APPROXIMATE MINIMIZATION OF FUZZY AUTOMATA

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The paper presents a contribution to minimization of fuzzy automata. Traditionally, the problem of minimization of fuzzy automata results directly from the problem of minimization of ordinary automata. That is, given a fuzzy automaton, describe an automaton with the minimal number of states which recognizes the same language as the given one. In this paper, we formulate a different problem. Namely, the minimal fuzzy automaton we are looking for is required to recognize a language which is similar to the language of the given fuzzy automaton to a certain degree a, such as $a = 0.9$, prescribed by a user. That is, we relax the condition of being equal to a weaker condition of being similar to degree a.

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1. Problem setting

The idea of extending ordinary automata by principles of fuzzy logic goes back to the early eage of fuzzy logic, see e.g. Santos,[10] Wee and Fu[11] for some of the early papers. Since then, many papers on fuzzy automata and their applications appeared, see Mordeson and Malik[9] for an overview, including recent approaches based on using general structures of truth degrees, instead of $[0, 1]$ equipped with fixed fuzzy logical connectives. The main motivation for studying fuzzy automata is the fact that it is quite natural to consider fuzzy languages, i.e. languages to which words belong in possibly intermediate degrees rather than just 0 (belongs) and 1 (does not belong).

One of the classical problems of finite automata is that of a minimization. Given an input automaton $\mathcal{M}$, one seeks an equivalent automaton

$\mathcal{M}'$ with as small number of states as possible. By "equivalent", one means "recognizing the same language". In all of the papers we have found, the problem of minimization of fuzzy automata is formulated essentially the same way as for the ordinary automata. That is, given a fuzzy automaton $\mathcal{M}$ and denoting by $\mathcal{L}(\mathcal{M})$ the language of $\mathcal{M}$, i.e. the fuzzy set of words recognized by $\mathcal{M}$, one is looking for a fuzzy automaton $\mathcal{M}'$ with as small number of states as possible such that $\mathcal{L}(\mathcal{M}) = \mathcal{L}(\mathcal{M}')$. Such a requirement might be considered too strong. Namely, one might require instead that $\mathcal{L}(\mathcal{M})$ and $\mathcal{L}(\mathcal{M}')$ be highly similar but not necessarily equal. With an appropriate definition of similarity of fuzzy languages, one might require that $\mathcal{L}(\mathcal{M})$ and $\mathcal{L}(\mathcal{M}')$ be similar in degree at least 0.9, for instance. Relaxing the requirement of equality of languages by replacing it with a weaker requirement of similarity (approximate equality), presents a new problem. The rationale behind is that

1. from the point of view of user's needs, an automaton recognizing approximately the same language may be acceptable,
2. with the weaker requirement of approximately equal languages, the number of states of the resulting minimal automaton may decrease compared to when we require equal languages.

The present paper presents several results regarding approximate minimization of fuzzy automata including a description of a minimal automaton which recognizes a language similar to the language of a given automaton in a given degree a or higher. Due to lack of space, we omit proofs.

2. Preliminaries

Due to lack of space, we just survey basic notions and refer to, e.g., Belohlavek,[1] Hájek[3] for further details and properties. Our basic structure of truth degrees is a complete residuated lattice, i.e., an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$; for each $a, b, c \in L$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". "Fuzzy negation" $\neg$ is defined by $\neg a = a \rightarrow 0$ for $a \in L$.

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \to b = \min(1 - a + b, 1)$), Gödel: ($a \otimes b = \min(a, b)$, $a \to b = 1$ if $a \leq b$, $a \to b = b$ else), Goguen (product): ($a \otimes b = a \cdot b$, $a \to b = 1$ if $a \leq b$, $a \to b = \frac{b}{a}$ else). Examples of finite structures: Take a finite subset $L \subseteq [0, 1]$ which is closed under Łukasiewicz or Gödel operations. If we take $L = \{0, 1\}$, this gives us a two-element Boolean algebra (structure of truth degrees of classical logic).

Given $\mathbf{L}$ which serves as a structure of truth degrees, we define usual notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A\colon U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. Given $A, B \in \mathbf{L}^U$, we define a degree $A \approx B$ to which A and B are equal by

$$A \approx B = \textstyle\bigwedge_{u \in U} \big(A(u) \leftrightarrow B(u)\big), \tag{1}$$

where $a \leftrightarrow b = (a \leftarrow b) \wedge (b \to a)$. (1) generalizes ordinary equality since $A = B$ iff $A \approx B = 1$. Described verbally, $A \approx B$ is a degree to which for each $u \in U$, u belongs to A iff u belongs to B.

3. Approximate minimization

3.1. *Fuzzy automata*

We use a definition of fuzzy automata from Belohlavek.[1] For a complete residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$, an $\mathbf{L}$-*automaton* $\mathcal{M}$ over a finite alphabet Σ is defined as a tuple $\langle Q, \Sigma, Q_I, Q_F, \delta\rangle$ of a finite set Q of states, the alphabet Σ, an $\mathbf{L}$-set Q_I in Q of initial states, an $\mathbf{L}$-set Q_F in Q of final states, and $\mathbf{L}$-relation δ between Q, Σ, and Q (for $q, q' \in Q$, $s \in \Sigma$, $\delta(q, s, q')$ is the degree to which the $\mathbf{L}$-automaton $\mathcal{M}$ can transfer from q to q' if the actual input symbol is s).

For any input word $\alpha = s_1 \dots s_n$ we set

$$\delta(q, \alpha, q') = \bigvee_{\substack{q_0, \dots, q_n \in Q \\ q_0 = q, q_n = q'}} \delta(q_0, s_1, q_1) \wedge \dots \wedge \delta(q_{n-1}, s_n, q_n),$$
$$(\mathcal{L}(\mathcal{M}))(\alpha) = \textstyle\bigvee_{q, q' \in Q} Q_I(q) \wedge \delta(q, \alpha, q') \wedge Q_F(q').$$

$\delta(q, \alpha, q')$ is the degree to which $\mathcal{M}$ can transfer from q to q' having α at input. For $\alpha = \varepsilon$, this yields

$$\delta(q, \varepsilon, q') = \begin{cases} 1 \text{ if } q = q' \\ 0 \text{ otherwise.} \end{cases}$$

The **L**-set $\mathcal{L}(\mathcal{M})$ is called the **L**-language *recognized* by $\mathcal{M}$.

The degree $Q_I(\alpha, q_0)$ to which $\mathcal{M}$ will reach the state q_0 by the input word α, and the degree $Q_F(q_0, \alpha)$ to which $\mathcal{M}$ will accept the input word α when starting from q_0 are defined as follows:

$$Q_I(\alpha, q_0) = \bigvee_{q \in Q} Q_I(q) \wedge \delta(q, \alpha, q_0), \; Q_F(q_0, \alpha) = \bigvee_{q \in Q} \delta(q_0, \alpha, q) \wedge Q_F(q).$$

Obviously,

$$Q_I(q_0) = Q_I(\varepsilon, q_0), \; Q_F(q_0) = Q_F(q_0, \varepsilon).$$

3.2. *Deterministic fuzzy automata*

We use a definition of a deterministic fuzzy automaton from Blohlavek.[2] An **L**-automaton $\mathcal{M}$ is *deterministic* if there is exactly one state $q_0 \in Q$, called the *initial state*, such that

$$Q_I(q) = \begin{cases} 1 \text{ if } q = q_0, \\ 0 \text{ otherwise,} \end{cases}$$

and for any state $q_1 \in Q$ and symbol $s \in \Sigma$ there is exactly one state $q_2 \in Q$ such that

$$\delta(q_1, s, q) = \begin{cases} 1 \text{ if } q = q_2, \\ 0 \text{ otherwise.} \end{cases}$$

For q_1, q_2 above we write $q_2 = \delta(q_1, s)$, which defines an ordinary function δ, called *transition function*, in place of the fuzzy relation δ used earlier. If $\mathcal{M}$ is deterministic with the initial state q_0 and transition function δ then we write $\mathcal{M} = \langle Q, \Sigma, q_0, Q_F, \delta \rangle$. Note that it was proved in Belohlavek[2] that if our complete residuated lattice **L** satisfies that every complete sublattice generated by a finite $L' \subseteq L$ is finite, then every **L**-automaton can be replaced by an equivalent deterministic automaton.

3.3. *The problem of approximate minimization*

Our problem can be formulated as follows. Given an **L**-automaton $\mathcal{M}$ and a similarity threshold $a \in L$, find an **L**-automaton $\mathcal{M}'$ such that

$$(\mathcal{L}(\mathcal{M}) \approx \mathcal{L}(\mathcal{M}')) \geq a, \tag{2}$$

which is the case iff for each input word α we have

$$([\mathcal{L}(\mathcal{M})](\alpha) \leftrightarrow [\mathcal{L}(\mathcal{M}')](\alpha)) \geq a,$$

cf. (1). Note that measuring similarity of languages by $\leftrightarrow$ is the technical reason we consider logical connectives on the lattice of truth degrees.

In the ordinary setting, minimization of an automaton involves removal of inaccessible states followed by minimization (e.g., factorization by an equivalence relation which represents indistinguishability of states). In our setting, (in)accessibility comes in degrees, i.e. there are degrees to which a state is (in)accessible. In Section 3.4, we show an appropriate "graded version" of a well-known fact saying that removing inaccessible states does not change the language recognized by an automaton. As to minimization, the situation is more complex in the setting of an approximate equality. We restrict ourselves to the case of deterministic fuzzy automata and present a result describing minimal fuzzy automaton $\mathcal{M}'$ for a given $\mathcal{M}$ satisfying (2). The following example shows that approximate minimization is non-trivial in that it can lead to a decrease of number of states.

Example 3.1. Let n be a positive integer and $L = [0,1]$ be equipped with the Łukasiewicz structure. Consider a deterministic **L**-automaton $\langle Q, \Sigma, q_0, Q_F, \delta\rangle$ with $Q = \{q_0, q_1, \ldots, q_n\}$, $\Sigma = \{s\}$, $Q_F(q_i) = \frac{1}{i+1}$ for $i < n$, $Q_F(q_n) = 0$, and δ defined by $\delta(q_i, s) = q_{i+1}$ for $i < n$ and $\delta(q_n, s) = q_n$. It is easy to see that the language $\mathcal{L} = \mathcal{L}(\mathcal{M})$ of $\mathcal{M}$ is given by

$$\mathcal{L}(s^i) = \begin{cases} \frac{1}{i+1} & \text{for } i < n, \\ 0 & \text{for } i \geq n. \end{cases}$$

Since for $i \neq j$ we have $Q_F(q_i) \neq Q_F(q_j)$ there does not exist a deterministic fuzzy automaton $\mathcal{M}'$ with less states than $n + 1$ recognizing the same language as $\mathcal{M}$. Therefore, the automaton is minimal in the classical sense. Consider now $a = \frac{3}{4}$. Then there exists and **L**-automaton $\mathcal{M}'$ with just two states satisfying (2). Namely, one can put $Q' = \{q'_0, q'_1\}$, $\Sigma = \{s\}$, $Q'_F(q'_0) = 1$, $Q'_F(q'_1) = \frac{1}{4}$ and define δ' by $\delta'(q'_0, s) = q'_1$, $\delta(q'_1, s) = q'_1$. Therefore, approximate minimization can, indeed, decrease the number of states of a fuzzy automaton which is minimal in the classical sense.

3.4. *Inaccessible states*

For any subset $Y \subset Q$, we can construct a new **L**-automaton $\mathcal{M}' = \langle Q', \Sigma, Q'_I, Q'_F, \delta'\rangle$ by *removing states* belonging to Y from the automaton $\mathcal{M}$. The set of states Q' of $\mathcal{M}'$ is equal to the set $Q \setminus Y$, the **L**-sets Q'_I and Q'_F are constructed by restriction of the **L**-sets Q_I, and Q_F to the set Q', and the transition **L**-relation δ' between Q', Σ, and Q' is constructed by restricting the **L**-relation δ to the set $Q' \times \Sigma \times Q'$.

For any state $q \in Q$ we define the *accessibility degree* $(\mathrm{Acc}(\mathcal{M}))(q)$ of q by

$$(\mathrm{Acc}(\mathcal{M}))(q) = \bigvee_{\alpha \in \Sigma^*} Q_I(\alpha, q). \tag{3}$$

This defines the $\mathbf{L}$-set $\mathrm{Acc}(\mathcal{M})$ of accessible states of $\mathcal{M}$.

Theorem 3.1. *If the* $\mathbf{L}$*-automaton* $\mathcal{M}'$ *results from an* $\mathbf{L}$*-automaton* $\mathcal{M}$ *by removing all states* q *such that* $(\mathrm{Acc}(\mathcal{M}))(q) \leq \neg a$ *then condition* (2) *is satisfied.* □

Corollary 3.1. *If* $\mathcal{M}'$ *results from* $\mathcal{M}$ *by removing a state* q*, then*

$$\neg \mathrm{Acc}(\mathcal{M})(q) \leq (\mathcal{L}(\mathcal{M}) \approx \mathcal{L}(\mathcal{M}')).$$

Corollary 3.1 says that if we remove a state q, then the statement "if q is inaccessible then $\mathcal{L}(\mathcal{M})$ is equal to $\mathcal{L}(\mathcal{M}')$" is true in degree 1 if interpreted in a fuzzy logic with $\mathbf{L}$ as the structure of truth degrees. Obviously, if $\mathbf{L}$ is the two-element Boolean algebra, this brings us to the realm of ordinary automata and Corollary 3.1 becomes the well known statement. Note that if $\mathcal{M}$ is deterministic then the accessibility degree of any of its states is equal to 0 or 1 only. In this case, states with accessibility degree 0 and 1 are called *inaccessible* and *accessible*, respectively. If $\mathcal{M}$ contains accessible states only, it is called *accessible*, see e.g. Hopcroft and Ullman.[4]

3.5. *Approximate minimization*

Suppose that $\mathcal{M} = \langle \mathbf{L}, \Sigma, q_0, Q_F, \delta \rangle$ is an accessible deterministic $\mathbf{L}$-automaton.

For $a \in L$ we call a set $P \subseteq Q$ to be *a set of* a*-similar states* if for any word $\alpha \in \Sigma^*$ there is a $c \in L$ such that

$$\bigwedge_{q \in P} (Q_F(q, \alpha) \leftrightarrow c) \geq a. \tag{4}$$

Let Q' be a covering of Q. Q' is called an a*-covering of* $\mathcal{M}$ if it contains only sets of a-similar states. Q' is called *invariant* if for any $q_1 \in Q'$, $s \in \Sigma$ there is a $q_2 \in Q'$ such that $\delta_s(q_1) \subseteq q_2$. Q' is called *minimal invariant* a*-covering* if it is an invariant a-covering with the minimal number of elements.

Since the partition $\{\{q\} \mid q \in Q\}$ is an invariant a-covering of $\mathcal{M}$ then there always exists a minimal invariant a-covering of $\mathcal{M}$.

Let Q' be a minimal invariant a-covering of $\mathcal{M}$. We construct a new deterministic $\mathbf{L}$-automaton $\mathcal{M}' = \langle Q', \Sigma, q'_0, Q'_F, \delta' \rangle$ as follows. We choose q'_0 to be any element of Q' containing the initial state q_0 of $\mathcal{M}$, set $Q'_F(q') = c$, where $c \in L$ satisfies $\bigwedge_{q \in q'}(Q_F(q) \leftrightarrow c) \geq a$, and, finally, set $\delta'(q'_1, s) = q'_2$ where $q'_2 \in Q'$ is any element such that $\delta_s(q'_1) \subseteq q'_2$.

Theorem 3.2 (similarity of $\mathcal{M}$ and $\mathcal{M}'$). *Automata $\mathcal{M}$ and $\mathcal{M}'$ satisfy condition* (2). $\square$

Theorem 3.3 (minimality of $\mathcal{M}'$). *Let $\mathcal{M}''$ be a deterministic* **L**-*automaton such that $\mathcal{L}(\mathcal{M}'') \approx \mathcal{L}(\mathcal{M}) \geq a$. Then $|Q''| \geq |Q'|$.* $\square$

4. Future research

Our future research will focus on finding an efficient algorithm for a construction of minimal approximately equivalent fuzzy automata and to several other problems related to minimization which are known from the ordinary case.

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