline{D}$ being the dual isomorphism. Lattices $\mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ have been studied by Markowsky, see e.g. [17] (see Section 2).

Corollary 3.1. *(1)* For Boolean matrices A and B, the row space of $A \circ B$ is a subset of the row space of B.

(2) For a Boolean matrix A, the row space of A has the same number of elements as the columns space of A.

Proof:
(1) is a particular case of (11) for $L = \{0, 1\}$.

(2): By Theorem 3.1, $R_i(A) = \mathrm{Int}(X^{\cap}, Y^{\cup}, A)$ and $C_i(A) = \mathrm{Int}(Y^{\cap}, X^{\cup}, A^{\mathrm{T}})$. As is mentioned in Remark 3.2, $\mathrm{Int}(X^{\cap}, Y^{\cup}, A)$ is dually isomorphic to $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, \overline{A})$ and hence isomorphic to $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, \overline{A})$. Thus, $\mathrm{Int}(Y^{\cap}, X^{\cup}, A^{\mathrm{T}})$ is isomorphic to $\mathcal{B}(Y^{\uparrow}, X^{\downarrow}, \overline{A}^{\mathrm{T}})$. As is well-known from FCA [10], $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, \overline{A})$ is dually isomorphic to $\mathcal{B}(Y^{\uparrow}, X^{\downarrow}, \overline{A}^{\mathrm{T}})$, proving the claim. □

Remark 3.3. (1) From Theorem 3.1 we have $|R_c(I)| = |C_c(I)|$ for any $I \in L^{n\times m}$ since $C_c(I) = \mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$ and, as is well known, $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$ is dually isomorphic to $R_c(I) = \mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$.

(2) Contrary to Corollary 3.1 (2), $|R_i(I)| = |C_i(I)|$ does not hold for general L. As an example, consider L being a finite chain containing $a < b$ with $\otimes$ defined as follows:

$$x \otimes y = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for each $x, y \in L$. One can easily see that $x \otimes \bigvee_j y_j = \bigvee_j (x \otimes y_j)$ and thus an adjoint operation $\rightarrow$ exists such that $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ is a complete residuated lattice (see e.g. [12]). Namely, $\rightarrow$ is given as follows:

$$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x = 1, \\ b & \text{otherwise,} \end{cases}$$

for each $x, y \in L$. Now, for the matrix $I = \begin{pmatrix} a & b \end{pmatrix}$, we have $R_i(I) = \{(a, b), (0, 0)\}$ and $C_i(I) = \{(0), (a), (b)\}$.

The next theorem shows that Schein ranks of matrices with the same row or column spaces are equal.

Theorem 3.4. Let I and J be $n \times m$ matrices.

(a) If $R_i(I) = R_i(J)$ then $\rho_{\mathrm{s}\circ}(I) = \rho_{\mathrm{s}\circ}(J)$.

(b) If $C_i(I) = C_i(J)$ then $\rho_{\mathrm{s}\circ}(I) = \rho_{\mathrm{s}\circ}(J)$.

(c) If $R_c(I) = R_c(J)$ then $\rho_{\mathrm{s}\triangleleft}(I) = \rho_{\mathrm{s}\triangleleft}(J)$ and $\rho_{\mathrm{s}\triangleright}(I) = \rho_{\mathrm{s}\triangleright}(J)$.

(d) If $C_c(I) = C_c(J)$ then $\rho_{\mathrm{s}\triangleleft}(I) = \rho_{\mathrm{s}\triangleleft}(J)$ and $\rho_{\mathrm{s}\triangleright}(I) = \rho_{\mathrm{s}\triangleright}(J)$.

Proof:
(a): Let $I = A \circ B$ for an $n \times k$ matrix A and a $k \times m$ matrix B. According to Theorem 3.3 (a), there exists an $n \times k$ matrix A' such that $J = A' \circ B$, proving $\rho_{\mathrm{s}\circ}(I) \geq \rho_{\mathrm{s}\circ}(J)$. In a similar way one shows $\rho_{\mathrm{s}\circ}(I) \leq \rho_{\mathrm{s}\circ}(J)$.

(b): Let $I = A \circ B$ for an $n \times k$ matrix A and a $k \times m$ matrix B. According to Theorem 3.3 (b), there exists a $k \times m$ matrix B' such that $J = A \circ B'$, proving $\rho_{\mathrm{s}\circ}(I) \geq \rho_{\mathrm{s}\circ}(J)$. In a similar way one shows $\rho_{\mathrm{s}\circ}(I) \leq \rho_{\mathrm{s}\circ}(J)$.

(c): Let $I = A \triangleleft B$ for an $n \times k$ matrix A and a $k \times m$ matrix B. According to Theorem 3.3 (c), there exists an $n \times k$ matrix A' such that $J = A' \triangleleft B$, proving $\rho_{\mathrm{s}\triangleleft}(I) \geq \rho_{\mathrm{s}\triangleleft}(J)$. In a similar way one shows $\rho_{\mathrm{s}\triangleleft}(I) \leq \rho_{\mathrm{s}\triangleleft}(J)$.

Similarly, Let $I = A \triangleright B$ for an $n \times k$ matrix A and a $k \times m$ matrix B. By Theorem 3.3 (f), there exists a $k \times m$ matrix B' such that $J = A \triangleright B'$, proving $\rho_{\mathrm{s}\triangleright}(I) \geq \rho_{\mathrm{s}\triangleright}(J)$. In a similar way one shows $\rho_{\mathrm{s}\triangleright}(I) \leq \rho_{\mathrm{s}\triangleright}(J)$.

(d): Similar to (c). □

4. Matrices with Isomorphic Row Spaces

Because of their particular role in $\circ$-decompositions and because of the established results in the Boolean case, row i-spaces are investigated in more detail in this section. In particular, we characterize matrices with isomorphic row i-spaces and isomorphic concept lattices, and show that isomorphism of row i-spaces (or, equivalently, of concept lattices) implies equality of the Schein rank $\rho_{\mathrm{s}\circ}$. Clearly, one obtains results regarding column i-spaces by going to transpose matrices.

Definition 4.1. A mapping $h : V \to W$ from an i-subspace $V \subseteq L^p$ into an i-subspace $W \subseteq L^q$ is called an *i-morphism* if it is a $\otimes$- and $\bigvee$-morphism, i.e. if

- $h(a \otimes C) = a \otimes h(C)$ for each $a \in L$ and $C \in V$;
- $h(\bigvee_{k \in K} C_k) = \bigvee_{k \in K} h(C_k)$ for every collection of $C_k \in V$ $(k \in K)$.

An i-morphism $h : V \to W$ is called an

- *extendable i-morphism* if h can be extended to an i-morphism of L^p to L^q, i.e. if there exists an i-morphism $h' : L^p \to L^q$ such that for every $C \in V$ we have $h'(C) = h(C)$;
- an *i-isomorphism* if h is bijective and both h and h^{-1} are extendable i-morphisms; if such h exists, we write $V \cong W$ and call V and W i-isomorphic.

A mapping $h : V \to W$ from a c-subspace $V \subseteq L^p$ into a c-subspace $W \subseteq L^q$ is called a *c-morphism* if it is a $\to$- and $\bigwedge$-morphism, i.e. if

- $h(a \to C) = a \to h(C)$ for each $a \in L$ and $C \in V$;
- $h(\bigwedge_{k \in K} C_k) = \bigwedge_{k \in K} h(C_k)$ for every collection of $C_k \in V$ $(k \in K)$.

The notions of *extendable c-morphism* and *c-isomorphism* are defined similarly as in the case of i-morphisms.

For our purpose, we need the following two lemmas, establishing an important fact that i-morphisms are just the mappings obtained from matrices using the $\circ$-product.

Lemma 4.1. For every matrix $A \in L^{p \times q}$, the mapping $h_A : L^p \to L^q$ defined by

$$h_A(C) = C \circ A$$

is an extendable i-morphism.

Proof:
Follows easily from the properties of residuated lattices. □

Lemma 4.2. If for $V \subseteq L^p$, $h : V \to L^q$ is an extendable i-morphism then there exists a matrix $A \in L^{p \times q}$ such that $h(C) = C \circ A$ for every $C \in L^p$.

Proof:
Since h is extendable, we may safely assume that $h : L^p \to L^q$, i.e. that h is defined for every $C \in L^p$. Let $A \in L^{p \times q}$ be defined by

$$A_{ij} = \bigwedge_{C \in V}(C(i) \to (h(C))(j)).$$

That is, $A_{i\_} = \bigwedge_{C \in V}(C(i) \to h(C))$, i.e. the row $A_{i\_}$ contains a vector of degrees that can be interpreted as the intersection of images of those vectors C from V for which the corresponding fuzzy set contains i (in Boolean case: for which the i-th component is 1).

We now check $h(C) = C \circ A$ for every $C \in L^p$. First,

$$\begin{aligned}(C \circ A)(j) &= \textstyle\bigvee_{i=1}^{p}[C(i) \otimes A_{ij}] = \\ &= \textstyle\bigvee_{i=1}^{p}[C(i) \otimes (\bigwedge_{C' \in V}(C'(i) \to (h(C'))(j)))] \leq (h(C))(j).\end{aligned}$$

Second, to establish $(h(C))(j) \leq (C \circ A)(j)$, we first show

$$(h(E_k))(j) \leq (E_k \circ A)(j) \tag{23}$$

for every $k = 1, \ldots, p$, where E_k is defined by

$$E_k(i) = \begin{cases} 0 & \text{for } i \neq k, \\ 1 & \text{for } i = k, \end{cases}$$

for every $i = 1, \ldots, p$. Notice that for any $C \in V$, as $C(k) \otimes E_k \leq C$, we have $C(k) \otimes h(E_k) = h(C(k) \otimes E_k) \leq h(C)$, whence $h(E_k) \leq C(k) \to h(C)$. Using this inequality, we get

$$\begin{aligned}(E_k \circ A)(j) &= \textstyle\bigvee_{i=1}^{p}[E_k(i) \otimes (\bigwedge_{C \in V}(C(i) \to (h(C))(j)))] = \\ &= \textstyle\bigwedge_{C \in V}(C(k) \to (h(C))(j)) \geq (h(E_k))(j).\end{aligned}$$

Using (23), we now get

$$\begin{aligned}(h(C))(j) &= (h(\textstyle\bigvee_{i=1}^{p}(C(i) \otimes E_i)))(j) = \textstyle\bigvee_{i=1}^{p}(C(i) \otimes h(E_i)(j)) \leq \\ &\leq \textstyle\bigvee_{i=1}^{p}(C(i) \otimes (E_i \circ A)(j)) = (\bigvee_{i=1}^{p}(C(i) \otimes E_i) \circ A)(j)) = (C \circ A)(j),\end{aligned}$$

finishing the proof. □

Remark 4.1. (1) As a result of Lemma 4.1 and Lemma 4.2, extendable i-morphisms may be represented by matrices by means of $\circ$-products.

(2) In case of Boolean matrices, every i-morphism is extendable. Namely, due to [15, Lemma 1.3.2], for every i-morphism $h : V \to \{0,1\}^q$ there exists a Boolean matrix $A \in \{0,1\}^{p \times q}$ such that $h(C) = C \circ A$ for every $C \in V$. Clearly, $h' : \{0,1\}^p \to \{0,1\}^q$ defined by $h'(C) = C \circ A$ for any $C \in \{0,1\}^p$ is the required extension of h which is an i-morphism.

(3) For general residuated lattices, however, there exist i-morphisms that are not extendable. Consider any finite chain L with $a < b$ being two elements of L. Let $\otimes$ be defined as in Remark 3.3 (2). For $p = q = 1$, put $V = \{(0),(a)\}$, $W = \{(0),(b)\}$. Clearly, both V and W are i-subspaces for which $h((0)) = (0)$ and $h((a)) = (b)$ defines an i-morphism h. If h was extendable, there would exist a matrix $A = (c)$ for which $h(C) = C \circ A$ (Lemma 4.2). In particular, this would mean $(b) = h((a)) = (a) \circ (c)$, i.e. $b = a \otimes c$ which is impossible because $b > a$. Therefore, h is not extendable.

The following theorem provides us with a criterion for two row spaces of matrices to be isomorphic as i-spaces.

Theorem 4.1. Let $I \in L^{n \times m}$ and $J \in L^{p \times r}$ be matrices. Then $R_i(I) \cong R_i(J)$ (row spaces of I and J are i-isomorphic) if and only if there exists a matrix $K \in L^{p \times m}$ such that $R_i(I) = R_i(K)$ and $C_i(K) = C_i(J)$.

Proof:
"⇒": Let $h : R_i(J) \to R_i(I)$ be an i-isomorphism. According to Lemma 4.2, there exist matrices $X \in L^{r\times m}$ and $Y \in L^{m\times r}$ such that

$$h(C) = C \circ X \text{ and } h^{-1}(D) = D \circ Y$$

for every $C \in R_i(J)$ and $D \in R_i(I)$. Because every row of J is an element of $R_i(J)$, it follows that $J \circ X \circ Y = J$. Therefore, according to Theorem 3.2 (18), $C_i(J) \subseteq C_i(J \circ X)$. Since, according to Theorem 3.2 (18) again, $C_i(J) \supseteq C_i(J \circ X)$, we conclude $C_i(J \circ X) = C_i(J)$. Furthermore, if $D \in R_i(I)$, then $D \circ Y = h^{-1}(D) \in R_i(J)$, hence $D \circ Y = C \circ J$ for some $C \in L^p$. Since $D = (D \circ Y) \circ X$, we get $D = (C \circ J) \circ X = C \circ (J \circ X)$, showing $D \in R_i(J \circ X)$. We established $R_i(I) \subseteq R_i(J \circ X)$. If $D \in R_i(J \circ X)$ then $D = C \circ (J \circ X) = (C \circ J) \circ X$ for some $C \in L^p$. Since $C \circ J \in R_i(J)$, we get

$$D = (C \circ J) \circ X = h(C \circ J) \in R_i(I),$$

proving $R_i(J \circ X) \subseteq R_i(I)$. Summing up, we proved $R_i(I) = R_i(J \circ X)$. Now, $J \circ X$ yields the required matrix K.

"⇐": Since $C_i(K) = C_i(J)$, an application of Theorem 3.2 (18) to $C_i(K) \supseteq C_i(J)$ and $C_i(K) \subseteq C_i(J)$ yields a matrix $S \in L^{m\times r}$ for which $K \circ S = J$ and a matrix $T \in L^{r\times m}$ for which $J \circ T = K$, respectively. Consider now mappings $f : R_i(K) \to R_i(J)$ and $g : R_i(J) \to R_i(K)$ defined for $D \in R_i(K)$ and $F \in R_i(J)$ by

$$f(D) = D \circ S \qquad \text{and} \qquad g(F) = F \circ T. \tag{24}$$

Notice that every $D \in R_i(K)$ is in the form $D = C \circ K$ for some $C \in L^p$ and that every $F \in R_i(J)$ is in the form $F = E \circ J$ for some $E \in L^p$. The mappings f and g are defined correctly. Indeed,

$$f(D) = D \circ S = (C \circ K) \circ S = C \circ (K \circ S) = C \circ J$$

for some C, and because $C \circ J \in R_i(J)$, we have $f(D) \in R_i(J)$. In a similar way one obtains $g(F) \in R_i(K)$. Next, since D is in the form $D = C \circ K$ for some C, we have

$$g(f(D)) = ((C \circ K) \circ S) \circ T = (C \circ (K \circ S)) \circ T = C \circ (J \circ T) = C \circ K = D$$

and, similarly, $f(g(F)) = F$, proving that f and g are mutually inverse bijections. Finally, due to (24), Lemma 4.1 implies that f and g are extendable i-morphisms. This shows that $R_i(K) \cong R_i(J)$, and hence $R_i(I) \cong R_i(J)$. □

Remark 4.2. For Boolean matrices, Theorem 4.1 is known for $n = m = p = r$, i.e. for square matrices [15]. Note that for Boolean matrices, $R_i(I) \cong R_i(J)$ means that $R_i(I)$ and $R_i(J)$ are isomorphic as lattices. Namely, the $\otimes$-morphism property is not required because it is satisfied for free in the Boolean case.

Next, we show how Theorem 4.1 may be used to prove a characterization of isomorphism of concept lattices induced by the $^\cap$ and $^\cup$ operators. We consider mappings of concept lattices. Since every extent of a formal concept is uniquely determined by the corresponding intent and vice versa (using operators

${}^{\cap}$ and ${}^{\cup}$), a mapping $h : \mathcal{B}(X_1^{\cap}, Y_1^{\cup}, I_1) \to \mathcal{B}(X_2^{\cap}, Y_2^{\cup}, I_2)$ may be thought of as consisting of a pair $\langle h_{\mathrm{Ext}}, h_{\mathrm{Int}}\rangle$ of mappings, such that $h(A,B) = \langle h_{\mathrm{Ext}}(A), h_{\mathrm{Int}}(B)\rangle$. That is, h consists of

$$h_{\mathrm{Ext}} : \mathrm{Ext}(X_1^{\cap}, Y_1^{\cup}, I_1) \to \mathrm{Ext}(X_2^{\cap}, Y_2^{\cup}, I_2)$$

and

$$h_{\mathrm{Int}} : \mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1) \to \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2).$$

Since $\mathrm{Ext}(X_i^{\cap}, Y_i^{\cup}, I_i)$ are c-spaces and $\mathrm{Int}(X_i^{\cap}, Y_i^{\cup}, I_i)$ are i-spaces, the following definition provides natural requirements for h to be a morphism.

Definition 4.2. A mapping $h = \langle h_{\mathrm{Ext}}, h_{\mathrm{Int}}\rangle : \mathcal{B}(X_1^{\cap}, Y_1^{\cup}, I_1) \to \mathcal{B}(X_2^{\cap}, Y_2^{\cup}, I_2)$ is called an *(extendable) morphism* if h_{Ext} is an (extendable) c-morphism and h_{Int} is an (extendable) i-morphism (cf. Definition 4.1). h is called an *isomorphism* if h_{Ext} is a c-isomorphism and h_{Int} is an i-isomorphism; if such h exists, we write $\mathcal{B}(X_1^{\cap}, Y_1^{\cup}, I_1) \cong \mathcal{B}(X_2^{\cap}, Y_2^{\cup}, I_2)$.

Lemma 4.3. If $h_{\mathrm{Int}} : \mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1) \to \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$ is an i-isomorphism then the corresponding mapping $h_{\mathrm{Ext}} : \mathrm{Ext}(X_1^{\cap}, Y_1^{\cup}, I_1) \to \mathrm{Ext}(X_2^{\cap}, Y_2^{\cup}, I_2)$ is a c-isomorphism, hence $\mathcal{B}(X_1^{\cap}, Y_1^{\cup}, I_1) \cong \mathcal{B}(X_2^{\cap}, Y_2^{\cup}, I_2)$.

Proof:
Due to Lemma 4.2, there exists a matrix A_h such that

$$h_{\mathrm{Int}}(C) = C \circ A_h,$$

i.e. $h_{\mathrm{Int}}(C) = C^{\cap_{A_h}}$ for every $C \in \mathrm{Int}(X_1^{\cap}, Y_1^{\cup}, I_1)$. As a result,

$$h_{\mathrm{Ext}}(E) = (h_{\mathrm{Int}}(E^{\cap_{I_1}}))^{\cup_{I_2}} = E^{\cap_{I_1}\cap_{A_h}\cup_{I_2}} \tag{25}$$

for every $E \in \mathrm{Ext}(X_1^{\cap}, Y_1^{\cup}, I_1)$. Since $R_i(I) = \mathrm{Int}(X^{\cap}, Y^{\cup}, I)$ for every I due to Theorem 3.1 (7), Theorem 4.1 and its proof (put $I = I_2$, $J = I_1$, and $X = A_h$) imply that the matrix $K = I_1 \circ A_h$ satisfies $R_i(K) = \mathrm{Int}(X_1^{\cap}, Y_2^{\cup}, K) = \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2) = R_i(I_2)$. Since $R_i(K) \subseteq R_i(I_2)$, there is a matrix J such that $K = J \circ I_2$ (Theorem 3.2 (17)). Note that due to [8, Theorem 3 (11)], ${}^{\cap_{J\circ I_2}} = {}^{\cap_J\cap_{I_2}}$ and ${}^{\cup_{J\circ I_2}} = {}^{\cup_{I_2}\cup_J}$. As a result, (25) implies

$$h_{\mathrm{Ext}}(E) = E^{\cap_{I_1}\cap_{A_h}\cup_{I_2}} = E^{\cap_{I_1\circ A_h}\cup_{I_2}} = E^{\cap_K\cup_{I_2}} = E^{\cap_{J\circ I_2}\cup_{I_2}} = E^{\cap_J\cap_{I_2}\cup_{I_2}}. \tag{26}$$

Observe now that since h_{Ext} is a bijection, we have

$$E^{\cap_J\cap_{I_2}\cup_{I_2}\cup_J} = E \tag{27}$$

for every $E \in \mathrm{Ext}(X_1^{\cap}, Y_1^{\cup}, I_1)$. Indeed, since $E^{\cap_J\cap_{I_2}\cup_{I_2}\cup_J} = E^{\cap_{J\circ I_2}\cup_{J\circ I_2}}$, it follows from the general properties of isotone Galois connections that

$$E^{\cap_{J\circ I_2}\cup_{J\circ I_2}} \supseteq E. \tag{28}$$

If in (28), $E^{\cap_{J\circ I_2}\cup_{J\circ I_2}} \supset E$, i.e. $E^{\cap_{J\circ I_2}\cup_{J\circ I_2}} \neq E$ then applying ${}^{\cap_{J\circ I_2}\cup_{I_2}}$ to both sides of the inequality and taking into account that ${}^{\cap_{J\circ I_2}\cup_{I_2}} = h_{\mathrm{Ext}}$ is a bijection, we get

$$E^{\cap_{J\circ I_2}\cup_{J\circ I_2}\cap_{J\circ I_2}\cup_{I_2}} \neq E^{\cap_{J\circ I_2}\cup_{I_2}}, \tag{29}$$

which yields a contradiction because using ${}^{\cap_{J\circ I_2}\cup_{J\circ I_2}\cap_{J\circ I_2}} = {}^{\cap_{J\circ I_2}}$, both sides of (29) are equal.

We established (26) and (27) from which it follows that ${}^{\cup_J}$ is inverse to h_{Ext}, i.e.

$$h_{\mathrm{Ext}}^{-1}(E_2) = E_2^{\cup_J} \tag{30}$$

for each $E_2 \in \mathrm{Ext}(X_2^{\cap}, Y_2^{\cup}, I_2)$.

Now, in a similar way, one may show that there exists a matrix J' such that

$$h_{\mathrm{Ext}}(E_1) = E_1^{\cup_{J'}} \tag{31}$$

for each $E_1 \in \mathrm{Ext}(X_1^{\cap}, Y_1^{\cup}, I_1)$. Namely, just start as in the beginning of this proof with h_{Int}^{-1} instead of h_{Int}, i.e. start by claiming the existence of A'_h for which $h_{\mathrm{Int}}^{-1}(D) = D \circ A'_h$ and proceed dually to how we have proceeded above.

Observe now that (30) implies that h_{Ext}^{-1} is a $\rightarrow$-morphism:

$$h_{\mathrm{Ext}}^{-1}(E_2)(x) = (a \rightarrow D)^{\cup_J}(x) = \bigwedge_{y\in Y} J(x,y) \rightarrow (a \rightarrow D(y)) =$$
$$\bigwedge_{y\in Y} a \rightarrow (J(x,y) \rightarrow D(y)) = a \rightarrow \bigwedge_{y\in Y} J(x,y) \rightarrow D(y) = a \rightarrow D^{\cup_J}(x).$$

For the same reason, (31) implies that h_{Ext} is a $\rightarrow$-morphism. Since h_{Int} is a bijective $\bigvee$-morphism, it is a lattice isomorphism and hence, in particular, a $\bigwedge$-morphism. Since mapping an extent to the corresponding intent is a lattice isomorphism of the lattice of extents to the lattice of intents, h_{Ext} is a $\bigwedge$-morphism. To sum up, h_{Ext} is a c-morphism. Furthermore, it follows from (31) and (30) that h_{Ext} and h_{Ext}^{-1} are extendable. As a result, h_{Ext} is a c-isomorphism, finishing the proof. □

For a positive integer n, we put

$$\hat{n} = \{1, \ldots, n\}.$$

Theorem 4.2. Let $I \in L^{n\times m}$ and $J \in L^{p\times r}$ be matrices. $\mathcal{B}(\hat{n}^{\cap}, \hat{m}^{\cup}, I) \cong \mathcal{B}(\hat{p}^{\cap}, \hat{q}^{\cup}, J)$ if and only if there exists a matrix $K \in L^{p\times m}$ such that $R_i(I) = R_i(K)$ and $C_i(K) = C_i(J)$.

Proof:
"⇒": If $\mathcal{B}(\hat{n}^{\cap}, \hat{m}^{\cup}, I) \cong \mathcal{B}(\hat{p}^{\cap}, \hat{r}^{\cup}, J)$ then $R_i(I) \cong R_i(J)$. Due to Theorem 3.1 (7), $\mathrm{Int}(\hat{n}^{\cap}, \hat{m}^{\cup}, I) = R_i(I)$ and $\mathrm{Int}(\hat{p}^{\cap}, \hat{r}^{\cup}, J) = R_i(J)$. A matrix K satisfying the required conditions exists due to Theorem 4.1.

"⇐": Due to Theorem 4.1, $R_i(I) \cong R_i(J)$. Since $\mathrm{Int}(\hat{n}^{\cap}, \hat{m}^{\cup}, I) = R_i(I)$ and $\mathrm{Int}(\hat{p}^{\cap}, \hat{r}^{\cup}, J) = R_i(J)$, this means that there exists an i-isomorphism $h_{\mathrm{Int}} : \mathrm{Int}(\hat{n}^{\cap}, \hat{m}^{\cup}, I) \rightarrow \mathrm{Int}(\hat{p}^{\cap}, \hat{r}^{\cup}, J)$. Lemma 4.3 now implies $\mathcal{B}(\hat{n}^{\cap}, \hat{m}^{\cup}, I) \cong \mathcal{B}(\hat{p}^{\cap}, \hat{r}^{\cup}, J)$. □

As a corollary, we obtain the following theorem.

Theorem 4.3. Let $I \in L^{n\times m}$ and $J \in L^{p\times r}$ be matrices. If $\mathcal{B}(\hat{n}^{\cap}, \hat{m}^{\cup}, I) \cong \mathcal{B}(\hat{p}^{\cap}, \hat{q}^{\cup}, J)$ then $\rho_{\mathrm{so}}(I) = \rho_{\mathrm{so}}(J)$.

Proof:
Let $\mathcal{B}(\hat{n}^{\cap}, \hat{m}^{\cup}, I) \cong \mathcal{B}(\hat{p}^{\cap}, \hat{q}^{\cup}, J)$, let K be a matrix for which $R_i(I) = R_i(K)$ and $C_i(K) = C_i(J)$ which exists due to Theorem 4.2. Due to Theorem 3.4 (a) and (b), $\rho_{\mathrm{so}}(I) = \rho_{\mathrm{so}}(K) = \rho_{\mathrm{so}}(J)$. □

5. Conclusions

We investigated the notions of a row and column space, known for Boolean matrices, for matrices with entries from complete residuated lattices. We showed that in the more general, non-Boolean setting, two kinds of spaces naturally appear, namely, interior- and closure- row and column spaces. We provided properties of these spaces and established links to concept-forming operators and concept lattices known from formal concept analysis. We provided connections between the Schein ranks of two matrices and their row and column spaces. Topics left for future research include further characterizations of isomorphic row and column spaces, in particular c-spaces, and in general, further investigation of the calculus of matrices over residuated lattices with the focus on results regarding matrix decompositions. A particularly interesting topic seems to be an investigation of structures related to approximate decompositions, i.e. decompositions in which matrix I is approximately equal to $A * B$.

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