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Grouping fuzzy sets by similarity

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ABSTRACT

The paper presents results on factorization of systems of fuzzy sets. The factorization consists in grouping those fuzzy sets which are pairwise similar at least to a prescribed degree a. An obstacle to such factorization, well known in fuzzy set theory, is the fact that "being similar at least to degree a" is not an equivalence relation because, in general, it is not transitive. As a result, ordinary factorization using equivalence classes cannot be used. This obstacle can be overcome by considering maximal blocks of fuzzy sets which are pairwise similar at least to degree a. We show that one can introduce a natural complete lattice structure on the set of all such maximal blocks and study this lattice. This lattice plays the role of a factor structure for the original system of fuzzy sets. Particular examples of our approach include factorization of fuzzy concept lattices and factorization of residuated lattices.

1. Introduction

Factorization represents a fundamental construction in mathematics. Its main aim is to capture the process of simplification by abstraction. An input to a factorization is a mathematical structure, typically a system of elements equipped possibly with relations and functions. An output of a factorization consists of another structure, called a factor structure (or a quotient structure), which can be considered a simplified version of the input structure. Elements of the factor structure are groups of elements of the original structure, which are indistinguishable from a certain point of view. The indistinguishability is usually represented by an equivalence relation and the groups of elements are the corresponding equivalence classes. To be able to introduce a naturally inherited structure on the groups of indistinguishable elements, the equivalence relation needs to be compatible with functions and relations of the original structure.

In this paper, we present a general framework for factorization of systems of fuzzy sets by similarity. The input structure consists of a system of fuzzy sets equipped with a subsethood relation. The indistinguishability relation which we use for factorization is represented by the relation "being similar at least to degree a" where similarity degrees are assessed by means of a well-known Leibniz similarity relation, see Section 2, i.e. the indistinguishability is represented by an a-cut of a particular fuzzy equivalence relation. We assume that the fuzzy sets are fixpoints of some fuzzy closure operator. Examples of such systems are fuzzy concept lattices, fuzzy sets in a given universe, or complete residuated lattices.

Such assumptions are natural: We deal with a system $\mathscr{S}$ of fuzzy sets and if $\mathscr{S}$ is considered too large, we want to simplify it by putting together those fuzzy sets from $\mathscr{S}$ which are pairwise similar to a prescribed degree a (threshold, a parameter to the factorization). However, the ordinary factorization cannot be used. Namely, an obstacle consists in the fact, well known in fuzzy set theory, that "being similar at least to degree a" is not a transitive relation and hence not an equivalence relation. We overcome this obstacle by utilizing results from lattice theory on factorization of complete lattices by compatible reflex-

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ive and symmetric relations. We show that "being similar at least to degree a" is a compatible relation and thus, one can introduce the structure of a complete lattice on the set of all maximal blocks of this relation. We study this lattice and provide an efficient description of the blocks which can be used to compute the factor system. Namely, we show that the upper bounds of the maximal blocks are just fixpoints of a particular fuzzy closure operator for which we present an explicit description.

2. Preliminaries from fuzzy logic

In classical logic, the structure $\mathbf{L}$ of truth degrees consists of the two-element set $L = \{0, 1\}$ of truth degrees and the truth functions of logical connectives. In fuzzy logic, there are more options, both for the set L of truth degrees and for the functions of logical connectives. As the structures of truth degrees, we use complete residuated lattices in our approach. Complete residuated lattices are general structures of truth degrees and several variants of them are used in fuzzy logic. A complete residuated lattice [5,13] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy the adjointness property: $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$ for every $a, b, c \in L$. The fact that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice means that the infimum $\bigwedge_{i \in I} a_i$ and the supremum $\bigvee_{i \in I} a_i$ exist for every subset $\{a_i \mid i \in I\} \subseteq L$. Elements $a \in L$ are called truth degrees. Operations $\otimes$ and $\rightarrow$, called multiplication and residuum, are the truth functions of logical connectives "fuzzy conjunction" and "fuzzy implication". A biresiduum of $\mathbf{L}$ is a binary operation $\leftrightarrow$ defined by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a).$$

We denote by $\leqslant$ the lattice order induced by $\mathbf{L}$. Examples of residuated lattices are well known. A generic one is: Take $L = [0, 1]$ and a left-continuous t-norm $\otimes$. That is, $\otimes$ is binary operation, which is left-continuous in its first argument (as a real function of two variables), commutative, associative, monotone, and has 1 as its neutral element [13]. Put $a \rightarrow b = \bigvee\{c \in L \mid a \otimes c \leqslant b\}$. Then $\langle [0, 1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ is a complete residuated lattice.

Particular well-known examples include the following t-norms and residua: Łukasiewicz ($a \otimes b = \max(0, a + b - 1), a \rightarrow b = \min(1, 1 - a + b)$); Gödel (minimum) ($a \otimes b = a \wedge b$, $a \rightarrow b = b$ for $a > b$ and $a \rightarrow b = 1$ for $a \leqslant b$); Goguen (product) ($a \otimes b = a \cdot b, a \rightarrow b = \frac{b}{a}$ for $a > b$ and $a \rightarrow b = 1$ for $a \leqslant b$). Another well-known class of examples includes residuated lattices which are finite chains, e.g. $L = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ equipped with restrictions of the abovementioned Łukasiewics or Gödel operations. Throughout the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Given $\mathbf{L}$, we define the usual notions regarding fuzzy sets and fuzzy relations: a fuzzy set (an $\mathbf{L}$-set) A in universe X is a mapping $A : X \rightarrow L, A(x)$ being interpreted as "the degree to which x belongs to A". The set of all fuzzy sets in X is denoted by L^X. Operations with fuzzy sets are defined componentwise. For instance, the intersection of fuzzy sets $A, B \in \mathbf{L}^X$ is a fuzzy set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc. For fuzzy sets $A, B \in L^X$, put

$$S(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x)), \tag{1}$$

$$A \approx B = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x)). \tag{2}$$

$S(A, B)$ and $A \approx B$ are called the degree of subsethood of A in B and the degree of equality of A and B, respectively. Note that $S(A, B)$ can be seen as the truth degree of "for each $x \in X$: if x belongs to A then x belongs to B". Similarly, $A \approx B$ can be seen as the truth degree of "for each $x \in X$: x belongs to A if and only if x belongs to B". $\approx$ is a fuzzy equivalence relation, i.e. $A \approx A = 1$ (reflexivity), $A \approx B = B \approx A$ (symmetry), and $(A \approx B) \otimes (B \approx C) \leqslant A \approx C$, which is called the Leibniz similarity. Furthermore, $S(A, B) = 1$ iff $A(x) \leqslant B(x)$ for each $x \in X$ (A is fully contained in B). This fact is denoted by

$$A \subseteq B.$$

For more details we refer to [5,12,13].

3. Factorization by similarity

Suppose $\mathscr{S}$ is a system of fuzzy sets in X, i.e. $\mathscr{S} \subseteq L^X$. Suppose furthermore that $\mathscr{S}$ is a system of fixpoints of an $\mathbf{L}$-closure operator (fuzzy closure operator) C in X. Recall [2,5,15] that an $\mathbf{L}$-closure operator C in X is a mapping $C : L^X \rightarrow L^X$ satisfying

$$A \subseteq C(A), \tag{3}$$

$$S(A_1, A_2) \leqslant S(C(A_1), C(A_2)), \tag{4}$$

$$C(A) = C(C(A)), \tag{5}$$

for every $A, A_1, A_2 \in L^X$. As a consequence, we also have

$$(A_1 \approx A_2) \leqslant (C(A_1) \approx C(A_2)). \tag{6}$$

The set $\text{fix}(C)$ of all fixpoints of C is defined by

$$\text{fix}(C) = \{A \in L^X \mid A = C(A)\}.$$

Remark 1. The concept of an **L**-closure operator generalizes the well-known concept of a closure operator. Namely, for $L = \{0,1\}$, **L**-closure operators coincide with ordinary closure operators.

We thus assume $\mathscr{S} = \text{fix}(C)$ for an **L**-closure operator C. Next, we present three examples of such systems.

Example 1. The first example comes from formal concept analysis of data with fuzzy attributes [5–7,17]. Let $\langle X, Y, I\rangle$ be a formal fuzzy context, i.e. X and Y are sets of objects and attributes, and $I : X \times Y \to L$ is a fuzzy relation between X and Y. For $x \in X$ and $y \in Y, I(x,y)$ is interpreted as the degree to which object x has attribute y. Let $^{\uparrow} : L^X \to L^Y$ and $^{\downarrow} : L^Y \to L^X$ denote the associated operators, i.e.

$$A^{\uparrow}(y) = \bigwedge_{x\in X}(A(x) \to I(x,y)), \quad B^{\downarrow}(x) = \bigwedge_{y\in Y}(B(y) \to I(x,y)).$$

Let $\mathscr{B}(X,Y,I) = \{\langle A,B\rangle | A^{\uparrow} = B, B^{\downarrow} = A\}$ denote the associated concept lattice. $\mathscr{B}(X,Y,I)$ equipped with $\leqslant$ defined by $\langle A_1,B_1\rangle \leqslant \langle A_2,B_2\rangle$ iff $A_1 \subseteq A_2$ (iff $B_1 \supseteq B_2$) is indeed a complete lattice. Elements $\langle A,B\rangle \in \mathscr{B}(X,Y,I)$ are called formal concepts and represent particular clusters in the data described by $\langle X,Y,I\rangle$. A and B are called the extent and the intent of $\langle A,B\rangle$ and represent the collection of all objects and attributes covered by the formal concept $\langle A,B\rangle$. Consider the set

$$\text{Ext}(X,Y,I) = \{A \mid \langle A,B\rangle \in \mathscr{B}(X,Y,I) \text{ for some } B \in L^Y\}$$

of all extents of $\langle X,Y,I\rangle$. It is well-known [2] that the mapping $C : L^X \to L^X$ defined by $C(A) = A^{\uparrow\downarrow}$ is an **L**-closure operator for which $\text{fix}(C) = \text{Ext}(X,Y,I), \text{Ext}(X,Y,I)$ is the set of all fixpoints of C. Note that $\langle \text{Ext}(X,Y,I), \subseteq\rangle$ is isomorphic to $\langle \mathscr{B}(X,Y,I), \leqslant\rangle$.

Now, putting $\mathscr{S} = \text{Ext}(X,Y,I)$, we have our first example of a system of fuzzy sets. Since $\mathscr{B}(X,Y,I) = \{\langle A, A^{\uparrow}\rangle \mid A \in \text{Ext}(X,Y,I)\}, \mathscr{B}(X,Y,I)$ can be identified with $\text{Ext}(X,Y,I)$. Therefore, loosely speaking, $\mathscr{S}$ is the concept lattice associated to the input data $\langle X,Y,I\rangle$. Note also that every **L**-closure operator in X can be obtained this way, i.e. from some $\langle X,Y,I\rangle$, see [2,5].

Example 2. Every complete residuated lattice **L** can be thought of as a system $\mathscr{S}$ of fuzzy sets. Namely, putting $X = \{x\}$, we can identify L with the set L^X of all fuzzy sets in X (just identify a with $\{a/x\}$). Consider the identity mapping $C : L \to L$, i.e. $C(a) = a$. Obviously, C is an **L**-closure operator on X and $\text{fix}(C) = L$. $\mathscr{S} = L$ is our second example. Note that in this example, $S(a,b) = a \to b$, $a \approx b = a \leftrightarrow b$, see Section 2.

Example 3. Let $\equiv$ be a fuzzy equivalence on X and put $[C_{\equiv}(A)](x) = \bigvee_{y\in X} A(y) \otimes (x \equiv y)$. $C_{\equiv}$ is an **L**-closure operator which is well known in fuzzy set theory. Putting $\mathscr{S} = \text{fix}(C), \mathscr{S}$ contains just the fuzzy sets in X which are called extensional w.r.t. $\equiv$, i.e. those satisfying $A(x) \otimes (x \equiv y) \leqslant A(y)$ (reads: if x is in A and x is similar to y then y is in A).

It is easily seen that $\text{fix}(C)$ equipped with inclusion $\subseteq$, see Section 2, is a complete lattice in which

$$\bigwedge_{j\in J} A_j = \bigcap_j A_j, \quad \bigvee_{j\in J} A_j = C\left(\bigcup_j A_j\right).$$

Define a binary relation ${}^a\!\approx$ on $\text{fix}(C)$ by

$$A\,{}^a\!\approx B \quad \text{iff } (A \approx B) \geqslant a.$$

That is, ${}^a\!\approx$ is the a-cut of $\approx$. $A\,{}^a\!\approx B$ means that A and B are similar at least to degree a. The following lemma follows from [4]:

Lemma 1. *${}^a\!\approx$ is a complete tolerance on $\langle \text{fix}(C), \subseteq\rangle$. That is, ${}^a\!\approx$ is a reflexive and symmetric relation on $\text{fix}(C)$ which is compatible with infima and suprema in $\langle \text{fix}(C), \subseteq\rangle$.*

Note that compatibility of ${}^a\!\approx$ with infima and suprema means that for any $A_j, B_j \in \text{fix}(C)$ $(j \in J)$, if $A_j\,{}^a\!\approx B_j$ for all $j \in J$, then

$$\bigwedge_{j\in J} A_j\,{}^a\!\approx \bigwedge_{j\in J} B_j \quad \text{and} \quad \bigvee_{j\in J} A_j\,{}^a\!\approx \bigvee_{j\in J} B_j.$$

Because ${}^a\!\approx$ is a complete tolerance on the complete lattice $\langle \text{fix}(C), \subseteq\rangle$ (Lemma 1), we can apply the construction of factorization of complete lattices by complete tolerances described in [10], see also [9,16], and define the factor lattice of $\text{fix}(C)$ by ${}^a\!\approx$.

Denote by $\text{fix}(C)/{}^a\!\approx$ the collection of all maximal blocks of ${}^a\!\approx$, i.e.

$$\text{fix}(C)/{}^a\!\approx = \{B \subseteq \text{fix}(C) \mid (B \times B) \subseteq {}^a\!\approx \quad \text{and} \quad (B' \times B') \not\subseteq {}^a\!\approx \text{ whenever } B' \supset B\}.$$

That is, blocks B of $\mathrm{fix}(C)/{}^a\!\approx$ are maximal sets of fixpoints of C which are pairwise similar at least to degree a. These blocks are particular intervals in the complete lattice $\langle \mathrm{fix}(C), \subseteq\rangle$. Namely, put

$$A_a = \bigwedge_{B\in\mathrm{fix}(C), A\,{}^a\!\approx B} B, \quad A^a = \bigvee_{B\in\mathrm{fix}(C), A\,{}^a\!\approx B} B.$$

for $A \in \mathrm{fix}(C)$. The following theorem follows from [10, Proposition 55, Theorem 14]. Namely, parts (1) and (2) are particular cases of [10, Proposition 55] and [10, Theorem 14] for a a complete tolerance ${}^a\!\approx$ on the complete lattice $\langle \mathrm{fix}(C), \subseteq\rangle$.

Theorem 2

(1) $\mathrm{fix}(C)/{}^a\!\approx = \{[A_a, (A_a)^a] \mid A \in \mathrm{fix}(C)\}$, *i.e. the blocks of* $\mathrm{fix}(C)/{}^a\!\approx$ *are certain intervals in* $\langle \mathrm{fix}(C), \subseteq\rangle$.
(2) With respect to a partial order $\leqslant$ *on* $\mathrm{fix}(C)/{}^a\!\approx$, *defined for* $[u_1, u_2], [v_1, v_2] \in \mathrm{fix}(C)/{}^a\!\approx$ *by*

$$[u_1, u_2] \leqslant [v_1, v_2] \quad \text{iff } u_1 \subseteq v_1 (\text{iff } u_2 \subseteq v_2),$$

$\langle \mathrm{fix}(C)/{}^a\!\approx, \leqslant\rangle$ *is a complete lattice, called the factor lattice of* $\langle \mathrm{fix}(C), \subseteq\rangle$ *by tolerance* ${}^a\!\approx$.

Note that for $A_1, A_2 \in \mathrm{fix}(C)$, the interval $[A_1, A_2]$ is defined by $[A_1, A_2] = \{A \in \mathrm{fix}(C) | A_1 \subseteq A \subseteq A_2\}$. Note also that it follows from [10, Theorem 14] that a mapping sending A to $[A_a, (A_a)^a]$ is a $\bigvee$-morphism of $\mathrm{fix}(C)$ to $\mathrm{fix}(C)/{}^a\!\approx$, i.e. preserves arbitrary suprema (but not arbitrary infima); dually, a mapping sending A to $[(A^a)_a, A^a]$ is a $\bigwedge$-morphism of $\mathrm{fix}(C)$ to $\mathrm{fix}(C)/{}^a\!\approx$, i.e. preserves arbitrary infima (but not arbitrary suprema).

Our particular setting enables us to describe maximal blocks in a simple way. Note that for a truth degree $a \in L$ and a fuzzy set $A \in L^X$, the fuzzy sets $a \otimes A \in L^X$ and $a \to A \in L^X$ are defined by $(a \otimes A)(x) = a \otimes A(x)$ and $(a \to A)(x) = a \to A(x)$. We start with following auxiliary result.

Lemma 3. *For* $A \in \mathrm{fix}(C), A_a = C(a \otimes A), A^a = a \to A$.

Proof. $A_a = C(a \otimes A)$: Since

$$a \leqslant (A \approx a \otimes A) \leqslant (C(A) \approx C(a \otimes A)) = (A \approx C(a \otimes A)),$$

$C(a \otimes A)$ is a fixpoint of C similar to A at least to degree a. Furthermore, if $B \in \mathrm{fix}(C)$ satisfies $a \leqslant (A \approx B)$ then $a \otimes A \subseteq B$ (due to adjointness), hence $C(a \otimes A) \subseteq C(B) = B$ by monotony of C. As a result, $C(a \otimes A)$ is the least fixpoint of C similar to A at least to degree a which immediately yields $A_a = C(a \otimes A)$.

$A^a = a \to A$: [2] yields that $\mathrm{fix}(C)$ is closed under $\to$-shifts, i.e. if $A \in \mathrm{fix}(C)$ then $a \to A \in \mathrm{fix}(C)$. Let $B \in \mathrm{fix}(C)$ be similar to A at least to degree a, i.e. $a \leqslant (A \approx B)$. Then $B \subseteq a \to A$ (due to adjointness), i.e. $a \to A$ is the largest fixpoint of C similar to A at least to degree a, whence $A^a = a \to A$. □

Example 4

(1) Consider Example 1. That is, $\mathrm{fix}(C) = \mathrm{Ext}(X, Y, I)$ is the set of all extents of formal concepts of $\mathcal{B}(X, Y, I)$. As mentioned above, $\langle \mathrm{fix}(C), \subseteq\rangle$ is isomorphic to $\langle \mathcal{B}(X, Y, I), \leqslant\rangle$. One can easily see that the factor lattice $\mathrm{fix}(C)/{}^a\!\approx$ is isomorphic to $\mathcal{B}(X, Y, I)/{}^a\!\approx$. That is, in this example, the factor lattice yields the factor concept lattice given by similarity threshold a, see [1,6].
(2) Consider Example 2. That is, $\mathrm{fix}(C) = L$ is a support set of a complete residuated lattice $\mathbf{L}$. For $c \in L, c_a = a \otimes c$ and $c^a = a \to c$. The factor lattice $L/{}^a\!\approx$ coincides with the lattice part of a factor algebra of the complete residuated lattice $\mathbf{L}$ modulo ${}^a\!\approx$, see [14].

The following lemma describes the mappings sending A to A_a, and to A^a.

Lemma 4. *The mappings* $f : A \mapsto C(a \otimes A)$ *and* $g : B \mapsto a \to B$ *satisfy*

$$S(A_1, A_2) \leqslant S(f(A_1), f(A_2)), \tag{7}$$
$$S(B_1, B_2) \leqslant S(g(B_1), g(B_2)), \tag{8}$$
$$f(g(A)) \subseteq A, \tag{9}$$
$$B \subseteq g(f(B)). \tag{10}$$

Proof. By routine verification using standard properties of complete residuated lattices. □

Remark 2. Note that mappings satisfying (7)–(10). were studied in [11].

Lemma 4 provides us with useful properties. For example, as a direct consequence of Lemma 4, $f(A) = fgf(A)$ and $g(B) = gfg(B)$, i.e. $A_a = ((A_a)^a)_a$ and $A^a = ((A^a)_a)^a$. Because of this, $[A_a, (A_a)^a] = [((A_a)^a)_a, (A_a)^a]$, i.e. every $[A_a, (A_a)^a]$ is of the form $[(B^a)_a, B^a]$. By similar arguments, every $[(B^a)_a, B^a]$ is of the form $[A_a, (A_a)^a]$. This implies a possibility of a dual description of the blocks, namely, $\mathrm{fix}(C)/^a\!\approx\, = \{[(A^a)_a, A^a] \mid A \in L^X\}$, Let us now turn to a description of blocks of the factor lattice $\mathrm{fix}(C)/^a\!\approx$ which provides a way to compute the factor lattice efficiently.

Let us first note that if we can efficiently compute closures $C(A)$ for $A \in L^X$ (and if "everything is finite"), we can compute $\mathrm{fix}(C)/^a\!\approx$ by computing first $\mathrm{fix}(C)$ and then computing the blocks of $\mathrm{fix}(C)/^a\!\approx$. Namely, the set $\mathrm{fix}(C)$ of fixpoints of a fuzzy closure operator C can be computed by a modification [3] of Ganter's NEXTCLOSURE algorithm [10]. In addition, the blocks are blocks of a tolerance relation and they can be computed by available algorithms (these algorithms come from graph theory; namely, maximal blocks of a tolerance relation T are just maximal independent sets in the graph of the complement of T). However, there is a better way than this "naive" one. We present it below.

As we know from Theorem 2, the elements of $\mathrm{fix}(C)/^a\!\approx$ are $^a\!\approx$-blocks and every such a block is an interval of the form $[A_a, (A_a)^a]$ for $A \in \mathrm{fix}(C)$. By the previous results, each such block is determined by its upper bound $(A_a)^a$. To compute all elements of $\mathrm{fix}(C)/^a\!\approx$, it is therefore sufficient to compute the set

$$\mathrm{UB} = \{B \in L^X \mid [A, B] \in \mathrm{fix}(C)/^a\!\approx \text{ for some } A\}$$

of all upper bounds of blocks from $\mathrm{fix}(C)/^a\!\approx$. Taking into account Lemma 4, the following claim is easy to check.

Lemma 5. *The mapping*

$$C_a : A \mapsto a \to C(a \otimes A)$$

sending A to $(A_a)^a = a \to C(a \otimes A)$ is an **L***-closure operator in X.*

Now, C_a provides a useful description of UB.

Theorem 6. $\mathrm{UB} = \mathit{fix}(C_a)$.

Proof. Let $B \in \mathrm{UB}$. Then there exists an A such that $[A, B] \in \mathrm{fix}(C)/^a\!\approx$. By Theorem 2 (1) and Lemma 4 and its consequences, $B = (B_a)^a$, i.e. by Lemma 3, $B = a \to C(a \otimes B)$, i.e. $B \in \mathrm{fix}(C_a)$.

Conversely, let $B \in \mathrm{fix}(C_a)$, i.e. $B = a \to C(a \otimes B) = (B_a)^a$. Due to Theorem 2 (1), in order to see that $B \in \mathrm{UB}$, it suffices to verify that $B \in \mathrm{fix}(C)$. This is, indeed, true: Clearly, $C(a \otimes B) \in \mathrm{fix}(C)$. Therefore, $B = a \to C(a \otimes B) \in \mathrm{fix}(C)$ due to the fact that $\mathrm{fix}(C)$ is closed under a-shifts [2]. □

Therefore, the fixpoints of C_a are just the upper bounds UB. Since, as mentioned above, $\mathrm{fix}(C)/^a\!\approx$ can be restored from UB, we reduced the problem of computing the factor lattice $\mathrm{fix}(C)/^a\!\approx$ to the problem of computing a set of fixpoints of a fuzzy closure operator, namely, of C_a. This way is more efficient than the "naive" one because we need not compute all the fixpoints of C (note that $\mathrm{fix}(C_a) \subseteq \mathrm{fix}(C)$) and, moreover, we need not compute the maximal blocks of fixpoints which is a time-consuming step even when one employs specialized algorithms. Note that some of the formulas and results obtained in this section generalize those from [6] where a particular case of concept lattices was considered, see Example 1. In particular, it was demonstrated in [6] that the speed-up in computing $\mathrm{fix}(C)/^a\!\approx$ using C_a is high and depends in a natural way on the threshold a.

Alternatively, one can proceed in a dual way and use the lower bounds of blocks from $\mathrm{fix}(C)/^a\!\approx$. In the rest of this section, we briefly describe the approach. Recall [8], [11] that an **L**-interior operator I in X is a mapping $I : L^X \to L^X$ satisfying

$$I(A) \subseteq A, \tag{11}$$
$$S(A_1, A_2) \leqslant S(I(A_1), I(A_2)), \tag{12}$$
$$I(A) = I(I(A)), \tag{13}$$

for every $A, A_1, A_2 \in L^X$. The set $\mathrm{fix}(I)$ of all fixpoints of I is defined by

$$\mathrm{fix}(I) = \{A \in L^X | I(A) = A\}.$$

In addition, define

$$\mathrm{LB} = \{A \in L^X \mid [A, B] \in \mathrm{fix}(C)/^a\!\approx \text{ for some } A\}.$$

LB is the set of lower bounds of blocks from $\mathrm{fix}(C)/^a\!\approx$. The following lemma follows easily from Lemma 4.

Lemma 7. *The mapping*

$$I_a : A \mapsto C(a \otimes (a \to A))$$

sending A to $(A^a)_a = C(a \otimes (a \to A))$ is an **L***-interior operator in X.*

Theorem 8. $\mathrm{LB} = \mathit{fix}(I_a)$,

Proof. Let $A \in \mathrm{LB}$. Then there exists a B such that $[A, B] \in \mathrm{fix}(C)/^a\!\approx$. From Theorem 2 (1) and consequences of Lemma 4 we have $A = B_a = C(a \otimes B)$, $B = A^a = a \to A$. Put together, $A = C(a \otimes (a \to A)) = I_a(A)$ and $A \in \mathrm{fix}(I_a)$,

Conversely, let $A \in \mathrm{fix}(I_a)$, i.e. $A = C(a \otimes (a \to A)) = (A^a)_a$. This means that A is a fixpoint of C. Since a-shifts of fixpoints are also fixpoints [2], we get that $B = a \to A = A^a$ is a fixpoint of C. Now, $[A, B] = [B_a, (B_a)^a]$ (Lemma 4 and its consequences), which is an element of $\mathrm{fix}(C)/^a\!\approx$ (Theorem 2). This shows $A \in \mathrm{LB}$. $\square$

4. Conclusions

We have demonstrated that the straightforward idea of grouping fuzzy sets by putting together those which are sufficiently similar, i.e. similar at least to a prescribed degree a, leads to feasible structures in spite of the fact that "similar to degree at least a" is not an equivalence relation. Particular examples of the general procedure presented here include factorization of concept lattices and factorization of complete residuated lattices. Note also that our results are degenerate in case of ordinary sets. Namely, similarity to degree 1 means equality of ordinary sets, while similarity to degree 0 presents no constraint. Correspondingly, $\mathrm{fix}(C)/^1\!\approx$ is isomorphic to $\mathrm{fix}(C)$ and $\mathrm{fix}(C)/^0\!\approx$ consists of a single block containing all fixpoints from $\mathrm{fix}(C)$. From this point of view, this paper points out a phenomenon which is hidden in case of ordinary sets.

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