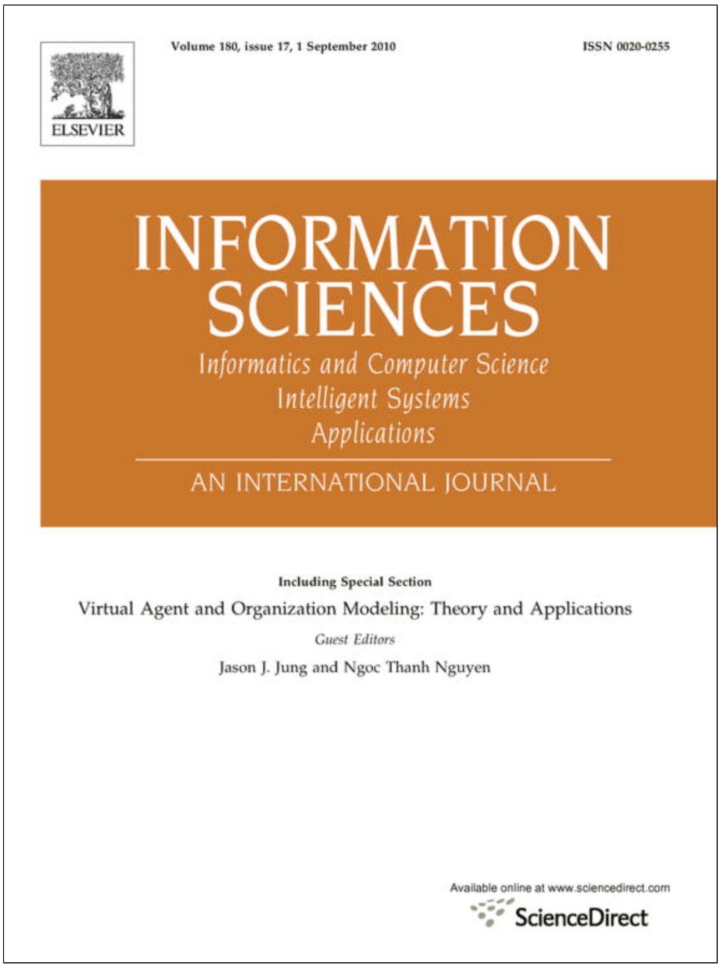
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Confluence and termination of fuzzy relations

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ABSTRACT

Confluence and termination are essential properties connected to the idea of rewriting and substituting which appear in abstract rewriting systems. The aim of the present paper is to investigate confluence, termination, and related properties from the point of view of fuzzy logic leaving the ordinary notions a particular case when the underlying structure of truth degrees is two-valued Boolean algebra. The main motivation of this study is the fact that in several natural situations, the notion of substitutability is inherently fuzzy rather than crisp.

1. Introduction and problem setting

Both confluence and termination are properties of binary relations related to the idea of performing substitutions specified by the respective binary relation. The notions have been introduced in the theory of abstract rewriting systems and have been the subject of extensive investigations. Since then, many applications of abstract rewriting systems have evolved. For instance, term rewriting systems can be used as a theoretical background for functional programming, various logical deductive systems can be formalized by rewriting systems, simplification of mathematical expressions can be seen as a reduction, rewriting plays an important role in the theory of formal grammars, etc. A good overview of rewriting systems can be found in [1,17].

The ordinary confluence and termination can be introduced as follows. Let R be a binary relation on a set X and assume that $\langle x,y\rangle \in R$ means that one may substitute y for x. The substitution "y for x" can be informally explained so that, from a certain point of view, whenever x does a certain job, y does it as well. An element $x \in X$ is called reducible if $\langle x,y\rangle \in R$ for some $y \in X$; otherwise, x is called irreducible. By a reduction we mean any sequence $x_0,\ldots,x_n$ such that $\langle x_{i-1},x_i\rangle \in R$ $(i = 1,\ldots,n)$; a reduction is called terminating if x_n is irreducible. In this case, x_0 is said to be reducible to x_n. Relation R is called *terminating* if it has only terminating reductions. Relation R is called *confluent* whenever x is reducible to both y and y' then there is some z such that both y and y' are reducible to z. Termination and confluence obey several interesting properties. There is a synergy between termination and confluence the most known example of which is that a relation which is both terminating and confluent has normal forms, i.e. each element is reducible to a unique irreducible element.

The aim of the present paper is to look at substitutability and the properties of confluence and termination from the point of view of fuzzy logic and fuzzy set theory [18,19]. Our basic motivation is the fact that the phenomenon of substitutability may not be bivalent. It is a common practice of everyday life to substitute y for x whenever x is too complex to handle and y does the job of x sufficiently well. In a similar way, one often substitutes an option y for option x whenever y

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is less costly than x and y does the job of x sufficiently well. For example, instead of working with a large dictionary which, although comprehensive, is too large to fit into the memory of a small computer system at use (i.e., a mobile system with limited resources), one uses a smaller dictionary; instead of working with a whole article, one may wish to work with its summary only (which is sufficiently informative); instead of using an expensive option poll based on survey of a sample of 10,000 persons, one may use cheaper option poll based on a few hundreds of persons (which will give sufficiently similar result); instead of using a highly reliable but costly component, one may wish to use a cheaper one (whose reliability is sufficient for the job); instead of buying a 4 BR (bedroom) family house, one may wish to buy a 3 BR house which is more affordable, etc. At the conceptual level, the common of these examples is that one works with a substitutability relation R which is, however, a fuzzy relation rather than a bivalent one. That is, for every $x, y \in X$, $R(x,y)$ is the degree to which y can be substituted for x (not necessarily 0 or 1). $R(x,y)$ embodies the expert knowledge and depends on the particular situation.

The issue of a degree of substitutability is also relevant in the context of term rewriting. In [4–6] it is shown that the fundamental notions and results on the equational reasoning can be accomodated to the requirement of respecting in a natural way an underlying similarity relation (i.e. a fuzzy relation which is reflexive, symmetric, and transitive) on the universe set. Given an underlying similarity, one may require (or find out) two terms to be substitutable only to some degree which may result from the fact that they always evaluate to elements which are similar but not identical. Note also that various similarity issues are recently gaining interest, see e.g. [3,14].

This paper studies "fuzzy substitutability" (degrees of substitutability) on the abstract level. We assume that a fuzzy relation on an abstract set is given. The primary interpretation of the fuzzy relation is the substitutability as outlined above. We study analogies of confluence and termination – the two most important properties of substitutability relations in the bivalent case. Section 2 presents the necessary notions on fuzzy logic and fuzzy sets. Section 3 introduces basic notions. Sections 4 and 5 deal with confluence and related notions. Termination, related notions, and normal forms are investigated in Sections 6–8. Section 9 contains conclusions and a list of open problems.

2. Preliminaries

We use complete residuated lattices as the structures of truth values. Complete residuated lattices, being introduced in the 1930s in ring theory [8,16], were introduced into the context of fuzzy logic by Goguen [10,11]. A fundamental contribution to formal fuzzy logic using residuated lattices as the structures of truth values is due to Pavelka [15]. Later on, various logical calculi were investigated using residuated lattices or particular types of residuated lattices. A thorough information about the role of residuated lattices in fuzzy logic can be obtained from monographs [12,13].

Recall that a (complete) residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that (i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e., $\otimes$ is commutative, associative, and $a \otimes 1 = a$); (ii) $\otimes$ (conjuntion) and $\rightarrow$ (residuum) are binary operations satisfying the adjointness property, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$ is true for all $a, b, c \in L$. In what follows, $\mathbf{L}$ always refers to a complete residuated lattice. Note that the class of complete residuated lattices includes structures of truth degrees on the real unit interval with $\otimes$ and $\rightarrow$ being a left-continuous t-norm and its corresponding residuum, respectively.

Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz: $a \otimes b = \max(0, a + b - 1)$, $a \rightarrow b = \min(1, 1 - a + b)$; Gödel (minimum): $a \otimes b = a \wedge b$, $a \rightarrow b = b$ for $a > b$ and $a \rightarrow b = 1$ for $a \leqslant b$; Goguen (product): $a \otimes b = a \cdot b$, $a \rightarrow b = \frac{b}{a}$ for $a > b$ and $a \rightarrow b = 1$ for $a \leqslant b$. Recall that the two-element Boolean algebra which plays an important role in the classical logic is a particular case of a complete residuated lattice where $L = \{0, 1\}$, $\wedge$ and $\vee$ being the minimum and maximum, respectively, $\otimes = \wedge$, and $\rightarrow$ being truth function of the two-valued implication. In the sequel, the two-element Boolean algebra will be denoted by $\mathbf{2}$.

In this paper we will use additional properties of truth degrees $a \in L$. Recall that $a \in L$ is called idempotent if $a \otimes a = a$. Furthermore, $0 \neq a \in L$ is called a zero-divisor of $\otimes$ if there is $0 \neq b \in L$ such that $a \otimes b = 0$. We say that $\mathbf{L}$ has no zero-divisors if for any $a,b \in L$: if $a \otimes b = 0$ then $a = 0$ or $b = 0$. Directly from the definition, if $\otimes$ is a Gödel conjunction then each $a \in L$ is idempotent; if $\otimes$ is the Łukasiewicz conjunction on $L = [0, 1]$ then each $0 < a < 1$ is a zero divisor. A complete residuated lattice $\mathbf{L}$ is called a chain if it is linearly ordered. If for any $a_i \in L$ $(i \in I)$ there is a finite subset $I' \subseteq I$ such that $\bigvee_{i\in I} a_i = \bigvee_{i \in I'} a_i$ then $\mathbf{L}$ is called a Noetherian [7] (complete) residuated lattice. All properties of complete residuated lattices used in the sequel are well known and can be found, e.g., in [2,9,12,13].

We now recall basic notions of fuzzy sets and fuzzy relations. Let $\mathbf{L}$ be a complete residuated lattice. An $\mathbf{L}$-set (or fuzzy set with truth degrees in $\mathbf{L}$) in a universe set X is any map $A : X \rightarrow L$, $A(x) \in L$ being interpreted as the truth value of "x belongs to A". Analogously, an n-ary $\mathbf{L}$-relation (or fuzzy relation with truth degrees in $\mathbf{L}$) on a universe set X is an $\mathbf{L}$-set in the universe set X^n. For instance, a binary $\mathbf{L}$-relation R on X is a map $R : X \times X \rightarrow L$. Binary $\mathbf{L}$-relations will be denoted by capital letters R, $R', \ldots$ or symbols $\rightharpoonup, \leftharpoondown, \ldots$ in which case we write $u \rightharpoonup v$ and $u \leftharpoondown v, \ldots$, instead of $\rightharpoonup(u, v)$ and $\leftharpoondown(u, v), \ldots$, respectively.

For an $\mathbf{L}$-set A in X we define its strong 0-cut $A^+ \subseteq X$ by $A^+ = \{x \in X | A(x) > 0\}$. A strong 0-cut of $\mathbf{L}$-relation $\rightharpoonup$ will be denoted by $\rightsquigarrow$. For binary $\mathbf{L}$-relations R_1, R_2 on X the $\circ$-composition of R_1 and R_2 is a binary $\mathbf{L}$-relation $R_1 \circ R_2$ on X defined by

$$(R_1 \circ R_2)(x,y) = \bigvee_{z \in X} (R_1(x,z) \otimes R_2(z,y)). \tag{1}$$

For **L**-sets A and B in X we define degrees $S(A,B) \in L$ and $A \approx B \in L$ as follows:

$$S(A,B) = \bigwedge_{x \in X} (A(x) \to B(x)), \tag{2}$$

$$A \approx B = \bigwedge_{x \in X} (A(x) \leftrightarrow B(x)). \tag{3}$$

$S(A,B)$ is called a degree of subsethood of A in B; $A \approx B$ is called a degree of equality of A and B. Note, that $a \leftrightarrow b$ in (3) is an abbreviation for $(a \to b) \wedge (b \to a)$. Clearly, $A \approx B = S(A,B) \wedge S(B,A)$. Furthermore, we write $A \subseteq B$ (A is a subset of B) if $S(A,B) = 1$, i.e. if $A(x) \leqslant B(x)$ is true for each $x \in X$.

Recall that an **L**-equivalence (fuzzy equivalence) relation E on a set X is a binary **L**-relation on X satisfying (i) $E(x,x) = 1$ (reflexivity), (ii) $E(x,y) = E(y,x)$ (symmetry), and (iii) $E(x,y) \otimes E(y,z) \leqslant E(x,z)$ ($\otimes$-transitivity) for all $x, y, z \in X$. An **L**-equivalence on X where $E(x,y) = 1$ implies $x = y$ is called an **L**-equality.

3. Basic notions

In the following we denote by $\rightharpoonup$ a binary **L**-relation on X. Given $\rightharpoonup$, we define a degree to which a sequence of elements from X is a reduction with respect to $\rightharpoonup$:

Definition 1. For $x_0,\ldots,x_n \in X$ we define a degree re $(x_0,\ldots,x_n) \in L$ by

$$\mathrm{re}(x_0,\ldots,x_n) = \begin{cases} 1, & \text{if } n = 0, \\ \mathrm{re}(x_0,\ldots,x_{n-1}) \otimes x_{n-1} \rightharpoonup x_n, & \text{otherwise.} \end{cases}$$

re $(x_0,\ldots,x_n)$ is called a degree to which $x_0,\ldots,x_n$ is a reduction (w.r.t. $\rightharpoonup$).

Remark 2. Using Definition 1, $\mathrm{re}(x_0,\ldots,x_n) = x_0 \rightharpoonup x_1 \otimes x_1 \rightharpoonup x_2 \otimes \cdots \otimes x_{n-1} \rightharpoonup x_n$. Thus, if $x \rightharpoonup y$ is interpreted as a degree to which x reduces to y then re $(x_0,\ldots,x_n)$ can be seen as a degree to which "x_0 reduces to x_1 and x_1 reduces to x_2 and,…,and x_{n-1} reduces to x_n". Clearly, if $\mathbf{L} = \mathbf{2}$, re $(x_0,\ldots,x_n) = 1$ iff $x_0,\ldots,x_n$ is a reduction in the usual sense.

Example 3. We are going to use weighted oriented graphs to depict **L**-relations. In Fig. 1, a solid arrow from an element x_i to an element x_j means that these two elements are in the **L**-relation represented by the corresponding graph to a nonzero degree. The degree is represented by the weight of the x_ix_j edge. Weights of dotted arrows in Fig. 1 represent the degrees $\mathrm{re}(x_0,\ldots,x_i)$ to which the element x_0 can be reduced to the element x_i. These values are computed according to the definition as conjunctions of weights of all edges in the reduction. Obviously, the values depend also on the used complete residuated lattice. In our case, we assume that $\rightharpoonup_1$ uses the standard Łukasiewicz algebra of truth degrees and that $\rightharpoonup_2$ and $\rightharpoonup_3$ use the Gödel and Goguen conjunctions, respectively. The degree re $(x_0,\ldots,x_5) = 0$ which appears in the graph of $\rightharpoonup_1$ means that the element x_0 cannot be reduced to the element x_5, i.e. the element x_5 cannot substitute the element x_0 to a nonzero degree. If we use Gödel or Goguen conjunctions, $\mathrm{re}(x_0,\ldots,x_i) = 0$ iff $x_j \rightharpoonup x_{j+1} = 0$ for some $j \in \{0,\ldots,i-1\}$. When using Łukasiewicz con-

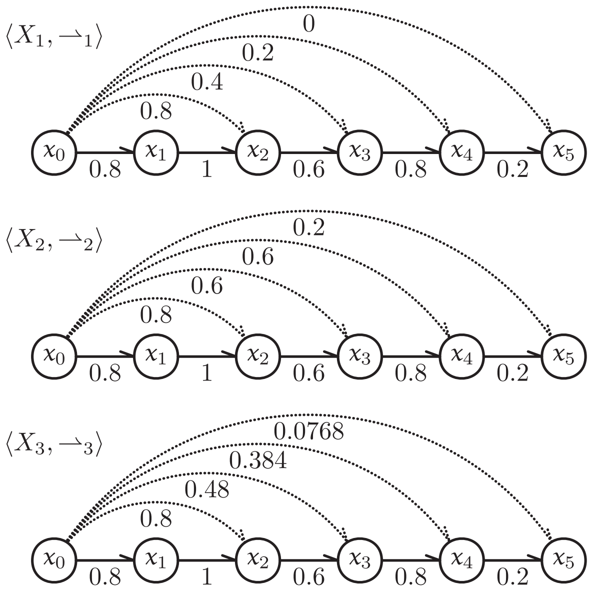


Fig. 1. Reductions.

junction, which has zero-divisors, there may be reductions with re $(x_0,\ldots,x_i) = 0$ and $x_j \rightharpoonup x_{j+1} \neq 0$ for each $j \in \{0,\ldots,i-1\}$ as shown in Fig. 1 (top).

Let us introduce further notation. By $\leftharpoondown$ we denote the inverse **L**-relation to $\rightharpoonup$, i.e. $x \leftharpoondown y = y \rightharpoonup x$. Furthermore, by $\rightleftharpoons$ we denote the symmetric closure of $\rightharpoonup$, i.e. $x \rightleftharpoons y = x \rightharpoonup y \vee y \rightharpoonup x = x \rightharpoonup y \vee x \leftharpoondown y$ (or, in short, $\rightleftharpoons\, = \,\rightharpoonup \cup \leftharpoondown$). In addition to that, by $\rightharpoonup^*$ we denote the reflexive and transitive closure of $\rightharpoonup$ (see, e.g. [2]):

$$x \rightharpoonup^* y = \begin{cases} 1, & \text{if } x = y, \\ \bigvee_{\langle z_1, z_2, \ldots, z_k\rangle \in X^*} \mathrm{re}(x, z_1, \ldots, z_k, y), & \text{otherwise,} \end{cases}$$

where $X^* = \bigcup_{n \in N_0} X^n$, i.e. X^* is a union of all Cartesian powers of the set X (recall that $X^0 = \{\emptyset\}$, i.e. $X^* = \{\emptyset\} \cup X \cup (X \times X) \cup (X \times X \times X) \cup \cdots$). Alternatively, we can write $\rightharpoonup^* = \bigcup_{n=0}^{\infty} \rightharpoonup^n$, where $\rightharpoonup^n$ is defined by $\rightharpoonup^n = \rightharpoonup \circ \rightharpoonup^{n-1} (n \geqslant 1)$, $\rightharpoonup^0$ being the identity **L**-relation (i.e., $x \rightharpoonup^0 x = 1$ for all $x \in X$; $x \rightharpoonup^0 y = 0$ for $x \neq y$). As in the ordinary case, $\rightharpoonup^*$ is the least reflexive and transitive fuzzy relation containing $\rightharpoonup$. The equivalence closure of $\rightharpoonup$, denoted by $\rightleftharpoons^*$, is the reflexive and transitive closure of $\rightleftharpoons$.

4. Convergence and Church–Rosser property

In this section we define convergence of elements and Church–Rosser property of binary **L**-relations. These notions will be defined as graded generalizations of their classical counterparts. Recall that in the classical case, elements x and y are convergent if they are reducible (according to a binary relation $\rightsquigarrow$) to a common element. That means there is z such that $x \rightsquigarrow^* z$ and $y \rightsquigarrow^* z$. In graded setting, we define degree of convergence of x and y as a degree to which there exists a z such that both x and y are reducible to z. The graded Church–Rosser property will be based on graded convergence and convertibility:

Definition 4. For $x, y \in X$ we define degree $x \downharpoonleft y \in L$ by

$$x \downharpoonleft y = \bigvee_{z \in X} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z).$$

$x \downharpoonleft y$ is called the *degree of convergence* of x and y. We also say that x and y are convergent to degree $x \downharpoonleft y$. If $x \downharpoonleft y = 1$ we say that x and y are convergent. For x and y, $x \rightleftharpoons^* y$ is called the *degree to which x and y are convertible*. The degree CR $(\rightharpoonup)$ to which $\rightharpoonup$ has the *Church–Rosser property* is defined by CR $(\rightharpoonup) = S(\rightleftharpoons^*, \downharpoonleft)$. If CR $(\rightharpoonup) = 1$ we say that $\rightharpoonup$ has the Church–Rosser property.

Remark 5. Directly from definitions, $\downharpoonleft\, = \,\rightharpoonup^* \circ \leftharpoondown^*$ i.e. $\downharpoonleft$ is a $\circ$-composition of $\rightharpoonup^*$ and $\leftharpoondown^*$, see (1). By definition, $x \downharpoonleft y$ is a degree to which there is z such that $x \rightharpoonup^* z$ and $y \rightharpoonup^* z$. According to the definition of the degree of subsethood, we have

$$\mathrm{CR}(\rightharpoonup) = \bigwedge_{x,y \in X} (x \rightleftharpoons^* y \rightarrow x \downharpoonleft y).$$

Note that if **L** = **2** then (i) $x \downharpoonleft y = 1$ iff x and y are convergent in the usual sense and (ii) CR $(\rightharpoonup) = 1$ iff $\rightharpoonup$ has the (ordinary) Church–Rosser property.

The converbility is illustrated in Fig. 2. In this particular case, the degree $x_0 \rightleftharpoons^* x_7$ to which the elements x_0 and x_7 are convertible is computed as a conjunction of weights of all arrows on the only path from x_0 to x_7 regardless of directions of the arrows, i.e. $x_0 \rightleftharpoons^* x_7 = a_0 \otimes a_1 \otimes a_2 \otimes a_3 \otimes a_4 \otimes a_5 \otimes a_6$.

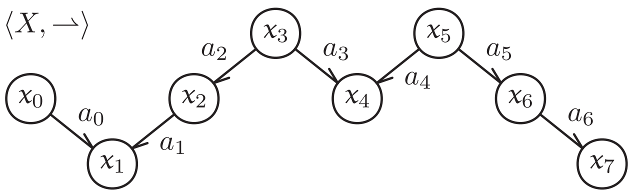
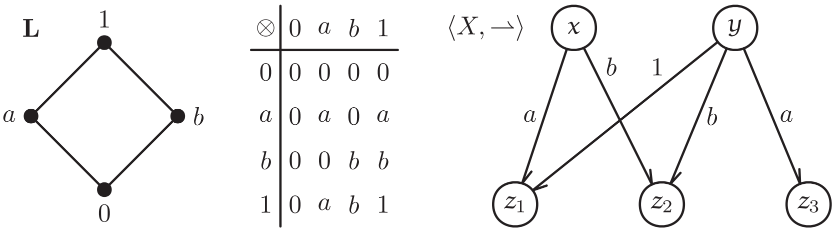


Fig. 2. Illustration of graded convertibility.



⊗	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Fig. 3. Relation with not linearly ordered residuated lattice.

Example 6. Note that the degree $x{\downharpoonleft}y$ of convergence may be strictly greater than $x \rightharpoonup^* z \otimes y \rightharpoonup^* z$ for all $z \in X$. Indeed, it is easy to find an example with $x{\downharpoonleft}y > x \rightharpoonup^* z \otimes y \rightharpoonup^* z$ for any $z \in X$. If **L** is not linearly ordered, it suffices to take a finite X. For instance taking $\rightharpoonup$ from Fig. 3, we get $x \rightharpoonup^* z_1 \otimes y \rightharpoonup^* z_1 = a$, $x \rightharpoonup^* z_2 \otimes y \rightharpoonup^* z_2 = b$ and $x \rightharpoonup^* z_3 \otimes y \rightharpoonup^* z_3 = 0$, but $x \downharpoonleft y = \bigvee\{0, a, b\} = 1$.

If **L** is a chain and $\rightharpoonup^*$ takes only a finite number of values from L then $x \downharpoonleft y = x \rightharpoonup^* z \otimes y \rightharpoonup^* z$ for some $z \in X$; this is the case if either L or X are finite. In a more general setting, if **L** is a Noetherian chain then for any x and y there is z' such that

$$x{\downharpoonleft}y = \bigvee_{z\in X}(x \rightharpoonup^* z \otimes y \rightharpoonup^* z) = x \rightharpoonup^* z' \otimes y \rightharpoonup^* z'.$$

This is a consequence of the facts that $\bigvee_{z\in X}(x \rightharpoonup^* z \otimes y \rightharpoonup^* z)$ can be expressed as a supremum over finitely many elements $z \in X$ and that **L** is linearly ordered.

The following assertion shows that the degree to which a binary **L**-relation $\rightharpoonup$ has the Church–Rosser property equals to the degree to which the convertibility **L**-relation $\rightleftharpoons^*$ equals the convergence **L**-relation $\downharpoonleft$.

Theorem 7. *CR* $(\rightharpoonup) = \rightleftharpoons^* \approx \downharpoonleft$.

Proof. Since CR $(\rightharpoonup) = S(\rightleftharpoons^*, \downharpoonleft)$ and $\rightleftharpoons^* \approx \downharpoonleft$ equals $S(\rightleftharpoons^*, \downharpoonleft) \wedge S(\downharpoonleft, \rightleftharpoons^*)$, in order to prove CR $(\rightharpoonup) = \rightleftharpoons^* \approx \downharpoonleft$, we have to check that $S(\downharpoonleft, \rightleftharpoons^*) = 1$, i.e. $\downharpoonleft \subseteq \rightleftharpoons^*$.

First, observe that $\rightharpoonup \subseteq \rightleftharpoons$ and $\leftharpoondown \subseteq \rightleftharpoons$, i.e. $x \rightharpoonup y \leqslant x \rightleftharpoons y$ and $x \leftharpoondown y \leqslant x \rightleftharpoons y$ are true for every $x, y \in X$. The latter inequalities and the monotony of the reflexive and transitive closure * yield $\rightharpoonup^* \subseteq \rightleftharpoons^*$ and $\leftharpoondown^* \subseteq \rightleftharpoons^*$.

Furthermore, since $\rightleftharpoons^*$ is transitive, we have $\rightleftharpoons^* \circ \rightleftharpoons^* \subseteq \rightleftharpoons^*$. Using $\downharpoonleft = \rightharpoonup^* \circ \leftharpoondown^*$ (see Remark 5) we get $\downharpoonleft = \rightharpoonup^* \circ \leftharpoondown^* \subseteq \rightleftharpoons^* \circ \rightleftharpoons^* \subseteq \rightleftharpoons^*$, proving the claim. □

Remark 8. Theorem 7 generalizes the classical theorem which is well known in the theory of abstract rewriting systems [1,17] in the following way. If we let $\mathbf{L} = \mathbf{2}$, Theorem 7 is equivalent to saying that CR $(\rightharpoonup) = 1$ iff $\rightleftharpoons^* = \downharpoonleft$, i.e. $\rightharpoonup$ has the Church–Rosser property iff the convertibility and convergence relations $\rightleftharpoons^*$ and $\downharpoonleft$ coincide.

5. Divergence and confluence

In this section we first introduce a graded divergence and then a graded confluence of binary **L**-relations and investigate their properties. The main goal of the section is to show a correspondence between the graded confluence and the graded Church–Rosser property.

Definition 9. For $x, y \in X$ we define degree $x \upharpoonleft y \in L$ by

$$x{\upharpoonleft}y = \bigvee_{z\in X}(z \rightharpoonup^* x \otimes z \rightharpoonup^* y).$$

$x \upharpoonleft y$ is called the *degree of divergence* of x and y. We also say that x and y are divergent to degree $x \upharpoonleft y$. If $x \upharpoonleft y = 1$ we say that x and y are divergent. The degree CFL $(\rightharpoonup)$ to which $\rightharpoonup$ is *confluent* is defined by CFL $(\rightharpoonup) = S(\upharpoonleft, \downharpoonleft)$. If CFL $(\rightharpoonup) = 1$ we say that $\rightharpoonup$ is confluent.

Remark 10. Analogously as in case of convergence, we have $\upharpoonleft = \leftharpoondown^* \circ \rightharpoonup^*$, i.e. $\upharpoonleft$ is a $\circ$-composition of $\leftharpoondown^*$ and $\rightharpoonup^*$. The degree $x{\upharpoonleft}y$ can be interpreted as a degree to which there is z such that $z \rightharpoonup^* x$ and $z \rightharpoonup^* y$. Using graded subsethood,

$$\mathrm{CFL}(\rightharpoonup) = \bigwedge_{x,y\in X}(x{\upharpoonleft}y \to x{\downharpoonleft}y).$$

As a consequence, if $\rightharpoonup$ is confluent then $x \upharpoonleft y \leqslant x{\downharpoonleft}y$ for all $x, y \in X$, i.e. $\upharpoonleft \subseteq \downharpoonleft$. Described verbally, the degree of confluence is a degree to which the following is true: "if any x and y are divergent then x and y are convergent". Note that if $\rightharpoonup$ is an **L**-equivalence then $\rightharpoonup = \rightleftharpoons^* = \downharpoonleft = \upharpoonleft$, i.e. CR $(\rightharpoonup) =$ CFL $(\rightharpoonup) = 1$.

The following assertion says that under an additional condition of idempotency, the degree CFL $(\rightharpoonup)$ to which $\rightharpoonup$ is confluent equals to the degree CR $(\rightharpoonup)$ to which $\rightharpoonup$ has the Church–Rosser property.

Theorem 11. *If CFL* $(\rightharpoonup)$ *is an idempotent element of* **L** *then CR* $(\rightharpoonup) =$ *CFL* $(\rightharpoonup)$.

Proof. "CR $(\rightharpoonup) \leqslant$ CFL $(\rightharpoonup)$": We have to show $S(\rightleftharpoons^*, \downharpoonleft) \leqslant S(\upharpoonleft, \downharpoonleft)$. This follows easily from the antitony of the graded subsethood (2) in its first argument and from $\upharpoonleft = \leftharpoondown^* \circ \rightharpoonup^* \subseteq \rightleftharpoons^*$.

"CFL $(\rightharpoonup) \leqslant$ CR $(\rightharpoonup)$": We have to show $S(\upharpoonleft, \downharpoonleft) \leqslant S(\rightleftharpoons^*, \downharpoonleft)$ which is true due to the adjointness iff for each $x, y \in X$ we have $x \rightleftharpoons^* y \otimes S(\upharpoonleft, \downharpoonleft) \leqslant x{\downharpoonleft}y$. By definition of $\rightleftharpoons^*$, the latter is true iff for each $z_1, \ldots, z_k \in X$ we have

$$x \rightleftharpoons z_1 \otimes z_1 \rightleftharpoons z_2 \otimes \cdots \otimes z_k \rightleftharpoons y \otimes S(\upharpoonleft, \downharpoonleft) \leqslant x \rightharpoonup^* \circ \leftharpoonup^* y.$$

Due to the definition of $\rightleftharpoons$ we have

$$\bigvee_{*_1 \in \{\rightharpoonup, \leftharpoonup\}} (x *_1 z_1) \otimes \bigvee_{*_2 \in \{\rightharpoonup, \leftharpoonup\}} (z_1 *_2 z_2) \otimes \cdots \otimes \bigvee_{*_{k+1} \in \{\rightharpoonup, \leftharpoonup\}} (z_k *_{k+1} y) \otimes S(\upharpoonleft, \downharpoonleft) \leqslant x \rightharpoonup^* \circ \leftharpoonup^* y,$$

which is equivalent to

$$\bigvee_{*_1, \ldots, *_{k+1} \in \{\rightharpoonup, \leftharpoonup\}} ((x *_1 z_1) \otimes (z_1 *_2 z_2) \otimes \cdots \otimes (z_k *_{k+1} y)) \otimes S(\upharpoonleft, \downharpoonleft) \leqslant x \rightharpoonup^* \circ \leftharpoonup^* y,$$

which holds true iff for every $*_1, \ldots, *_{k+1} \in \{\rightharpoonup, \leftharpoonup\}$ we have

$$(x *_1 z_1) \otimes (z_1 *_2 z_2) \otimes \cdots \otimes (z_k *_{k+1} y) \otimes S(\upharpoonleft, \downharpoonleft) \leqslant x \rightharpoonup^* \circ \leftharpoonup^* y. \tag{4}$$

We verify this inequality for $*_1$ being $\leftharpoonup$ and $*_{k+1}$ being $\rightharpoonup$ (the other cases are either analogous or easy extensions of our case). Indicating the consecutive sequences of $\leftharpoonup$'s and $\rightharpoonup$'s in the left-hand side of inequality (4), we can write

$$\begin{aligned}(x *_1 z_1) \otimes (z_1 *_2 z_2) \otimes \cdots \otimes (z_k *_{k+1} y) \otimes S(\upharpoonleft, \downharpoonleft) = {} & ((x \leftharpoonup z_{12}) \otimes \cdots \otimes (z_{1l_1} \leftharpoonup z_{21})) \otimes ((z_{21} \rightharpoonup z_{22}) \otimes \cdots \otimes (z_{2l_2} \\ & \rightharpoonup z_{31})) \otimes ((z_{31} \leftharpoonup z_{32}) \otimes \cdots \otimes (z_{3l_3} \leftharpoonup z_{41})) \otimes \cdots \otimes ((z_{2m,1} \\ & \rightharpoonup z_{2m,2}) \otimes \cdots \otimes (z_{2m,l_{2m}} \rightharpoonup y)) \otimes S(\upharpoonleft, \downharpoonleft),\end{aligned}$$

where $2m$ is the number of the sequences and l_i (for $i \in \{1, 2, \ldots, 2m\}$) is the length of the ith sequence. Notice that since we have initially assumed that $*_1$ is $\leftharpoonup$ and $*_{k+1}$ is $\rightharpoonup$, we indeed have an even number of sequences therefore the notation $2m$. So, the left-hand side of the inequality (4) is less than or equal to

$$(x \leftharpoonup^* z_{21}) \otimes (z_{21} \rightharpoonup^* z_{31}) \otimes (z_{31} \leftharpoonup^* z_{41}) \otimes \cdots \otimes (z_{2m,1} \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft).$$

Using the definition of the $\circ$-composition, the latter is less than or equal to

$$(x \leftharpoonup^* \circ \rightharpoonup^* z_{31}) \otimes (z_{31} \leftharpoonup^* \circ \rightharpoonup^* z_{51}) \otimes \cdots \otimes (z_{2m-1,1} \leftharpoonup^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft).$$

Therefore, in order to show the required inequality, it is enough to verify that

$$(x \leftharpoonup^* \circ \rightharpoonup^* z_1) \otimes (z_1 \leftharpoonup^* \circ \rightharpoonup^* z_2) \otimes \cdots \otimes (z_n \leftharpoonup^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) \leqslant x \rightharpoonup^* \circ \leftharpoonup^* y$$

is true for any $z_1, z_2, \ldots, z_n \in X$. We show this by induction over n.

For $n = 0$ we have to show

$$(x \leftharpoonup^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) \leqslant x \rightharpoonup^* \circ \leftharpoonup^* y.$$

According to the definition of the degree of subsethood (2) and using the fact that in every complete residuated lattice $a \otimes (a \to b) \leqslant b$ holds, we have

$$\begin{aligned}(x \leftharpoonup^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) = (x \leftharpoonup^* \circ \rightharpoonup^* y) \otimes \bigwedge_{x,y \in X} ((x \leftharpoonup^* \circ \rightharpoonup^* y) \to (x \rightharpoonup^* \circ \leftharpoonup^* y)) \leqslant (x \leftharpoonup^* \circ \rightharpoonup^* y) \otimes ((x \leftharpoonup^* \circ \rightharpoonup^* y) \\ \to (x \rightharpoonup^* \circ \leftharpoonup^* y)) \leqslant x \rightharpoonup^* \circ \leftharpoonup^* y,\end{aligned}$$

which proves the inequality for $n = 0$.

For $n + 1$, provided the assertion is valid for n and using the idempotency of CFL $(\rightharpoonup) = S(\upharpoonleft, \downharpoonleft)$, we have

$$\begin{aligned}&(x \leftharpoonup^* \circ \rightharpoonup^* z_1) \otimes \cdots \otimes (z_n \leftharpoonup^* \circ \rightharpoonup^* z_{n+1}) \otimes (z_{n+1} \leftharpoonup^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) \\ &\quad = (x \leftharpoonup^* \circ \rightharpoonup^* z_1) \otimes \cdots \otimes (z_n \leftharpoonup^* \circ \rightharpoonup^* z_{n+1}) \otimes S(\upharpoonleft, \downharpoonleft) \otimes (z_{n+1} \leftharpoonup^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) \\ &\quad \leqslant (x \rightharpoonup^* \circ \leftharpoonup^* z_{n+1}) \otimes (z_{n+1} \leftharpoonup^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft).\end{aligned}$$

Using the definition of the $\circ$-composition,

$$\begin{aligned}(x \rightharpoonup^* \circ \leftharpoonup^* z_{n+1}) \otimes (z_{n+1} \leftharpoonup^* \circ \rightharpoonup^* y) \otimes S(\upharpoonleft, \downharpoonleft) &= \bigvee_{u \in X} ((x \rightharpoonup^* u) \otimes (u \leftharpoonup^* z_{n+1})) \otimes \bigvee_{v \in X} ((z_{n+1} \leftharpoonup^* v) \otimes (v \rightharpoonup^* y)) \otimes S(\upharpoonleft, \downharpoonleft) \\ &= \bigvee_{u,v \in X} ((x \rightharpoonup^* u) \otimes (u \leftharpoonup^* z_{n+1}) \otimes (z_{n+1} \leftharpoonup^* v) \otimes (v \rightharpoonup^* y)) \otimes S(\upharpoonleft, \downharpoonleft) \\ &\leqslant \bigvee_{u,v \in X} ((x \rightharpoonup^* u) \otimes (u \leftharpoonup^* v) \otimes (v \rightharpoonup^* y)) \otimes S(\upharpoonleft, \downharpoonleft) \\ &= \bigvee_{u \in X} ((x \rightharpoonup^* u) \otimes \bigvee_{v \in X} ((u \leftharpoonup^* v) \otimes (v \rightharpoonup^* y))) \otimes S(\upharpoonleft, \downharpoonleft) \\ &= \bigvee_{u \in X} ((x \rightharpoonup^* u) \otimes (u \leftharpoonup^* \circ \rightharpoonup^* y)) \otimes S(\upharpoonleft, \downharpoonleft).\end{aligned}$$

Using properties of the subsethood and the facts that $\rightharpoonup^*$ is transitive and $a \otimes (a \to b) \leqslant b$,

$$\begin{aligned}\bigvee_{u\in X}((x\rightharpoonup^* u)\otimes(u\leftharpoondown^*\circ\rightharpoonup^* y))\otimes S(\upharpoonleft,\downharpoonleft) &= \bigvee_{u\in X}((x\rightharpoonup^*u)\otimes(u\leftharpoondown^*\circ\rightharpoonup^*y)\otimes S(\upharpoonleft,\downharpoonleft))\\
&\leqslant \bigvee_{u\in X}((x\rightharpoonup^*u)\otimes(u\leftharpoondown^*\circ\rightharpoonup^*y)\otimes((u\leftharpoondown^*\circ\rightharpoonup^*y)\to(u\rightharpoonup^*\circ\leftharpoondown^*y)))\\
&\leqslant \bigvee_{u\in X}((x\rightharpoonup^*u)\otimes(u\rightharpoonup^*\circ\leftharpoondown^*y)) = \bigvee_{u\in X}((x\rightharpoonup^*u)\otimes\bigvee_{w\in X}((u\rightharpoonup^*w)\otimes(w\leftharpoondown^*y)))\\
&= \bigvee_{u,w\in X}((x\rightharpoonup^*u)\otimes(u\rightharpoonup^*w)\otimes(w\leftharpoondown^*y)) = \bigvee_{w\in X}\left(\bigvee_{u\in X}((x\rightharpoonup^*u)\otimes(u\rightharpoonup^*w))\otimes(w\leftharpoondown^*y)\right)\\
&\leqslant \bigvee_{w\in X}((x\rightharpoonup^*w)\otimes(w\leftharpoondown^*y)) = (x\rightharpoonup^*\circ\leftharpoondown^*y).\end{aligned}$$

Altogether, $(x\rightharpoonup^*\circ\leftharpoondown^* z_{n+1})\otimes(z_{n+1}\leftharpoondown^*\circ\rightharpoonup^* y)\otimes S(\upharpoonleft,\downharpoonleft)\leqslant x\rightharpoonup^*\circ\leftharpoondown^* y$ which concludes the proof. □

Corollary 12. *If in L we have $\otimes = \wedge$, then CR $(\rightharpoonup)$ = CFL $(\rightharpoonup)$.*

Proof. Directly by Theorem 11 using the fact that if $\otimes = \wedge$ then each $a \in L$ is idempotent. □

Theorem 13. *$\rightharpoonup$ has the Church–Rosser property iff it is confluent.*

Proof. Directly by Theorem 11 using the fact that 1 is idempotent. □

Example 14. Fig. 4 contains diagrams of three **L**-relations $\rightharpoonup_1$, $\rightharpoonup_2$, and $\rightharpoonup_3$. Let **L** be the standard Goguen algebra of truth degrees. Then, one can show that CFL $(\rightharpoonup_1)$ = 1, i.e. that $\rightharpoonup_1$ is confluent. In case of $\rightharpoonup_2$, we have CFL $(\rightharpoonup_2)$ = 0.75, i.e. we can say that $\rightharpoonup_2$ is "more or less confluent". On the other hand, CFL $(\rightharpoonup_3)$ = 0.084, i.e. $\rightharpoonup_3$ is practically not confluent at all.

Example 15. Fig. 5 contains a diagram of an **L**-relation $\rightharpoonup$ on a set X of selected colors. The degree $x \rightharpoonup y$ to which x is $\rightharpoonup$-related with y can be interpreted as a degree to which "color y can be substituted for color x". The degrees were taken from an eleven-valued Łukasiewicz algebra, i.e., a finite subchain of the standard Łukasiewicz algebra. The degrees to which colors can be substituted by each other can be determined based on their wavelength and the human perception of "color similarity". Let us note that color perception is a complex neuro-chemical process with a psychological feedback that varies from person to person. The illustration in Fig. 5 and the corresponding $\langle X, \rightharpoonup\rangle$ show how the authors perceive a substitutability of lighter color for darker ones. It can be shown that $\rightharpoonup$ is an **L**-relation with a high degree of confluence. Namely, CFL $(\rightharpoonup)$ = 0.9.

6. Termination

In this section we turn our attention to termination which is considered an important property of abstract rewriting systems. Intuitively, termination can be seen as a natural property of "algorithms", saying that each reduction should terminate

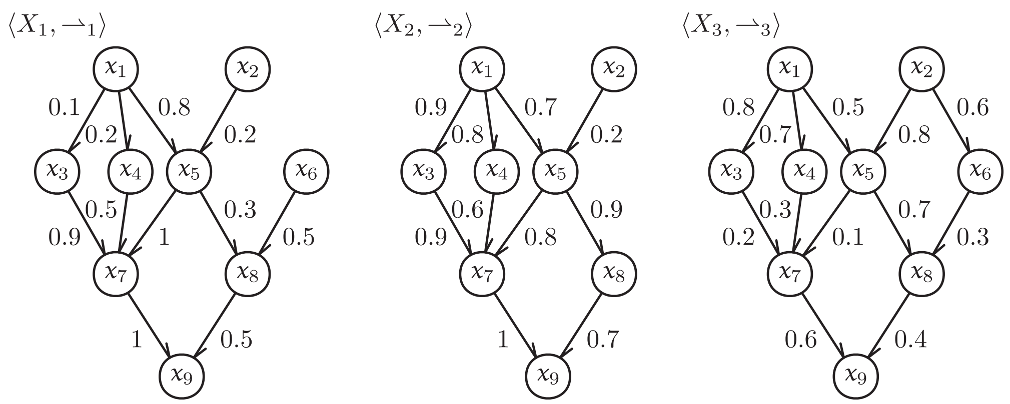


Fig. 4. Relations with different degree of confluence values.

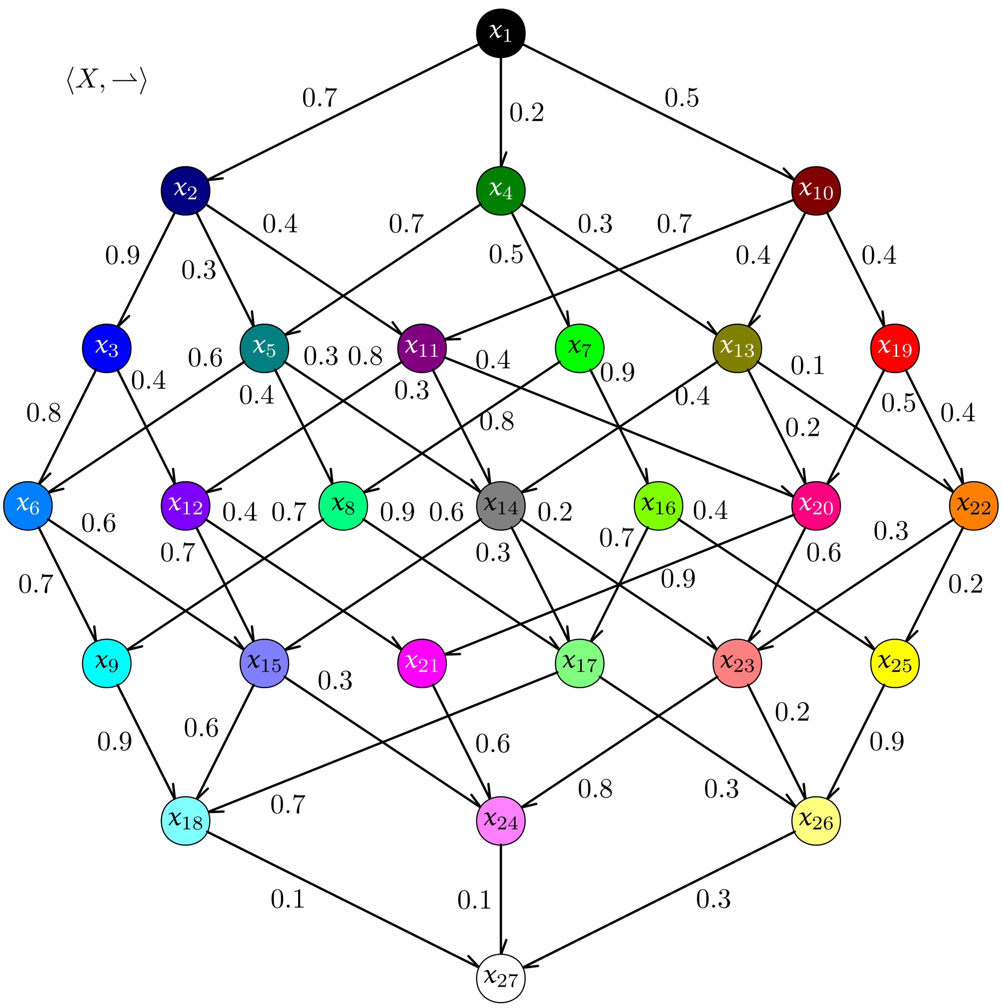


Fig. 5. Highly confluent fuzzy relation "color y can be substituted for color x".

after finitely many steps. We would like to have the same intuitive interpretation of termination in case of fuzzy relations. Therefore, unlike the notions introduced in the previous sections which were naturally graded, termination seems to be a bivalent notion: "A reduction either terminates or not". From the computational point of view, it is desirable that each reduction that is supposed to be handled algorithmically stops after finitely many steps. Following these motivations, we introduce several notions of termination and inspect their properties.

We first introduce a new notation. For any sequence $x_0,\ldots,x_n$ of elements from X, let nt $(x_0,\ldots,x_n) \in L$ denote a degree defined by

$$\mathrm{nt}(x_0,\ldots,x_n) = \bigvee_{y\in X} \mathrm{re}(x_0,\ldots,x_n,y). \tag{5}$$

Remark 16. Following the definition of re $(x_0,\ldots,x_n)$, (5) is equivalent to

$$\mathrm{nt}(x_0,\ldots,x_n) = \mathrm{re}(x_0,\ldots,x_n) \otimes \bigvee_{y\in X}(x_n \rightharpoonup y).$$

Hence, nt $(x_0,\ldots,x_n)$ can be seen as a degree to which the reduction $x_0,\ldots,x_n$ can be extended by an additional element y. If $x \rightharpoonup y$ is interpreted as a degree to which x reduces to y then nt $(x_0,\ldots,x_n)$ is a degree to which "x_0 reduces to x_1 and, …, and x_{n-1} reduces to x_n, and there is $y \in X$ such that x_n reduces to y". If re $(x_0,\ldots,x_n) \neq 0$ and nt $(x_0,\ldots,x_n) = 0$, we may say that $x_0,\ldots,x_n$ cannot be further extended with a nonzero degee of reduction. In a special case for $n = 0$, the latter is true if

$$\mathrm{nt}(x_0) = \bigvee_{y\in X} \mathrm{re}(x_0,y) = \bigvee_{y\in X}(x_0 \rightharpoonup y) = 0.$$

These observations motivate the following definition of termination.

Definition 17. An element $x \in X$ has a *terminating reduction* if there is a finite sequence $x = x_0, x_1, \ldots, x_n$ ($n \geqslant 0$) such that re $(x_0, \ldots, x_n) \neq 0$, and nt $(x_0, \ldots, x_n) = 0$. An element $x \in X$ is called *irreducible* if nt $(x) = 0$. An element $x \in X$ has a *strictly terminating reduction* if there is a terminating reduction $x = x_0, x_1, \ldots, x_n$, where x_n is irreducible. An element $x \in X$ has a *nonterminating reduction* if there is an infinite sequence $x = x_0, x_1, \ldots$, such that for each $n \in \mathbb{N}_0$, nt $(x_0, \ldots, x_n) \neq 0$.

Remark 18

(1) The first requirement in the definition of the terminating reduction, namely re $(x_0, \ldots, x_n) \neq 0$, postulates that the element x_0 can be reduced to the element x_n to a nonzero degree. On the other hand, the second condition, i.e. nt $(x_0, \ldots, x_n) = 0$, says that there is no element, which can further extend this reduction.
(2) Using (5), condition nt $(x) = 0$ can be equivalently written as

$$\mathrm{re}(x) \otimes \bigvee_{y \in X} x \rightharpoonup y = 1 \otimes \bigvee_{y \in X} x \rightharpoonup y = \bigvee_{y \in X} x \rightharpoonup y = 0.$$

Hence, if x is an irreducible element then $x \rightharpoonup y = 0$ is true for each $y \in X$.
(3) An element $x \in X$ has a nonterminating reduction iff there is an infinite sequence $x = x_0, x_1, \ldots$ such that for each $n \in \mathbb{N}_0$, re $(x_0, \ldots, x_n) \neq 0$. Intuitively, the element x can be reduced infinitely many times.
(4) Each strictly terminating reduction is terminating. The converse claim does not hold.

Example 19. Consider the **L**-relation $\rightharpoonup$ from Fig. 6 and let **L** be the Łukasiewicz structure. The element x_0 has a reduction $x_0x_1x_2$ which is terminating, i.e. nt $(x_0, x_1, x_2) = 0$. On the other hand, the element x_2 is not irreducible since nt $(x_2) = 0.4 \neq 0$. Hence, this reduction is not strictly terminating.

Since termination and strict termination of fuzzy relations have been introduced as bivalent notions, we should investigate their relationship to the (ordinary) termination of bivalent relations. The following assertion shows that termination of fuzzy relations can be observed from their strong 0-cuts.

Theorem 20. *The following are true for any binary* **L***-relation* $\rightharpoonup$ *on X.*

(i) If an element x has a strictly terminating reduction, then x has a terminating reduction in the strong 0-cut $\rightsquigarrow$ *of* $\rightharpoonup$.
(ii) If an element x has a nontermination reduction, then x has a nonterminating reduction in the strong 0-cut $\rightsquigarrow$ *of* $\rightharpoonup$.

Proof

(i): Denote by $\rightsquigarrow$ the strong 0-cut of $\rightharpoonup$. Let x have a strictly terminating reduction $x = x_0, x_1, \ldots, x_n$. By definition, we have re$(x_0, \ldots, x_n) \neq 0$ and nt $(x_n) = 0$. The inequality re$(x_0, \ldots, x_n) \neq 0$ yields $(x_0 \rightharpoonup x_1) \otimes \ldots \otimes (x_{n-1} \rightharpoonup x_n) \neq 0$, i.e. $x_i \rightharpoonup x_{i+1} \neq 0$ for each $i \in \{0, \ldots, n-1\}$. Therefore, in the strong 0-cut $\rightsquigarrow$ of $\rightharpoonup$, we have $x_i \rightsquigarrow x_{i+1}$ for each $i \in \{0, \ldots, n-1\}$, i.e. there is a reduction $x = x_0, x_1, \ldots, x_n$. From nt $(x_n) = 0$ it follows that $x_n \rightharpoonup y = 0$ for each $y \in X$. Thus, there is no element $y \in X$ such that $x_n \rightsquigarrow y$, i.e. x_n is an irreducible element in the strong 0-cut $\rightsquigarrow$. Altogether, the reduction $x_0, x_1, \ldots, x_n$ in $\rightsquigarrow$ is terminating.
(ii): Let x have a nonterminating reduction $x = x_0, x_1, \ldots$ By Remark 18, for each $n \in \mathbb{N}_0$, re $(x_0, \ldots, x_n) \neq 0$. Hence, $(x_0 \rightharpoonup x_1) \otimes (x_1 \rightharpoonup x_2) \otimes, \cdots, \neq 0$, i.e. $x_i \rightharpoonup x_{i+1} \neq 0$ is true for each $i \in \mathbb{N}_0$. Therefore, for each $i \in \mathbb{N}_0$, we get $x_i \rightsquigarrow x_{i+1}$, i.e. $x_0, x_1, \ldots$, is a nonterminating reduction in the strong 0-cut $\rightsquigarrow$. □

Remark 21. The converse claim to Theorem 20 does not hold in general. Consider $\rightharpoonup_1$ and $\rightharpoonup_2$ from Fig. 7 and their strong 0-cuts $\rightsquigarrow_1$ and $\rightsquigarrow_2$, respectively. Suppose that **L** is the Łukasiewicz structure of truth degrees. In these two cases, $x_0 \in X_1$ has a terminating reduction $x_0x_1x_2x_3$ in the strong 0-cut but re $(x_0, x_1, x_2, x_3) = 0$, i.e. it is not a reduction with respect to $\rightharpoonup_1$. Furthermore, $x_0 \in X_2$ has a nonterminating reduction $x_0, x_1, x_2, \ldots$ in the strong 0-cut but it does not have any nonterminating reduction with respect to $\rightharpoonup_2$.

We now introduce (strict) termination as a property of **L**-relations:

Definition 22. An **L**-relation $\rightharpoonup$ on X is called *terminating* if no $x \in X$ has a nonterminating reduction. Moreover, $\rightharpoonup$ is called *strictly terminating* if (i) $\rightharpoonup$ is terminating, and (ii) for each $x \in X$: if x has a terminating reduction $x = x_0, \ldots, x_n$ then x_n is irreducible.

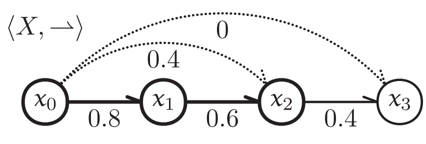


Fig. 6. Terminating reduction which is not strictly terminating.

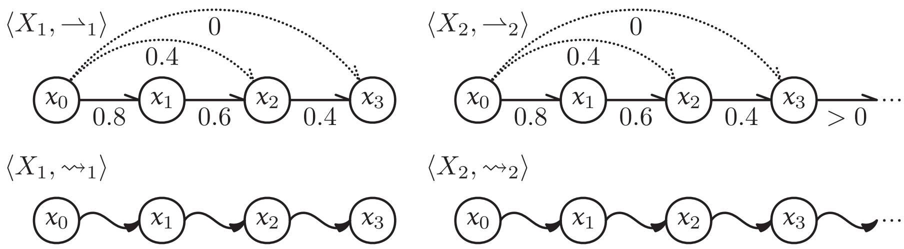


Fig. 7. Counterexamples to the converse claim to Theorem 20.

Remark 23. By definition, each strictly terminating **L**-relation is terminating, the converse does not hold in general (see Example 19).

Theorem 24. *If* $\rightharpoonup$ *is terminating then each* $x \in X$ *has a terminating reduction.*

Proof. The proof is done by showing that x has a terminating reduction $y_0,\ldots,y_k$. We construct the sequence $y_0,\ldots,y_k$ incrementally provided that in each jth step we already have $y_0,\ldots,y_j$ with re $(y_0,\ldots,y_j) \neq 0$.

Put $y_0 = x$. Directly from the definition, re $(y_0) = 1$. Let us have $y_0,\ldots,y_j$ with re $(y_0,\ldots,y_j) \neq 0$. If nt $(y_0,\ldots,y_j) = 0$, we are done for $y_0,\ldots,y_j$ is a terminating reduction. In the other case, we have $\bigvee_{y\in X}$re $(y_0,\ldots,y_j,y) \neq 0$, i.e. there is $y \in X$ such that re $(y_0,\ldots,y_j,y) \neq 0$. We can put $y_{j+1} = y$. Clearly, we cannot repeat the above procedure infinitely many times without getting a sequence with nt $(y_0,\ldots,y_k) = 0$, because x does not have any nonterminating reductions.

7. Inductive properties and well-foundedness

We shall investigate further properties of fuzzy relations which are closely related to termination. We are motivated by the fact that in the ordinary case, a relation terminates iff it is well-founded which is iff the relation obeys the principle of Noetherian induction. In this section we first present a generalization of inductive properties and well-foundedness and then investigate their relationship to our notions of termination. We conclude the section by presenting applications of well-foundedness. Namely, we introduce a notion of a local confluence and prove its relationship to confluence which is analogous to that known from the ordinary case.

Definition 25. An **L**-set $\mathcal{P} \in L^X$ is called a *property*. For any subset $B \subseteq X$ we define a degree $\|\mathcal{P}\|_B \in L$ to which $\mathcal{P}$ holds in B by $\|\mathcal{P}\|_B = \bigwedge_{b\in B}\mathcal{P}(b)$. An **L**-set $\mathcal{P} \in L^X$ is called an *inductive property* with the respect to an **L**-relation $\rightharpoonup$, if

$$\bigwedge_{y\in X}(x \rightharpoonup y \rightarrow \mathcal{P}(y)) \leqslant \mathcal{P}(x), \qquad \text{for each } x \in X. \tag{6}$$

Remark 26. An inductive property as it is established in Definition 25 is a graded generalization of the classical inductive property. Indeed, the formula used in the previous definition corresponds to the first-order predicate formula

$$(\forall x)((\forall y)(r(x,y) \Rightarrow p(y)) \Rightarrow p(x)), \tag{7}$$

which occurs in the definition of the classical inductive property (denoted by symbol) p on a set X with the respect to a relation (denoted by symbol) r. If (7) is required to be true to degree 1 in a general complete residuated lattice **L**, we obtain exactly the concept from Definition 25. Thus, (6) can be read as follows: "In order to prove that x has property $\mathcal{P}$, it suffices to show that each reduct of x has property $\mathcal{P}$". A finer reading, using degrees of truth may be: "The degree to which x has $\mathcal{P}$ is at least the degree to which all reducts of x have $\mathcal{P}$". Obviously, for **L** = **2**, we obtain the ordinary concept of an inductive property.

Definition 27. For $\emptyset \neq B \subseteq X$, an element $m \in B$ is called $\rightharpoonup$-*minimal* in B if, for each $x \in X$, $m \rightharpoonup x \neq 0$ implies $x \notin B$. Let min $(B) = \{m \in B | m$ is $\rightharpoonup -$ minimal in $B\}$. $\rightharpoonup$ is called *well-founded* **L**-relation on X if, for each $\emptyset \neq B \subseteq X$, min $(B) \neq \emptyset$.

Remark 28. Well-foundedness is again a generalization of the corresponding ordinary notion for fuzzy relations. In addition to that, the fact that $\rightharpoonup$ is well-founded can be observed from its strong 0-cut. Indeed, $\rightharpoonup$ is a well-founded **L**-relation on X iff each $B \subseteq X$ has at least one $\rightharpoonup$-minimal element which is true iff each $B \subseteq X$ has at least one $\rightsquigarrow$-minimal element in the strong 0-cut $\rightsquigarrow$ of $\rightharpoonup$. This follows directly from the fact that $m \rightharpoonup x \neq 0$ is true iff $m \rightsquigarrow x$, i.e. iff m is related with x in the strong 0-cut $\rightsquigarrow$ of $\rightharpoonup$. For **L** being **2**, Definition 27 yields the classical notion of well-foundedness.

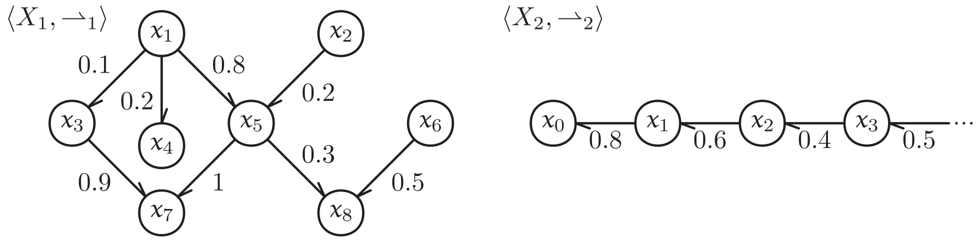


Fig. 8. Well-founded **L**-relations.

Example 29. Take $\rightharpoonup_1$ and $\rightharpoonup_2$ from Fig. 8. Both $\rightharpoonup_1$ and $\rightharpoonup_2$ are well-founded. It is easy to show that every subset of X_1 has a $\rightharpoonup_1$-minimal element. The **L**-relation $\rightharpoonup_2$ is an example of a well-founded **L**-relation on an infinite set. Observe that the well-foundedness is not influenced by the choice of a structure of truth degrees. The **L**-relation $\rightharpoonup_1$ from Fig. 9 is an example of an **L**-relation which is not well-founded.

The following theorem generalizes the well-known principle of Noetherian induction:

Theorem 30. *Let* $\rightharpoonup$ *be well-founded. If* $\mathcal{P}$ *is an inductive property then* $\|\mathcal{P}\|_X = 1$.

Proof. Put $B = \{x \in X | \mathcal{P}(x) = 1\}$ and assume $X - B \neq \emptyset$. The well-foundedness of $\rightharpoonup$ yields that $\min(X - B) \neq \emptyset$, i.e. there is a $\rightharpoonup$-minimal element $m \in \min(X - B) \subseteq X - B$. Observe that if $m \rightharpoonup y = 0$, we get $m \rightharpoonup y \rightarrow \mathcal{P}(y) = 1$, because $0 \rightarrow a = 1$ is true for any $a \in L$. Suppose that $m \rightharpoonup y \neq 0$. Since m is a $\rightharpoonup$-minimal element in $X - B$, $m \rightharpoonup y \neq 0$ yields $y \notin X - B$, i.e. $y \in B$. By definition of B, we get $\mathcal{P}(y) = 1$. As a consequence, $m \rightharpoonup y \rightarrow \mathcal{P}(y) = 1$, because $a \rightarrow 1 = 1$ is true for any $a \in L$. Altogether, the equality $m \rightharpoonup y \rightarrow \mathcal{P}(y) = 1$ is true for any $y \in X$. Therefore, $\bigwedge_{y \in X}(m \rightharpoonup y \rightarrow \mathcal{P}(y)) = 1$. Since $\mathcal{P}$ is supposed to be an inductive property, $1 = \bigwedge_{y \in X}(m \rightharpoonup y \rightarrow \mathcal{P}(y)) \leqslant \mathcal{P}(m)$, i.e. $\mathcal{P}(m) = 1$ which violates $m \in X - B$. Thus, we have $B = X$ and hence $\|\mathcal{P}\|_X = 1$. □

We now investigate the relationship between well-foundedness, principle of Noetherian induction, and termination.

Theorem 31. *Each well-founded fuzzy relation is terminating.*

Proof. Let $\rightharpoonup$ be well-founded. By contradiction, suppose $\rightharpoonup$ is not terminating, i.e. there is $x \in X$, which has a nonterminating reduction $x = x_0, x_1, \ldots$ Since $B = \{x_i | i \in \mathbb{N}_0\} \neq \emptyset$ and $\rightharpoonup$ is well-founded, there is $j \in \mathbb{N}_0$ such that $x_j \in \min(B)$. Since $x = x_0, \ldots, x_j, \ldots$ is a nonterminating reduction, from $\mathrm{nt}(x_0, \ldots, x_j, x_{j+1}) \neq 0$ it follows that $x_j \rightharpoonup x_{j+1} \neq 0$, i.e. we have $x_{j+1} \notin B$ by $\rightharpoonup$-minimality of x_j which is a contradiction. □

Theorem 32. *Each strictly terminating fuzzy relation is well-founded.*

Proof. The claim follows from the properties of termination of the bivalent relations. If $\rightharpoonup$ is strictly terminating then the corresponding strong 0-cut $\rightsquigarrow$ is terminating in the classical sense. This follows from Theorem 20. As a consequence, $\rightsquigarrow$ is well-founded in the classical sense. By definition, for every subset $\emptyset \neq Y \subseteq X$, there is a $\rightsquigarrow$-minimal element m, i.e. $m \rightsquigarrow x$ implies $x \notin Y$. Therefore, $m \rightharpoonup x \neq 0$ implies $x \notin Y$, i.e. m is $\rightharpoonup$-minimal in Y and $\rightharpoonup$ is well-founded. □

Example 33. The converse claims to Theorems 31 and 32 do not hold in general. Counterexamples can be found in Fig. 9. For instance, consider $\rightharpoonup_2$ from Fig. 9 and the Łukasiewicz structure of truth degrees. We got that $\rightharpoonup_2$ is well-founded and is not strictly terminating. It is easy to show that every subset of X_2 has a $\rightharpoonup_2$-minimal element but terminating reduction $x_0x_1x_2$ is not strictly terminating.

The following corollary summarizes previous observations on termination of fuzzy relations and their corresponding strong 0-cuts.

Corollary 34. *If* **L** *has no zero-divisors then* $\rightharpoonup$ *is terminating iff* $\rightharpoonup$ *is strictly terminating iff* $\rightharpoonup$ *is well-founded iff* $\rightsquigarrow$ *is well-founded iff* $\rightsquigarrow$ *is terminating.*

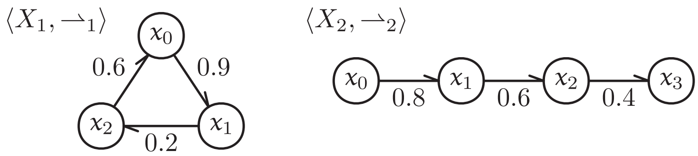


Fig. 9. Terminating **L**-relation which is not well-founded, well-founded **L**-relation which is not strictly terminating (**L** is a Łukasiewicz structure).

Proof. The claim follows from the fact that if **L** has no zero-divisors then re $(x_0,\ldots,x_n) \neq 0$ and nt $(x_0,\ldots,x_n) = 0$ mean that, for each $y \in X$, $x_n \rightharpoonup y = 0$. Hence, $\rightharpoonup$ is terminating strictly. The rest follows from Theorems 32, 20, and properties of the classical termination. □

Finally, we present an assertion showing the equivalence of well-foundedness, termination, and Noetherian induction under the assumption of no zero divisors.

Theorem 35. *Let* **L** *has no zero-divisors. Then the following conditions are equivalent.*

(i) $\rightharpoonup$ *is well-founded;*
(ii) *for each inductive property* $\mathcal{P}$ *with respect to* $\rightharpoonup$ *we have* $\|\mathcal{P}\|_X = 1$;
(iii) $\rightharpoonup$ *is terminating.*

Proof

"$(i) \Rightarrow (ii)$": Apply Theorem 30.

"$(ii) \Rightarrow (iii)$": Consider property $\mathcal{P} \in L^X$ such that $\mathcal{P}(x) = 1$ if x does not have a nonterminating reduction; $\mathcal{P}(x) = 0$ otherwise. We first show that the property $\mathcal{P}$ is inductive, i.e. $\bigwedge_{y\in X}(x \rightharpoonup y \to \mathcal{P}(y)) \leqslant \mathcal{P}(x)$ is true for any $x \in X$. Since $\mathcal{P}(x) \in \{0,1\}$, it suffices to check the inequality for all $x \in X$ such that $\mathcal{P}(x) = 0$. Thus, take any $x \in X$ such that $\mathcal{P}(x) = 0$. By definition of $\mathcal{P}$, x has a nonterminating reduction $x, x_1, x_2, \ldots$ Obviously, x_1 has a nonterminating reduction as well, meaning $\mathcal{P}(x_1) = 0$. In addition to that, $x \rightharpoonup x_1 \neq 0$. As a consequence,

$$x \rightharpoonup x_1 \to \mathcal{P}(x_1) = x \rightharpoonup x_1 \to 0 = \bigvee\{c \in L | x \rightharpoonup x_1 \otimes c \leqslant 0\}.$$

Since **L** has no zero-divisors, $x \rightharpoonup x_1 \otimes c \leqslant 0$ iff $x \rightharpoonup x_1 \otimes c = 0$ iff $c = 0$ because $x \rightharpoonup x_1 \neq 0$. Therefore, $\bigvee\{c \in L | x \rightharpoonup x_1 \otimes c \leqslant 0\} = 0$, showing $x \rightharpoonup x_1 \to \mathcal{P}(x_1) = 0$. From this we further get $\bigwedge_{y\in X}(x \rightharpoonup y \to \mathcal{P}(y)) = 0$, proving $\bigwedge_{y\in X}(x \rightharpoonup y \to \mathcal{P}(y)) \leqslant \mathcal{P}(x)$. Hence, we have shown that $\mathcal{P}$ is an inductive property. Therefore, using Theorem 30, we have $\|\mathcal{P}\|_X = 1$, i.e. no $x \in X$ has a nonterminating reduction, i.e. $\rightharpoonup$ is terminating.

"$(iii) \Rightarrow (i)$": Consequence of Corollary 34. □

We conclude this section with observations on confluent and terminating relations. The confluence of classical relations has a simpler characterization if the relation in question is terminating. Namely, there is a notion of a local confluence and each terminating relation is known to be confluent if and only if it is locally confluent. This characterization is beneficial since local confluence is much easier to check. We are going to show that in our setting there is also a notion of local confluence with similar properties.

Definition 36. We define a degree LCFL $(\rightharpoonup)$ to which $\rightharpoonup$ is *locally confluent* by LCFL $(\rightharpoonup) = S(\leftharpoondown \circ \rightharpoonup, \downarrow)$. If LCFL $(\rightharpoonup) = 1$ we say that $\rightharpoonup$ is locally confluent.

Remark 37. Observe the difference between confluence and local confluence. The only technical difference is that the definition of LCFL $(\rightharpoonup)$ involves $\leftharpoondown \circ \rightharpoonup$ instead of $\uparrow = \leftharpoondown^* \circ \rightharpoonup^*$. Clearly, $x \leftharpoondown \circ \rightharpoonup y$ is a degree to which there is z such that $z \rightharpoonup x$ and $z \rightharpoonup y$. Using graded subsethood, LCFL $(\rightharpoonup)$ can be restated as follows:

$$\begin{aligned}\mathrm{LCFL}(\rightharpoonup) &= \bigwedge_{x,y\in X}\left(\bigvee_{z\in X}(z \rightharpoonup x \otimes z \rightharpoonup y) \to x{\downarrow}y\right)\\ &= \bigwedge_{x,y,z\in X}((z \rightharpoonup x \otimes z \rightharpoonup y) \to x{\downarrow}y).\end{aligned}$$

Since $\leftharpoondown \circ \rightharpoonup \ \subseteq \ \leftharpoondown^* \circ \rightharpoonup^* = \uparrow$, we get CFL $(\rightharpoonup) \leqslant$ LCFL $(\rightharpoonup)$ because $\to$ is antitone in the first argument.

Theorem 38. *Let* $\rightharpoonup$ *be a well-founded* **L***-relation. Then CFL* $(\rightharpoonup) = 1$ *iff LCFL* $(\rightharpoonup) = 1$*, i.e.,* $\rightharpoonup$ *is confluent iff* $\rightharpoonup$ *is locally confluent.*

Proof. The "$\Rightarrow$"-part of the claim follows from $\leftharpoondown \circ \rightharpoonup \ \subseteq \ \uparrow$. In order to prove the "$\Leftarrow$"-part, consider the following property:

$$\mathcal{P}(u) = \bigwedge_{x,y\in X}((u \rightharpoonup^* x \otimes u \rightharpoonup^* y) \to x{\downarrow}y).$$

We first prove that $\mathcal{P}$ is an inductive property. The proof is illustrated in Fig. 10. Take arbitrary $u \in X$. If u is irreducible (i.e., u is $\rightharpoonup$-minimal in X) then clearly $\mathcal{P}(u) = 1$. If u is not irreducible, we can proceed as follows. Assume that $\mathcal{P}(v) = 1$ for each v such that $u \rightharpoonup v \neq 0$. We are going to prove that the latter assumption implies $\mathcal{P}(u) = 1$. First, notice that $u \rightharpoonup^* x \otimes u \rightharpoonup^* y \leqslant x{\downarrow}y$ is trivially true if x, y, and u are not distinct. Indeed, if $x = u$, we have

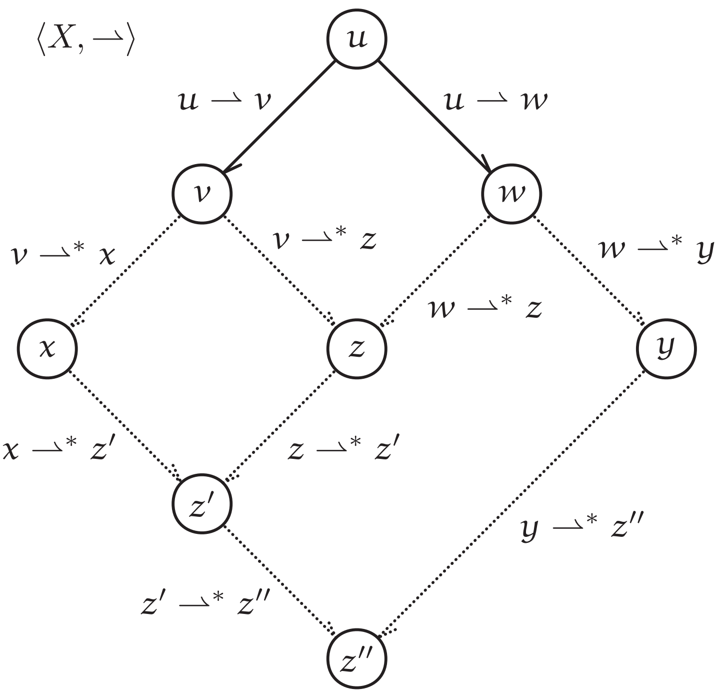


Fig. 10. Illustration for the proof of Theorem 38.

$$u\rightharpoonup^* x \otimes u\rightharpoonup^* y = 1 \otimes u\rightharpoonup^* y = u\rightharpoonup^* y \otimes 1 = u\rightharpoonup^* y \otimes y\rightharpoonup^* y \leqslant \bigvee_{z\in X}(u\rightharpoonup^* z \otimes y\rightharpoonup^* z) = u{\downarrow}y = x{\downarrow}y.$$

The other cases are analogous. Thus, consider $x \neq u \neq y$. Then,

$$u\rightharpoonup^* x \otimes u\rightharpoonup^* y = \bigvee_{v\in X}(u \rightharpoonup v \otimes v\rightharpoonup^* x) \otimes \bigvee_{w\in X}(u \rightharpoonup w \otimes w\rightharpoonup^* y) = \bigvee_{v,w\in X}(u \rightharpoonup v \otimes v\rightharpoonup^* x \otimes u \rightharpoonup w \otimes w\rightharpoonup^* y).$$

Notice that we can assume that the supremum "$\bigvee_{v,w\in X}$" in the last formula ranges over all $v,w \in X$ such that $u \rightharpoonup v \neq 0$ and $u \rightharpoonup w \neq 0$. Indeed, if either $u \rightharpoonup v = 0$ or $u \rightharpoonup w = 0$, we get $u \rightharpoonup v \otimes v \rightharpoonup^* x \otimes u \rightharpoonup w \otimes w \rightharpoonup^* y = 0$, i.e. such v and w are not essential.

Since $\rightharpoonup$ is locally confluent, $u \rightharpoonup v \otimes u \rightharpoonup w \leqslant v{\downarrow}w = \bigvee_{z\in X}(v \rightharpoonup^* z \otimes w \rightharpoonup^* z)$, i.e.

$$u\rightharpoonup^* x \otimes u\rightharpoonup^* y \leqslant \bigvee_{v,w\in X}(\bigvee_{z\in X}(v\rightharpoonup^* z \otimes w\rightharpoonup^* z) \otimes v\rightharpoonup^* x \otimes w\rightharpoonup^* y) = \bigvee_{v,w,z\in X}(v\rightharpoonup^* z \otimes w\rightharpoonup^* z \otimes v\rightharpoonup^* x \otimes w\rightharpoonup^* y).$$

Since $u \rightharpoonup v \neq 0$, $\mathcal{P}(v) = 1$ yields $v\rightharpoonup^* z \otimes v\rightharpoonup^* x \leqslant z{\downarrow}x = \bigvee_{z'\in X}(z\rightharpoonup^* z' \otimes x\rightharpoonup^* z')$. Therefore,

$$u\rightharpoonup^* x \otimes u\rightharpoonup^* y \leqslant \bigvee_{w,z\in X}\left(\bigvee_{z'\in X}(z\rightharpoonup^* z' \otimes x\rightharpoonup^* z') \otimes w\rightharpoonup^* z \otimes w\rightharpoonup^* y\right) = \bigvee_{w,z,z'\in X}(z\rightharpoonup^* z' \otimes x\rightharpoonup^* z' \otimes w\rightharpoonup^* z \otimes w\rightharpoonup^* y)$$
$$\leqslant \bigvee_{w,z'\in X}(w\rightharpoonup^* z' \otimes x\rightharpoonup^* z' \otimes w\rightharpoonup^* y).$$

Moreover, $\mathcal{P}(w) = 1$, i.e. $w\rightharpoonup^* z' \otimes w\rightharpoonup^* y \leqslant z'{\downarrow}y = \bigvee_{z''\in X}(z'\rightharpoonup^* z'' \otimes y\rightharpoonup^* z'')$, i.e.

$$u\rightharpoonup^* x \otimes u\rightharpoonup^* y \leqslant \bigvee_{z'\in X}\left(\bigvee_{z''\in X}(z'\rightharpoonup^* z'' \otimes y\rightharpoonup^* z'') \otimes x\rightharpoonup^* z'\right) = \bigvee_{z',z''\in X}(z'\rightharpoonup^* z'' \otimes y\rightharpoonup^* z'' \otimes x\rightharpoonup^* z') \leqslant \bigvee_{z''\in X}(x\rightharpoonup^* z'' \otimes y\rightharpoonup^* z'') = x{\downarrow}y,$$

showing $(u \rightharpoonup^* x \otimes u \rightharpoonup^* y) \rightarrow x{\downarrow}y = 1$. Since $x,y \in X$ have been taken arbitrarily, we get $\mathcal{P}(u) = \bigwedge_{x,y\in X}((u\rightharpoonup^* x \otimes u\rightharpoonup^* y) \rightarrow x{\downarrow}y) = 1$. Hence, assuming $\mathcal{P}(v) = 1$ for each v such that $u \rightharpoonup v \neq 0$, we have shown $\mathcal{P}(u) = 1$. From the latter observation we directly obtain $\bigwedge_{v\in X}(u \rightharpoonup v \rightarrow \mathcal{P}(v)) = 1 = \mathcal{P}(u)$, meaning that $\mathcal{P}$ is an inductive property. Since $\rightharpoonup$ is well founded, we get $\|\mathcal{P}\|_X = 1$, i.e. $(u \rightharpoonup^* x \otimes u \rightharpoonup^* y) \leqslant x{\downarrow}y$ is true for all $u,x,y \in X$. Hence, $x{\uparrow}y = \bigvee_{u\in X}(u \rightharpoonup^* x \otimes u \rightharpoonup^* y) \leqslant x{\downarrow}y$ is true for all $x,y \in X$, proving that $\rightharpoonup$ is confluent. □

Corollary 39. *If* **L** *has no zero-divisors then a terminating* **L***-relation* $\rightharpoonup$ *is confluent iff* $\rightharpoonup$ *is locally confluent.*

Proof. Follows from Corollary 34 and Theorem 38. □

8. Normal forms and further issues

Terminating and confluent relations play a crucial role in abstract rewriting systems because each element can be rewritten to a unique element in finitely many steps. The unique element is called a normal form. In this section, we present a preliminary study of normal forms and related issues.

Definition 40. Let $x \in X$. An element $y \in X$ is called a *normal form* of x if it satisfies the following property: if x has a terminating reduction $x_0, \ldots, x_n$ then $x_n = y$. The normal form of x (if it exists) is denoted by $\mathrm{nf}(x)$.

Obviously, the normal form $\mathrm{nf}(x)$ of x, if it exists, is determined uniquely. We now show that under the assumption of no zero-divisors each terminating confluent fuzzy relation has a normal form for any element.

Theorem 41. *Let* **L** *have no zero-divisors, let* $\rightharpoonup$ *be terminating and confluent. Then*

(i) each $x \in X$ has the normal form;
(ii) $x \rightharpoonup^* y \neq 0$ implies $\mathrm{nf}(x) = \mathrm{nf}(y)$.

Proof

(i): According to Lemma 24, any $x \in X$ has a terminating reduction. Suppose that x has terminating reductions $x = y_0, \ldots, y_n$ and $x = z_0, \ldots, z_m$. Since **L** has no zero-divisors, $\mathrm{re}(y_0, \ldots, y_n) \neq 0$ and $\mathrm{re}(z_0, \ldots, z_m) \neq 0$ yield $y_n \uparrow z_m \neq 0$. In addition to that, y_n and z_m are irreducible (this also is a consequence of the fact that **L** has no zero-divisors). Using the fact that $\rightharpoonup$ is a confluence, $0 \neq y_n \uparrow z_m \leqslant y_n \downharpoonleft z_m$. Hence, y_n and z_m are convergent to a nonzero degree $y_n \downharpoonleft z_m$ and both y_n and z_m are irreducible. This immediately gives $y_n = z_m = \mathrm{nf}(x)$, proving ($i$).

(ii): Let $x \rightharpoonup^* y \neq 0$. Then x has a terminating reduction

$$x = x_0, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n = \mathrm{nf}(x),$$

where $y = x_i$. Thus, y has a terminating reduction $y = x_i, \ldots, x_n = \mathrm{nf}(x)$. Applying (i), y has a normal form $\mathrm{nf}(y) = x_n$. Therefore, $\mathrm{nf}(x) = \mathrm{nf}(y)$. □

Remark 42. If **L** has no zero-divisors, nf can be seen as a map $\mathrm{nf}: X \to X$ which assigns to each element x from X its normal form $\mathrm{nf}(x)$. It is then convenient to consider an equivalence relation induced by such a map. Namely, we put $x \equiv_{\mathrm{nf}} y$ iff $\mathrm{nf}(x) = \mathrm{nf}(y)$. In the sequel, an equivalence class of $\equiv_{\mathrm{nf}}$ containing x will be denoted by $[x]_{\mathrm{nf}}$, i.e.

$$[x]_{\mathrm{nf}} = \{y \in X \mid \mathrm{nf}(x) = \mathrm{nf}(y)\}.$$

The following assertion shows how normal forms can be used to decide whether $x \rightleftharpoons^* y \neq 0$ and to estimate degrees $x \rightleftharpoons^* y$ to which x and y are convertible.

Theorem 43. *Let* **L** *have no zero-divisors, let* $\rightharpoonup$ *be terminating and confluent. Then*

(i) $x \rightleftharpoons^* y = \bigvee_{z \in [x]_{\mathrm{nf}}} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z)$, and
(ii) $x \rightleftharpoons^* y \neq 0$ iff $\mathrm{nf}(x) = \mathrm{nf}(y)$.

Proof

(i): Due to the Theorem 13, $\rightharpoonup$ has the Church–Rosser property, i.e. $\mathrm{CR}(\rightharpoonup) = 1$. Theorem 7 yields $\rightleftharpoons^* \approx \downharpoonleft = 1$, i.e. $x \rightleftharpoons^* y = x \downharpoonleft y$ for each $x, y \in X$. By Definition 4 and using the fact that $[x]_{\mathrm{nf}} \subseteq X$,

$$x \rightleftharpoons^* y = x \downharpoonleft y = \bigvee_{z \in X} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z) \geqslant \bigvee_{z \in [x]_{\mathrm{nf}}} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z).$$

To prove the converse inequality, observe that for each $z \in X$ with $x \rightharpoonup^* z \otimes y \rightharpoonup^* z \neq 0$ we have $x \rightharpoonup^* z \neq 0$. Using the latter fact, Theorem 41 (ii) yields $\mathrm{nf}(x) = \mathrm{nf}(z)$, i.e. $z \in [x]_{\mathrm{nf}}$. Since $z \in X$ has been taken arbitrarily, we get

$$\bigvee_{z \in X} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z) \leqslant \bigvee_{z \in [x]_{\mathrm{nf}}} (x \rightharpoonup^* z \otimes y \rightharpoonup^* z),$$

proving the equality in (i).

(ii): If $x \rightleftharpoons^* y \neq 0$ then there is $z \in X$ such that $x \rightharpoonup^* z \neq 0$, and $y \rightharpoonup^* z \neq 0$ by Theorems 13 and 7. Furthermore, Theorem 41 (ii) yields $\mathrm{nf}(x) = \mathrm{nf}(z) = \mathrm{nf}(y)$. Conversely, let $\mathrm{nf}(x) = \mathrm{nf}(y)$. Then there are reductions $x, \ldots, \mathrm{nf}(x)$ and $y, \ldots, \mathrm{nf}(y)$ such that $\mathrm{re}(x, \ldots, \mathrm{nf}(x)) \neq 0$ and $\mathrm{re}(y, \ldots, \mathrm{nf}(y)) \neq 0$, respectively. Thus, $x \rightharpoonup^* \mathrm{nf}(x) \neq 0$ and $y \rightharpoonup^* \mathrm{nf}(y) = y \rightharpoonup^* \mathrm{nf}(x) \neq 0$. Since **L** has no zero-divisors, $x \rightharpoonup^* \mathrm{nf}(x) \otimes y \rightharpoonup^* \mathrm{nf}(x) \neq 0$ from which we immediately get that $x \rightleftharpoons^* y = x \downharpoonleft y \neq 0$. □

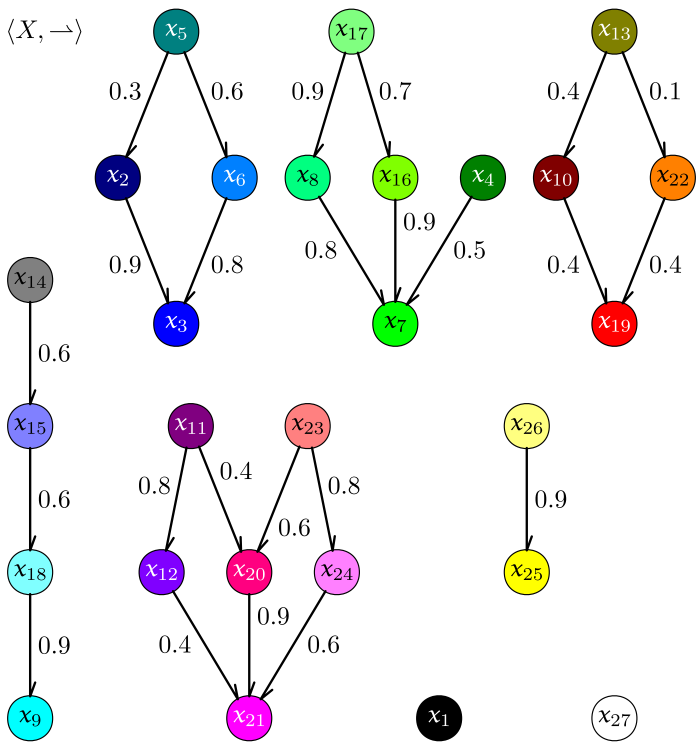


Fig. 11. Color depth reduction.

We get the following.

Corollary 44. *Let* **L** *have no zero-divisors. Then* $\leftrightsquigarrow^* = \equiv_{nf}$, *where* $\leftrightsquigarrow^*$ *is the strong 0-cut of* $\rightleftharpoons^*$ *and* $\equiv_{nf}$ *is defined as in Remark 42.*

Proof. Directly using Theorem 43 (ii). □

Remark 45. Theorem 43 and Corollary 44 allow us to decide whether $x \rightleftharpoons^* y = 0$. If x and y have the same normal form, $x \rightleftharpoons^* y \neq 0$ due to Theorem 43 (ii). In that case, if we wish to get the exact (nonzero) value of the convertibility $x \rightleftharpoons^* y$, we have to go through all elements $z \in [x]_{\mathrm{nf}} = [y]_{\mathrm{nf}}$, i.e. all elements that reduce to $\mathrm{nf}(x) = \mathrm{nf}(y)$ and compute the supremum of all $x \rightharpoonup^* z \otimes y \rightharpoonup^* z$. Since the equivalence class $[x]_{\mathrm{nf}}$ can be large (or even infinite), we can go over just a subset of $[x]_{\mathrm{nf}}$ to obtain a lower estimation of the convertibility degree $x \rightleftharpoons^* y$. Hence, normal forms can be used to compute estimates of degrees $x \rightleftharpoons^* y$ when computing the exact value is expensive.

Note that if $\mathbf{L} = \mathbf{2}$, Theorem 43 (ii) yields $x \rightleftharpoons^* y = 1$ iff $\mathrm{nf}(x) = \mathrm{nf}(y)$, which is the classical property of terminating and confluent relations. In case of $\mathbf{L} = \mathbf{2}$, Theorem 43 (i) collapses with Theorem 43 (ii) because $x \rightleftharpoons^* y = 1$ iff there is $z \in [x]_{\mathrm{nf}}$ such that $x \rightharpoonup^* z = 1$ and $y \rightharpoonup^* z = 1$, which is iff $\mathrm{nf}(x) = \mathrm{nf}(y)$.

Example 46. Consider the **L**-relation $\rightharpoonup$ on the set X of colors from Fig. 11 and let **L** be the Goguen structure of truth degrees. This **L**-relation describes a possible transformation of a picture color map provided one needs a new indexed color map containing 8 basic colors: x_1 (black), x_3 (blue), x_7 (green), x_9 (cyan), x_{19} (red), x_{21} (magenta), x_{25} (yellow), and x_{27} (white). The color x_i of each pixel in the picture will be substituted by its normal form $\mathrm{nf}(x_i)$. The similarity of an original color x_i and the substituted color $\mathrm{nf}(x_i)$ can be computed using Theorem 43 (i). Normal forms can be used to obtain a lower estimation of $x \rightleftharpoons^* y$. For instance, let $x = x_{11}$ and $y = x_{23}$. Since $\mathrm{nf}(x_{11}) = \mathrm{nf}(x_{23}) = x_{21}$ and $x_{21} \in [x_{11}]_{\mathrm{nf}}$, by Remark 45 we have $x_{11} \rightleftharpoons^* x_{23} \geqslant x_{11} \rightharpoonup^* x_{21} \otimes x_{23} \rightharpoonup^* x_{21} = 0.1944$.

9. Conclusions and open problems

We have introduced and studied properties of fuzzy relations which are connected to the idea of rewriting and substituting. The research is motivated by the fact that in many cases, relations which appear in rewriting systems are fuzzy rather than crisp. We have defined convergence, divergence, convertibility, Church–Rosser property, confluence, and local confluence of fuzzy relations and investigated their graded properties. We have shown that such fuzzy relations have analogous properties and mutual relationship as in the ordinary case. In addition to that, we have investigated termination, well-foundedness, and Noetherian induction from the point of view of fuzzy relations. We have outlined normal forms of fuzzy relations and their application in estimation of convertibility degrees. The present paper elaborates fundamental concepts in a theory that can be seen a multiple-valued generalization of results from abstract rewriting systems. Many problems related to this area have not been considered in this paper, e.g.:

- *similarity issues*: Do similar relations have similar degrees of convergence, divergence, confluence, etc.?
- *confluence and termination on sets with fuzzy equality*: Does it make sense to equip X with a relation of fuzzy equivalence and consider, e.g., degrees to which normal forms are similar?
- *properties of derived fuzzy relations*: Given $\rightharpoonup$, can we express, e.g., confluence degrees of $a \otimes \rightharpoonup$ (an a-multiple of $\rightharpoonup$) and $a \to \rightharpoonup$ (an a-shift of $\rightharpoonup$) in terms of the confluence of $\rightharpoonup$ and $a \in L$?
- *links to fuzzy equational logic and logic in narrow sense*: Can we use termination and confluence of fuzzy relations to estimate degree to which two identities are equal in a class of fuzzy structures (i.e., a variety of algebras with fuzzy equalities [4–6])?
- *term-rewriting systems*: Are there algorithms for converting a fuzzy relation $\rightharpoonup_1$ into a terminating confluence $\rightharpoonup_2$ such that $\rightleftharpoons_1^* = \rightleftharpoons_2^*$? Are there analogies of the Knuth-Bendix [17] procedure for fuzzy relations?
- *approximation algorithms and complexity issues*: Are there (efficient) approximation algorithms for computing convertibility degrees? What are their approximation ratios and complexities?

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