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On monotony of fuzzy closure

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ABSTRACT

Closure operators and systems play a significant role in a variety of areas of fuzzy logic. While the condition of monotony for ordinary closure operators has a straightforward form, two basic conditions of monotony naturally arise from the existing examples in a fuzzy setting. To unify these two conditions, two approaches can be found in the literature, one based on the notion of a filter of truth degrees and the other on the notion of a linguistic hedge. We present results connecting these approaches, explore their variants, and provide a notion of monotony that subsumes both the filter-based and hedge-based approaches. We study properties of this general concept of monotony and discuss open problems along with topics for future research.

1. Introduction

Closure operators and systems represent fundamental concepts which appear in many areas of mathematics. Naturally, closure structures play a significant role in a variety of areas of fuzzy logic and its applications; see, e.g., [1, chapter 5]. In particular, various notions of fuzzy closure operators and systems have been explored in particular in the studies of fuzzy topology, fuzzy consequence operators, foundational topics in fuzzy sets and relations, fuzzy orders, and formal concept analysis in a fuzzy setting. In these studies, two basic conditions of monotony for fuzzy closure operators have been proposed.

The first condition prevails, e.g., in fuzzy consequence operators [2–9], fuzzy topology [10] and related works on closure operators [11–14], and involves monotony based on ordinary, bivalent inclusion of fuzzy sets. This monotony condition demands that a fuzzy closure operator C on a universe X, i.e., a certain mapping C sending fuzzy sets in X to fuzzy sets in X, satisfies

$$A \subseteq B \text{ implies } C(A) \subseteq C(B), \tag{1}$$

for all fuzzy sets $A, B \in L^X$. Here, L^X denotes the set of all fuzzy sets in the universe X with membership degrees in an ordered set L, such as $L = [0, 1]$, and $\subseteq$ denotes the ordinary, bivalent inclusion defined for $D, E \in L^X$ by

$$D \subseteq E \quad \text{if and only if} \quad D(x) \leq E(x) \text{ for each element } x \text{ in } X.$$

The second condition prevails, e.g., in explorations of certain foundational topics of fuzzy sets and relations [15,16], in approximate reasoning [17], and formal concept analysis [16,18–20]. The corresponding monotony condition requires

$$S(A, B) \leq S(C(A), C(B)), \tag{2}$$

with S being the graded, many-valued inclusion defined for $D, E \in L^X$ by

$$S(D, E) = \bigwedge_{x \in X} (D(x) \to E(x)),$$

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i.e., as the infimum of the truth degrees $D(x) \to E(x)$ with an appropriate (truth-function of) a many-valued implication $\to$. The graded inclusion has been introduced in fuzzy logic by Goguen [21], but it has already been considered in the early studies of many-valued sets by Klaua [22], and later continued by Gottwald [23]. Notice that from a logical point of view, $S(D,E)$ has a natural interpretation. Namely, using the basic principles of a predicate many-valued logic, $S(D,E)$ is the truth degree of the proposition "for each x, if x is an element of D then x is an element of E."

While the ordinary inclusion $\subseteq$ of fuzzy sets represents an obvious generalization of the classical set inclusion, the graded inclusion S is more elaborate and carries more information. For the commonly used residuated implications, which we employ throughout the paper, the graded inclusion extends the ordinary inclusion in that $A \subseteq B$ if and only if $S(A,B) = 1$. It hence follows that (2) implies (1), i.e., the graded monotony (2) is stronger than the ordinary monotony (1). It is also immediate to observe that in the classical, bivalent framework, i.e., with $L = \{0,1\}$, both the ordinary and the graded monotony are equivalent and become the classical monotony condition. As we shall see, there exist natural fuzzy closure operators satisfying the graded monotony, and a fortiori also the ordinary monotony, as well as operators satisfying the ordinary but not the graded monotony.

The theories of fuzzy closure operators with the ordinary and the graded monotony differ; see, e.g., the properties of the corresponding systems of fixpoints [15]. To enable a unified treatment of these two theories, two ways to generalize the concept of a fuzzy closure oparator have bee proposed in the past. The first one [15] is based on the notion of a filter of truth degrees, i.e., a set of high truth degrees placed around the highest degree 1. The second one is based on the notion of a truth-stressing hedge [24], i.e., a unary logical connective representing intensifying expressions such as "very." We shall show below that these two proposals not only generalize the operators with the ordinary and the graded monotony, but also represent existing fuzzy closure operators that satisfy a monotony condition stronger than the ordinary but weaker than the graded monotony.

As we shall observe, both the notions of fuzzy closure operators—the one based on filters and the one based on hedges—may be regarded as basically arising from a common idea, namely, appropriate modeling of the condition "it is very true that A is included in B," or, "the truth degree of A being included in B is high." Yet, relationships between these two notions have not been studied in the past. The aim of our paper is to explore these relationships and thus contribute to the theory of fuzzy closure operators.

With basic notions recalled in Section 2, we proceed in Section 2.3 with examples justifying and also separating the existing notions of monotony. In Section 3, the filter approach to monotony is extended to more general situations. First, filters are replaced by a more natural and more expressive notion of a fuzzy filter, and a bijective relationship is established for the corresponding closure operators and systems. Next, we observe that the condition of being a fuzzy filter is too strict and present similar results with weaker structures. As the properties of these more general fuzzy sets suggest a relationship with hedges, a notion of a c-hedge is introduced, and the closure structures with monotony based on c-hedges are studied in Section 4. Among other properties, we show that the c-hedge approach represents the most general and thus the most expressive approach to monotony of fuzzy closure, and explore its properties. Conclusions and future work are the content of Section 5.

2. Preliminary notions and considerations

2.1. Fuzzy logic and fuzzy sets

Our formal framework of fuzzy logic is the commonly-used one based on residuated structures of truth degrees [8,16,25,26]. In particular, we use complete residuated lattices [27], introduced to fuzzy logic by Goguen [21], and the resulting calculus of fuzzy sets. This general setting covers many commonly used structures of truth degrees equipped with many-valued logical connectives and the corresponding calculi of fuzzy sets.

Throughout this paper, $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ denotes an arbitrary complete residuated lattice, i.e., a structure in which

- $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being its least and greatest elements, respectively;
- $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e., $\otimes$ is commutative, associative and has 1 as its neutral element;
- $\otimes$ and $\to$ form an adjoint pair, i.e.,

  $a \otimes b \leq c$ if and only if $a \leq b \to c$,

  for all $a, b, c \in L$.

Since complete residuated lattices are well-known, we refer, e.g., to the above-mentioned books for further information. Let us only note that from a logical viewpoint, $\otimes$ and $\to$ play the roles of (the truth functions of) a many-valued conjunction and implication, respectively. Also, the only residuated lattice on the classical set of truth degrees, $L = \{0,1\}$, coincides with the two-element Boolean algebra of classical logic in that $\otimes$ and $\to$ are the classical conjunction and implication, respectively.

For any $\mathbf{L}$, the set of all fuzzy sets in a universe set X, i.e., mappings of X to L, is denoted by L^X. The operations of $\mathbf{L}$ extend in a component-wise manner to the operations on the fuzzy sets in L^X; thus, $(A \otimes B)(x) = A(x) \otimes B(x)$, $(\bigwedge_i \in I A_i)(x) = \bigwedge_{i\in I} A_i(x)$, etc. Important for our considerations are the ordinary inclusion $\subseteq$ and the graded inclusion S defined in Section 1. For more information, we again refer to [8,16,25,26].

2.2. Two basic notions of fuzzy closure and their existing unifications

The two basic notions inspiring our study are the fuzzy closure operators on a set X with the ordinary and the graded monotony which are mentioned in Section 1. Both are mappings $C : L^X \to L^X$ satisfying extensivity, also known as inclusion or inflationarity in the literature, i.e.,

$$A \subseteq C(A) \tag{3}$$

and idempotency, i.e.,

$$C(A) = C(C(A)), \tag{4}$$

for any $A \in L^X$, and the correponding condition of monotony, i.e., (1) for the ordinary monotony and (2) for the graded monotony.

Let us note that in a more general perspective, one may consider closure operators in fuzzy ordered sets [28,29], of which the above fuzzy closure operators on a set X are particular cases if one considers L^X as the universe and the graded inclusion S as the fuzzy order. We, however, do not consider this abstract setting to keep the matter concrete.

Even though fuzzy closure operators with the ordinary monotony are more general than those with the graded monotony, some important phenomena naturally described by the graded monotony cannot be described by the ordinary monotony. There have hence been proposed more general notions of a fuzzy closure operator that generalize both monotonies and allow their unified treatment. We now recall these generalized notions and the basic related results.

The first generalized concept comes from [15] and is based on the notion of a filter in a complete residuated lattice $\mathbf{L}$. An $\leq$-filter in $\mathbf{L}$ is a non-empty subset K of L for which $a \in K$ and $a \leq b$ imply $b \in K$. An $\leq$-filter is called a filter in $\mathbf{L}$ if $a \otimes b \in K$ for all $a, b \in K$.

Definition 1 ([15]). Let K be a filter in $\mathbf{L}$. A mapping $C: L^X \to L^X$ is called an $\mathbf{L}_K$-closure operator if it satisfies (3), (4), and

$$S(A,B) \leq S(C(A),C(B)) \quad \text{whenever } S(A,B) \in K, \tag{5}$$

for all $A, B \in L^X$.

Remark 1. (a) The filter K plays a role of a parameter in the monotony condition (5).

(b) Obviously, for $K = \{1\}$, Definition 1 yields the notion of a fuzzy closure operator with the ordinary monotony. For $K = L$, Definition 1 yields the notion of a fuzzy closure operator with the graded monotony.

A basic property of ordinary closure operators is their natural relationship with closure systems, i.e., systems closed under arbitrary set intersections; see, e.g., [30]. For the present setting, this relationship obtains the following form.

Definition 2 ([15]). A system $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$, $\mathcal{S}$ is called an $\mathbf{L}_K$-closure system on X if for all $A \in L^X$,

$$\bigcap_{i \in I, S(A,A_i) \in K} \big(S(A,A_i) \to A_i\big) \in \mathcal{S}. \tag{6}$$

The one-to-one relationship between closure operators and systems in this setting has been obtained in [15]:

Theorem 1. *Let K be a filter in $\mathbf{L}$, C be an $\mathbf{L}_K$-closure operator on X, and $\mathcal{S} = \{A_i \mid i \in I\}$ be an $\mathbf{L}_K$-closure system on X. Then*

(a) $\mathcal{S}_C = \{A \in L^X \mid C(A) = A\}$ is an $\mathbf{L}_K$-closure system on X;
(b) $C_{\mathcal{S}}(A) = \bigcap_{i \in I, S(A,A_i) \in K} (S(A,A_i) \to A_i)$ is an $\mathbf{L}_K$-closure operator on X;
(c) $C = C_{\mathcal{S}_C}$ and $\mathcal{S}_{C_{\mathcal{S}}}$, i.e., the mappings $C \mapsto \mathcal{S}_C$ and $\mathcal{S} \mapsto C_{\mathcal{S}}$ are mutually inverse.

The second generalization comes from [24] and is based on the notion of a truth-stressing hedge in $\mathbf{L}$, introduced in [31] as a truth function $*: L \to L$ of a unary connective "very true" satisfying

$$1^* = 1,$$
$$a^* \leq a,$$
$$(a \to b)^* \leq a^* \to b^*,$$

for all $a, b \in L$. These conditions imply that $*$ is isotone, i.e., $a \leq b$ entails $a^* \leq b^*$, and also that $a^* \otimes b^* \leq (a \otimes b)^*$ [24,32]. The generalized notion of a closure operator is then defined as follows.

Definition 3 ([24]). Let $*: L \to L$ be a truth-stressing hedge on $\mathbf{L}$. A mapping $C: L^X \to L^X$ is called an $\mathbf{L}^*$-closure operator if it satisfies (3), (4), and

$$S(A,B)^* \leq S(C(A),C(B)) \tag{7}$$

for all $A, B \in L^X$.

The corresponding notion of a closure system is defined as follows.

Definition 4 ([24]). A system $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$, $\mathcal{S}$ is called an $\mathbf{L}^*$-closure system on X if for all $A \in L^X$,

$$\bigcap_{i \in I} \big(S(A,A_i)^* \to A_i\big) \in \mathcal{S}. \tag{8}$$

Remark 2. (a) As with filters, the hedge $*$ plays a role of a parameter in the monotony condition (7).

(b) Obviously, for $*$ being the so-called globalization, i.e., $1^* = 1$ and $a^* = 0$ for $a < 1$, Definition 3 becomes a definition of a fuzzy closure operator with the ordinary monotony. For $*$ being the identity, i.e., $a^* = a$, Definition 3 renders the notion of a fuzzy closure operator with the graded monotony. □

The mutual relationship between closure operators and systems has been obtained in [24]:

Theorem 2. *Let $*$ be a truth-stressing hedge on* $\mathbf{L}$*,* C *be an* $\mathbf{L}^*$*-closure operator on* X*, and* $\mathcal{S} = \{A_i \mid i \in I\}$ *be an* $\mathbf{L}^*$*-closure system on* X*. Then*

(a) $\mathcal{S}_C = \{A \in L^X \mid C(A) = A\}$ *is an* $\mathbf{L}^*$*-closure system on* X*;*
(b) $C_{\mathcal{S}}(A) = \bigcap_{i\in I}(S(A, A_i)^* \to A_i)$ *is an* $\mathbf{L}^*$*-closure operator on* X*;*
(c) $C = C_{\mathcal{S}_C}$ *and* $\mathcal{S}_{C_{\mathcal{S}}}$*, i.e., the mappings* $C \mapsto \mathcal{S}_C$ *and* $\mathcal{S} \mapsto C_{\mathcal{S}}$ *are mutually inverse.*

2.3. Separating and justifying examples

We first consider two well-known kinds of fuzzy closure operators, namely, logical consequence operators and topological operators, and demonstrate even though these operators satisfy the ordinary monotony (1), they fail to satisfy the graded monotony (2). Next, we consider fuzzy closure operators that appear in formal concept analysis of data with fuzzy attributes and show that they satisfy the hedge monotony and, in particular, the graded monotony. Our two last examples separate hedge and filter monotony in that they represent operators satyisfying non-trivial hedge monotony that cannot be expressed using filter monotony, and vice versa.

Example 1. Consider a Pavelka-style propositional logic with the Łukasiewicz connectives [7,8,26]. Recall that a theory in a Pavelka-style fuzzy logic is a fuzzy set T of formulas for which $T(\varphi)$ is interpreted as a truth degree to which formula φ is assumed valid, i.e., may be used in proofs from T etc. The corresponding syntactic entailment operator as defined in [9] is a fuzzy closure operator C that maps a fuzzy set T of formulas to the fuzzy set $C(T)$ of formulas, such that $[C(T)](\varphi)$ equals the degree of provability of the formula φ from the theory T.

Assume, as usual, the graded modus ponens as the deduction rule, i.e., assume the rule

$$\frac{{}^{a}/\varphi,\ {}^{b}/\varphi \Rightarrow \psi}{{}^{a\otimes b}/\psi},$$

which says: from the formula φ that is assumed as valid to the truth degree a and the formula $\varphi \Rightarrow \psi$ assumed to the truth degree b infer that the formula ψ is valid to the truth degree $a \otimes b$.

As is well known [9], C satisfies the ordinary monotony (1). However, as we now show, C does not satisfy the graded monotony (2). Consider $L = [0, 1]$ and the theories T_1 and T_2 given by

$$T_1(p) = 1, T_1(p \Rightarrow q) = 1, \text{ and } T_2(p) = 0.9, T_2(p \Rightarrow q) = 0.9.$$

It follows immediately from the definitions that $[C(T_1)](p) \geq 1$ and $[C(T_1)](p \Rightarrow q) \geq 1$, hence due to modus ponens, also $[C(T_1)](q) \geq 1$, i.e., we have

$$[C(T_1)](p) = 1, [C(T_1)](p \Rightarrow q) = 1, \text{ and } [C(T_1)](q) = 1.$$

Likewise, we obtain $[C(T_2)](p) \geq 0.9$, $[C(T_2)](p \Rightarrow q) \geq 0.9$, and $[C(T_2)](q) \geq 0.8$. To see that, indeed, one also has $[C(T_2)](p) \leq 0.9$, $[C(T_2)](p \Rightarrow q) \leq 0.9$, and $[C(T_2)](q) \leq 0.8$, it suffices to realize that the deduction is sound and that that the evaluation $e(p) = 0.9$, $e(p \Rightarrow q) = 0.9$, $e(q) = 0.8$ is a model of T_2. Thus,

$$[C(T_2)](p) = 0.9, [C(T_2)](p \Rightarrow q) = 0.9, \text{ and } [C(T_2)](q) = 0.8.$$

Now, as one easily checks,

$$S(T_1, T_2) = 0.9 \nleq 0.8 = S(C(T_1), C(T_2)).$$

Hence, C fails to satisfy the graded monotony (2). □

Example 2. Consider the real unit interval $L = [0, 1]$ with the Łukasiewicz connectives. For the set $X = \{a, b, c, d\}$, consider the mapping $C : L^X \to L^X$ defined by

$$C(A) = \begin{cases} \emptyset & \text{if } A = \emptyset, \\ \{a, b\} & \text{if } A \subseteq \{a, b\} \text{ and } A \neq \emptyset, \\ \{a, b, c\} & \text{if } A \subseteq \{a, b, c\} \text{ and } A \nsubseteq \{a, b\}, \\ \{a, b, d\} & \text{if } A \subseteq \{a, b, d\} \text{ and } A \nsubseteq \{a, b\}, \\ X & \text{otherwise.} \end{cases}$$

In this definition, we identify sets with crisp fuzzy sets; thus, e.g., $\{a, b\}$ stands for the crisp fuzzy set A for which $A(a) = 1$, $A(b) = 1$, $A(c) = 0$, and $A(d) = 0$. It is easy to check that—in the terminology of [33]—C is a topological fuzzy closure operator. In particular, this implies that C fulfills extensivity, idempotency, and the ordinary monotony. However, C, does not satisfy the graded monotony (2) as

$$\begin{aligned} S(\{a, b, {}^{0.1}/c\}, \{a, b\}) &= 0.1 \to 0 = 0.9 \nleq 0 = S(\{a, b, c\}, \{a, b\}) \\ &= S(C(\{a, b, {}^{0.1}/c\}), C(\{a, b\})). \end{aligned}$$

□

Example 3. We now present fuzzy closure operators arising in formal concept analysis of data with fuzzy attributes [16]. In particular, we consider the approach in which truth-stressing hedges are used to control the size of the concept lattice associated to the input data [34].

For an arbitrary $\mathbf{L}$ and hedges $*_X$ and $*_Y$ on $\mathbf{L}$, let $\langle X, Y, I\rangle$ be a formal fuzzy context. That is, X and Y are sets of the considered objects and attributes, respectively, and $I : X \times Y \to L$ is a fuzzy relation for which $I(x, y)$ is interpreted as the degree to which the object $x \in X$ has the attribute $y \in Y$. The concept lattice of $\langle X, Y, I\rangle$ with hedges $*_X$ and $*_Y$ is the set

$$\mathcal{B}(X^{*_X}, Y^{*_Y}, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B, B^{\downarrow} = A\},$$

where

$$A^{\uparrow}(y) = \bigwedge_{x\in X} A(x)^{*_X}(x) \to I(x, y),$$
$$B^{\downarrow}(x) = \bigwedge_{y\in Y} B(y)^{*_Y}(y) \to I(x, y).$$

Each pair $\langle A, B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is called a formal concept; A and B are called the extent and the intent of $\langle A, B\rangle$, respectively.[1]

Consider now a particularly important case in which $*_X$ is the identity. This subsumes the basic fuzzy concept lattices and the crisply generated fuzzy concept lattices; see [34]. In this setting, consider the operator $C : L^X \to L^X$ defined by

$$C(A) = (A^{\uparrow})^{\downarrow}$$

for $A \in L^X$. Due to (ii) and (ix) of Theorem 3.7 in [34], it holds $A \subseteq A^{\uparrow\downarrow}$ and $A^{\uparrow\downarrow} = A^{\uparrow\downarrow\uparrow\downarrow}$, i.e., $A \subseteq C(A)$ and $C(A) = C(C(A))$, which means that C is extensive and idempotent, respectively.

Moreover, according to (i) of Theorem 3.7 in [34], one has $S(A_1, A_2) \le S(A_2^{\uparrow}, A_1^{\uparrow})$ and $S(B_1^{*_Y}, B_2^{*_Y}) \le S(B_2^{\downarrow}, B_1^{\downarrow})$. Using the isotony of $*_Y$, $(\bigwedge_i a_i)^{*_Y} \le \bigwedge_i a_i^{*_Y}$, and $(a \to b)^{*_Y} \le a^{*_Y} \to b^{*_Y}$, this yields

$$\begin{aligned} S(A_1, A_2)^{*_Y} &\le S(A_2^{\uparrow}, A_1^{\uparrow})^{*_Y} = \Big(\bigwedge_{y\in Y} A_2^{\uparrow}(y) \to A_1^{\uparrow}(y)\Big)^{*_Y} \\ &\le \bigwedge_{y\in Y} \big(A_2^{\uparrow}(y) \to A_1(y)^{\uparrow}\big)^{*_Y} \le \bigwedge_{y\in Y} \big((A_2^{\uparrow})^{*_Y}(y) \to (A_1^{\uparrow})^{*_Y}(y)\big) \\ &= S((A_2^{\uparrow})^{*_Y}, (A_1^{\uparrow})^{*_Y}) \le S(A_1^{\uparrow\downarrow}, A_2^{\uparrow\downarrow}) = S(C(A_1), C(A_2)), \end{aligned}$$

verifying the monotony (7) w.r.t. the hedge $*_Y$. To conclude, C is a fuzzy closure operator satisfying the hedge monotony according to Definition 3. In particular, for $*_Y$ being the identity, C satisfies the graded monotony (2). □

If an operator does not satisfy the graded monotony, it may still satisfy a monotony condition stronger than the ordinary monotony. An example would be an operator satisfying the hedge monotony with a hedge $*$ only slightly inhibiting the truth values, i.e., with a^* being not too much smaller than a. Such operators may be found, e.g., as the fuzzy closure operators associated to formal concept analysis with hedges presented in Example 3. We now present a simple example demonstrating the existence of such fuzzy closure operators.

Example 4. (a) Consider the five-element Łukasiewicz chain $\mathbf{L}$ and $X = \{x\}$, i.e., $L = \{0, 1/4, 1/2, 3/4, 1\}$ equipped with the Łukasiewicz operations. In this case, L^X may be identified with L. The mapping $C : L^X \to L^X$, i.e., essentially $C : L \to L$, defined as

$$C(0) = 0,\ C(1/4) = 1/4,\ C(1/2) = 1/2,\ C(3/4) = C(1) = 1, \tag{9}$$

satisfies extensivity, idempotency, and the ordinary monotony, and thus represents a fuzzy closure operator with the ordinary monotony. In this case, $S(A, B) = A \to B$ for any $A, B \in L$, hence for $A = 3/4$ and $B = 1/2$ we obtain

$$\begin{aligned} S(A, B) &= S(3/4, 1/2) = 3/4 \nleq 1/2 = S(1, 1/2) \\ &= S(C(3/4), C(1/2)) = S(C(A), C(B)), \end{aligned}$$

which demonstrates that C does not satisfy the graded monotony (2). Intuitively, however, C is close to satisfying the graded monotony. Namely, while the graded monotony fails, it it fails only by a slight margin: for the above pair of $A = 3/4$ and $B = 1/2$, $S(A, B)$ is only slightly higher than $S(C(A), C(B))$, and the same is true for the other violating pairs, which are $A = 3/4$, $B = 1/4$ and $A = 3/4$, $B = 0$.

Consider now the mapping $*$ given by

$$0^* = 0, 1/4^* = 0, 1/2^* = 1/4, 3/4^* = 1/2, 1^* = 1. \tag{10}$$

As is easily seen, $*$ is a truth-stressing hedge on $\mathbf{L}$. Now, C happens to be an $\mathbf{L}^*$-closure operator. To check that C satisfies the hedge monotony (7), it suffices to verify (7) for A such that $A \neq C(A)$ because if $A = C(A)$, condition (7) is satisfied due to $B \subseteq C(B)$. The only A with $A \neq C(A)$ is $A = 3/4$. Since for $B \supseteq A$, (7) is obviously valid, we need to check the condition for $B = 1/2$, $1/4$, and 0, which leads to

$$S(A, 1/2)^* = 3/4^* = 1/2 = S(1, 1/2) = S(C(A), C(1/2)),$$

[1] In fact, the approach in [34] is slightly more general, in that it involves possibly different hedges for different objects, and the same for the attributes.

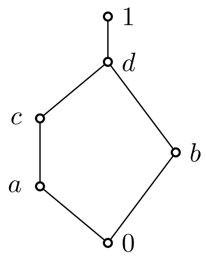

Fig. 1. The lattice in Example 1.

$$S(A, 1/4)^* = 1/2^* = 1/4 = S(1, 1/4) = S(C(A), C(1/4)),$$
$$S(A, 0)^* = 1/4^* = 0 = S(1, 0) = S(C(A), C(0)),$$

hence C satisfies the hedge monotony (7).

(b) Consider now a different mapping, namely,

$$C(0) = 0,\ C(1/4) = 1/4,\ C(1/2) = C(3/4) = C(1) = 1.$$

Again, C is a fuzzy closure operator satisfying the ordinary monotony but not the graded monotony. In this case, the graded monotony is violated by a larger margin since, e.g., for $A = 1/2$ and $B = 1/4$ we obtain $S(A, B) = 3/4$ and $S(C(A), C(B)) = 1/4$. Yet, even in this case, C satisfies a hedge monotony, but now for a more inhibiting hedge such as the following one:

$$0^* = 0, 1/4^* = 0, 1/2^* = 0, 3/4^* = 1/4, 1^* = 1.$$

(c) A fuzzy closure operator with the ordinary monotony that violates the graded monotony even more severely is given by

$$C(0) = 0,\ C(1/4) = C(1/2) = C(3/4) = C(1) = 1.$$

Namely, for $A = 1/4$ and $B = 0$, one has $S(A, B) = 3/4$ and $S(C(A), C(B)) = 0$. This operator again satisfies hedge monotony, but only for the most inhibiting truth-stressing hedge (globalization) given by

$$0^* = 0, 1/4^* = 0, 1/2^* = 0, 3/4^* = 0, 1^* = 1.$$

Since the hedge monotony for truth-stressing hedge of globalization coincides with ordinary monotony, the latter operator may be regarded as violating graded monotony the most. The details are checked as in part (a) of this example. □

Remark 3. We have seen in Example 4 (a) that C "almost satisfies" the graded monotony and that this property may be expressed by the hedge monotony using the slightly inhibiting hedge (10). On the other hand, this property cannot be expressed by the filter monotony because the only two filters on $\mathbf{L}$ in Example 4 are $K = \{1\}$ corresponding to the ordinary monotony and $K = L$ corresponding to the graded monotony. This means that using filter monotony one can only express that C is an $\mathbf{L}_{\{1\}}$-closure operator (satisfies ordinary monotony) that is not an $\mathbf{L}_L$-closure operator (does not satisfy graded monotony). □

Example 5. Let $X = \{x\}$ and consider the complete residuated lattice on $L = \{0, a, b, c, d, 1\}$ with its lattice part shown in Fig. 1 and its connectives are defined as follows:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	a	a
b	0	0	b	0	b	b
c	0	a	0	a	a	c
d	0	a	b	a	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	c	c	1	c	1	1
c	b	d	b	1	1	1
d	0	c	b	c	1	1
1	0	a	b	c	d	1

As in Example 4, we identify L^X with L, and hence consider a mapping $C : L^X \to L^X$ as $C : L \to L$.

(a) Consider the mapping $C\colon L^X \to L^X$, essentially $C\colon L \to L$ defined by $C(b) = d$ and $C(y) = y$ for $y \neq b$. Clearly, is a fuzzy closure operator with the ordinary monotony that does not fulfil graded monotony since

$$S(b, 0) = b \to 0 = c \nleq 0 = d \to 0 = S(d, 0) = S(C(b), C(0)).$$

Nevertheless, C satisfies a stronger form of monotony than ordinary monotony, and this stronger property may be expressed using filter monotony. Namely, C satisfies condition (5) of filter monotony for the filter $K = \{d, 1\}$. To check this fact, observe that since C satisfies the ordinary monotony, (5) is ensured for all $A, B \in L$ with $S(A, B) = 1$. It hence remains to check the two possible instances of (5) with $S(A, B) = d$. Since

$$S(c, a) = c \to a = d \leq d = S(c, a) = S(C(c), C(a)), \text{ and}$$
$$S(1, d) = 1 \to d = d \leq d = S(1, d) = S(C(1), C(d)),$$

C is an L_K-closure operator with a non-trivial filter K.

(b) Consider now $C: L \to L$ defined by $C(c) = d$ and $C(y) = y$ for $y \neq c$. Again, C is a fuzzy closure operator satisfying the ordinary monotony that does not satisfy the graded monotony since

$$S(c, a) = c \to a = d \nleq c = d \to a = S(d, a) = S(C(c), C(a)).$$

The latter property also reveals that C does not satisfy filter monotony for $K = \{d, 1\}$. Therefore, as the ordinary monotony is equivalent to filter monotony for $K = \{1\}$, the foregoing observation implies that the trivial filter $K = \{1\}$ is the only filter for which C satisfies the condition of filter monotony. In other words, C is can be considered as violating graded monotony the most. □

3. On filter-based monotony

In this section, we consider several approaches that extend the idea of monotony based on filters as proposed in [15] and codified by Definition 1.

3.1. Monotony based on fuzzy filters

As discussed above, one direction in extending monotony of fuzzy closure operators exploits filters K, i.e., ordinary subsets of L naturally representing sets of large truth degrees. Since "being a large truth degree" itself is a graded property, and hence admits degrees of truth, using ordinary sets K as filters may seem inappropriate. In this section, we hence explore the possibility to utilize fuzzy sets of truth degrees to represent filters. In addition to rendering a more natural notion of a filter and filter-based monotony, we shall see that the proposed idea results in a mathematically feasible approach with a strictly greater expressive power. The following definition provides a straightforward generalization of filters of degrees of truth; cf. Section 2.2.

Definition 5. A fuzzy set $\mathcal{K} \in L^L$ is called a fuzzy $\leq$-filter in $\mathbf{L}$ if

$$\mathcal{K}(1) = 1 \tag{11}$$

$$a \leq b \text{ implies } \mathcal{K}(a) \leq \mathcal{K}(b), \tag{12}$$

for all $a, b \in L$. If, moreover, $\mathcal{K}(a) \otimes \mathcal{K}(b) \leq \mathcal{K}(a \otimes b)$ for all $a, b \in L$, then $\mathcal{K}$ is called a fuzzy filter in $\mathbf{L}$.

The notion of a fuzzy ($\leq$-)filter obviously generalizes that of a ($\leq$-)filter: Denote for an arbitrary subset $K \subseteq L$ by $c_K : L \to \{0, 1\}$ the corresponding crisp fuzzy set, i.e., the characteristic function of K given by $c_K(a) = 1$ if and only if $a \in K$. Then K is a ($\leq$-)filter in $\mathbf{L}$ if and only if c_K is a fuzzy ($\leq$-)filter in $\mathbf{L}$.

While a crisp ($\leq$-)filter K splits truth degrees into large ones (those in K) and non-large ones (those not in K), a fuzzy filter naturally accommodates the graded nature of "being large" with $\mathcal{K}(a) \in L$ representing a truth degree to which any given truth degree $a \in L$ is considered large. The following example demonstrates that proper fuzzy filters exist.

Example 6. Consider $L = [0, 1]$ with the Łukasiewicz t-norm and its residuum. Let $\mathcal{K} \in L^L$ be defined by $\mathcal{K}(a) = a$. Clearly, $\mathcal{K}(1) = 1$ and for $a \leq b$ we have $\mathcal{K}(a) = a \leq b = \mathcal{K}(b)$, whence $\mathcal{K}$ is a fuzzy $\leq$-filter. In addition, $\mathcal{K}(a) \otimes \mathcal{K}(b) = a \otimes b = \mathcal{K}(a \otimes b)$. Hence, $\mathcal{K}$ is a proper fuzzy filter. With respect to $\mathcal{K}$, each $a \in L$ is considered large to the degree $\mathcal{K}(a) = a$. Therefore, 1 is considered large to the largest extent, 0 is not considered large at all, 0.5 is considered partly large, etc. □

We now provide the corresponding generalizations of Definitions 1 and 2:

Definition 6. Let $\mathcal{K}$ be a fuzzy filter in $\mathbf{L}$. A mapping $C: L^X \to L^X$ is called an $\mathbf{L}_{\mathcal{K}}$-closure operator on X if it satisfies (3), (4), and

$$\mathcal{K}(S(A, B)) \otimes S(A, B) \leq S(C(A), C(B)), \tag{13}$$

for all $A, B \in L^X$.

Notice that the new monotony condition (13) takes into account the degree $\mathcal{K}(S(A, B))$ to which the degree $S(A, B)$ of inclusion of A in B is considered large: The more $S(A, B)$ may be considered large, the stricter the lower bound for $S(C(A), C(B))$. That the monotony condition (13) indeed generalizes condition (5) is observed next:

Lemma 1. *Let $K \subseteq L$ and let $\mathcal{K}$ be the cirsp fuzzy set corresponding to K, i.e., $\mathcal{K}(a) = 1$ if and only if $a \in K$ for each $a \in L$. An operator $C: L^X \to L^X$ is K-monotone according to (5) if and only if it is $\mathcal{K}$-monotone according to (13).*

Proof. It is clear that $\mathcal{K}$-monotony (13) is equivalent to

$$S(A, B) \leq S(C(A), C(B)) \quad \text{if } S(A, B) \in K,$$

$$0 \leq S(C(A), C(B)) \quad \text{otherwise.}$$

Since the second condition is trivially satisfied, (13) is equivalent to the first condition, i.e., to K-monotony (5). □

The following example demonstrates that fuzzy filters are indeed more expressive than filters as regards monotony of closure operators.

Example 7. Consider again the setting of Example 4, i.e., the Łukasiewicz chain $L = \{0, 1/4, 1/2, 3/4, 1\}$, $X = \{x\}$, and the closure operators in (a), (b), and (c) in this example. Like in Example 6, the fuzzy set $\mathcal{K}$ given by $\mathcal{K}(a) = a$ for each $a \in L$ is a fuzzy filter. It is immediate that checking monotony (13) of an operator C with this $\mathcal{K}$ amounts to verifying

$$(a \to b) \otimes (a \to b) \leq C(a) \to C(b)$$

for each $a > b$ in L. A straightforward verification reveals that the operators in (a) and (b) satisfy this condition, and hence form $\mathbf{L}_{\mathcal{K}}$-closure operators. Now, the monotony expressed by $\mathcal{K}$ is stronger than the ordinary monotony for both these operators. For instance, for $a = 3/4$ and $b = 1/2$, (13) represents a requirement stronger than the ordinary monotony requirements as it becomes

$$1/2 = 3/4 \otimes 3/4 = (a \to b) \otimes (a \to b) = S(a, b) \otimes S(a, b) \leq S(C(a), C(b)).$$

This requirement is fulfilled by both the operators because in this case, both these operators yield

$$S(C(a), C(b)) = C(a) \to C(b) = 1 \to 1/2 = 1/2.$$

In view of Remark 3, this requirement cannot be expressed by a filter in $\mathbf{L}$, whence the greater expressive power of the monotony based on fuzzy filters.

Note also that the operator C in (c) fails to satisfy monotony (13) since for $a = 1/4$ and $b = 0$, one has

$$\begin{aligned} \mathcal{K}(S(a, b)) \otimes S(a, b) &= \mathcal{K}(a \to b) \otimes (a \to b) = \mathcal{K}(3/4) \otimes 3/4 = 3/4 \otimes 3/4 \\ &= 1/2 \nleq 0 = 1 \to 0 = C(a) \to C(b) = S(C(a), C(b)). \end{aligned}$$

□

Closure systems obtain the following form in the present setting:

Definition 7. A system $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$ is called an $\mathbf{L}_{\mathcal{K}}$-closure system on X if for all $A \in L^X$,

$$\bigcap_{i \in I} \big((\mathcal{K}(S(A, A_i)) \otimes S(A, A_i)) \to A_i\big) \in \mathcal{S}. \tag{14}$$

One easily checks that (14), expressing that $\mathcal{S}$ is closed under intersections, generalizes the corresponding condition of Definition 2.

In the rest of this section, we show that there exists a natural bijective correspondence between $\mathbf{L}_{\mathcal{K}}$-closure operators and systems. For the next lemma to be true, it suffices that $\mathcal{K}$ be an ≤-filter, i.e., $\mathcal{K}$ need not be a filter (the corresponding notions of a ≤-filter-based closure operator and system result from Definitions 6 and 7 in an obvious manner).[2]

Lemma 2. *Let $\mathcal{K} \in L^L$ be a fuzzy ≤-filter in $\mathbf{L}$ and $C : L^X \to L^X$ an $\mathbf{L}_{\mathcal{K}}$-closure operator. Then $\mathcal{S}_C = \{A \in L^X \mid A = C(A)\}$ is an $\mathbf{L}_{\mathcal{K}}$-closure system.*

Proof. Idempotency of C implies that $C(A) \in \mathcal{S}_C$ for each $A \in L^X$. It hence suffices to check that for all $A \in L^X$,

$$\bigcap_{i \in I} \big((\mathcal{K}(S(A, A_i)) \otimes S(A, A_i)) \to A_i\big) = C(A).$$

As $C(A) \in \mathcal{S}_C$, we have

$$\begin{aligned} &\bigcap_{i \in I} \big((\mathcal{K}(S(A, A_i)) \otimes S(A, A_i)) \to A_i\big) \\ &\quad \subseteq (\mathcal{K}(S(A, C(A))) \otimes S(A, C(A))) \to C(A) \\ &\quad = (\mathcal{K}(1) \otimes 1) \to C(A) = C(A). \end{aligned}$$

For the converse inclusion we need to check

$$C(A) \subseteq \bigcap_{i \in I} \big((\mathcal{K}(S(A, A_i)) \otimes S(A, A_i)) \to A_i\big), \qquad \text{or equivalently,}$$

$$C(A) \otimes \mathcal{K}(S(A, A_i)) \otimes S(A, A_i) \subseteq A_i \qquad \text{for all } i \in I,$$

which holds true since monotony (13) of C yields

$$C(A) \otimes \mathcal{K}(S(A, A_i)) \otimes S(A, A_i) \subseteq C(A) \otimes S(C(A), C(A_i)) \subseteq C(A_i) = A_i.$$

□

Lemma 3. *Let $\mathcal{K} \in L^L$ be a fuzzy filter in $\mathbf{L}$ and let $\mathcal{S} = \{A_i \mid i \in I\}$ be an $\mathbf{L}_{\mathcal{K}}$-closure system on X. The mapping $C_{\mathcal{S}} : L^X \to L^X$ defined as*

$$C_{\mathcal{S}}(A) = \bigcap_{i \in I} \big((\mathcal{K}(S(A, A_i)) \otimes S(A, A_i)) \to A_i\big) \tag{15}$$

is an $\mathbf{L}_{\mathcal{K}}$-closure operator on X.

[2] The fact that ≤-filters suffice to obtain closure systems from closure operators has not been explicitly mentioned in [15].

Proof. We check that C_S is extensive, idempotent and satisfies the monotony condition (13). Extensivity: Adjointness in $\mathbf{L}$ and basic properties of the graded inclusion S yield

$$S(A, C_{\mathcal{S}}(A)) = S\left(A, \bigcap_{i\in I}((\mathcal{K}(S(A,A_i)) \otimes S(A,A_i)) \to A_i)\right)$$
$$\subseteq \bigcap_{i\in I}((\mathcal{K}(S(A,A_i)) \otimes S(A,A_i)) \to S(A,A_i)) = 1,$$

whence $A \subseteq C_{\mathcal{S}}(A)$.

Idempotency: Since $\mathcal{S}$ forms an $\mathbf{L}_{\mathcal{K}}$-closure system, one has $C_{\mathcal{S}}(A) \in \mathcal{S}$ for each $A \in L^X$., i.e. , $C_{\mathcal{S}}(A) = A_i$ for some $i \in I$. Therefore,

$$C_{\mathcal{S}}(C_{\mathcal{S}}(A)) = \bigcap_{i\in I}((\mathcal{K}(S(C_{\mathcal{S}}(A), A_i)) \otimes S(C_{\mathcal{S}}(A), A_i)) \to A_i)$$
$$\subseteq (\mathcal{K}(S(C_{\mathcal{S}}(A), C_{\mathcal{S}}(A))) \otimes S(C_{\mathcal{S}}(A), C_{\mathcal{S}}(A))) \to C_{\mathcal{S}}(A)$$
$$= (\mathcal{K}(1) \otimes 1) \to C_{\mathcal{S}}(A) = 1 \to C_{\mathcal{S}}(A) = C_{\mathcal{S}}(A).$$

Since the converse inclusion follows by extensivity, we get $C_{\mathcal{S}}(A) = C_{\mathcal{S}}(C_{\mathcal{S}}(A))$.

Monotony (13): We have to prove

$$\mathcal{K}(S(A,B)) \otimes S(A,B) \leq S(C_{\mathcal{S}}(A), C_{\mathcal{S}}(B)), \quad \text{or equivalently,}$$
$$\mathcal{K}(S(A,B)) \otimes S(A,B) \otimes C_{\mathcal{S}}(A) \subseteq C_{\mathcal{S}}(B)), \quad \text{or equivalently,}$$
$$\mathcal{K}(S(A,B)) \otimes S(A,B) \otimes C_{\mathcal{S}}(A) \otimes \mathcal{K}(S(B,A_i)) \otimes S(B,A_i) \subseteq A_i \quad \text{for all } i \in I.$$

Rearranging the terms, we need to prove

$$S(A,B) \otimes S(B,A_i) \otimes \mathcal{K}(S(A,B)) \otimes \mathcal{K}(S(B,A_i)) \otimes C_{\mathcal{S}}(A) \subseteq A_i$$

for all $i \in I$. Since $\mathcal{K}$ is a fuzzy filter, we get

$$S(A,B) \otimes S(B,A_i) \otimes \mathcal{K}(S(A,B)) \otimes \mathcal{K}(S(B,A_i)) \otimes C_{\mathcal{S}}(A)$$
$$\subseteq S(A,B) \otimes S(B,A_i) \otimes \mathcal{K}(S(A,B) \otimes S(B,A_i)) \otimes C_{\mathcal{S}}(A).$$

Now since $S(A,B) \otimes S(B,A_i) \leq S(A,A_i)$, associativity and isotony of $\otimes$ along with $\mathcal{K}$ being a fuzzy $\leq$-filter yield

$$(S(A,B) \otimes S(B,A_i)) \otimes (\mathcal{K}(S(A,B) \otimes S(B,A_i))) \otimes C_{\mathcal{S}}(A)$$
$$\subseteq S(A,A_i) \otimes \mathcal{K}(S(A,A_i)) \otimes C_{\mathcal{S}}(A)$$
$$= S(A,A_i) \otimes \mathcal{K}(S(A,A_i)) \otimes \bigcap_{j\in I}\big((\mathcal{K}(S(A,A_j)) \otimes S(A,A_j)) \to A_j\big)$$
$$\subseteq S(A,A_i) \otimes \mathcal{K}(S(A,A_i)) \otimes \big((\mathcal{K}(S(A,A_i)) \otimes S(A,A_i)) \to A_i\big) \subseteq A_i,$$

proving that $C_{\mathcal{S}}$ is an $L_{\mathcal{K}}$-closure operator. □

The following theorem provides a generalization of Theorem 1 for fuzzy filters.

Theorem 3. *Let $\mathcal{K} \in L^L$ be a fuzzy filter in $\mathbf{L}$, $C : L^X \to L^X$ be an $\mathbf{L}_{\mathcal{K}}$-closure operator and $\mathcal{S}$ be an $\mathbf{L}_{\mathcal{K}}$-closure system on X. Then*

(a) $\mathcal{S}_C$ is an $\mathbf{L}_{\mathcal{K}}$-closure system;
(b) $C_{\mathcal{S}}$ is an $\mathbf{L}_{\mathcal{K}}$-closure operator;
(c) $C = C_{\mathcal{S}_C}$ and $\mathcal{S}_{C_{\mathcal{S}}}$.

Proof. (a) and (b) are direct consequences of Lemmas 2 and 3, respectively. To check (c), notice that for an $\mathbf{L}_{\mathcal{K}}$-closure operator $C : L^X \to L^X$, the proof of Lemma 2 yields that for the set $\mathcal{S}_C = \{A \in L^X \mid A = C(A)\}$ we have

$$C_{\mathcal{S}_C}(A) = \bigcap_{i\in I}(\mathcal{K}(S(A,A_i)) \otimes S(A,A_i)) \to A_i = C(A).$$

Moreover, the proof of Lemma 3 shows that for an $\mathbf{L}_{\mathcal{K}}$-closure system $\mathcal{S}$, we have $C_{\mathcal{S}}(A) \in \mathcal{S}$ for each $A \in L^X$. As a consequence, $\mathcal{S}_{C_{\mathcal{S}}} = \{A \in L^X \mid A = C_{\mathcal{S}}(A)\} \subseteq \mathcal{S}$. Conversely, it suffices to prove that for $A \in \mathcal{S}$ we have $A = C_{\mathcal{S}}(A)$. This is indeed the case. Namely, extensivity yields $A \subseteq C_{\mathcal{S}}(A)$, and since $A \in \mathcal{S}$,

$$C_{\mathcal{S}}(A) = \bigcap_{i\in I}(\mathcal{K}(S(A,A_i)) \otimes S(A,A_i)) \to A_i \subseteq (\mathcal{K}(S(A,A)) \otimes S(A,A)) \to A = A.$$

□

3.2. Monotony based on fuzzy sets

The property of $\mathcal{K}$ of being a fuzzy filter enables us to naturally incorporate the idea that "being a large truth degree," which underlies filter-based monotony, is a graded property. Checking the proofs, though, one observes that essential in the proofs is the condition that $a \le b$ implies $a \otimes \mathcal{K}(a) \le b \otimes \mathcal{K}(b)$. This suggests a possibility to extend the concept of monotony to a wider collection of fuzzy sets satisfying this weaker condition.

We shall hence consider fuzzy sets $\Phi \in L^L$ satisfying

$$\Phi(1) = 1, \tag{16}$$

$$a \le b \text{ implies } a \otimes \Phi(a) \le b \otimes \Phi(b). \tag{17}$$

and sometimes also the following form of compatibility with $\otimes$:

$$\Phi(a) \otimes \Phi(b) \le \Phi(a \otimes b) \tag{18}$$

Remark 4. Notice that the above conditions for Φ resemble the conditions for a truth-stressing hedge. Namely, (16) and (18) are just two of the conditions satisfied by truth-stressing hedges; cf. the text preceding Definition 3. Moreover, condition (17) resembles the condition of isotony. This is our first encounter with a natural resemblance between the (fuzzy-)set-based monotony and the hedge-based monotony. The connection between these approaches shall further be explored in Section 4.

Using such fuzzy sets Φ, we define the following notion of a closure operator:

Definition 8. Let $C: L^X \to L^X$ be a mapping. C is said to be an $\mathbf{L}_\Phi$-closure operator if it is idempotent, extensive, and for all $A, B \in L^X$ satisfies

$$\Phi(S(A, B)) \otimes S(A, B) \le S(C(A), C(B)). \tag{19}$$

Correspondingly, one defines closure systems:

Definition 9. A system $\mathcal{S} = \{A_i \in L^X \mid i \in I\}$, $\mathcal{S}$ is called an $\mathbf{L}_\Phi$-closure system if for all $A \in L^X$,

$$\bigcap_{i \in I} \big((\Phi(S(A, A_i)) \otimes S(A, A_i)) \to A_i\big) \in \mathcal{S}. \tag{20}$$

Using a similar reasoning as for fuzzy filters in Section 3.1, one now obtains a bijective correspondence between closure operators and systems for the present kind of monotony based on fuzzy sets $\Phi \in L^L$ satisfying (16), (17), and (18). For one:

Lemma 4. *Let $C: L^X \to L^X$ be an $\mathbf{L}_\Phi$-closure operator. Then $\mathcal{S} = \{A \in L^X \mid A = C(A)\}$ is an $\mathbf{L}_\Phi$-closure system.*

Conversely:

Lemma 5. *Let $\mathcal{S} = \{A_i \mid i \in I\}$ be an $\mathbf{L}_\Phi$-closure system. The mapping $C_\mathcal{S}: L^X \to L^X$ defined by*

$$C_\mathcal{S}(A) = \bigcap_{i \in I} \big((\Phi(S(A, A_i)) \otimes S(A, A_i)) \to A_i\big). \tag{21}$$

is an $\mathbf{L}_\Phi$-closure operator.

This yields:

Theorem 4. *Let $\Phi \in L^L$ be a fuzzy set satisfying (16), (17), and (18), $C: L^X \to L^X$ be an $\mathbf{L}_\Phi$-closure operator, and $\mathcal{S}$ be an $\mathbf{L}_\Phi$-closure system. Then $\mathcal{S}_C$ is an $\mathbf{L}_\Phi$-closure system, $C_\mathcal{S}$ is an $\mathbf{L}_\Phi$-closure operator, $C = C_{\mathcal{S}_C}$ and $\mathcal{S} = \mathcal{S}_{C_\mathcal{S}}$.*

It is easy to check that fuzzy $\le$-filters $\mathcal{K}$ are a particular case of fuzzy sets Φ satisfying (16) and (17). In addition, fuzzy filters $\mathcal{K}$ are a particular case of fuzzy sets Φ satisfying (16), (17), and (18). However, the converse statements are not true:

Example 8. Let $L = [0, 1]$ be equipped with the Łukasiewicz operations. Consider $\Phi \in L^L$ defined by

$$\Phi(a) = \begin{cases} 1 & \text{if } a \ge 0.5, \\ 1 - a & \text{otherwise,} \end{cases}$$

whose graph is depicted in Fig. 2. Then Φ satisfies (16) and (17): By definition $\Phi(1) = 1$. Furthermore, if $a \le b$ then for $a \ge 0.5$ we get

$$a \otimes \Phi(a) = a \le b = b \otimes \Phi(b)$$

while for $a < 0.5$ we get

$$a \otimes \Phi(a) = a \otimes (1 - a) = 0 \le b \otimes \Phi(b).$$

However, Φ is not a fuzzy $\le$-filter because, while $0 \le 0.4$, we have $\Phi(0) = 1 \not\le 0.6 = \Phi(0.4)$.
□

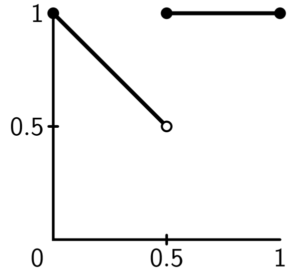

Fig. 2. Fuzzy set $\Phi : [0,1] \to [0,1]$ that is not a fuzzy $\le$-filter.

4. On monotony based on hedges

In Remark 4, we observed a resemblance of the conditions for the fuzzy sets that underly filter-based and fuzzy-set-based monotony on the one hand, with the conditions for the truth-stressing hedges underlying hedge-based monotony. In this section, we thus examine relationships between the two kinds to monotony. For this purpose, we utilize what we call crisply monotonic hedges, or c-hedges for short, which result as a natural generalization of truth-stressing hedges.

4.1. Closure structures with c-hedges

We start with an extension of the notion of a truth-stressing hedge:

Definition 10. A mapping $* : L \to L$ is called a c-hedge if it satisfies

$$1^* = 1, \tag{22}$$

$$a^* \le a, \tag{23}$$

$$a \le b \text{ implies } a^* \le b^*, \tag{24}$$

for all $a, b \in L$.

The term c-hedge derives from condition (24) of ordinary, or "crisp," isotony which replaces the original condition

$$(a \to b)^* \le a^* \to b^* \tag{25}$$

in the definition of a (truth-stressing) hedge; see Section 2.1. As mentioned in the paragraph preceding Definition 3, condition (24) is a consequence of (25), i.e., each hedge is a c-hedge. The next example demonstrates that (25) is indeed stronger than (24), i.e., the notion of a c-hedge is a proper generalization of the notion of a hedge.

Example 9. Consider $L = [0,1]$ equipped with the Łukasiewicz operations and consider the mapping $* : L \to L$ defined by

$$a^* = \begin{cases} a & \text{if } a \ge 0.5 \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that $*$ is a c-hedge. Namely, $1^* = 1$ and $a^* \le a$ are obvious; for $a \le b$, distinguish two cases: for $a \ge 0.5$ we get $a^* = a \le b = b^*$, while for $a < 0.5$ we have $a^* = 0 \le b^*$ for any value of b.

However, $*$ is not a truth-stressing hedge since

$$0.9 = 0.9^* = (0.5 \to 0.4)^* \nleq 0.5^* \to 0.4^* = 0.5 \to 0 = 0.5.$$

□

Closure operators and systems based on hedges as defined by Definitions 3 and 4, respectively, obviously extend to c-hedges $*$. Thus, an $\mathbf{L}^*$-closure operator $C : L^X \to L^X$ is extensive, idempotent, and satisfies the hedge monotony

$$S(A,B)^* \le S(C(A), C(B)), \tag{26}$$

for all $A, B \in L^X$, and an $\mathbf{L}^*$-closure system $\mathcal{S} \subseteq L^X$ satisfies for all $A \in L^X$ the condition

$$\bigcap_{i \in I} \left(S(A, A_i)^* \to A_i \right) \in \mathcal{S}. \tag{27}$$

Let us now explore the one-to-one relationship between closure operators and systems established for hedges in [24], which is discussed above in Section 2.2. For one, we obtain:

Lemma 6. *Let $*$ be a c-hedge. If $C : L^X \to L^X$ is an $\mathbf{L}^*$-closure operator, then $\mathcal{S}_C = \{A \in L^X \mid A = C(A)\}$ is an $\mathbf{L}^*$-closure system.*

Proof. The proof is obtained from the corresponding proof in [24]; namely, an inspection of that proof reveals that (22) is the only property of $*$ used. □

To show the converse of Lemma 6, however, we need an additional condition for $*$, namely,

$$a^* \otimes b^* \le (a \otimes b)^* \tag{28}$$

for each $a, b \in L$.

Lemma 7. *Let $*$ be a c-hedge satisfying (28). If $\mathcal{S} = \{A_i \mid i \in I\}$ be an $\mathbf{L}^*$-closure system, then the mapping $C_{\mathcal{S}} : L^X \to L^X$ defined by*

$$C_{\mathcal{S}}(A) = \bigcap_{i \in I} \left(S(A, A_i)^* \to A_i \right) \tag{29}$$

is an $\mathbf{L}^*$-closure operator.

Proof. We prove that $C_{\mathcal{S}}$ is extensive, satisfies monotony (26), and is idempotent. Extensivity follows due to

$$S(A, C_{\mathcal{S}}(A)) = S\left(A, \bigcap_{i \in I} (S(A, A_i)^* \to A_i) \right) \subseteq \bigcap_{i \in I} (S(A, A_i)^* \to S(A, A_i)) = 1.$$

Thanks to extensivity, we only need to show $C_{\mathcal{S}}(C_{\mathcal{S}}(A)) \subseteq C_{\mathcal{S}}(A)$ for each $A \in L^X$ to verify the idempotency of $C_{\mathcal{S}}$. Since $\mathcal{S}$ is an L^*-closure system, we get $C_{\mathcal{S}}(A) \in \mathcal{S}$. Therefore, there exists $j \in I$ such that $A_j = C_{\mathcal{S}}(A)$, whence

$$\begin{aligned} C_{\mathcal{S}}(C_{\mathcal{S}}(A)) &= \bigcap_{i \in I} (S(C_{\mathcal{S}}(A), A_i)^* \to A_i) \\ &\subseteq (S(C_{\mathcal{S}}(A), C_{\mathcal{S}}(A))^* \to C_{\mathcal{S}}(A) \\ &= 1^* \to C_{\mathcal{S}}(A) = C_{\mathcal{S}}(A). \end{aligned}$$

To verify monotony, we need to check

$$\begin{aligned} &S(A, B)^* \le S(C_{\mathcal{S}}(A), C_{\mathcal{S}}(B)), && \text{or equivalently} \\ &S(A, B)^* \otimes C_{\mathcal{S}}(A) \subseteq C_{\mathcal{S}}(B)), && \text{or equivalently} \\ &S(A, B)^* \otimes C_{\mathcal{S}}(A) \otimes S(B, A_i)^* \subseteq A_i && \text{for all } i \in I. \end{aligned}$$

Rearranging the terms, we shall prove $S(A, B)^* \otimes S(B, A_i)^* \otimes C_{\mathcal{S}}(A) \subseteq A_i$. As $*$ satisfies (28), we have

$$S(A, B)^* \otimes S(B, A_i)^* \otimes C_{\mathcal{S}}(A) \subseteq (S(A, B) \otimes S(B, A_i))^* \otimes C_{\mathcal{S}}(A).$$

Finally, $S(A, B) \otimes S(B, A_i) \le S(A, A_i)$ and monotony (24) of $*$ yield

$$\begin{aligned} &(S(A, B) \otimes S(B, A_i))^* \otimes C_{\mathcal{S}}(A) \\ &\quad \subseteq S(A, A_i)^* \otimes C_{\mathcal{S}}(A) \\ &\quad = S(A, A_i)^* \otimes \bigcap_{j \in I} \left(S(A, A_j)^* \to A_j \right) \\ &\quad \subseteq S(A, A_i)^* \otimes \left(S(A, A_i)^* \to A_i \right) \subseteq A_i, \end{aligned}$$

verifying the monotony of $C_{\mathcal{S}}$. □

Consider now again condition (28) demanding $a^* \otimes b^* \le (a \otimes b)^*$, and its relationship to condition (25) demanding $(a \to b)^* \le a^* \to b^*$. While (25) is one of the defining conditions for hedges, (28) is an additional condition for a c-hedge used in Lemma 7. Even though these conditions appear dual, they are substantially different with respect to the isotony (24) of $*$:

Lemma 8. *Let $* : L \to L$ satisfy $1^* = 1$ and $a^* \le a$ for each $a \in L$.*

(a) Condition (25) implies the isotony of $*$.
(b) Condition (28) does not imply the isotony of $*$.

Proof. (a) is known; cf. the text preceding Definition 3.

(b) is obtained using the following counterexample. Let $L = \{0, a, b, 1\}$ with $0 < a < b < 1$ and $\otimes$ defined by $x \otimes y = 0$ for $x, y < 1$ and $x \otimes y = \min(x, y)$ otherwise. While this definition yields a non-left-continuous t-norm on $[0, 1]$ (so-called drastic product), and thus does not yield a structure of a residuated lattice on $[0, 1]$, it does so for a finite chain L. The mapping $* : L \to L$ defined by

$$0^* = 0, \quad a^* = a, \quad b^* = 0, \quad 1^* = 1.$$

Clearly, $*$ satisfies both $1^* = 1$ and $x^* \le x$. Moreover, $*$ satisfies (28): if $x, y < 1$ then $x \otimes y = 0$, hence $x^* \otimes y^* \le x \otimes y = (x \otimes y)^*$; if $x = 1$ we get $x^* \otimes y^* = 1 \otimes y^* = y^* = (1 \otimes y)^* = (x \otimes y)^*$, and likewise for $y = 1$. Nevertheless, $*$ is not isotone since $a \le b$ but $a^* = a \nleq 0 = b^*$. □

Having now Lemma 6 and Lemma 7, one can obtain the one-to-one relationship between $\mathbf{L}^*$-closure operators and systems by easily adopting the proof of the corresponding result in [24], i.e., obtain a generalization of Theorem 2 for $*$ being a c-hedge satisfying (28). However, the following lemma—apparently surprising in view of Lemma 8—implies that such a theorem would collapse to the already known Theorem 2. Namely, the lemma entails that any c-hedge satisfying (28) already is a truth-stressing hedge:

Lemma 9. *For any c-hedge* $*$ *and all* $a, b \in L$ *we have* $(a \to b)^* \le a^* \to b^*$ *if and only if* $a^* \otimes b^* \le (a \otimes b)^*$.

Proof. The fact that $(a \to b)^* \le a^* \to b^*$ implies $a^* \otimes b^* \le (a \otimes b)^*$ is proved, e.g., in [24]. Conversely, assume $a^* \otimes b^* \le (a \otimes b)^*$. Clearly, $(a \to b)^* \le a^* \to b^*$ is equivalent to $a^* \otimes (a \to b)^* \le b^*$. Now, $a \otimes (a \to b) \le b$, isotony (24) of $*$, and the assumed condition (28) imply

$$a^* \otimes (a \to b)^* \le (a \otimes (a \to b))^* \le b^*,$$

proving the claim. □

While the one-to-one correspondence between closure operators and systems remains an open problem for c-hedges not satisfying (28), c-hedges provide a unifying notion of monotony for fuzzy closure operators, as explored in the next section.

4.2. Comparison between c-hedge monotony and previous approaches

We now show that all the monotony conditions for fuzzy closure considered so far are particular cases of monotonies based on c-hedges. For brevity, we use "K-monotony," "$\mathcal{K}$-monotony," "Φ-monotony," and "$*$-monotony" to refer to the monotony conditions involving K, $\mathcal{K}$, Φ, and $*$, respectively, as introduced above in this paper.

Theorem 5. *Let* $\mathbf{L}$ *be a complete residuated lattice and* $C : L^X \to L^X$ *be a mapping.*

(a) If $K \subseteq L$ *is a* $\le$*-filter then there exists a c-hedge* $*_K : L \to L$ *such that* K*-monotony of* C *is equivalent to* $*_K$*-monotony of* C*. In addition, if* K *is a filter then* $*_K$ *satisfies (28).*
(b) If $\mathcal{K} \in L^L$ *is a fuzzy* $\le$*-filter then there exists a c-hedge* $*_\mathcal{K} : L \to L$ *such that* $\mathcal{K}$*-monotony of* C *is equivalent to* $*_\mathcal{K}$*-monotony of* C*. In addition, if* $\mathcal{K}$ *is a fuzzy filter then* $*_\mathcal{K}$ *satisfies (28).*
(c) If $\Phi \in L^L$ *is a fuzzy set satisfying (16) and (17) then there exists a c-hedge* $*_\Phi : L \to L$ *such that* Φ*-monotony of* C *is equivalent to* $*_\Phi$*-monotony of* C*. In addition, if* Φ *satisfies (18) then* $*_\Phi$ *satisfies (28).*

Proof.
(a): Let $K \subseteq L$ be a $\le$-filter and let $*_K : L \to L$ be defined by

$$a^{*_K} = \begin{cases} a & \text{if } a \in K, \\ 0 & \text{otherwise.} \end{cases}$$

We first check that $*_K$ is a c-hedge. clearly, $1^{*_K} = 1$ is due to $1 \in K$, and $a^{*_K} \le a$ for all $a \in L$ is immediate. It hence remains to check that $*_K$ is isotone. Let $a, b \in L$ with $a \le b$. If $a \notin K$ then $a^{*_K} = 0 \le b^{*_K}$. On the other hand, if $a \in K$, then since K is a $\le$-filter, we have $b \in K$, whence $a^{*_K} = a \le b = b^{*_K}$.

We now check that C satisfies K-monotony if and only if it satisfies $*_K$-monotony. Observe that the $*_K$-monotony of C, demaning that for all $A, B \in L^X$ one has

$$S(A, B)^{*_K} \le S(C(A), C(B)),$$

is—due to the definition of $*_K$—equivalent to the following compound condition:

$$\begin{array}{ll} S(A, B) \le S(C(A), C(B)) & \text{if } S(A, B) \in K, \text{ and} \\ 0 \le S(C(A), C(B)) & \text{if } S(A, B) \notin K. \end{array}$$

But since $0 \le S(C(A), C(B))$ is always the case, the $*_K$-monotony of C is equivalent to the first part of the compound condition, which is but the K-monotony of C.

Assume now that K is a filter, and let $a, b \in L$. If $a \notin K$ or $b \notin K$

$$a^{*_K} \otimes b^{*_K} = 0 \le (a \otimes b)^{*_K}.$$

If $a \in K$ and $b \in K$ then $a \otimes b \in K$, whence

$$a^{*_K} \otimes b^{*_K} = a \otimes b = (a \otimes b)^{*_K}.$$

We have thus verified (28).

(b): Let $\mathcal{K} \in L^L$ be a fuzzy $\le$-filter and let $*_\mathcal{K} : L \to L$ be defined by

$$a^{*_\mathcal{K}} = a \otimes \mathcal{K}(a), \tag{30}$$

for all $a \in L$. To check that $*_\mathcal{K}$ is a c-hedge, observe that due to $\mathcal{K}(1) = 1$, $1^{*_\mathcal{K}} = 1 \otimes \mathcal{K}(1) = 1 \otimes 1 = 1$, and that $a^{*_\mathcal{K}} \le a$ for all $a \in L$ follows from $x \otimes y \le x$. To check the isotony of $*_\mathcal{K}$, take $a, b \in L$ such that $a \le b$. The upward closedness (12) of fuzzy filters then ensures $\mathcal{K}(a) \le \mathcal{K}(b)$, whence the isotony of $\otimes$ yields

$$a^{*_\mathcal{K}} = a \otimes \mathcal{K}(a) \le b \otimes \mathcal{K}(b) = b^{*_\mathcal{K}},$$

verifying that $*_\mathcal{K}$ is a c-hedge.

The $*_{\mathcal{K}}$-monotony of C requires that for all $A, B \in L^X$ one has

$$S(A,B)^{*_{\mathcal{K}}} \leq S(C(A), C(B)),$$

which—due to (30)—is equivalent to

$$\mathcal{K}(S(A,B)) \otimes S(A,B) \leq S(C(A), C(B)),$$

i.e., to the $\mathcal{K}$-monotony of C.

Let now $\mathcal{K}$ be a fuzzy filter, and let $a, b \in L$. Then

$$\begin{aligned} a^{*_{\mathcal{K}}} \otimes b^{*_{\mathcal{K}}} &= a \otimes \mathcal{K}(a) \otimes b \otimes \mathcal{K}(b) \\ &= a \otimes b \otimes \mathcal{K}(a) \otimes \mathcal{K}(b) \\ &\leq a \otimes b \otimes \mathcal{K}(a \otimes b) \\ &= (a \otimes b)^{*_{\mathcal{K}}}. \end{aligned}$$

(c): The proof proceeds the same way as in (b), taking $*_\Phi : L \to L$ defined by $a^{*_\Phi} = a \otimes \Phi(a)$ and observing that the conditions of fuzzy filters used in (b) are available due to the assumed properties of Φ. □

Note that part (a) in the previous theorem can also be obtained as a consequence of (b) using simple relationships between ordinary filters and fuzzy filters; we nevertheless included a direct proof due to the significance of ordinary filters.

The above results now render the question of a relationship between the set Ξ of all fuzzy sets $\Phi \in L^L$ satisfying (16) and (17) and partially ordered by inclusion, and the set Ω of all c-hedges $*$ on L equipped with a pointwise order. These ordered sets shall be denoted $\langle \Xi, \subseteq \rangle$ and $\langle \Omega, \leq \rangle$; i.e., $\Phi_1 \subseteq \Phi_2$ means $\Phi_1(a) \leq \Phi_2(a)$ for each $a \in L$, and $*_1 \subseteq *_2$ means $a^{*_1} \leq a^{*_2}$ for each $a \in L$.

Theorem 6. *Let $\otimes$ be divisible, i.e., satisfy $a \wedge b = a \otimes (a \to b)$. The pair of mappings $f : \Xi \to \Omega$ defined by $f(\Phi)(a) = a \otimes \Phi(a)$ and $g : \Omega \to \Xi$ defined by $g(*)(a) = a \to a^*$ is an adjunction between $\langle \Xi, \subseteq \rangle$ and $\langle \Omega, \leq \rangle$ for which $f \circ g = \mathrm{id}_\Omega$.*

Proof. If $\Phi \in \Xi$, due to (c) of Theorem 5, $f(\Phi)$ is a c-hedge, i.e., $f(\Phi) \in \Omega$.

Conversely, take a c-hedge $* \in \Omega$ and consider the fuzzy set $\Phi = g(*)$, i.e., $\Phi(a) = a \to a^*$ for $a \in L$. We need to check that Φ satisfies (16) and (17). Since $1^* = 1$, we get $\Phi(1) = 1 \to 1^* = 1 \to 1 = 1$, verifying (16). Let now $a, b \in L$ such that $a \leq b$. Since $a^* \leq a$, $*$ is isotone, and $\otimes$ is divisible,

$$\begin{aligned} a \otimes \Phi(a) &= a \otimes (a \to a^*) = a \wedge a^* = a^* \leq b^* \\ &= b \wedge b^* = b \otimes (b \to b^*) = b \otimes \Phi(b), \end{aligned}$$

checking (17). Therefore, both f and g are well-defined.

To prove that f and g form the adjunction, the definition requires us to verify that $\Phi \subseteq g(*)$ if and only if $f(\Phi) \leq *$ for all $\Phi \in \Xi$ and $* \in \Omega$. This is, indeed, the case as

$$\begin{aligned} \Phi \subseteq g(*) \ &\text{if and only if} \ \Phi(a) \leq g(*)(a) = a \to a^* \text{ for all } a \in A, \\ &\text{if and only if} \ f(\Phi)(a) = a \otimes \Phi(a) \leq a^* \text{ for all } a \in A, \\ &\text{if and only if} \ f(\Phi) \leq * \,. \end{aligned}$$

Finally, let us check that $(f \circ g)(*) = *$. For any $a \in L$

$$\begin{aligned} [(f \circ g)(*)](a) &= [f(g(*))](a) = a \otimes (g(*))(a) \\ &= a \otimes (a \to a^*) = a \wedge a^* = a^*, \end{aligned}$$

due to the divisibility of $\otimes$. □

The following example shows that $g \circ f = \mathrm{id}_\Xi$ need not hold.

Example 10. Consider the complete residuated lattice on $L = [0, 1]$ given by the Łukasiewicz operations and the fuzzy set $\Phi \in L^L$ given by $\Phi(a) = a$. Clearly, Φ satisfies all the required conditions. Then since $(g \circ f)(\Phi)(a) = a \to f(\Phi)(a) = a \to (a \otimes \Phi(a)) = a \to (a \otimes a)$, we obtain

$$(g \circ f)(\Phi)(0) = 0 \to (0 \otimes 0) = 1 \neq 0 = \Phi(0).$$

□

Remark 5. (a) Example 10 makes it clear that the adjunction given by f and g in Theorem 5 is a proper adjunction, and does not provide a bijective correspondence.

(b) If Φ additionally satisfies (18), then $* = f(\Phi)$ satisfies (28). Indeed, since $a^* = a \otimes \Phi(a)$, the required inequality of (28), i.e.,

$$a^* \otimes b^* = a \otimes \Phi(a) \otimes b \otimes \Phi(b) \leq (a \otimes b) \otimes \Phi(a \otimes b) = (a \otimes b)^*,$$

follows readily from (18).

(c) Conversely, if $*$ additionally satisfies (28), then $\Phi = g(*)$ satisfies (18). Indeed, verification of (18), i.e., $\Phi(a) \otimes \Phi(b) \leq \Phi(a \otimes b)$, amounts to verifying

$$(a \to a^*) \otimes (b \to b^*) \leq (a \otimes b) \to (a \otimes b)^*,$$

which by adjointness of $\otimes$ and $\to$ is equivalent to

$$(a \otimes b) \otimes (a \to a^*) \otimes (b \to b^*) \leq (a \otimes b)^*.$$

Now, the last inequality holds due to (28) since

$$\begin{aligned}(a \otimes b) \otimes (a \to a^*) \otimes (b \to b^*) \\ &= a \otimes (a \to a^*) \otimes b \otimes (b \to b^*) \\ &\leq a^* \otimes b^* \leq (a \otimes b)^*.\end{aligned}$$

□

Let us now turn to the assumption of divisibility of $\otimes$ in Theorem 5. The proof of Theorem 5 implies that we face two possible problems with a non-divisible $\otimes$: First, $g(*) \notin \Xi$, second, $g(*) \in \Xi$ but $(f \circ g)(*) = *$ fails to hold. We now demonstrate that both options may indeed occur. The first example provides a non-divisible $\otimes$ for which $g(*) \in \Xi$ but $(f \circ g)(*) \neq *$:

Example 11. Let $L = [0, 1]$ and $\otimes$ be the nilpotent t-norm and $\to$ its residuum, i.e.,

$$a \otimes b = \begin{cases} \min(a, b) & \text{if } a + b > 1, \\ 0 & \text{otherwise} \end{cases} \qquad a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ \max(1 - a, b) & \text{if } a > b. \end{cases}$$

This t-norm is not divisible since

$$0.5 \wedge 0.1 = 0.1 \neq 0.5 = 0.5 \otimes (0.5 \to 0.1).$$

For the c-hedge $*$ given by

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ a - 0.5 & \text{if } 0.5 \leq a < 1, \\ 0 & \text{if } a < 0.5, \end{cases}$$

the corresponding fuzzy set $\Phi = g(*)$, i.e., $\Phi(a) = a \to a^*$, satisfies (16) and (17), whence $g(*) \in \Xi$. This is indeed easily seen from the following explicit formula for Φ:

$$\Phi(a) = \begin{cases} 1 & \text{if } a = 1, \\ a - 0.5 & \text{if } 0.75 \leq a < 1, \\ 1 - a & \text{if } a < 0.75. \end{cases}$$

On the other hand, $(f \circ g)(*) \neq *$ since the explicit formula for $(f \circ g)(*)$, which itself is defined by

$$[(f \circ g)(*)](a) = a \otimes \Phi(a),$$

reads

$$[(f \circ g)(*)](a) = \begin{cases} 1 & \text{if } a = 1, \\ a - 0.5 & \text{if } 0.75 < a < 1, \\ 0 & \text{if } a \leq 0.75. \end{cases}$$

□

The second example provides a non-divisible $\otimes$ for which g is not well-defined:

Example 12. Consider the four-element residuated lattice with $L = \{0, a, b, 1\}$ used in the proof of Theorem 8. Consider the c-hedge $* : L \to L$ given by $0^* = 0$, $a^* = b^* = a$ and $1^* = 1$. The fuzzy set $g(*)$ defined by $(g(*))(x) = x \to x^*$ for $x \in L$ is given as $g(*) = \{{}^1/0, {}^1/a, {}^b/b, {}^1/1\}$. But this fuzzy set does not satisfy (17) because while $a \leq b$, we have

$$a \otimes g(*)(a) = a \otimes 1 = a \nleq 0 = b \otimes b = b \otimes g(*)(b).$$

□

5. Conclusions and further work

We considered two basic approaches to the property of monotony of fuzzy closure operators that exist in the literature, namely, a monotony based on filters of degrees of truth and a monotony based on truth-stressing hedges, and examined their expressive power and mutual relationships. We then proposed several generalizations of the two existing concepts along with justifying examples and theorems regarding their properties and mutual relationships. We then proved that all the considered kinds of monotony turn out to be particular cases of monotony condition utilizing certain generalized hedges, and examined properties of the involved concepts. The topics for future research include the following.

- The present paper leaves several open problems that are presented in the remarks above. Solving these problems shall enhance our comprehension of monotony of fuzzy closure.
- Our work presents considerable finer notions of monotony compared to those previously available. The presented notions provide a conceptual tool to classify fuzzy closure operators. Such a classification of concrete fuzzy closure operators and systems known from various areas of fuzzy set theory appears a feasible step toward a better understanding of closure structures in fuzzy logic.
- Considering the examples and results presented in the paper, the following question naturally arises. Given an operator C, how to measure an extent to which C is monotone? We suggest two basic approaches to answer this question.

  The first naturally derives from our observations according to which the larger the filter or fuzzy filter, the stronger the monotony defined by the filter, and, analogously, the larger the hedge, the stronger the corresponding monotony. One may hence attempt to define the extent of operator's monotony as the largest filter/hedge for which monotony is satisfied, and explore ramifications of this concept.
- The second approach to the question consists in utilizing what may be termed a degree of operator's monotony which may be defined by

$$\mathrm{mon}(C) = \bigwedge_{A,B\in L^X} (S(A,B) \to S(C(A),C(B))).$$

  In a broader perspective, this opens way to considerations regarding graded properties of objects, as pursued in a general perspective, e.g., in [16,25,35].

CRediT authorship contribution statement

Radim Belohlavek: Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization; **Manuel Ojeda-Hernández:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Data availability

No data was used for the research described in the article.

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The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] R. Belohlavek, J.W. Dauben, G.J. Klir, Fuzzy Logic and Mathematics: A Historical Perspective, Oxford University Press, New York, New York, 2017.
[2] J.L. Castro, E. Trillas, S. Cubillo, On consequence in approximate reasoning, J Appl. Non-Class. Logics 4 (1994) 91–103.
[3] M.K. Chakraborty, Use of fuzzy set theory in introducing graded consequence in multiple valued logic, in: M.M. Gupta, T. Yamakawa (Eds.), Fuzzy Logic in Knowledge-Based Systems, Decision, and Control, Amsterdam, North-Holland, 1988, pp. 247–257.
[4] M.K. Chakraborty, Graded consequence: further studies, J Appl. Non-Class. Logics 5 (1995) 227–238.
[5] G. Gerla, Graded consequence relations and closure operators, J. Appl. Non-Class. Logics 6 (1996) 369–379.
[6] G. Gerla, Closure operators, fuzzy logic and constraints, in: D. Dubois, H. Prade, E.P. Klement (Eds.), Fuzzy Sets, Logics, and Reasoning about Knowledge, Dordrecht, Kluwer, 1999.
[7] G. Gerla, Fuzzy Logic: Mathematical Tools for Approximate Reasoning, Kluwer, Dordrecht, Dordrecht, 2001.
[8] V. Novák, I. Perfilieva, J. Močkoř, Mathematical Principles of Fuzzy Logic, Kluwer, Boston, Boston, 1999.
[9] J. Pavelka, On fuzzy logic I: many-valued rules of inference, Z. Math. Logik Grundlagen Math. 25 (1979) 45–52.
[10] R. Lowen, Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl. 56 (3) (1976) 621–633.
[11] L. Biacino, G. Gerla, An extension principle for fuzzy closure operators, J. Math. Anal. Appl. 198 (1996) 1–24.
[12] J.R. Elorza, J. Fuentes-González, P. Bragard, Burillo, On the relation between fuzzy closing morphological operators, fuzzy consequence operators induced by fuzzy preorders and fuzzy closure and co-closure systems, Fuzzy Sets Syst. 218 (2013) 73–89.
[13] J. Elorza, P. Burillo, Connecting fuzzy preorders, fuzzy consequence operators and fuzzy closure and co-closure systems, Fuzzy Sets Syst. 139 (3) (2003) 601–613.
[14] V. Murali, Lattice of fuzzy subalgebras and closure systems in I^X, Fuzzy Sets Syst. 41 (1991) 101–111.
[15] R. Belohlavek, Fuzzy closure operators, J. Mathematical Analysis and Applications 262 (2001) 473–489.
[16] R. Belohlavek, Fuzzy Relational Systems: Foundations and Principles, Kluwer, New York, New York, 2002.
[17] R.O.R. ǵuez, F. Esteva, P. Garcia, L. Godo, On implicative closure operators in approximate reasoning, Int. J. Approximate Reasoning 33 (2) (2003) 159–184.
[18] R. Belohlavek, Fuzzy Galois connections, Math. Logic Quart. 45 (1999) 497–504.
[19] R. Belohlavek, Concept lattices and order in fuzzy logic, Annal. Pure Appl. Logic 128 (2004) 277–298.
[20] S. Pollandt, Fuzzy-Begriffe, Formale Begriffsanalyse Unscharfer Daten, Springer-verlag, 2013.

[21] J.A. Goguen, The logic of inexact concepts, Synthese 19 (1968) 325–373.
[22] D. Klaua, Über einen zweiten Ansatz zur mehrwertigen Mengenlehre [On a second approach to many-valued set theory], Monatsberichte der Deutschen Akademie der Wissenschaften zu Berlin 8 (1966) 161–177.
[23] S. Gottwald, A cumulative system of fuzzy sets, Lect. Notes Math. 537 (1976) 109–119.
[24] R. Belohlavek, T. Funioková, V. Vychodil, Fuzzy closure operators with truth stressers, Logic J. IGPL 13 (5) (2005) 503–513.
[25] S. Gottwald, A Treatise on Many-Valued Logic, Research Studies Press, Baldock, UK, Baldock, UK, 2001.
[26] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer, Dordrecht, Dordrecht, 1998.
[27] M. Ward, R.P. Dilworth, 45, American Mathematical Society, 1939. Residuated lattices. Transactions of the. 335–354
[28] R. Belohlavek, Lattice type fuzzy order and closure operators in fuzzy ordered sets, in: Proc. Joint 9th IFSA World Congress and 20Th NAFIPS International Conference, Vancouver, Canada, IEEE Press, 2001, pp. 2281–2286.
[29] M. Ojeda-Hernández, I.P. Cabrera, P. Cordero, E. Muñoz-Velasco, Fuzzy closure systems: motivation, definition and properties, Int. J. Approximate Reason. 148 (2022) 151–161.
[30] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, Cambridge University Press, Cambridge, UK, second edition edition, Cambridge, UK, 2002.
[31] P. Hájek, On very true, Fuzzy Sets Syst. 124 (3) (2001) 329–333.
[32] R. Belohlavek, V. Vychodil, Fuzzy Equational Logic, Berlin, Springer, 2005.
[33] M. Ghanim, F.S. Al-Sirehy, Topological modification of a fuzzy closure space, Fuzzy Sets Syst. 27 (2) (1988) 211–215.
[34] R. Belohlavek, V. Vychodil, Formal concept analysis and linguistic hedges, Int. J. General Syst. 41 (5) (2012) 503–532.
[35] L. Běhounek, Bodenhofer, P. Cintula, Relations in fuzzy class theory: initial steps, Fuzzy Sets Syst. 159 (2008) 1729–1772.