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Central points and approximation in residuated lattices

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ABSTRACT

The paper presents results on approximation in residuated lattices given that closeness is assessed by means of biresiduum. We describe central points and optimal central points of subsets of residuated lattices and examine several of their properties. In addition, we present algorithms for two problems regarding optimal approximation.

1. Introduction and preliminaries

Functions on lattices related to concepts of magnitude and distance, such as valuations and metrics, received considerable attention in lattice theory, see e.g. [4, Chap. II and X]. In this paper, we study certain problems related to closeness in complete residuated lattices, which is represented by the so-called biresiduum. Recall that a complete residuated lattice [16] is a structure $\mathbf{L} = \langle L, \otimes, \to, \wedge, \vee, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes$ (multiplication) and $\to$ (residuum) form an adjoint pair, i.e. $a \otimes b \le c$ iff $a \le b \to c$. Examples of residuated lattices are abundant in mathematics and logic [8,13]. In what follows, we use the following three well-known examples of complete residuated lattices on $L = [0, 1]$ induced by continuous t-norms [14]: the standard Łukasiewicz algebra ($a \otimes b = \max(0, a + b - 1)$, $a \to b = \min(1, 1 - a + b)$), the standard product algebra ($a \otimes b = a \cdot b$, $a \to b = 1$ if $a \le b$, and $a \to b = b/a$ otherwise), also called the standard Goguen algebra, the standard Gödel algebra ($a \otimes b = \min(a, b)$, $a \to b = 1$ if $a \le b$, and $a \to b = b$ otherwise). It is well known [1,12] that a biresiduum $\leftrightarrow$, defined in any residuated lattice by

$$a \leftrightarrow b = (a \to b) \wedge (b \to a),$$

satisfies

$$a \leftrightarrow b = 1 \quad \text{iff} \quad a = b, \tag{1}$$

$$a \leftrightarrow b = b \leftrightarrow a, \tag{2}$$

$$(a \leftrightarrow b) \otimes (b \leftrightarrow c) \le a \leftrightarrow c. \tag{3}$$

Hence, $a \leftrightarrow b$ may naturally be interpreted as an element in L representing a degree of closeness (similarity) of a and b, with $(a_1 \leftrightarrow b_1) \le (a_2 \leftrightarrow b_2)$ meaning that a_2 and b_2 are closer (more similar) to each other than a_1 and b_1. Note that (1)–(3) resemble dual versions of the axioms of a metric with a generalized triangular inequality. Indeed, for the standard

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Łukasiewicz algebra, $d_\leftrightarrow(a,b) = 1 - (a \leftrightarrow b)$ is a $[0,1]$-valued metric on $[0,1]$. This metric coincides with the usual Euclidean metric and is called the Chang distance function [6]. More generally, using $d_\leftrightarrow(a,b) = f(a \leftrightarrow b)$ one obtains a metric from a biresiduum of a continuous Archimedean t-norm with an additive generator f [14]. For the standard Gödel algebra, in which case the t-norm is not Archimedean, $d_\leftrightarrow(a,b) = 1 - (a \leftrightarrow b)$ is a $[0,1]$-valued ultrametric on $[0,1]$. General relationships between fuzzy equivalences and metric structures along this line, including metrics, ultrametrics, and in general so-called G-metrics, are found in [15]. These relationships show that similarities represented by biresidua are closely related to metric-like structures and are richer than the ordinary metrics.

Note also that residuated lattices and their generalizations, developed initially within the studies of ring ideals [16], are the main structures of truth degrees used in many-valued logic [9–11] in which biresiduum is the truth function of the logical connective of equivalence.

In this paper, we present results motivated by the following problem. Given a set of elements of a residuated lattice, what are its central points, i.e. elements which are close/similar (as much as possible or to some specified level) to every element of the set, provided closeness/similarity is assessed by means of biresiduum?

2. Preliminaries

We assume familiarity with some basic properties of residuated lattices [16] and basic concepts from tolerance relations on complete lattices [7,17]. In this section, we recall briefly what we use in the paper.

Each residuated lattice satisfies the following conditions:

$$(a \otimes b) \to c = a \to (b \to c), \tag{4}$$

$$a \otimes (a \to b) \le b, \tag{5}$$

$$a \to (a \otimes b) \ge b. \tag{6}$$

Moreover,

$$a_1 \le a_2 \text{ and } b_1 \le b_2 \quad \text{implies} \quad a_1 \otimes b_1 \le a_2 \otimes b_2 \tag{7}$$

$$\text{and} \quad a_2 \to b_1 \le a_1 \to b_2. \tag{8}$$

The following conditions are satisfied in each complete residuated lattice:

$$a \to (\textstyle\bigwedge_{b\in B} b) = \textstyle\bigwedge_{b\in B}(a \to b), \tag{9}$$

$$(\textstyle\bigvee_{b\in B} b) \to a = \textstyle\bigwedge_{b\in B}(b \to a), \tag{10}$$

$$a \to (\textstyle\bigvee_{b\in B} b) \ge \textstyle\bigvee_{b\in B}(a \to b), \tag{11}$$

$$(\textstyle\bigwedge_{b\in B} b) \to a \ge \textstyle\bigvee_{b\in B}(b \to a), \tag{12}$$

$$a \otimes (\textstyle\bigvee_{b\in B} b) = \textstyle\bigvee_{b\in B}(a \otimes b), \tag{13}$$

$$a \otimes (\textstyle\bigwedge_{b\in B} b) \le \textstyle\bigwedge_{b\in B}(a \otimes b). \tag{14}$$

A *tolerance* is a reflexive and symmetric binary relation. A *block* of a tolerance T on a set U is a subset B of U for which $B \times B \subseteq T$. A maximal block of T is a block B of T which is maximal with respect to set inclusion. A collection of maximal tolerance blocks of T is denoted by U/T and forms a covering of U. A *class* of T given by $u \in U$ is the set $[u]_T = \{v \in U \mid u\,T\,v\}$. Clearly, if T is an equivalence, maximal blocks as well as classes of T are just the equivalence classes of T.

Throughout the paper, $\mathbf{L}$ denotes a complete residuated lattice and e an element of its support set L. By $\approx_e$, we denote the tolerance on L defined by

$$a \approx_e b \quad \text{iff} \quad a \leftrightarrow b \ge e.$$

3. Central points, central sets, and maximal blocks

For $B \subseteq L$, we set

$$C_e(B) = \{c \in L \mid \text{for each } b \in B, c \leftrightarrow b \ge e\}. \tag{15}$$

We call $C_e(B)$ the *e-central set of B* and its elements *e-central points of B*.

Lemma 3.1. *$c \in C_e(B)$ iff $(c \to \bigwedge B) \wedge (\bigvee B \to c) \ge e$.*

Proof. (9) and (10) yield $c \to (\bigwedge_{b\in B} b) = \bigwedge_{b\in B}(c \to b)$ and $(\bigvee_{b\in B} b) \to c = \bigwedge_{b\in B}(b \to c)$. □

Denoting by $[p,q]$ the interval $\{x \in L \mid p \le x \le q\} \subseteq L$, we get:

Theorem 3.2. $C_e(B) = [e \otimes \bigvee B, e \to \bigwedge B]$.

Proof. By adjointness, $e \le c \to \bigwedge B$ is equivalent to $c \le e \to \bigwedge B$ and $e \le \bigvee B \to c$ is equivalent to $e \otimes \bigvee B \le c$. The assertion thus follows from Lemma 3.1. □

For $c \in L$, we let

$$B_e(c) = \{b \in L \mid c \leftrightarrow b \ge e\} \tag{16}$$

and call $B_e(c)$ the *closed ball with center c and radius e*. $B_e(c)$ is called *maximal* if there is no $c' \ne c$ such that $B_e(c) \subset B_e(c')$.

Example 3.3. In the standard Łukasiewicz algebra, $B_e(c) = [c - (1 - e), c + (1 - e)] \cap [0, 1]$ (the closed ball with center c and radius $1 - e$ w.r.t. the standard Euclidean metric on $[0, 1]$). The closed ball $B_{0.5}(0) = [0, 0.5]$ is not maximal, since $B_{0.5}(0) \subset B_{0.5}(0.5) = [0, 1]$.

Note that a closed ball $B_e(c)$ is just the class of tolerance $\approx_e$ determined by c. In addition, $C_e(B) = \bigcap_{c \in B} B_e(c)$ and $B_e(c) = C_e(\{c\})$. The following assertion thus follows from Theorem 3.2.

Theorem 3.4. $B_e(c) = [e \otimes c, e \to c]$.

Let for $a \in L$,

$$a_e = e \otimes a, \quad a^e = e \to a, \quad [a]_e = [a_e, (a_e)^e], \quad [a]^e = [(a^e)_e, a^e].$$

Using (9), (10), (11), and (12), it is easy to verify that $\approx_e$ is a complete tolerance relation on the complete lattice $\langle L, \le \rangle$, i.e. $a_i \approx_e b_i$ $(i \in I)$ implies $\bigwedge_{i \in I} a_i \approx_e \bigwedge_{i \in I} b_i$ and $\bigvee_{i \in I} a_i \approx_e \bigvee_{i \in I} b_i$. Since Theorem 3.4 says that a_e and a^e are the least and the greatest elements of L which are $\approx_e$-related to a, [17] yields the following description of maximal blocks of $\approx_e$.

Theorem 3.5. $L/\!\approx_e = \{[a]_e \mid a \in L\} = \{[a]^e \mid a \in L\}$.

The next lemma shows further properties of balls.

Lemma 3.6. $[c]^e \cap [c]_e = [(c^e)_e, (c_e)^e]$ *is the set of all d for which* $B_e(d) \supseteq B_e(c)$. *Moreover,*

$$B_e((c^e)_e) = [e \otimes e \otimes (e \to c), e \to c], \tag{17}$$

$$B_e((c_e)^e) = [e \otimes c, e \to (e \to (e \otimes c))]. \tag{18}$$

Proof. According to Theorem 3.4, $B_e(d) \supseteq B_e(c)$ is equivalent to $e \otimes d \le e \otimes c$ and $e \to c \le e \to d$. By adjointness, the first inequality is equivalent to $d \le e \to (e \otimes c) = (c_e)^e$, the second one to $(c^e)_e = e \otimes (e \to c) \le d$, proving the first part. (17) and (18) easily follow from Theorem 3.4, and from $e \to c = e \to (e \otimes (e \to c))$ and $e \otimes c = e \otimes (e \to (e \otimes c))$, which themselves follow from (5), (6), (7), and (8). □

In general, it may happen that if $B_e(c)$ is not maximal, there exists a maximal ball $B_e(d)$ such that $(c^e)_e < d < (c_e)^e$ (i.e. $B_e(d) \supseteq B_e(c)$), $e \otimes d < e \otimes c$ (smallest element of $B_e(d)$ < smallest element of $B_e(c)$), and $e \to d > e \to c$ (largest element of $B_e(d)$ > largest element of $B_e(c)$). This is shown in the following example.

Example 3.7. Let **L** be the direct product of two standard Łukasiewicz algebras, $e = \langle 0.6, 0.6 \rangle$, $c = \langle 0.2, 0.8 \rangle$, and $d = \langle 0.4, 0.6 \rangle$. Then $(c^e)_e = \langle 0.2, 0.6 \rangle$, $(c_e)^e = \langle 0.4, 0.8 \rangle$, $B_e(c) = [\langle 0, 0.4 \rangle, \langle 0.6, 1 \rangle]$, $B_e(d) = [\langle 0, 0.2 \rangle, \langle 0.8, 1 \rangle]$. The situation is depicted in Fig. 1.

This, however, does not happen in linearly ordered complete residuated lattices where $(c^e)_e$ and $(c_e)^e$ play an important role in describing the maximal balls containing $B_e(c)$.

Theorem 3.8. *Let* **L** *be linearly ordered and assume* $B_e(d) \supseteq B_e(c)$. *Then*

1. $e \otimes d = e \otimes c$ *or* $e \to d = e \to c$, *i.e.* $B_e(d)$ *and* $B_e(c)$ *have the same lower bound or the same upper bound;*
2. *if* $B_e(d)$ *is maximal then* $B_e(d) = B_e((c^e)_e)$ *in which case* $d \in [e \otimes (e \to c), e \to (e \otimes e \otimes (e \to c))]$, *or* $B_e(d) = B_e((c_e)^e)$ *in which case* $d \in [e \otimes (e \to (e \to (e \otimes c))), e \to (e \otimes c)]$;
3. $B_e((c^e)_e)$ *is maximal or* $B_e((c^e)_e) = B_e(c)$, $B_e((c_e)^e)$ *is maximal or* $B_e((c_e)^e) = B_e(c)$.

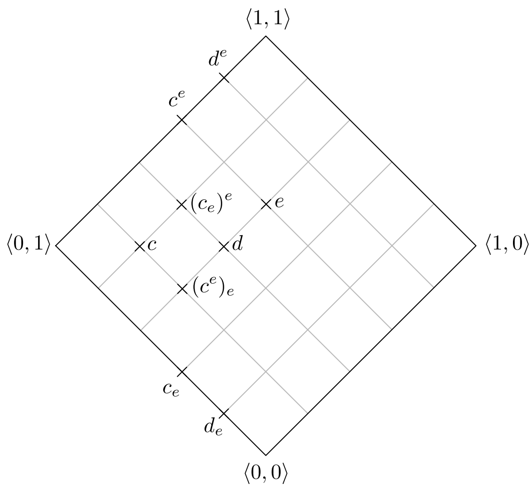

Fig. 1. Residuated lattice **L** from Example 3.7. We have $d_e < c_e$, $d^e > c^e$ and $B_e(d) \supset B_e(c)$.

Proof. 1. Since $B_e(d) \supseteq B_e(c)$, $e \otimes d \le e \otimes c$ and $e \to d \ge e \to c$ by Theorem 3.4. Due to linearity, (7), and (8), $e \otimes d < e \otimes c$ implies $d < c$ and $e \to d > e \to c$ implies $d > c$, hence the claim.

2. According to 1., $e \otimes d = e \otimes c$ or $e \to d = e \to c$. Assume $e \otimes d = e \otimes c$ (the proof is similar for $e \to d = e \to c$). Lemma 3.6 implies $d \le (c_e)^e$, hence (8) $e \to d \le e \to (c_e)^e = e \to (e \to (e \otimes c))$. Therefore, $B_e(d) = [e \otimes c, e \to d] \subseteq [e \otimes c, e \to (e \to (e \otimes c))] = B_e((c_e)^e)$. Maximality of $B_e(d)$ now implies $B_e(d) = B_e((c_e)^e)$. The fact that $d \in [e \otimes (e \to (e \to (e \otimes c))), e \to (e \otimes c)]$ follows directly from Lemma 3.6 and from $(((c_e)^e)_e)^e = (c_e)^e$.

3. Assume $B_e((c^e)_e)$ is not maximal. Then $B_e(c) \subseteq B_e((c^e)_e) \subset B_e(d)$ for some maximal $B_e(d)$. According to 2., $B_e(d) = B_e((c_e)^e)$. From (17) and (18) we get $B_e((c^e)_e) = [e \otimes e \otimes (e \to c), e \to c] \subseteq [e \otimes c, e \to (e \to (e \otimes c))] = B_e((c_e)^e)$, i.e. $e \otimes c \le e \otimes e \otimes (e \to c)$. Since $e \otimes c \ge e \otimes e \otimes (e \to c)$ is always the case (5), we conclude $e \otimes c = e \otimes e \otimes (e \to c)$ which means $B_e((c^e)_e) = [e \otimes c, e \to c] = B_e(c)$. □

We now turn our attention to the relationship between closed balls with radius e and blocks of the tolerance $\approx_{e^2}$ (where $e^2 = e \otimes e$) and show that maximal closed balls with radius e coincide with maximal blocks of this tolerance. It is easy to check that $B \subseteq B'$ implies $C_e(B) \supseteq C_e(B')$ and that $B \subseteq C_e(C_e(B))$ for every $B \subseteq L$. As a consequence, taking into account that $B_e(c) = C_e(\{c\})$, we get the following lemma.

Lemma 3.9. 1. *For any $B \subseteq L$, $B \subseteq B_e(c)$ for every $c \in C_e(B)$. For any $c \in L$, $c \in C_e(B_e(c))$.*

2. *$C_e(C_e(B))$ is the largest subset of L which has the same e-central points as B.*

Lemma 3.10. *$C_e(B)$ is non-empty if and only if B is a block of $\approx_{e^2}$. Hence, $B_e(c)$ is a block of $\approx_{e^2}$ for each $c \in L$.*

Proof. According to Theorem 3.2, $C_e(B)$ is non-empty iff $e \otimes \bigvee B \le e \to \bigwedge B$ which is equivalent to $e \otimes e \le \bigvee B \to \bigwedge B$. Since $\bigvee B \to \bigwedge B = \bigwedge_{a,b \in B} a \to b = \bigwedge_{a,b \in B} a \leftrightarrow b$ due to (9) and (10), we conclude that $C_e(B)$ is non-empty iff $e \otimes e \le a \leftrightarrow b$ for every $a, b \in B$, proving the claim. As $c \in B_e(c)$, $B_e(c)$ is non-empty and since $B_e(c) = C_e(\{c\})$, $B_e(c)$ is a block of $\approx_{e^2}$. □

Theorem 3.11. *The following conditions are equivalent.*

1. *B is a maximal closed ball with radius e.*
2. *B is a maximal block of $\approx_{e^2}$.*
3. *$C_e(B) \ne \emptyset$ and $C_e(B) = \{c \in L \mid B = B_e(c)\}$.*

Proof. "1. ⇒ 2.": According to Lemma 3.10, a maximal closed ball $B_e(c)$ is a block of $\approx_{e^2}$. Therefore, $B_e(c) \subseteq B$ for some maximal block B of $\approx_{e^2}$. From Lemma 3.10 it follows that $C_e(B)$ is non-empty. Let thus $c' \in C_e(B)$. Lemma 3.9 implies $B \subseteq B_e(c')$, hence $B_e(c) \subseteq B \subseteq B_e(c')$. Maximality of $B_e(c)$ thus yields $B_e(c) = B$.

"2. ⇒ 3.": $C_e(B)$ is non-empty due to Lemma 3.10. On the one hand, if $c \in C_e(B)$ then, due to Lemma 3.10 and Lemma 3.9, $B_e(c)$ is a block of $\approx_{e^2}$ and $B \subseteq B_e(c)$. Since B is maximal, we conclude $B = B_e(c)$. On the other hand, if $B = B_e(c)$ then clearly $c \in C_e(B)$.

"3. ⇒ 1.": Suppose $B = B_e(c)$ is not maximal. Then $B \subset B_e(c')$ for some c'. As a consequence, $c' \in C_e(B)$, i.e. $B = B_e(c')$, a contradiction. □

4. Optimal central points

An *optimal central point* of $B \subseteq L$ is an element $c \in L$ such that for every $d \in L$,

$$\bigwedge_{a\in B}(a \leftrightarrow d) \leq \bigwedge_{a\in B}(a \leftrightarrow c). \tag{19}$$

That is, the infimum of similarity degrees $a \leftrightarrow c$ of $a \in L$ to c is the largest possible. Since for any d, $\bigwedge_{a\in B}(a \leftrightarrow d) = (d \rightarrow \bigwedge B) \wedge (\bigvee B \rightarrow d)$ (see, for example, the proof of Lemma 3.1), c is an optimal central point iff for every $d \in L$,

$$\left(d \rightarrow \bigwedge B\right) \wedge \left(\bigvee B \rightarrow d\right) \leq \left(c \rightarrow \bigwedge B\right) \wedge \left(\bigvee B \rightarrow c\right). \tag{20}$$

We say that $e \in L$ is

- an *admissible radius* of B if $C_e(B) \neq \emptyset$;
- a *radius of B for $a \in L$* if e is the largest element for which $a \in C_e(B)$.

One can see that the radius of B for a equals $\bigwedge_{a'\in B}(a' \leftrightarrow a)$.

Theorem 4.1. *The following three conditions are equivalent for any $B \subseteq L$.*

1. *The set of all optimal central points of B is non-empty and e is the radius of B for some optimal central point c of B.*
2. *The set of all optimal central points of B is non-empty and e is the radius of B for any of the optimal central points.*
3. *e is the largest admissible radius of B.*

Any of conditions 1., 2., *and* 3. *implies condition* 4.

4. *The set of all optimal central points is equal to $C_e(B)$.*

Proof. "1. $\Rightarrow$ 2.": (19) implies that the radii of B for any two optimal central points c_1 and c_2 are equal.

"2. $\Rightarrow$ 3.": Clearly, e is an admissible radius of B. If e' is an admissible radius of B then for any $d \in C_{e'}(B)$, we have $e' \leq \bigwedge_{a\in B}(a \leftrightarrow d)$. For any optimal central point c of B, the assumption yields $\bigwedge_{a\in B}(a \leftrightarrow c) = e$. Therefore, (19) implies $\bigwedge_{a\in B}(a \leftrightarrow d) \leq e$, whence $e' \leq e$, proving 3.

"3. $\Rightarrow$ 1.": For $c \in C_e(B)$, $e \leq \bigwedge_{a\in B}(a \leftrightarrow c)$. On the other hand, since $\bigwedge_{a\in B}(a \leftrightarrow c)$ is an admissible radius (the radius of B for c), we have $\bigwedge_{a\in B}(a \leftrightarrow c) \leq e$, whence $\bigwedge_{a\in B}(a \leftrightarrow c) = e$. Since for any d, $\bigwedge_{a\in B}(a \leftrightarrow d)$ is an admissible radius, we get $\bigwedge_{a\in B}(a \leftrightarrow d) \leq e = \bigwedge_{a\in B}(a \leftrightarrow c)$. Therefore, c is an optimal central point of B, proving 1.

"2. $\Rightarrow$ 4.": Clearly, every optimal central point of B is in $C_e(B)$. If d is not optimal then $\bigwedge_{a\in B}(a \leftrightarrow d) < e$ and hence $d \notin C_e(B)$. $\square$

Remark 4.2. A. Note that e being the largest admissible radius of B is equivalent to the following condition:

$$\text{for every } c', e' \in L : \; B \subseteq B_{e'}(c') \text{ implies } e' \leq e. \tag{21}$$

Indeed, if (21) is the case and e' is an admissible radius of B then $\emptyset \neq C_{e'}(B)$ and thus for $c' \in C_{e'}(B)$ we have $B \subseteq B_{e'}(c')$ and hence $e' \leq e$ by (21). On the other hand, if e is the largest admissible radius of B and $B \subseteq B_{e'}(c')$ then $c' \in C_{e'}(B)$, i.e. e' is an admissible radius of B, whence $e' \leq e$.

B. Condition 4. of Theorem 4.1 does not imply conditions 1., 2., and 3. Indeed, consider the standard product algebra on $L = [0, 1]$. The set of optimal central points of $B = \{0\}$ equals B and $C_a(B) = B$ for every $a \in (0, 1]$. Thus, for $e = 0.5$ condition 3. does not hold, but condition 4. holds.

Theorem 4.3. *If e is the largest admissible radius of B then e is the largest admissible radius of $C_e(C_e(B))$. $C_e(C_e(B))$ is the largest subset of L which has the same set of optimal central points as B and for which e is an admissible radius.*

Proof. According to 2. of Lemma 3.9, $C_e(C_e(B))$ is the largest subset whose set of e-central points is $C_e(B)$. Hence, since $C_e(B) \neq \emptyset$, e is an admissible radius of $C_e(C_e(B))$. If e' is an admissible radius of $C_e(C_e(B))$ then since $B \subseteq C_e(C_e(B))$, we get $\emptyset \neq C_{e'}(C_e(C_e(B))) \subseteq C_{e'}(B)$, i.e. e' is an admissible radius of B, whence $e' \leq e$. Thus, e is the largest admissible radius of $C_e(C_e(B))$ and Theorem 4.1 implies that $C_e(C_e(B))$ has the same optimal central points as B. Let B' be another set with the same optimal central points as B for which e is an admissible radius and let e' be the largest admissible radius of B'. Then $C_e(B) = C_{e'}(B')$ and, since $e \leq e'$, $C_{e'}(B') \subseteq C_e(B')$. Hence $C_e(B) \subseteq C_e(B')$ from which we get $C_e(C_e(B)) \supseteq C_e(C_e(B')) \supseteq B'$ by Lemma 3.9, completing the proof. $\square$

Lemma 4.4. 1. *e is an admissible radius of B iff $e \otimes e \leq \bigvee B \rightarrow \bigwedge B$.*

2. *For any $a \in L$, $e = a \wedge (a \rightarrow (\bigvee B \rightarrow \bigwedge B))$ is an admissible radius of B.*

3. *e is an admissible radius of B iff* $e = e \wedge (e \to (\bigvee B \to \bigwedge B))$.

4. $$R = \{a \wedge (a \to (\bigvee B \to \bigwedge B)) \mid a \in L\} \tag{22}$$

is the set of all admissible radii of B.

Proof. 1. According to Theorem 3.2, $C_e(B) \neq \emptyset$ iff $[e \otimes \bigvee B, e \to \bigwedge B] \neq \emptyset$ iff $e \otimes \bigvee B \leq e \to \bigwedge B$ iff $e \otimes e \leq \bigvee B \to \bigwedge B$.

2. Putting $d = \bigvee B \to \bigwedge B$, we have by (7) and (5), $(a \wedge (a \to d)) \otimes (a \wedge (a \to d)) \leq a \otimes (a \to d) \leq d$, hence the claim follows from 1.

3. From 1. and adjointness we get that e is an admissible radius of B iff $e \leq e \to (\bigvee B \to \bigwedge B)$, which is equivalent to $e = e \wedge (e \to (\bigvee B \to \bigwedge B))$.

4. A consequence of 2. and 3. □

Theorem 4.5 *(Optimal central points). B has optimal central points if and only if the set R from (22) has a largest element. This element is the largest admissible radius e of B and the set of optimal central points of B equals* $C_e(B)$.

Proof. Follows directly from 4. of Lemma 4.4 and Theorem 4.1. □

Remark 4.6. The following observations concern the relationship between optimal central points and centers of maximal balls.

A. Let B have optimal central points and let e be the corresponding largest admissible radius of B. Then for some optimal central point c of B, $B_e(c)$ is a maximal ball with radius e. Indeed, in this case $C_e(B)$ is the set of optimal central points and for any maximal ball $B_e(c) \supseteq B$ (such maximal balls exist because for any $d \in C_e(B)$, $B_e(d) \supseteq B$ is contained in some maximal $B_e(c)$), we have $c \in C_e(B_e(c)) \subseteq C_e(B)$.

B. However, it may be the case that for an optimal central point c of $B = B_e(c)$ with the largest admissible radius e, $B_e(c)$ is not a maximal ball with radius e. Namely, consider the Łukasiewicz operations on $L = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$, and let $c = \frac{3}{4}$, $e = \frac{2}{4}$. Then c is an optimal central point of $B_e(c) = \{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$ and e is the largest admissible radius. However, $B_e(c)$ is not maximal since $B_e(c) \subset B_e(\frac{2}{4}) = L$.

C. Neither is it true that if $B_e(c)$ is a maximal ball with radius e then c is an optimal central point of $B_e(c)$. Consider the complete residuated lattice given by chain $L = 0 < a < b < 1$ equipped with the following operations.

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

$B_a(1) = \{a, b, 1\}$ is a maximal ball since $B_a(1) = B_a(b) = B_a(a)$ and $B_a(0) = \{0\}$. However, 1 is not an optimal central point of $B_a(1)$. Namely, the only optimal central point of $B_a(1)$ is b because $\bigwedge_{x \in B_a(1)}(x \leftrightarrow 1) = \bigwedge_{x \in B_a(1)}(x \leftrightarrow a) = a < b = \bigwedge_{x \in B_a(1)}(x \leftrightarrow b)$.

D. The example from C. also shows that if $B_e(c)$ is a maximal ball with radius e and c is an optimal central point of $B_e(c)$, e need not be the largest admissible radius of $B_e(c)$. Namely, for $c = b$ and $e = a$, the largest admissible radius of $B_e(c) = \{a, b, 1\}$ is b which is larger than e.

Example 4.7. There exists a complete residuated lattice $\mathbf{L}$ and subset $B \subseteq L$ without optimal central points. Let $L = \{0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, 1\}$ be equipped with the ordering specified by the Hasse diagram in Fig. 2 and operations $\otimes$ and $\to$ on L defined as follows.

$\otimes$	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	1
0	0	0	0	0	0	0	0	0	0
a_1	0	0	0	0	0	0	0	0	a_1
a_2	0	0	0	0	0	0	0	0	a_2
a_3	0	0	0	0	0	0	0	0	a_3
a_4	0	0	0	0	0	0	0	0	a_4
a_5	0	0	0	0	0	a_1	a_4	a_4	a_5
a_6	0	0	0	0	0	a_4	a_2	a_4	a_6
a_7	0	0	0	0	0	a_4	a_4	a_4	a_7
1	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	1

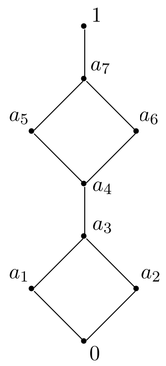

Fig. 2. Ordered set **L** from Example 4.7.

$\to$	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	1
0	1	1	1	1	1	1	1	1	1
a_1	a_7	1	a_7	1	1	1	1	1	1
a_2	a_7	a_7	1	1	1	1	1	1	1
a_3	a_7	a_7	a_7	1	1	1	1	1	1
a_4	a_7	a_7	a_7	a_7	1	1	1	1	1
a_5	a_4	a_5	a_4	a_5	a_7	1	a_7	1	1
a_6	a_4	a_4	a_6	a_6	a_7	a_7	1	1	1
a_7	a_4	a_4	a_4	a_4	a_7	a_7	a_7	1	1
1	0	a_1	a_2	a_3	a_4	a_5	a_6	a_7	1

Set $B = [a_3, 1]$. The set R from (22) of all admissible radii of B is equal to $\{0, a_1, a_2, a_3, a_4, a_5, a_6\}$ and does not have a greatest element. Therefore, there is no optimal radius and, consequently, no optimal central point of B.

As the next theorem shows, every set in a linearly ordered complete residuated lattice has optimal central points.

Theorem 4.8. *Let* **L** *be linearly ordered complete residuated lattice, $B \subseteq L$. The set R of admissible radii of B is closed under suprema. Hence, R has a largest element, the largest admissible radius of B, and B has optimal central points.*

Proof. Let $\{e_i \mid i \in I\} \subseteq R$. By 1. of Lemma 4.4, $e_i \otimes e_i \leq \bigvee B \to \bigwedge B$ for every $i \in I$. Since **L** is linearly ordered, $e_i \leq e_j$ or $e_j \leq e_i$ and thus, by (7), either $e_i \otimes e_j \leq e_i \otimes e_i$, or $e_i \otimes e_j \leq e_j \otimes e_j$. Thus, $e_i \otimes e_j \leq (e_i \otimes e_i) \vee (e_j \otimes e_j)$ for every $i, j \in I$. Therefore (13), $(\bigvee_{i \in I} e_i) \otimes (\bigvee_{i \in I} e_i) = \bigvee_{i,j \in I} e_i \otimes e_j = \bigvee_{i \in I} e_i \otimes e_i \leq \bigvee B \to \bigwedge B$, proving $\bigvee_{i \in I} e_i \in R$, i.e. R is closed under suprema. Since $0 \in R$, R is nonempty and hence has a largest element. □

Example 4.9. We show that the requirement of completeness of **L** is essential even in the case **L** is linearly ordered. Let **C** be the Chang MV-algebra [5], i.e. a residuated lattice with the support set C consisting of elements $1 = a_0 > a_1 > a_2 > a_3 > \ldots > \ldots > b_3 > b_2 > b_1 > b_0 = 0$ with product and residuum given by

$$
\begin{aligned}
a_m \otimes a_n &= a_{m+n}, & a_m \otimes b_n &= b_{\max(0,n-m)}, & b_m \otimes b_n &= 0,\\
a_m \to a_n &= a_{\max(0,n-m)}, & a_m \to b_n &= b_{m+n}, & &\\
b_m \to a_n &= 1, & b_m \to b_n &= a_{\max(0,m-n)}. & &
\end{aligned}
$$

The set R of admissible radii of C is $\{0, b_1, b_2, \ldots\}$ and does not have a greatest element. Thus, there is no largest admissible radius and, consequently, C does not have optimal central points. Notice that **C** is not complete, since neither the supremum $\bigvee\{0, b_1, b_2, \ldots\}$ nor the infimum $\bigwedge\{1, a_1, a_2, \ldots\}$ exist.

Next we derive an explicit description of largest admissible radii and optimal central points for complete residuated lattices with square roots [12]: A complete residuated lattice **L** has square roots if there is a function $\sqrt{\ }: L \to L$ satisfying

$$\sqrt{a} \otimes \sqrt{a} = a, \tag{23}$$

$$b \otimes b \leq a \quad \text{implies} \quad b \leq \sqrt{a}, \tag{24}$$

for every $a, b \in L$. One easily checks that such function is unique if it exists. As an example, the standard Łukasiewicz, product, and Gödel algebras on $[0, 1]$ have square roots. These are given by

$$\sqrt{a} = \frac{a+1}{2} \quad \text{for Łukasiewicz},$$
$$\sqrt{a} = \text{ordinary number-theoretic square root of } a \text{ for product},$$
$$\sqrt{a} = a \quad \text{for Gödel}.$$

Lemma 4.10. *If* **L** *has square roots then it satisfies*

$$\sqrt{a} \otimes (\sqrt{a} \to a) = a. \tag{25}$$

Proof. (5) yields "≤". "≥" follows from monotony of ⊗ (7) and from $\sqrt{a} \to a \geq \sqrt{a}$, which holds true by adjointness and (23). □

Theorem 4.11. *If* **L** *has square roots then every subset $B \subseteq L$ has optimal central points. For the corresponding largest admissible radius e of B it holds*

$$e = \sqrt{\bigvee B \to \bigwedge B}. \tag{26}$$

Proof. According to 1. of Lemma 4.4, (23), and (24), e is the largest admissible radius of B. The rest follows from Theorem 4.1. □

In general, the existence of optimal central points does not imply the existence of square roots. As an example, for $L = \{0, 0.5, 1\}$ equipped with the Łukasiewicz operations, every $B \subseteq L$ has optimal central points. However, 0.5 does not have a square root.

Theorem 4.12. **L** *has square roots iff for every $a \in L$, $[a, 1]$ has an optimal central point such that for the corresponding admissible radius e we have $e \otimes e = a$.*

Proof. If **L** has square roots then according to Theorem 4.11, the largest admissible radius of $[a, 1]$ is $e = \sqrt{1 \to a} = \sqrt{a}$ and we have $\sqrt{a} \otimes \sqrt{a} = a$. Conversely, it follows from the definitions and Theorem 4.1 that the admissible radius corresponding to an optimal central point of $[a, 1]$ is the square root of a. □

Corollary 4.13. *If for every $a \in L$, $[a, 1]$ has an optimal central point such that for the corresponding admissible radius e we have $e \otimes e = a$, then every $B \subseteq L$ has optimal central points.*

In the last part of this section we discuss uniqueness of central points. We say $c \in L$ is a *center of* $B \subseteq L$ if c is a unique optimal central point of B, i.e. if for the largest admissible radius e of B (Theorem 4.5) it holds $C_e(B) = \{c\}$. By Theorem 3.2 we have

$$c = e \to \bigwedge B = e \otimes \bigvee B. \tag{27}$$

A complete residuated lattice is said to have *strong square roots* if it has square roots and satisfies

$$\sqrt{a} = \sqrt{a} \to a. \tag{28}$$

The standard product and Gödel algebras do not have strong square roots. Indeed, in these algebras we have $\sqrt{0} = 0$, but $0 \to 0 = 1$. For the same reason, complete Boolean algebras do not have strong square roots either. It can be verified easily that the standard Łukasiewicz algebra has strong square roots.

Theorem 4.14. *If* **L** *has square roots then* (28) *is equivalent to*

$$b \otimes \sqrt{a} = a \quad \text{implies} \quad b \leq \sqrt{a}. \tag{29}$$

Proof. If $b \otimes \sqrt{a} = a$ then adjointness and (28) yield $b \leq \sqrt{a} \to a = a$. Thus, (28) implies (29). Now suppose (29) holds. As $(\sqrt{a} \to a) \otimes \sqrt{a} = a$ (Lemma 4.10), we have $\sqrt{a} \to a \leq \sqrt{a}$ by (29). The converse inequality, $\sqrt{a} \to a \geq \sqrt{a}$, follows by adjointness from (23). □

Recall that a residuated lattice is called *divisible* if it satisfies

$$a \wedge b = a \otimes (a \to b).$$

Divisible residuated lattices form an important subclass of residuated lattices. For $L = [0, 1]$, **L** is divisible iff ⊗ is continuous as a function $[0, 1]^2 \to [0, 1]$ [12]. Thus, the standard Łukasiewicz, Gödel and product algebras are all divisible.

Lemma 4.15. *If* **L** *is divisible then for any function* $\sqrt{\ }: L \to L$, (28) *implies* (23).

Proof. Due to (8), $\sqrt{a} \leq 1$ implies $1 \to a \leq \sqrt{a} \to a$ and hence $a = 1 \to a$ and (28) imply $a \leq \sqrt{a}$. Applying divisibility, $\sqrt{a} \otimes \sqrt{a} = \sqrt{a} \otimes (\sqrt{a} \to a) = \sqrt{a} \wedge a = a$. □

Theorem 4.16. *The following two conditions are equivalent if* **L** *is divisible.*

1. *Each nonempty subset $B \subseteq L$ has a center.*
2. **L** *has strong square roots.*

Proof. "1. ⇒ 2.": Let $a \in L$ and $B = \{a, 1\}$. We show that the largest admissible radius e of B is the strong square root of a. By (27), $e = e \otimes 1 = e \to b$, proving (28). (23) now follows from Lemma 4.15. Let $b \otimes b \leq a$. By 1. of Lemma 4.4, b is an admissible radius of B. Hence $b \leq e$, proving (24).

"2. ⇒ 1.": Let $\emptyset \neq B \subseteq L$. By Theorem 4.11, there exists the largest admissible radius e of B. Let $B' = C_e(C_e(B))$. Theorem 4.3 implies that e is the largest admissible radius of B' and $C_e(B') = C_e(B)$. It hence suffices to prove that B' has a unique e-central point. By Theorem 3.2, $\bigvee B' = e \to (e \otimes \bigvee B)$. Thus, by the same reasoning as at the end of the proof of Lemma 3.6,

$$e \to (e \otimes \bigvee B') = e \to (e \otimes (e \to (e \otimes \bigvee B))) = e \to (e \otimes \bigvee B) = \bigvee B'. \tag{30}$$

According to Theorem 4.11, $e = \sqrt{\bigvee B' \to \bigwedge B'}$. By divisibility and (23), $e \otimes e \otimes \bigvee B' = (\bigvee B' \to \bigwedge B') \otimes \bigvee B' = \bigwedge B' \wedge \bigvee B' = \bigwedge B'$, whence $\bigwedge B' \leq e \otimes \bigvee B'$. Hence, isotony of $\to$ in the second argument and (30) yield

$$e \to \bigwedge B' \leq e \to (e \otimes \bigvee B') = \bigvee B'. \tag{31}$$

Using the above fact $e = \sqrt{\bigvee B' \to \bigwedge B'}$, (28), (4), divisibility, and (31) we obtain $\bigvee B' \otimes e = \bigvee B' \otimes (e \to (\bigvee B' \to \bigwedge B')) = \bigvee B' \otimes (\bigvee B' \to (e \to \bigwedge B')) = \bigvee B' \wedge (e \to \bigwedge B') = e \to \bigwedge B'$. Thus the assertion follows from Theorem 3.2. □

5. Optimal approximation algorithms

We now consider the following type of problems. Given a set $M \subseteq L$, find a small set $K \subseteq L$ which approximates M. The *degree* $\text{appr}(M, K)$ *to which* $M \subseteq L$ *is approximated by* $K \subseteq L$ is defined by

$$\text{appr}(M, K) = \textstyle\bigwedge_{a \in M} \bigvee_{b \in K} (a \leftrightarrow b). \tag{32}$$

$\text{appr}(M, K)$ can be seen as a truth degree of "for every $a \in M$ there is $b \in K$ such that a and b are similar". Clearly, $\text{appr}(M, K) = 1$ for $M \subseteq K$, and $K_1 \subseteq K_2$ implies $\text{appr}(M, K_1) \leq \text{appr}(M, K_2)$. Consider the following problems.

Problem 1. Given a finite $M \subseteq L$ and a threshold $e \in L$, find $K \subseteq L$ such that

(1) K approximates M to degree at least e, i.e.

$$\text{appr}(M, K) \geq e, \tag{33}$$

(2) there is no K' with $|K'| < |K|$ for which $\text{appr}(M, K') \geq e$.

Problem 2. Given a finite $M \subseteq L$ and a threshold $e \in L$, find $K \subseteq L$ satisfying (1) and (2) of Problem 1, and

(3) For any K' with $|K'| = |K|$,

$$\text{appr}(M, K) \geq \text{appr}(M, K'), \tag{34}$$

i.e. among the sets with $|K|$ elements, K provides the best approximation of M.

In the rest of this section, we assume that the complete residuated lattice **L** is linearly ordered, i.e. $a \leq b$ or $b \leq a$ for every $a, b \in L$. The following theorem provides a universal description of sets K satisfying (33).

Theorem 5.1. *Let* **L** *be linearly ordered.*

1. *Let $\Omega \subseteq 2^L$ be a covering of $M \subseteq L$ and $\varphi: \Omega \to L$ a mapping assigning to each $B \in \Omega$ an e-central point of B: $\varphi(B) \in C_e(B)$. Then Ω consists of blocks of $\approx_{e^2}$ and $K = \varphi(\Omega)$ satisfies* (33).

2. *If a finite $K \subseteq L$ satisfies (33) then $\Omega = \{B_e(b) \mid b \in K\} \subseteq 2^L$ is a set of blocks of $\approx_{e^2}$ that forms a covering of M. Moreover $\varphi\colon \Omega \to L$ defined by $\varphi(B_e(b)) = b$ satisfies $\varphi(B) \in C_e(B)$.*

Proof. 1. Since $C_e(B) \neq \emptyset$, the first assertion follows from Lemma 3.10. Since Ω is a covering of M, for every $a \in M$ there exists $B \in \Omega$ containing a. From the definition of $C_e(B)$ we get $a \leftrightarrow \varphi(B) \geq e$ and from $\varphi(B) \in K$, $\bigvee_{b\in K} a \leftrightarrow b \geq e$, proving (33).

2. Every $B \in \Omega$ is a closed ball with radius e, hence also a block of $\approx_{e^2}$ due to Lemma 3.10. For $a \in M$ let $c \in K$ be such that $a \leftrightarrow c = \bigvee_{b\in K} a \leftrightarrow b$ (such c exists since K is finite and **L** is linearly ordered). (33) implies $a \leftrightarrow c \geq e$, hence $a \in B_e(c)$ and Ω is a covering of M. $\varphi(B) \in C_e(B)$ means $c \in C_e(B_e(c))$ which is true due to 1. of Lemma 3.9. □

Example 5.2. Let $L = [0, 1]^2$ with Łukasiewicz structure, $M = L$, $e = \langle 0.75, 0.75\rangle$. Then

$$K = \{\langle 0, 0\rangle, \langle 0.5, 0\rangle, \langle 1, 0\rangle, \langle 0, 0.5\rangle, \langle 0, 1\rangle\}$$

satisfies (33) because $\mathrm{appr}(M, K) = \langle 0.75, 0.75\rangle = e$. However, $\Omega = \{B_e(b) \mid b \in K\}$ is not a covering of M because $\langle 1, 1\rangle$ does not belong to any $B_e(b)$, $b \in K$. Hence, 2. of Theorem 5.1 does not hold for non-linear complete residuated lattices.

We now present two algorithms which provide solutions to Problem 1. The first algorithm constructs K going through L upwards.

Algorithm 5.3.

INPUT: M, e
OUTPUT: K satisfying (1) and (2) of Problem 1
$K := \emptyset$
while M is not empty do
  $min := \min(M)$
  add $e \to min$ to K
  remove from M every element $\leq (e \otimes e) \to min$
endwhile
return K

The second one constructs K going through L downwards.

Algorithm 5.4.

INPUT: M, e
OUTPUT: K satisfying (1) and (2) of Problem 1
$K := \emptyset$
while M is not empty do
  $max := \max(M)$
  add $e \otimes max$ to K
  remove from M every element $\geq e \otimes e \otimes max$
endwhile
return K

Lemma 5.5. 1. *Let the set K^u produced by Algorithm 5.3 consist of elements $k_1^u < \cdots < k_p^u$, let $K \subseteq L$, consisting of elements $k_1 < \cdots < k_q$, approximate M to degree at least e. Then $q \geq p$ and for any i, $1 \leq i \leq p$, $k_i^u \geq k_i$.*

2. *Let the set K^l produced by Algorithm 5.4 consist of elements $k_1^l < \cdots < k_p^l$, let $K \subset L$, consisting of elements $k_1 < \cdots < k_q$, approximate M to degree at least e. Then $q \geq p$ and for any i, $1 \leq i \leq p$, $k_i^l \leq k_i$.*

Proof. 1. Let $\Omega = \{B_e(b) \mid b \in K\}$. By part 2. of Theorem 5.1, Ω is a covering of M.

Suppose that the set of all $i \leq \min(p, q)$ such that $k_i^u < k_i$ is not empty and denote its least element by j. Since $k_1^u = e \to \bigwedge M$ then k_1^u is equal to the greatest $a \in L$ such that $a \leftrightarrow \bigwedge M \geq e$ (Theorem 3.4), which means that $j > 1$. Denote by min the least element remaining in M after $j - 1$ steps. From $min > k_{j-1}^u \geq k_{j-1}$ and $min \notin B_e(k_{j-1}^u)$ we obtain $min \notin B_e(k_{j-1})$. Since k_j^u is equal to the greatest $b \in L$ such that $min \leftrightarrow b \geq e$ then also $min \notin B_e(k_j)$. Thus, Ω is not a covering of M, which is a contradiction.

It remains to be shown that $q \geq p$. Indeed, if $q \leq p - 1$ then $k_q \leq k_{p-1}^u < \bigvee M$. Since $\bigvee M \notin B(k_{p-1}^u)$ then also $\bigvee M \notin B(k_q)$ and Ω is not a covering of M.

Part 2. can be proved similarly. □

As the next two theorems show, the algorithms indeed produce solutions to Problem 1.

Theorem 5.6. *Algorithms 5.3 and 5.4 are correct. They terminate after at most $O(|M|)$ steps.*

Proof. Since $(e \otimes e) \to min \geq min$, at least one element of M, namely min, is removed from M in every step in Algorithm 5.3. For Algorithm 5.4, since $e \otimes e \otimes max \leq max$, max is removed from M in every step. As a result, the algorithms terminate after at most $O(|M|)$ steps.

For Algorithm 5.3: In every step, all elements $a \in M \cap B$, where $B = [min, (e \otimes e) \to min]$, are removed from M and, at the same time, $e \to min$ is added to K. By Theorem 3.4, $e \otimes e \to min$ is the greatest among all $a \in L$ for which $min \leftrightarrow a \geq e \otimes e$. Hence, B is a block of $\approx_{e^2}$ and by Lemma 3.10, $C_e(B)$ is non-empty. Set $\varphi(B) = e \to min$. By Theorem 3.2, $\varphi(B) \in C_e(B)$. Thus, condition (1) of Problem 1 follows from part 1. of Theorem 5.1. Condition (2) follows directly from Lemma 5.5.

The proof for Algorithm 5.4 is similar. □

Furthermore, the algorithms provide upper and lower bounds for every set K which is a correct output for Problem 1.

Theorem 5.7. *Let the sets K^u and K^l produced by Algorithm 5.3 and Algorithm 5.4 consist of elements $k_1^u < \cdots < k_p^u$ and $k_1^l < \cdots < k_p^l$, respectively. If K consisting of $k_1 < \cdots < k_p$ satisfies* $\mathrm{appr}(M, K') \geq e$*, then*

$$k_1^l \leq k_1 \leq k_1^u, \ldots, k_p^l \leq k_p \leq k_p^u.$$

Proof. Follows directly from Lemma 5.5. □

The following example shows that not every $K = \{k_1, \ldots, k_p\}$ for which $k_i^l \leq k_i \leq k_i^u$ satisfies $\mathrm{appr}(M, K) \geq e$.

Example 5.8. Consider the standard Łukasiewicz algebra on $L = [0, 1]$, $M = \{0.5, 0.7, 0.8\}$, and $e = 0.9$. Then $K^l = \{0.4, 0.7\}$ and $K^u = \{0.6, 0.9\}$. Let $K = \{0.4, 0.9\}$. Then $\mathrm{appr}(M, K) = 0.8 < e$.

Although the set K produced by Algorithm 5.3 or Algorithm 5.4 is optimal in that it is one of the smallest sets for which $\mathrm{appr}(M, K) \geq e$, there can be a set K' of the same size, i.e. $|K'| = |K|$, for which $\mathrm{appr}(M, K') > \mathrm{appr}(M, K)$, i.e. K' provides a better approximation of M than K. From this point of view, the output set K from Algorithm 5.3 and Algorithm 5.4 can be improved. Namely, it is easily seen from the description of Algorithm 5.3 and Algorithm 5.4 that the set

$$\{B_e(k) \cap M \mid k \in K\}$$

forms a partition of M. In general, k is not an optimal central point of $B_e(k) \cap M$. Therefore, we can improve K by replacing every $k \in K$ by an optimal central point of $B_e(k) \cap M$.

By Theorem 4.11, if $\mathbf{L}$ has square roots, then any element from

$$\left[\sqrt{\bigwedge(B_e(k) \cap M)} \otimes \bigvee(B_e(k) \cap M), \sqrt{\bigwedge(B_e(k) \cap M)} \to \bigwedge(B_e(k) \cap M)\right]$$

can be used to replace k. For instance, for $M = \{0.5, 0.7, 0.8\}$ and $K = K^l = \{0.4, 0.7\}$ from Example 5.8, $B_{0.9}(0.7) \cap M = \{0.7, 0.8\}$ and $B_{0.9}(0.4) \cap M = \{0.5\}$. Hence, 0.4 can be replaced by 0.5 (optimal central point of $\{0.5\}$) and 0.7 can be replaced by 0.75 (optimal central point of $\{0.7, 0.8\}$). As a result, we get a set $\mathrm{opt}(K^l) = \{0.5, 0.75\}$ for which $\mathrm{appr}(M, \mathrm{opt}(K^l)) = 0.95 > 0.9 = \mathrm{appr}(M, K)$. In addition, we have $K^u = \{0.6, 0.9\}$, $\mathrm{opt}(K^u) = \{0.6, 0.8\}$, but this time $\mathrm{appr}(M, \mathrm{opt}(K^u)) = 0.9 = \mathrm{appr}(M, K)$.

As the next example shows, such improvement does not, in general, satisfy condition (3) of Problem 2. That is, replacement of points k in K by better points k' which cover the same part of M, i.e. for which $B_e(k) \cap M = B_e(k') \cap M$, does not result in the best approximating set with size $|K|$.

Example 5.9. Consider the standard Łukasiewicz algebra on $L = [0, 1]$, $M = \{0, 0.1, 0.3, 0.7, 0.9, 1\}$, and $e = 0.9$. Then $K^l = \{0, 0.2, 0.6, 0.9\}$ and $K^u = \{0.1, 0.4, 0.8, 1\}$, $\mathrm{opt}(K^l) = \{0, 0.2, 0.7, 0.95\}$, $\mathrm{opt}(K^u) = \{0.05, 0.3, 0.8, 1\}$, $\mathrm{appr}(M, K^l) = 0.9$, $\mathrm{appr}(M, K^u) = 0.9$, $\mathrm{appr}(M, \mathrm{opt}(K^l)) = 0.9$, $\mathrm{appr}(M, \mathrm{opt}(K^u)) = 0.9$. However, $K = \{0.05, 0.3, 0.7, 0.95\}$ is a solution to Problem 2 for which $\mathrm{appr}(M, K) = 0.95$.

In what follows, we present an algorithm that provides a solution to Problem 2 provided the largest admissible radii of subsets of L exist and may be determined. Due to Theorem 4.11, an important class of residuated lattices that satisfies this assumption consists of complete residuated lattices with square roots.

Let thus $M = \{m_1 \leq \cdots \leq m_n\}$ and denote by $r(A)$ the largest admissible radius of $A \subseteq L$.

Algorithm 5.10.

1: INPUT: M, e
2: OUTPUT: K solving Problem 2
3: $K' :=$ output of Algorithm 5.3 for M, e
4: $q := |K'|$
5: $e' := e$
6: repeat
7: $K := K'$
8: for $i = 1$ to $n - 1$ do
9: $r' := \min\{r([m_i, m]) \mid m \in M,\ r([m_i, m]) > e\}$
10: if $e' < r'$ then $e' := r'$
11: endfor
12: $e := e'$
13: $K' :=$ output of Algorithm 5.3 for M, e
14: until $|K'| > q$
15: return K

Theorem 5.11. *Algorithm 5.10 is correct. It terminates after at most* $O(|M|^4)$ *steps.*

Proof. The algorithm starts by computing a set K' with q elements that approximates M to degree at least e. The aim of loop 6–14 is to compute another set of q elements that approximates M to a largest possible degree d, larger than original e. To this end, the loop in 8–11 determines the least candidate e' for such d. This is easily seen by realizing that $r([m_i, m])$ is the radius of $[m_i, m]$ for any optimal central point of $[m_i, m]$, see Theorem 4.1. Any such optimal central point covers $[m_i, m])$ and is a potential element of a new set of q elements that approximates M to a higher degree than the lastly computed K'. For e', Algorithm 5.3 in line 13 computes a new set K' that approximates M to degree at least e'. Due to Lemma 5.5, this set has more than q elements if and only if no set with q elements approximates M to degree e' (or larger). In this case, the candidate e' may not be used a lower approximation of d and the loop 6–14 terminates. In the other case, e' is taken as a lower approximation of d and the loop 6–14 is entered again to produce and test another, larger candidate, until such a candidate exists. The last computed set K' with q elements is therefore the output to Problem 2.

Within loop 8–11, $r([m_i, m])$ is evaluated at most $\frac{|M|\cdot(|M|-1)}{2} = O(|M|^2)$ times. Loop 8–11 is run within loop 6–14 which itself is run at most $\frac{|M|\cdot(|M|-1)}{2} = O(|M|^2)$ times, yielding the total of $O(|M|^4)$ steps in the worst case. □

Note that after Algorithm 5.10 finishes, we may run Algorithm 5.4 on M and e' where e' is the value in Algorithm 5.10 for which the output of Algorithm 5.10 was computed. According to Theorem 5.7, the outputs produced by Algorithm 5.10 and Algorithm 5.4 provide us with lower and upper bounds for every solution to Problem 2.

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