Triadic fuzzy Galois connections as ordinary connections

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***Abstract*—The paper presents results on representation of the basic structures related to ternary fuzzy relations by the structures related to ordinary ternary relations, such as Galois connections, closure operators, and trilattices (structures of maximal Cartesian subrelations). These structures appear as the fundamental structures in relational data analysis such as formal concept analysis or association rules. We prove several representation theorems that allow us to automatically transfer some of the known results from the ordinary case to fuzzy case. The transfer is demonstrated by examples.**

I. Introduction

Relations play a fundamental role in mathematics, computer science, and their applications. Many results about ordinary relations have been generalized to the setting of fuzzy relations in the past. There has always been a fundamental question of how the various fuzzifications are related to the ordinary notions and results. Needless to say, this question is important both from a practical and theoretical point of view and is treated to some extent in textbooks, see e.g. [15].

In this paper we deal with basic structures associated to ternary relations that appear as fundamental ones in the methods of relational data analysis, namely the closure-like structures such as Galois connections, closure operators, structures of their fixpoints and the like. Such structures appear e.g. in formal concept analysis [11], association rules [25], relational databases [19], or relational equations [10]. We focus on ternary relations and provide hints to generalize the results in a straightforward way to general n-ary relations. Note that ternary and higher order relations are becoming increasingly important for their role in the analysis of three-way and n-way data [9], [14], [16], [17]. In particular, we look at the structures associated to ordinary ternary relations and their relationships to their counterparts associated to ternary fuzzy relations. Since the very basic structures are induced by Galois connections, we focus on these and the structures of their fixpoints. Such structures appear directly in triadic concept analysis [18], [23], triadic association rules [14], or in factor analysis of triadic data [5], [7].

The most common way of looking at the relationship between ordinary notions and their fuzzy counterparts is in terms of a-cuts of fuzzy relations (see e.g. [15]) but there are additional possible views at the question as well. One of them, utilized in this paper, is provided in [3, Section 3.1.2].

Our paper is organized as follows. We first provide preliminaries in Section II. In Section III, we introduce the Galois connections induced by ternary fuzzy relations and provide their axiomatization. In Section IV, we describe two kinds of representation of these Galois connections by means of Galois connections induced by ordinary relations and present an application of our results, namely a theorem showing that every fuzzy concept trilattice is isomorphic to some ordinary concept trilattice. Due to lack of space, the proofs are omited and left for the full version of this paper.

II. Preliminaries

A. Fuzzy logic and fuzzy sets

This section contains preliminaries regarding the structures of truth degrees and basic notions used in fuzzy logic and fuzzy sets. More information can be found in [3], [13].

We assume that the degrees form a bounded partially ordered set L which is a complete lattice. Furthermore, we assume that L is equipped with certain aggregation operators which are known from mathematical fuzzy logic [12], [13]. In particular, we assume that the scale L of degrees forms a complete residuated lattice, i.e. an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$$

such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy the so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

for each $a, b, c \in L$. In fuzzy logic, elements a of L are called truth degrees and $\otimes$ and $\rightarrow$ are considered as the (truth functions of) many-valued conjunction and implication.

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a continuous (or at least left-continuous) t-norm (i.e. a commutative, associative, and isotone operation on $[0, 1]$ with 1 acting as a neutral element) with the corresponding $\rightarrow$ given by $a \rightarrow b = \max\{c \,|\, a \otimes c \leq b\}$. The three most important pairs of adjoint operations on the unit interval are: Łukasiewicz ($a \otimes b = \max(a+b-1, 0)$, $a \rightarrow b = \min(1-a+b, 1)$; Gödel ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$, $= b$ if $a > b$; Goguen ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$, $= b/a$ if $a > b$). Another common choice is a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of L and restrictions of Gödel operations from $[0,1]$ to L. A special case of a complete residuated lattice is the two-element Boolean algebra $\langle\{0,1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of classical logic. This is important because for the particular case $\mathbf{L} = \mathbf{2}$, the notions and results become the ones regarding ordinary sets and relations.

Given a complete residuated lattice $\mathbf{L}$, we define the usual notions [3], [13]: an $\mathbf{L}$-set (fuzzy set, graded set) A in a universe U is a mapping $A\colon U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Let L^U denote the collection of all $\mathbf{L}$-sets in U. Operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in L^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. $\mathbf{2}$-sets and operations with $\mathbf{2}$-sets can be identified with ordinary sets and operations with ordinary sets, respectively. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$; similarly for ternary relations. Given $A, B \in L^U$, we define the degree $S(A,B)$ of inclusion of A in B by

$$S(A,B) = \textstyle\bigwedge_{u \in U}\big(A(u) \rightarrow B(u)\big) \qquad (2)$$

and the degree of equality of A and B by

$$A \approx B = \textstyle\bigwedge_{u \in U}\big(A(u) \leftrightarrow B(u)\big) \qquad (3)$$

Note that (2) generalizes the ordinary subsethood relation $\subseteq$. Described verbally, $S(A,B)$ represents a degree to which every element of A is an element of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. Likewise, (3) generalizes the ordinary equality relation $=$. $A \approx B$ represents a degree to which every element belongs to A iff it belongs to B. Clearly, $A = B$ iff $A \approx B = 1$.

B. Triadic concept analysis and Galois connections

In this section, we provide the basic notions regarding the structures related to ternary relations, mainly in terms of formal concept analysis which is the main intended application area of our results. More information can be found in [18], [23], see also [11].

A *triadic context* is a quadruple $\langle X, Y, Z, I\rangle$ where X, Y, and Z are non-empty sets, and I is a ternary relation between X, Y, and Z, i.e. $I \subseteq X \times Y \times Z$. The sets X, Y, and Z are interpreted as the sets of objects, attributes, and conditions, respectively; I is interpreted as the incidence relation ("to have-under relation"). That is, $\langle x, y, z\rangle \in I$ is interpreted as: object x has attribute y under condition z. In this case, we say that x, y, z (or x, z, y, or the result of listing x, y, z in any other sequence) are related by I. For convenience, a triadic context is denoted by $\langle X_1, X_2, X_3, I\rangle$.

Let $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ be a triadic context. For $\{i,j,k\} = \{1,2,3\}$ and $C_k \subseteq X_k$, consider the dyadic context [11]

$$\mathbf{K}^{ij}_{C_k} = \langle X_i, X_j, I^{ij}_{C_k}\rangle$$

defined by

$$\begin{aligned}&\langle x_i, x_j\rangle \in I^{ij}_{C_k} \text{ iff}\\ &\qquad \text{for each } x_k \in C_k : x_i, x_j, x_k \text{ are related by } I.\end{aligned}$$

The concept-forming operators induced by $\mathbf{K}^{ij}_{C_k}$ are denoted by ${}^{(i,j,C_k)}$. That is, for $C_i \subseteq X_i$ and $C_j \subseteq X_j$ we have

$$\begin{aligned}C_i^{(i,j,C_k)} &= \{x_j \in X_j : \text{ for each } x_i \in C_i : \langle x_i, x_j\rangle \in I^{ij}_{C_k}\},\\ C_j^{(i,j,C_k)} &= \{x_i \in X_i : \text{ for each } x_j \in C_j : \langle x_i, x_j\rangle \in I^{ij}_{C_k}\}.\end{aligned}$$

A *triadic concept* of $\langle X_1, X_2, X_3, I\rangle$ is a triplet $\langle C_1, C_2, C_3\rangle$ of $C_1 \subseteq X_1$, $C_2 \subseteq X_2$, and $C_3 \subseteq X_3$, such that for every $\{i,j,k\} = \{1,2,3\}$ we have

$$C_i = C_j^{(i,j,C_k)},\ C_j = C_k^{(j,k,C_i)}, \ \text{ and } C_k = C_i^{(k,i,C_j)}.$$

Note that the latter conditions hold for every 3-element set $\{i,j,k\} = \{1,2,3\}$ if and only if they hold for any single 3-element set $\{i,j,k\} = \{1,2,3\}$. C_1, C_2, and C_3 are called the *extent*, *intent*, and *modus* of $\langle C_1, C_2, C_3\rangle$. Geometrically, triadic concepts are just the maximal cuboids contained in I, i.e. maximal subrelations of I that result as Cartesian products of sets of objects, attributes, and modi. The set of all triadic concepts of $\langle X_1, X_2, X_3, I\rangle$ is denoted by $\mathcal{T}(X_1, X_2, X_3, I)$ and is called the concept trilattice of $\langle X_1, X_2, X_3, I\rangle$; we refer to Section IV for the notion of a trilattice. Note that complete trilattices (and n-lattices) are the appropriate generalizations of complete lattices (i.e. dyadic lattices) that result as naturally structured sets of fixpoints of the connections induced by ternary (and n-ary) relations.

For $\{i,j,k\} = \{1,2,3\}$, a triadic context $\langle X_1, X_2, X_3, I\rangle$ induces the operators

$$(i)_I : 2^{X_j} \times 2^{X_k} \rightarrow 2^{X_i}$$

defined by

$$(C_j, C_k)^{(i)_I} = C_j^{(j,i,C_k)}$$

for any $C_j \subseteq X_j$ and $C_k \subseteq X_k$. The triplet $\langle {}^{(1)_I}, {}^{(2)_I}, {}^{(3)_I}\rangle$, denoted also simply by $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle$, forms an (ordinary) triadic Galois connection [8], see also Section III. That is, for every $C_i \subseteq X_i$, $C_j \subseteq X_j$, and $C_k \subseteq X_k$, one has

$$\begin{aligned}C_3 \subseteq (C_1, C_2)^{(3)} &\text{ iff } C_1 \subseteq (C_2, C_3)^{(1)}\\ &\text{ iff } C_2 \subseteq (C_1, C_3)^{(2)}.\end{aligned} \qquad (4)$$

Conversely, every triplet $\langle {}^{(1)_I}, {}^{(2)_I}, {}^{(3)_I}\rangle$ satisfying (4) is induced by some triadic context [8].

III. Triadic fuzzy Galois connections

A. Triadic fuzzy contexts and their Galois connections

The basic notions of triadic concept analysis have been generalized for data with fuzzy attributes, i.e. for ternary fuzzy relations, in [6] and have been utilized for factor analysis of three-way data in [7]. We now recall the notions needed.

A *triadic* $\mathbf{L}$*-context* (triadic fuzzy context, or just triadic context) is a quadruple $\langle X, Y, Z, I\rangle$ where X, Y, and Z are non-empty sets, and I is a ternary fuzzy relation between X, Y, and Z, i.e. $I : X \times Y \times Z \to L$. Again, X, Y, and Z are interpreted as the sets of objects, attributes, and conditions, respectively, and for $x \in X$, $y \in Y$, and $z \in Z$, the degree $I(x, y, z) \in L$ is interpreted as the degree to which object x has attribute y under condition z. In this case, we also say that $I(x, y, z)$ is the degree to which x, y, z (or x, z, y or z, x, y, etc.) are related and, for convenience, denote $I(x, y, z)$ also by $I\{x, y, z\}$ or $I\{x, z, y\}$ or $I\{z, x, y\}$, etc. As in the ordinary case, we denote a triadic fuzzy context by $\langle X_1, X_2, X_3, I\rangle$. Except for mathematical arguments, the motivation for considering triadic fuzzy context is that in several situations, the relationship between objects, attributes, and modi appear. For example, a degree to which object x has feature y under condition z may be interpreted as the degree to which customer z considers product x as having feature y (e.g., the degree $3/4$ means that customer z considers food product x as having a good taste), see [6].

For every $\{i, j, k\} = \{1, 2, 3\}$ and a fuzzy set $A_k \in L^{X_k}$, a triadic $\mathbf{L}$-context $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ induces a dyadic $\mathbf{L}$-context

$$\mathbf{K}^{ij}_{A_k} = \langle X_i, X_j, I^{ij}_{A_k}\rangle$$

in which the fuzzy relation $I^{ij}_{A_k}$ between X_i and X_j is defined by

$$I^{ij}_{A_k}(x_i, x_j) = \bigwedge_{x_k \in X_k} (A_k(x_k) \to I\{x_i, x_j, x_k\})$$

for every $x_i \in X_i$ and $x_j \in X_j$. Dyadic $\mathbf{L}$-contexts and the associated structures including concept-forming operators and concept lattices were studied in a series of papers, see e.g. [3], [4], [20]. The concept-forming operators induced by $\mathbf{K}^{ij}_{A_k}$ are denoted by ${}^{(i,j,A_k)}$. That is, for a fuzzy set A_i in X_i, we define a fuzzy set $A_i^{(i,j,A_k)}$ in X_j by

$$A_i^{(i,j,A_k)}(x_j) = \textstyle\bigwedge_{x_i \in X_i} A_i(x_i) \to I^{ij}_{A_k}(x_i, x_j).$$

Similarly,

$$A_j^{(i,j,A_k)}(x_i) = \textstyle\bigwedge_{x_j \in X_j} A_j(x_j) \to I^{ij}_{A_k}(x_i, x_j).$$

A *triadic* $\mathbf{L}$*-concept* (triadic fuzzy concept) of $\langle X_1, X_2, X_3, I\rangle$ is a triplet $\langle A_1, A_2, A_3\rangle$ consisting of fuzzy sets $A_1 \in L^{X_1}$, $A_2 \in L^{X_2}$, and $A_3 \in L^{X_3}$, such that for every $\{i, j, k\} = \{1, 2, 3\}$ we have $A_i = A_j^{(i,j,A_k)}$, $A_j = A_k^{(j,k,A_i)}$, and $A_k = A_i^{(k,i,A_j)}$. In this case, A_1, A_2, and A_3 are called the *extent*, *intent*, and *modus* of $\langle A_1, A_2, A_3\rangle$. The set of all triadic concepts of $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ is denoted by $\mathcal{T}(X_1, X_2, X_3, I)$ and is called the $\mathbf{L}$*-concept trilattice* (fuzzy concept trilattice) of $\mathbf{K}$.

Remark 1. Clearly, the notions introduced in this section generalize the corresponding ordinary notions reviewed in Section II. Namely, putting $L = \{0, 1\}$, the notions of a triadic $\mathbf{L}$-context, the induced operators and so on may be identified with the ordinary notions.

B. Axiomatizing Galois connections of triadic fuzzy contexts

As in the ordinary case, a triadic $\mathbf{L}$-context $\langle X_1, X_2, X_3, I\rangle$ induces three operators

$$(i)_I : L^{X_j} \times L^{X_k} \to L^{X_i}$$

for $\{i, j, k\} = \{1, 2, 3\}$ which are defined by

$$(A_j, A_k)^{(i)_I} = A_j^{(j,i,A_k)} \tag{5}$$

for any $A_j \in L^{X_j}$ and $A_k \in L^{X_k}$. The triplet $\langle {}^{(1)_I}, {}^{(2)_I}, {}^{(3)_I}\rangle$, denoted also just by $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle$, is axiomatized below. In fact, we provide an axiomatization of a wider class of operators for reasons that become apparent later.

Remark 2. For convenience, we use also $(A_2, A_1)^{(3)}$ with the meaning $(A_2, A_1)^{(3)} = (A_1, A_2)^{(3)}$; same for ${}^{(1)}$ and ${}^{(2)}$.

Recall that an order filter in a partially ordered set $\langle L, \leq\rangle$ is any subset $K \subseteq L$ for which $a \in K$ and $a \leq b$ imply $b \in K$ for any $a, b \in L$.

Definition 1. *Let K be an order filter in $\langle L, \leq\rangle$. A* triadic $\mathbf{L}_K$-Galois connection *between sets X_1, X_2, and X_3 is a triplet $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle$ of mappings ${}^{(1)} : L^{X_2} \times L^{X_3} \to L^{X_1}$, ${}^{(2)} : L^{X_1} \times L^{X_3} \to L^{X_2}$, and ${}^{(3)} : L^{X_1} \times L^{X_2} \to L^{X_3}$, satisfying for every $A_1 \in L^{X_1}$, $A_2 \in L^{X_2}$, and $A_3 \in L^{X_3}$, that if $S(A_3, (A_1, A_2)^{(3)}) \in K$ or $S(A_1, (A_2, A_3)^{(1)}) \in K$ or $S(A_2, (A_1, A_3)^{(2)}) \in K$, then*

$$\begin{aligned} S(A_3, (A_1, A_2)^{(3)}) &= S(A_1, (A_2, A_3)^{(1)}) = \\ &= S(A_2, (A_1, A_3)^{(2)}). \end{aligned} \tag{6}$$

Remark 3. (a) One can easily see that for $L = \{0, 1\}$, triadic $\mathbf{L}_K$-Galois connections become ordinary triadic Galois connections (observe that in this case, there are only two filters, namely $K = L$ and $K = \{1\}$ and both lead to the same notion of an $\mathbf{L}_K$-Galois connection).

(b) In accordance with [1], we use the term $\mathbf{L}$-Galois connections for $\mathbf{L}_L$-Galois connections.

The following theorem provides an alternative characterization of $\mathbf{L}_K$-Galois connections in terms of extensivity ond antitony.

Theorem 1. *For $\{i, j, k\} = \{1, 2, 3\}$, a triplet $\langle {}^{(1)}, {}^{(2)}, {}^{(3)}\rangle$ is a triadic $\mathbf{L}_K$-Galois connection iff the following conditions hold for all $A_i, A_i' \in L^{X_i}$, $A_j \in L^{X_j}$, $A_k \in L^{X_k}$:*

(a) $A_i \subseteq (A_j, (A_i, A_j)^{(k)})^{(i)}$ *(extensivity),*

(b) *if $S(A_i, A_i') \in K$ then*
$S(A_i, A_i') \leq S((A_i', A_j)^{(k)}, (A_i, A_j)^{(k)})$ *(antitony).*

Next, we provide some properties that are needed to show a bijective correspondence between ternary fuzzy relations and triadic $\mathbf{L}$-Galois connections, i.e. proving that triadic $\mathbf{L}$-Galois connections are represented by ternary fuzzy relations.

Lemma 1. *For $\{i,j,k\} = \{1,2,3\}$, index sets P, Q, and fuzzy sets $A_{ip} \in L^{X_i}$, and $A_{jp} \in L^{X_j}$ the following equality holds:*

$$(\bigvee_{p\in P} A_{ip}, \bigvee_{q\in Q} A_{jq})^{(k)} = \bigwedge_{p\in P, q\in Q} (A_{ip}, A_{jq})^{(k)} \tag{7}$$

Lemma 2. *Let $(^{(1)}, ^{(2)}, ^{(3)})$ be a triadic $\mathbf{L}$-Galois connection. Then for $\{i,j,k\} = \{1,2,3\}$ and $A_i \in L^{X_i}$ let the mappings $\uparrow_{A_i}: L^{X_j} \to L^{X_k}$ and $\downarrow_{A_i}: L^{X_k} \to L^{X_j}$ be defined as*

$$\begin{aligned} A_k^{\uparrow_{A_i}} &= (A_k, A_i)^{(j)} \\ A_j^{\downarrow_{A_i}} &= (A_j, A_i)^{(k)}. \end{aligned}$$

Then $\langle \uparrow_{A_i}, \downarrow_{A_i} \rangle$ forms a dyadic $\mathbf{L}$-Galois connection between X_j and X_k [1].

Lemma 3. *For $\{i,j,k\} = \{1,2,3\}$ it holds*

(a) $a \to (\{1/x_i\}, \{1/x_j\})^{(k)} = (\{a/x_i\}, \{1/x_j\})^{(k)}$,
(b) $\bigwedge_{x_i\in X_i} A_i(x_i) \to (\{1/x_i\}, \{1/x_j\})^{(k)} = (A_i, \{1/x_j\})^{(k)}$.

The next theorem, which can be proved using the above lemmas, shows that triadic $\mathbf{L}$-Galois connections are just the mappings obtained from ternary fuzzy relations by (5).

Theorem 2. *Let $I \in L^{X_1\times X_2\times X_3}$. Let $\langle ^{(1)}, ^{(2)}, ^{(3)} \rangle$ be a triadic $\mathbf{L}$-Galois connection between X_1, X_2, and X_3 and define a ternary relation $I_{\langle ^{(1)}, ^{(2)}, ^{(3)} \rangle} \in L^{X_1\times X_2\times X_3}$ by*

$$\begin{aligned} & I_{\langle ^{(1)}, ^{(2)}, ^{(3)} \rangle}(x_1, x_2, x_3) = (\{1/x_1\}, \{1/x_2\})^{(3)} = \\ = \; & (\{1/x_1\}, \{1/x_3\})^{(2)} = (\{1/x_2\}, \{1/x_3\})^{(1)}. \end{aligned}$$

Then

(a) *The triplet $\langle ^{(1)_I}, ^{(2)_I}, ^{(3)_I} \rangle$ forms a triadic $\mathbf{L}$-Galois connection.*
(b) $I = I_{\langle ^{(1)_I}, ^{(2)_I}, ^{(3)_I} \rangle}$.
(c) $\langle ^{(1)}, ^{(2)}, ^{(3)} \rangle = \langle ^{(1)_{I_{\langle (1),(2),(3) \rangle}}}, ^{(2)_{I_{\langle (1),(2),(3) \rangle}}}, ^{(3)_{I_{\langle (1),(2),(3) \rangle}}} \rangle$.

Therefore, (6) provides an axiomatization of the mappings induced by ternary fuzzy relations by (5).

IV. Representation of triadic fuzzy Galois connections by ordinary connections

In this section, we provide two kinds of representation of triadic fuzzy Galois connections using ordinary tradic Galois connections. In Section IV-A, we present a representation which is based on looking at a fuzzy set A in U as the area below the membership function, i.e. a subset of the Cartesian product $U \times L$ of U and the set L of truth degrees. In Section IV-B, we present another representation, a cut-like one, using which a triadic fuzzy Galois connection is represented as a nested system of ordinary triadic connections. In Section IV-C, we present an application of the Cartesian representation in showing that every fuzzy concept trilattice is isomorphic to some ordinary concept trilattice.

A. Cartesian representation

For the first type of representation, we utilize the following mappings, studied in [2] and further developed in [3] (note that these mappings were independently introduced in [20]). For a fuzzy set $A \in L^U$ put

$$\lfloor A \rfloor = \{\langle u, a \rangle \in U \times L \mid a \leq A(u)\};$$

For an ordinary set $B \subseteq U \times L$, define a fuzzy set $\lceil B \rceil$ in U by

$$\lceil B \rceil(u) = \bigvee_{\langle u,a \rangle \in B} a.$$

$\lfloor A \rfloor$ may be thought of as the area below A while $\lceil B \rceil$ may be thought of as an upper envelope of B. In what follows, we use the properties of $\lfloor\ \rfloor$ and $\lceil\ \rceil$ which may be found in [3].

Definition 2. *An (ordinary) triadic Galois connection $\langle ^{\langle 1 \rangle}, ^{\langle 2 \rangle}, ^{\langle 3 \rangle} \rangle$ between $X_1 \times L$, $X_2 \times L$, $X_3 \times L$ is called* commutative with respect to $\lfloor\lceil\rceil\rfloor$ *iff*

$$(\lfloor\lceil A_i \rceil\rfloor, \lfloor\lceil A_j \rceil\rfloor)^{\langle k \rangle} = \lfloor\lceil (A_i, A_j) \rceil\rfloor^{\langle k \rangle} \tag{8}$$

holds for any $\{i,j,k\} = \{1,2,3\}$ and any sets $A_1 \in X_1 \times L$, $A_2 \in X_2 \times L$, and $A_3 \in X_3 \times L$.

The following definition shows how triplets of mappings on fuzzy sets in X_is may be defined from triplets of mappings on subsets of $X_i \times L$s and vice versa. (By small abuse of notation we utilize $(i)_{(i)}$ to denote the mapping induced by (i).)

Definition 3. *Let $\{i,j,k\} = \{1,2,3\}$. For a triadic Galois connection $\langle ^{\langle 1 \rangle}, ^{\langle 2 \rangle}, ^{\langle 3 \rangle} \rangle$ between $X_1 \times L$, $X_2 \times L$, $X_3 \times L$, and fuzzy sets $A_i \in L^{X_i}$, $A_j \in L^{X_j}$, and $A_k \in L^{X_k}$ we define mappings $^{(i)_{\langle i \rangle}}: L^{X_j} \times L^{X_k} \to L^{X_i}$ by*

$$(A_j, A_k)^{(i)_{\langle i \rangle}} = \lceil (\lfloor A_j \rfloor, \lfloor A_k \rfloor)^{(i)} \rceil \tag{9}$$

Let $\langle ^{(1)}, ^{(2)}, ^{(3)} \rangle$ be a triadic $\mathbf{L}$-Galois connection between X_1, X_2, and X_3. Then for sets $A_i \in X_i \times L$, $A_j \in X_j \times L$, and $A_k \in X_k \times L$, we define mappings $^{\langle i \rangle_{(i)}}: (X_j \times L) \times (X_k \times L) \to X_i \times L$ by

$$(A_j, A_k)^{\langle i \rangle_{(i)}} = \lfloor (\lceil A_j \rceil, \lceil A_k \rceil)^{(i)} \rfloor \tag{10}$$

The following theorem provides the first way to represent triadic fuzzy Galois connections using ordinary connections.

Theorem 3. *Let $\langle ^{(1)}, ^{(2)}, ^{(3)} \rangle$ be a triadic $L_{\{1\}}$-Galois connection between X_1, X_2, and X_3 and $\langle ^{\langle 1 \rangle}, ^{\langle 2 \rangle}, ^{\langle 3 \rangle} \rangle$ be a triadic Galois connection. between $X_1 \times L$, $X_2 \times L$, and $X_3 \times L$.*

Then the following holds:

(a) $\langle ^{\langle 1 \rangle_{(1)}}, ^{\langle 2 \rangle_{(2)}}, ^{\langle 3 \rangle_{(3)}} \rangle$ *is a triadic Galois connection commutative with respect to $\lfloor\lceil\rceil\rfloor$.*
(b) $\langle ^{(1)_{\langle 1 \rangle}}, ^{(2)_{\langle 2 \rangle}}, ^{(3)_{\langle 3 \rangle}} \rangle$ *is a triadic $\mathbf{L}_{\{1\}}$-Galois connection.*
(c) *The map $\langle ^{(1)}, ^{(2)}, ^{(3)} \rangle \mapsto \langle ^{\langle 1 \rangle}, ^{\langle 2 \rangle}, ^{\langle 3 \rangle} \rangle$ is an one-to-one map between the set of all triadic $\mathbf{L}_{\{1\}}$-Galois connections between X_1, X_2, and X_3 and the set of all triadic Galois connections between $X_1 \times L$, $X_2 \times L$, $X_3 \times L$ that are commutative with respect to $\lfloor\lceil\rceil\rfloor$.*

B. Cut-like representation

The second representation is inspired by the notion of an a-cut of a fuzzy set. Recall that for a fuzzy set $A \in L^U$ and a degree $a \in L$, the a-cut ${}^aA$ of A is the ordinary subset of U defined by

$$^aA = \{u \in U \mid a \leq A(u)\}.$$

It is well known that each fuzzy set is uniquely represented by the system of its a-cuts. Depending on the properties of the scale of truth degrees, one may introduce an appropriate notion of a nested system of subsets of U in such a way that nested systems become just the system of a-cuts of fuzzy sets, see e.g. [3].

One may easily verify that straightforward conditions such as $({}^aA_1, {}^aA_2)^{(3)} = {}^a((A_1, A_2)^{(3)})$ do not hold for tradic fuzzy Galois connections. Nevertheless, a cut-like representation of triadic fuzzy Galois connections is possible, as shown in the rest of this section. The representation is based on the following notion.

Definition 4. *Let $\{i,j,k\} = \{1,2,3\}$. A system $\{\langle^{(1_a)}, ^{(2_a)}, ^{(3_a)}\rangle \mid a \in L\}$ of (ordinary) triadic Galois connections is called* $\mathbf{L}$-nested *iff*

1) *for each $a, b \in L$ such that $a \leq b$, and $A_i \in L^{X_i}$, $A_j \in L^{X_j}$ it holds $(A_i, A_j)^{(k_a)} \supseteq (A_i, A_j)^{(k_b)}$*
2) *for all $x_i \in X_i$, $x_j \in X_j$, $x_k \in X_k$ the set $\{a \in L \mid x_i \in (\{x_j\}, \{x_k\})^{(i_a)}\}$ has a greatest element.*

We need the following lemmas.

Lemma 4. *For $\{i,j,k\} = \{1,2,3\}$, let $I \in L^{X_1 \times X_2 \times X_3}$ be an $\mathbf{L}$-relation, $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$ be the triadic $\mathbf{L}$-Galois connection induced by I and for $a \in L$ let $\langle^{(1_a)}, ^{(2_a)}, ^{(3_a)}\rangle$ be the triadic Galois connections induced by the cuts ${}^aI$. Then*

(a) *for every $A_i \in 2^{X_i}$, $A_j \in 2^{X_j}$, and $a \in L$ we have*

$$^a(A_i, A_j)^{(k)} = (A_i, A_j)^{(k_a)},$$

(b) *for all fuzzy sets $A_i \in L^{X_i}$, $A_j \in L^{X_j}$, and $b, c \in L$ we have*

$$^a(A_i, A_j)^{(k)} = \bigcap_{b,c \in L} ({}^bA_i, {}^cA_j)^{(k_{a \otimes b \otimes c})}.$$

Lemma 5. *Let $\langle^{(1)_1}, ^{(2)_1}, ^{(3)_1}\rangle$ and $\langle^{(1)_2}, ^{(2)_2}, ^{(3)_2}\rangle$ be triadic $\mathbf{L}$-Galois connections, let I_1 and I_2 be the corresponding $\mathbf{L}$-relations between X_1, X_2, and X_3. Then for $\{i,j,k\} = \{1,2,3\}$ it holds that $I_1 \subseteq I_2$ iff for each $A_i \in L^{X_i}$, $A_j \in L^{X_j}$ it holds $(A_i, A_j)^{(k)_1} \subseteq (A_i, A_j)^{(k)_2}$.*

Using the above lemmas, one may proove the following theorem which provides the cut-like representation of triadic fuzzy Galois connections.

Theorem 4. *For a triadic $\mathbf{L}$-Galois connection $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$ between X_1, X_2, and X_3 denote*

$$\mathcal{C}_{\langle^{(1)}, ^{(2)}, ^{(3)}\rangle} = \{\langle^{(1_a)}, ^{(2_a)}, ^{(3_a)}\rangle \mid a \in L\}.$$

For an $\mathbf{L}$-nested system $\{\langle^{(1_a)}, ^{(2_a)}, ^{(3_a)}\rangle \mid a \in L\}$ of triadic Galois connections between X_1, X_2, and X_3 denote by $\langle^{(1)\mathcal{C}}, ^{(2)\mathcal{C}}, ^{(3)\mathcal{C}}\rangle$ the mappings defined for $\{i,j,k\} = \{1,2,3\}$, and $A_i \in L^{X_i}$, $A_j \in L^{X_j}$ by

$$(A_i, A_j)^{(k)\mathcal{C}}(x_k) = \bigvee\{a \mid x_k \in \bigcap_{b,c \in L} ({}^bA_i, {}^cA_j)^{(k_{a \otimes b \otimes c})}\}.$$

Then

(a) *$\mathcal{C}_{\langle^{(1)}, ^{(2)}, ^{(3)}\rangle}$ is an $\mathbf{L}$-nested system of triadic Galois connections,*
(b) *$\langle^{(1)\mathcal{C}}, ^{(2)\mathcal{C}}, ^{(3)\mathcal{C}}\rangle$ is a triadic $\mathbf{L}$-Galois connection,*
(c) *$\langle^{(1)}, ^{(2)}, ^{(3)}\rangle = \langle^{(1)\mathcal{C}_{\langle(1),(2),(3)\rangle}}, ^{(2)\mathcal{C}_{\langle(1),(2),(3)\rangle}}, ^{(3)\mathcal{C}_{\langle(1),(2),(3)\rangle}}\rangle$, and $\mathcal{C} = \mathcal{C}_{\langle^{(1)\mathcal{C}}, ^{(2)\mathcal{C}}, ^{(3)\mathcal{C}}\rangle}$, i.e. the mappings between the sets of all triadic $\mathbf{L}$-Galois connections and all nested systems of triadic Galois connections are mutually inverse bijections.*

C. Application of the Cartesian representation

In this section we present, as an application of the previous results, a theorem saying that every fuzzy concept trilattice is isomorphic to some ordinary concept trilattice via a natural isomorphism. A particular consequence of this result is the claim that a fuzzy concept trilattice is indeed a trilattice (see below for the notion of a trilattice).

Let us recall the following notions [23]. Let V be a non-empty set, and for $i \in \{1,2,3\}$ let $\lesssim_i$ be quasiorder relations on V. $\langle V, \lesssim_1, \lesssim_2, \lesssim_3\rangle$ is called a *triordered set* if and only if the following two conditions hold for every $\{i,j,k\} = \{1,2,3\}$:

1) $v \lesssim_i w$ and $v \lesssim_j w$ implies $w \lesssim_k v$ for every $v, w \in V$;
2) $\sim_i \cap \sim_j \cap \sim_k$ is an identity relation on V.

For triordered sets $\langle V, \lesssim_1^V, \lesssim_2^V, \lesssim_3^V\rangle$ and $\langle W, \lesssim_1^W, \lesssim_2^W, \lesssim_3^W\rangle$, a mapping $h : V \to W$ is called an isomorphism if it is bijective and satisfies $v_1 \lesssim_i^V v_2$ iff $h(v)_1 \lesssim_i^W h(v_2)$ for every $v_1, v_2 \in V$ and every $i \in \{1,2,3\}$.

Let $V_i, V_k \subseteq V$. An element $v \in V$ is called an *ik-bound* of $\langle V_i, V_k\rangle$ if $v_i \lesssim_i v$ and $v_k \lesssim_k v$ for every $v_i \in V_i$ and $v_k \in V_k$. An ik-bound v is called an *ik-limit* of $\langle V_i, V_k\rangle$ if $u \lesssim_j v$ for every ik-bound u of $\langle V_i, V_k\rangle$. In every triordered set $(V, \lesssim_1, \lesssim_2, \lesssim_3)$ there is at most one ik-limit v of $\langle V_i, V_k\rangle$ satisfying $v \lesssim_k u$ for every ik-limit u of $\langle V_i, V_k\rangle$. If such v exists, we call v an *ik-join* of $\langle V_i, V_k\rangle$ and denote it by $\nabla_{ik}(V_i, V_k)$. A triordered set $(V, \lesssim_1, \lesssim_2, \lesssim_3)$ in which the ik-join exists for all $i \neq k$ ($i, k \in \{1,2,3\}$) and all pairs $\langle V_i, V_k\rangle$ of subsets of V is called a *complete trilattice*.

It is easy to see that the set of all triadic concepts of a triadic $\mathbf{L}_K$-Galois connection forms a triordered set w.r.t. inclusions $\subseteq$ of fuzzy sets as the quasiorders. We utilize the facts that the set of all triadic concepts of an ordinary triadic Galois connection $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$ forms a trilattice, denoted here by $\mathcal{T}(X_1, X_2, X_3, \langle^{(1)}, ^{(2)}, ^{(3)}\rangle)$.

Lemma 6. *For a triadic $\mathbf{L}_K$-Galois connection $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$ the triordered sets $\mathcal{T}(X_1, X_2, X_3, \langle^{(1)}, ^{(2)}, ^{(3)}\rangle)$ and $\mathcal{T}(X_1 \times L, X_2 \times L, X_3 \times L, \langle^{\langle 1\rangle_{(1)}}, ^{\langle 2\rangle_{(2)}}, ^{\langle 3\rangle_{(3)}}\rangle)$ are isomorphic. Moreover, $\mathcal{T}(X_1 \times L, X_2 \times L, X_3 \times L, \langle^{\langle 1\rangle_{(1)}}, ^{\langle 2\rangle_{(2)}}, ^{\langle 3\rangle_{(3)}}\rangle) =$*

$\mathcal{T}(X_1 \times L, X_2 \times L, X_3 \times L, I^\times)$*, where*

$$\langle (x_1, a), (x_2, b), (x_3, c) \rangle \in I^\times \text{ iff } c \leq (\{a/x_1\}, \{b/x_2\})^{(3)}.$$

The following theorem, which can be proven by the previous lemma, shows an important fact that every fuzzy concept trilattice is isomorphic to a certain concept trilattice.

Theorem 5. *Any* $\mathbf{L}$*-concept trilattice* $\mathcal{T}(X_1, X_2, X_3, I)$ *is isomorphic as a triordered set to the (ordinary) concept trilattice* $\mathcal{T}(X_1 \times L, X_2 \times L, X_3 \times L, I^\times)$*, where*

$$\langle (x_1, a), (x_2, b), (x_3, c) \rangle \in I^\times \text{ iff } a \otimes b \otimes c \leq I(x_1, x_2, x_3).$$

As a consequence, it follows that every fuzzy concept trilattice is indeed a trilattice in the sense defined above. Moreover, one may use the above results to obtain a short proof of the basic theorem of fuzzy concept trilattices [6] by reduction to the basic theorem of ordinary concept trilattices [23]. This result is left for the extended version of this paper.

V. Conclusions and Further Issues

We provided an axiomatic characterization of triadic fuzzy Galois connections and two ways to represent them by ordinary tradic connections. These connections appear in data analysis of three-way relational data. Most importantly, the fixpoints of these mappings are maximal cuboids contained in the data (maximal Cartesian subrelations of the relation representing the data).

The results establish important connections between the ordinary and the fuzzy case that enable us to easily carry over results (theorems, algorithms) for triadic fuzzy data from those for ordinary triadic data. As an example, we presented a theorem showing that every fuzzy concept trilattice is isomorphic, via a natural isomorphism, to some ordinary concept trilattice.

The following topics are left for future research:

– Identify, formally if possible, the types of results that may be automatically carried over from the ordinary case to fuzzy case.
– Develop other possible types of reduction. Extend the applicability of the presented representation to a wider class of relational methods (see [3] for a general cut-like semantics for predicate fuzzy logic).
– Study the computational efficiency of the representation results with the aim of obtaining algorithms for fuzzy relations from those for ordinary relations.

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