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RESEARCH ARTICLE

Triadic concept lattices of data with graded attributes

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We present foundations for triadic concept analysis of data with graded attributes. Triadic concept analysis departs from the dyadic case by taking into account modi, such as time instances or conditions, under which objects have attributes. That is, instead of a two-dimensional table filled with 0s and 1s (equivalently, binary relation or two-dimensional binary matrix), the input data in triadic concept analysis consists of a three-dimensional table (equivalently, ternary relation or three-dimensional binary matrix). In the ordinary triadic concept analysis, one assumes that the ternary relationship between objects, attributes, and modi, which specifies whether a given object has a given attribute under a given modus, is a yes-or-no relationship. In the present paper, we show how triadic concept analysis may be developed in a setting in which the ternary relationship between objects, attributes, and modi is a matter of degree rather than a yes-or-no relationship. We present the basic notions for the new, "graded" setting, an illustrative example, and generalize basic results including the basic theorem of triadic concept analysis.

Keywords: triadic concept analysis, data with graded attributes, fuzzy concept trilattice, formal concept analysis

1. Introduction

In recent years, there has been a growing interest in analyzing three-way and, in general, multi-way data. Various methods, established for the analysis of two-way data, i.e. data describing certain objects and their attributes, were extended to the case of multi-way data, most importantly the methods of matrix analysis. Kolda and Bader (2009) provide an up-to-date survey with 244 references, see also (Cichocki *at al.* 2009, Kroonenberg 2008, Smilde *at al.* 2004). Our aim in this paper is to provide fundamental notions and results for a particular method for analyzing three-way relational data, namely for triadic concept analysis of data with graded (fuzzy) attributes.

Formal concept analysis (FCA) (Ganter and Wille 1999) is a method of analysis of relational data with applications in various areas. FCA has strong mathematical and computational foundations. FCAs appeal derives from the fact that it provides a user with an easy-to-understand diagram derived from data, so-called concept lattice, in which a user can see interesting clusters in data as well as dependencies among attributes which hold true in the data. Moreover, FCA is rooted in a traditional approach to (human) concepts. Namely, FCA is based on the notion of a formal concept—a simple formalization of a dyadic understanding of concepts

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according to which a concept consists of a collection of objects and a collection of attributes which are called the extent and the intent of the concept. Formal concepts are identified in the input data which is in the form of a table with rows corresponding to objects, columns corresponding to attributes, and entries representing which objects have which attributes.

In (Lehmann and Wille 1995, Wille 1995), the authors extended the basic notions of FCA to a triadic case in which one assumes that the input data is in the form of a three-dimensional table with the dimensions corresponding to objects, attributes, and modi. An example of a modus is a time instance or a condition. The table entries represent which objects have which attributes under which conditions. In the triadic case, the counterpart of the notion of a formal concept (or, dyadic concept), is the notion of a triadic concept. While a dyadic concept is a pair $\langle A, B\rangle$ with A and B being sets of objects and attributes, respectively, that satisfy certain conditions (see later), triadic concepts are certain triplets $\langle A, B, C\rangle$ where in addition to sets A and B of objects and attributes, respectively, we have a set C of modi. C is understood as the set of conditions under which objects from A have attributes from B. Needless to say, many data come in the form in which various conditions appear that determine an object-attribute relationship.

The ordinary triadic concept analysis assumes that a given object either has or does not have a given attribute under a given condition, i.e. one assumes that the data is binary. We are interested in extending this case to a more general setting in which one assumes that a given object has a given attribute to a certain degree in a given condition. We proceed in a general setting and assume that the degrees form a partially ordered scale abounded from below by 0 (representing *false*, i.e. *does not have (at all)*) and from above by 1 (representing *true*, or *(fully) has*), and equipped with certain aggregation operations. In particular, we assume that the the degrees form a complete residuated lattice (Ward and Dilworth 1939) which is the main structure used as the structure of truth degrees in (mathematical) fuzzy logic (Hajek 1998) and covers the most important particular scales such as the real unit interval $[0, 1]$ and finite chains.

The paper is organized as follows. Section 2 provides preliminaries from FCA and fuzzy logic. In Section 3, we present the fundamental notions and basic results of triadic concept analysis in the general framework for three-way data with degrees. Section 4 provides a small illustrative example. In Section 5, we consider in detail the question of the structure of the set of all triadic concepts associated to a given data. Section 6 concludes the paper and outlines future research topics.

2. Preliminaries

2.1 *Dyadic and Triadic Formal Concept Analysis*

This section introduces the notions needed in our paper. For further information we refer to (Ganter and Wille 1999) (dyadic FCA) and (Lehmann and Wille 1995, Wille 1995) (triadic FCA).

A *formal context* (or *dyadic context*) is a triplet $\langle X, Y, I\rangle$ where X and Y are non-empty sets and I is a binary relation between X and Y, i.e. $I \subseteq X \times Y$. X and Y are interpreted as the sets of objects and attributes, respectively; I is interpreted as the incidence relation ("to have relation"). That is, $\langle x, y\rangle \in I$ is interpreted as: object x has attribute y. A formal context $\mathbf{K} = \langle X, Y, I\rangle$ induces a pair of operators

$^{\uparrow}: 2^X \to 2^Y$ and $^{\downarrow}: 2^Y \to 2^X$ defined for $C \subseteq X$ and $D \subseteq Y$ by

$$C^{\uparrow} = \{y \in Y \mid \text{for each } x \in C : \langle x, y\rangle \in I\},$$
$$D^{\downarrow} = \{x \in X \mid \text{for each } y \in D : \langle x, y\rangle \in I\}.$$

These operators, called *concept-forming operators*, form a Galois connection (Ganter and Wille 1999) between X and Y. A *formal concept* (or *dyadic concept*) of $\langle X, Y, I\rangle$ is a pair $\langle C, D\rangle$ consisting of sets $C \subseteq X$ and $D \subseteq Y$ such that $C^{\uparrow} = D$ and $D^{\downarrow} = C$; C and D are called the *extent* and *intent* of $\langle C, D\rangle$. The collection of all formal concepts of $\langle X, Y, I\rangle$ is denoted by $\mathcal{B}(X, Y, I)$ and is called the *concept lattice* of $\langle X, Y, I\rangle$. That is,

$$\mathcal{B}(X, Y, I) = \{\langle C, D\rangle \mid C^{\uparrow} = D,\ D^{\downarrow} = C\}.$$

A concept lattice equipped with a partial order corresponding to a subconcept-superconcept hierarchy is indeed a complete lattice (Ganter and Wille 1999). A formal context may be represented by a binary matrix: rows and columns correspond to objects and attributes; the entry corresponding to $x \in X$ and $y \in Y$ contains 1 iff $\langle x, y\rangle \in I$. Geometrically, formal concepts of $\langle X, Y, I\rangle$ are just maximal rectangular areas in the corresponding binary matrix which are full of 1s (Ganter and Wille 1999).

A *triadic context* is a quadruple $\langle X, Y, Z, I\rangle$ where X, Y, and Z are non-empty sets, and I is a ternary relation between X, Y, and Z, i.e. $I \subseteq X \times Y \times Z$. X, Y, and Z are interpreted as the sets of objects, attributes, and conditions, respectively; I is interpreted as the incidence relation ("to have-under relation"). That is, $\langle x, y, z\rangle \in I$ is interpreted as: object x has attribute y under condition z. In this case, we say that x, y, z (or x, z, y, or the result of listing x, y, z in any other sequence) are related by I. For convenience, a triadic context is denoted by $\langle X_1, X_2, X_3, I\rangle$.

Let $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ be a triadic context. For $\{i, j, k\} = \{1, 2, 3\}$ and $C_k \subseteq X_k$, we define a dyadic context

$$\mathbf{K}^{ij}_{C_k} = \langle X_i, X_j, I^{ij}_{C_k}\rangle$$

by

$$\langle x_i, x_j\rangle \in I^{ij}_{C_k} \quad \text{iff} \quad \text{for each } x_k \in C_k : x_i, x_j, x_k \text{ are related by } I.$$

The concept-forming operators induced by $\mathbf{K}^{ij}_{C_k}$ are denoted by $^{(i,j,C_k)}$.

A *triadic concept* of $\langle X_1, X_2, X_3, I\rangle$ is a triplet $\langle C_1, C_2, C_3\rangle$ of $C_1 \subseteq X_1$, $C_2 \subseteq X_2$, and $C_3 \subseteq X_3$, such that for every $\{i, j, k\} = \{1, 2, 3\}$ we have $C_i = C_j^{(j,i,C_k)}$; C_1, C_2, and C_3 are called the *extent*, *intent*, and *modus* of $\langle C_1, C_2, C_3\rangle$. The set of all triadic concepts of $\langle X_1, X_2, X_3, I\rangle$ is denoted by $\mathcal{T}(X_1, X_2, X_3, I)$ and is called the concept trilattice of $\langle X_1, X_2, X_3, I\rangle$; we refer to Section 5 where the notion of a trilattice is defined.

2.2 *Residuated Lattices and Fuzzy Logic*

We assume that the degrees form a bounded partially ordered set L which is a complete lattice. Furthermore, we assume that L is equipped with certain aggregation operators which are known from mathematical fuzzy logic (Goguen 1968, Hajek

1998). In particular, we assume that the scale L of degrees forms a complete residuated lattice (Hajek 1998) i.e. an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$$

such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

for each $a, b, c \in L$. Residuated lattices are used in several areas of mathematics, notably in mathematical fuzzy logic. In fuzzy logic, elements a of L are called truth degrees and $\otimes$ and $\rightarrow$ are considered as the (truth functions of) many-valued conjunction and implication.

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a continuous (or at least left-continuous) t-norm (i.e. a commutative, associative, and isotone operation on $[0, 1]$ with 1 acting as a neutral element) with the corresponding $\rightarrow$ given by $a \rightarrow b = \max\{c \mid a \otimes c \leq b\}$. The three most important pairs of adjoint operations on the unit interval are: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$; Gödel ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$, $= b$ if $a > b$; Goguen ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$, $= b/a$ if $a > b$). Another common choice is a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of L and restrictions of Gödel operations from $[0, 1]$ to L. A special case of a complete residuated lattice is the two-element Boolean algebra $\langle\{0, 1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of classical logic. This is important because for the particular case $\mathbf{L} = \mathbf{2}$, the notions and results obtained in this paper become the ones of ordinary triadic concept analysis.

Given a complete residuated lattice $\mathbf{L}$, we define the usual notions (Belohlavek 2002, Hajek 1998): an $\mathbf{L}$-set (fuzzy set, graded set) A in a universe U is a mapping $A : U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. Operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. $\mathbf{2}$-sets and operations with $\mathbf{2}$-sets can be identified with ordinary sets and operations with ordinary sets, respectively. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$; similarly for ternary relations.

A binary $\mathbf{L}$-relation $R : U \times U \rightarrow L$ is called reflexive, symmetric, and transitive, if the following properties are satisfied for every $u, v, w \in U$, respectively:

$$R(u, u) = 1,$$
$$R(u, v) = R(v, u),$$
$$R(u, v) \otimes R(v, w) \leq R(u, w).$$

R is called an $\mathbf{L}$-quasiorder if it is reflexive and transitive; an $\mathbf{L}$-equivalence if it is reflexive, symmetric, and transitive; an $\mathbf{L}$-equality if it is an $\mathbf{L}$-equivalence for which $R(u, v) = 1$ implies $u = v$.

In the following we use well-known properties of residuated lattices and fuzzy sets over residuated lattices which can be found, e.g., in (Belohlavek 2002, Hajek 1998).

3. Triadic Concept-Forming Operators and Concepts

We start with the assumption that the input data is given by a ternary relationship specifying which objects have which attributes under which conditions. However, we want to include situations where the relationship is a matter of degree. For example, let the objects be products (e.g., food products), attributes be product features (e.g., a good taste), modi be customers (e.g., participating in a customer survey), and let the scale L be a five element scale $\{0, 1/4, 1/2, 3/4, 1\}$ with the degrees representing "very bad", "bad", "neutral", "good", "excellent". Then a degree (e.g., 3/4) to which object x has feature y under condition z may be interpreted as the degree to which customer z considers product x as having feature y (e.g., degree 3/4 means that customer z considers food product x as having a good taste).

A *triadic* **L**-*context* (triadic fuzzy context, or just triadic context) is a quadruple $\langle X, Y, Z, I\rangle$ where X, Y, and Z are non-empty sets, and I is a ternary fuzzy relation between X, Y, and Z, i.e. $I : X \times Y \times Z \to L$. X, Y, and Z are interpreted as the sets of objects, attributes, and conditions, respectively. For every $x \in X$, $y \in Y$, and $z \in Z$, $I(x, y, z) \in L$ is interpreted as the degree to which object x has attribute y under condition z. In this case, we also say that $I(x, y, z)$ is the degree to which x, y, z (or x, z, y or z, x, y, etc.) are related and, for convenience, denote $I(x, y, z)$ also by $I\{x, y, z\}$ or $I\{x, z, y\}$ or $I\{z, x, y\}$, etc. As in the ordinary case, we denote a triadic fuzzy context by $\langle X_1, X_2, X_3, I\rangle$.

We now define the concept-forming operators and the notion of a triadic concept in a triadic **L**-context. Our aim is to have triadic concepts defined as triplets of fuzzy sets of objects, attributes, and modi. Namely, we want a concept to apply to objects to degrees since we wish to have concepts that apply to different objects to possibly different degrees; similarly for attributes and conditions.

For every $\{i, j, k\} = \{1, 2, 3\}$ and a fuzzy set $C_k \in L^{X_k}$, a triadic **L**-context $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ induces a dyadic **L**-context

$$\mathbf{K}^{ij}_{C_k} = \langle X_i, X_j, I^{ij}_{C_k}\rangle$$

in which the fuzzy relation $I^{ij}_{C_k}$ between X_i and X_j is defined by

$$I^{ij}_{C_k}(x_i, x_j) = \bigwedge_{x_k \in X_k} (C_k(x_k) \to I\{x_i, x_j, x_k\})$$

for every $x_i \in X_i$ and $x_j \in X_j$. Note that dyadic **L**-contexts and the associated structures including concept-forming operators and concept lattices were studied in a series of papers, see e.g. (Belohlavek 2002, 2004, Pollandt 1997) for the notions that we need in the following. The concept-forming operators induced by $\mathbf{K}^{ij}_{C_k}$ are denoted by ${}^{(i,j,C_k)}$. That is, for a fuzzy set C_i in X_i, we define a fuzzy set $C_i^{(i,j,C_k)}$ in X_j by

$$C_i^{(i,j,C_k)}(x_j) = \bigwedge_{x_i \in X_i} C_i(x_i) \to I^{ij}_{C_k}(x_i, x_j).$$

Similarly, for a fuzzy set C_j in X_j, we define a fuzzy set $C_j^{(i,j,C_k)}$ in X_i by

$$C_j^{(i,j,C_k)}(x_i) = \bigwedge_{x_j \in X_j} C_j(x_j) \to I^{ij}_{C_k}(x_i, x_j).$$

A *triadic* **L**-*concept* (triadic fuzzy concept) of $\langle X_1, X_2, X_3, I\rangle$ is a triplet $\langle C_1, C_2, C_3\rangle$ consisting of fuzzy sets $C_1 \in L^{X_1}$, $C_2 \in L^{X_2}$, and $C_3 \in L^{X_3}$, such that for every $\{i,j,k\} = \{1,2,3\}$ we have $C_i = C_j^{(i,j,C_k)}$, $C_j = C_k^{(j,k,C_i)}$, and $C_k = C_i^{(k,i,C_j)}$. Note that the latter conditions hold for every 3-element set $\{i,j,k\} = \{1,2,3\}$ if and only if they hold for any single 3-element set $\{i,j,k\} = \{1,2,3\}$ (this follows e.g. from Lemma 3.1 (b)). In this case, C_1, C_2, and C_3 are called the *extent*, *intent*, and *modus* of $\langle C_1, C_2, C_3\rangle$. The set of all triadic concepts of $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ is denoted by $\mathcal{T}(X_1, X_2, X_3, I)$ and is called the **L**-*concept trilattice* (fuzzy concept trilattice) of **K**.

We now define important structural relations on the set of triadic concepts. These relations are based on the subsethood relations on the sets of objects, attributes, and modi, and are essential to understand the structure of the set of all triadic concepts. In a graded setting, subsethood is a matter of degree. Given fuzzy sets $A, B \in \mathbf{L}^U$, the degree $S(A,B)$ of inclusion of A in B is given by

$$S(A,B) = \textstyle\bigwedge_{u\in U}\big(A(u) \to B(u)\big) \tag{2}$$

and the degree of equality of A and B by

$$A \approx B = \textstyle\bigwedge_{u\in U}\big(A(u) \leftrightarrow B(u)\big). \tag{3}$$

Note that (2) generalizes the ordinary subsethood relation $\subseteq$. Described verbally, $S(A,B)$ represents a degree to which every element of A is an element of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. As a consequence, $A \subseteq B$ if and only if $A(u) \le B(u)$ for each $u \in U$. Likewise, (3) generalizes the ordinary equality relation $=$. $A \approx B$ represents a degree to which every element belongs to A iff it belongs to B. Clearly, $A = B$ if and only if $A \approx B = 1$.

Consider the following fuzzy relations on $\mathcal{T}(\mathbf{K})$, for $i = 1,2,3$:

$$\begin{aligned}\langle A_1, A_2, A_3\rangle \precsim_i \langle B_1, B_2, B_3\rangle &= S(A_i, B_i),\\ \langle A_1, A_2, A_3\rangle \approx_i \langle B_1, B_2, B_3\rangle &= A_i \approx B_i.\end{aligned}$$

We denote the 1-cuts of $\precsim_i$ and $\approx_i$ by $\lesssim_i$ and $\eqsim_i$, i.e.

$$\begin{aligned}\langle A_1, A_2, A_3\rangle \lesssim_i \langle B_1, B_2, B_3\rangle &\text{ iff } A_i \subseteq B_i,\\ \langle A_1, A_2, A_3\rangle \eqsim_i \langle B_1, B_2, B_3\rangle &\text{ iff } A_i = B_i.\end{aligned}$$

It is easy to check that $\precsim_i$ and $\approx_i$ are an **L**-quasiorder and an **L**-equivalence on $\mathcal{T}(\mathbf{K})$ (see Section 2). Consequently, $\lesssim_i$ and $\eqsim_i$ are a quasiorder and an equivalence on $\mathcal{T}(\mathbf{K})$. Denote by $\mathcal{T}(\mathbf{K})/\eqsim_i$ the corresponding factor set with equivalence classes denoted by $[\langle A_1, A_2, A_3\rangle]_i$. Letting

$$[\langle A_1, A_2, A_3\rangle]_i \preceq_i [\langle B_1, B_2, B_3\rangle]_i = \langle A_1, A_2, A_3\rangle \precsim_i \langle B_1, B_2, B_3\rangle,$$

$\preceq_i$ is an **L**-quasiorder on $\mathcal{T}(\mathbf{K})/\eqsim_i$ satisfying that $([\langle A_1, A_2, A_3\rangle]_i \preceq_i [\langle B_1, B_2, B_3\rangle]_i) = 1$ and $([\langle B_1, B_2, B_3\rangle]_i \preceq_i [\langle A_1, A_2, A_3\rangle]_i) = 1$ implies

$[\langle A_1, A_2, A_3\rangle]_i = [\langle B_1, B_2, B_3\rangle]_i$. Therefore, the 1-cut of $\preceq_i$, which we denote by $\leq_i$, is a partial order on $\mathcal{T}(\mathbf{K})/\approx_i$.

If $A \in \mathbf{L}^U$ and $B \in \mathbf{L}^V$ are two fuzzy sets, let $A \otimes B$ denote a fuzzy set in $U \times V$ defined by

$$(A \otimes B)(u, v) = A(u) \otimes B(v). \tag{4}$$

One can easily check that

$$S(A_1, A_2) \otimes S(B_1, B_2) \leq S(A_1 \otimes B_1, A_2 \otimes B_2). \tag{5}$$

The following lemma is needed later.

Lemma 3.1: *Let* $\{i, j, k\} = \{1, 2, 3\}$. *Then*

(a) $A_i^{(i,j,C_k)}(x_j) = \bigwedge_{\langle x_i, x_j\rangle \in X_i \times X_k}((A_i(x_i) \otimes C_k(x_k)) \to I\{x_i, x_j, x_k\})$,
(b) $A_i^{(i,j,C_k)} = C_k^{(k,j,A_i)}$,
(c) $S(A_i, B_i) \otimes S(C_k, D_k) \leq S(B_i^{(i,j,D_k)}, A_i^{(i,j,C_k)})$,
(d) *if* $A_i \subseteq B_i$ *and* $C_k \subseteq D_k$ *then* $B_i^{(i,j,D_k)} \subseteq A_i^{(i,j,C_k)}$,

for any $A_i, B_i \in \mathbf{L}^{X_i}$, $C_k, D_k \in \mathbf{L}^{X_k}$.

Proof: (a)

$$\begin{aligned}
&A_i^{i,j,C_k}(x_j) = \\
&= \textstyle\bigwedge_{x_i \in X_i} A_i(x_i) \to I_{C_k}^{ij}(x_i, x_j) = \\
&= \textstyle\bigwedge_{x_i \in X_i} \big(A_i(x_i) \to \bigwedge_{x_k \in X_k}(C_k(x_k) \to I\{x_i, x_j, x_k\})\big) = \\
&= \textstyle\bigwedge_{x_i \in X_i} \bigwedge_{x_k \in X_k} (A_i(x_i) \to (C_k(x_k) \to I\{x_i, x_j, x_k\})) = \\
&= \textstyle\bigwedge_{\langle x_i, x_k\rangle \in X_i \times X_k} (A_i(x_i) \otimes C_k(x_k) \to I\{x_i, x_j, x_k\})
\end{aligned}$$

(b) Follows from (a) by commutativity of $\otimes$.
(c) Consider the dyadic fuzzy context $\langle X_i \times X_k, X_j, I_{ik,j}\rangle$ defined by

$$I_{ik,j}(\langle x_i, x_k\rangle, x_j) = I\{x_i, x_j, x_k\},$$

and fuzzy sets $A_i \otimes C_k$ and $B_i \otimes D_k$ in $X_i \times X_k$ defined by (4). According to (a), $A_i^{(i,j,C_k)}(x_j) = (A_i \otimes C_k)^{I_{ik,j}}$ and $B_i^{(i,j,D_k)}(x_j) = (B_i \otimes D_k)^{I_{ik,j}}$. Using (5) $S(A_i, B_i) \otimes S(C_k, D_k) \leq S(A_i \otimes C_k, B_i \otimes D_k)$. Now, (Belohlavek 2000) implies $S(A_i \otimes C_k, B_i \otimes D_k) \leq S((B_i \otimes D_k)^{I_{ik,j}}, (A_i \otimes C_k)^{I_{ik,j}})$ from which the assertion readily follows.

(d) A consequence of (c). Namely, $A \subseteq B$ is equivalent to $S(A, B) = 1$; therefore, if $A_i \subseteq B_i$ and $C_k \subseteq D_k$ then $1 = 1 \otimes 1 \leq S(B_i^{(i,j,D_k)}, A_i^{(i,j,C_k)})$, whence $B_i^{(i,j,D_k)} \subseteq A_i^{(i,j,C_k)}$.

□

Theorem 3.2: *Let* $\{i, j, k\} = \{1, 2, 3\}$. *Then*

$$\begin{aligned}
&(\langle A_1, A_2, A_3\rangle \precsim_i \langle B_1, B_2, B_3\rangle) \otimes (\langle A_1, A_2, A_3\rangle \precsim_j \langle B_1, B_2, B_3\rangle) \leq \\
&\leq (\langle B_1, B_2, B_3\rangle \precsim_k \langle A_1, A_2, A_3\rangle),
\end{aligned}$$

for all triadic fuzzy concepts $\langle A_1, A_2, A_3\rangle$ *and* $\langle B_1, B_2, B_3\rangle$ *from* $\mathcal{T}(\mathbf{K})$*. Furthermore,* $\approx_i \cap \approx_j$ *is an* $\mathbf{L}$*-equality on* $\mathcal{T}(\mathbf{K})$*.*

Proof: As $\langle A_1, A_2, A_3\rangle, \langle B_1, B_2, B_3\rangle \in \mathcal{T}(\mathbf{K})$, we have $A_i^{(i,k,A_j)} = A_k$ and $B_i^{(i,k,B_j)} = B_k$. Lemma 3.1 (c) therefore yields

$$(\langle A_1, A_2, A_3\rangle \precsim_i \langle B_1, B_2, B_3\rangle) \otimes (\langle A_1, A_2, A_3\rangle \precsim_j \langle B_1, B_2, B_3\rangle) =$$
$$= S(A_i, B_i) \otimes S(A_j, B_j) \leq S(B_i^{(i,k,B_j)}, A_i^{(i,k,A_j)}) =$$
$$= S(B_k, A_k) = \langle A_1, A_2, A_3\rangle \precsim_k \langle B_1, B_2, B_3\rangle.$$

Since $\approx_i \cap \approx_j$ is an $\mathbf{L}$-equivalence (an intersection of two $\mathbf{L}$-equivalences), it suffices to show that if $\langle A_1, A_2, A_3\rangle(\approx_i \cap \approx_j)\langle B_1, B_2, B_3\rangle = 1$ then $\langle A_1, A_2, A_3\rangle = \langle B_1, B_2, B_3\rangle$. If $\langle A_1, A_2, A_3\rangle(\approx_i \cap \approx_j)\langle B_1, B_2, B_3\rangle = 1$ then $A_i \approx B_i = 1$ and $A_j \approx B_j = 1$, whence $A_i = B_i$ and $A_j = B_j$. But then $A_k = A_i^{(i,k,A_j)} = B_i^{(i,k,B_j)} = B_k$ because both $\langle A_1, A_2, A_3\rangle$ and $\langle B_1, B_2, B_3\rangle$ are triadic concepts.

□

Since $\lesssim_i$s are 1-cuts of $\precsim_i$s and $\eqsim_i$s are 1-cuts of $\approx_i$s, we get the following corollary of Theorem 3.2

Corollary 3.3: *Let* $\{i, j, k\} = \{1, 2, 3\}$*. For any triadic fuzzy concepts* $\langle A_1, A_2, A_3\rangle$ *and* $\langle B_1, B_2, B_3\rangle$ *from* $\mathcal{T}(\mathbf{K})$*,*
if $\langle A_1, A_2, A_2\rangle \lesssim_i \langle B_1, B_2, B_3\rangle$ *and* $\langle A_1, A_2, A_2\rangle \lesssim_j \langle B_1, B_2, B_3\rangle$*,*
then $\langle B_1, B_2, B_3\rangle \lesssim_k \langle A_1, A_2, A_3\rangle$*.*
Furthemore, $\eqsim_i \cap \eqsim_j$ *is the identity on* $\mathcal{T}(\mathbf{K})$*.*

The following theorem shows that starting with fuzzy sets C_i in X_i and C_k in X_k, one obtains a particular triadic concept $\langle A_i, A_j, A_k\rangle$ by three projections via I. One first projects C_i and C_k onto A_j, then A_j and C_k onto A_i, and then A_i and A_j onto A_k. This may be seen as an analogy of how to create a dyadic concept from a fuzzy set of objects (or, dually, of attributes), cf. (Wille 1995).

Theorem 3.4: *For* $C_i \in L^{X_i}, C_k \in L^{X_k}$ *with* $\{i, j, k\} = \{1, 2, 3\}$*, let* $A_j = C_i^{(i,j,C_k)}$*,* $A_i = A_j^{(i,j,C_k)}$*, and* $A_k = A_i^{(i,k,A_j)}$*. Then* $\langle A_1, A_2, A_3\rangle$ *is a triadic fuzzy concept, denoted by* $\mathfrak{b}_{ik}(C_i, C_k)$*.*

$\mathfrak{b}_{ik}(C_i, C_k)$ *has the smallest k-th component among all triadic fuzzy concepts* $\langle B_1, B_2, B_3\rangle$ *with the greatest j-th component satisfying* $C_i \subseteq B_i$ *and* $C_k \subseteq B_k$*. In particular,* $\mathfrak{b}_{ik}(A_i, A_k) = \langle A_1, A_2, A_3\rangle$ *for each triadic fuzzy concept* $\langle A_1, A_2, A_3\rangle$*.*

Proof: First, observe that $C_i \subseteq A_i$ and $C_k \subseteq A_k$. Indeed, $C_i \subseteq A_i$ holds true because A_i is the closure of C_i w.r.t. the closure operator on $\mathbf{K}^{ij}_{C_k}$. $C_k \subseteq A_k$ because by definition of A_k, Lemma 3.1(a) yields that the inclusion is equivalent to $C_k(x_k) \leq A_i(x_i) \otimes A_j(x_j) \to I\{x_i, x_j, x_k\}$ being true for every $\langle x_i, x_k\rangle$, which is equivalent to $A_i(x_i) \otimes A_j(x_j) \otimes C_k(x_k) \leq I\{x_i, x_j, x_k\}$. The last inequality holds true because

$$A_i(x_i) \otimes A_j(x_j) \otimes C_k(x_k) \leq$$
$$\leq (A_j(x_j) \otimes C_k(x_k) \to I\{x_i, x_j, x_k\}) \otimes A_j(x_j) \otimes C_k(x_k) \leq$$
$$\leq I\{x_i, x_j, x_k\}.$$

Next, we prove that $\langle A_1, A_2, A_3\rangle$ is a triadic fuzzy concept. $A_k = A_i^{(i,k,A_j)}$ is satisfied by definition. Consider A_j. Due to Lemma 3.1(d), $A_j = C_i^{(i,j,C_k)} \supseteq A_i^{(i,j,A_k)}$

and $A_j \subseteq (A_j^{(j,k,A_i)})^{(j,k,A_i)} = A_k^{(j,k,A_i)} = A_i^{(i,j,A_k)}$, thus $A_j = A_i^{(i,j,A_k)}$. The proof for A_i is similar.

Let $\langle B_1, B_2, B_3\rangle$ be a triadic fuzzy concept with $C_i \subseteq B_i$ and $C_k \subseteq B_k$. Then $B_j = B_i^{(i,j,B_k)} \subseteq C_i^{(i,j,C_k)} = A_j$. This shows that $\langle A_1, A_2, A_3\rangle$ has the greatest j-th component among all concepts $\langle B_1, B_2, B_3\rangle$ that satisfy $C_i \subseteq B_i$ and $C_k \subseteq B_k$. Let now $B_j = A_j$. Then $A_i = A_j^{(i,j,C_k)} \supseteq B_j^{(i,j,B_k)} = B_i$ thus $B_i \subseteq A_i$, whence $A_k = A_i^{(i,k,A_j)} \subseteq (B_i^{(i,k,B_j)} = B_k$.

Finally, if $\langle A_1, A_2, A_3\rangle$ is a triadic fuzzy concept, then $A_j = A_i^{(i,j,A_k)}$, $A_i = A_j^{(i,j,A_k)}$, and $A_k = A_i^{(i,k,A_j)}$ by definition. Hence, $\mathfrak{b}_{ik}(A_i, A_k) = \langle A_1, A_2, A_3\rangle$.

□

The next theorem shows that triadic concepts may be geometrically interpreted as cubical patterns in data. Namely, they may be seen as maximal Cartesian products of fuzzy sets of object, attributes, and conditions that are included in I. Note that the Cartesian product of fuzzy sets A, B, and C in X_1, X_2, and X_3, respectively, is the fuzzy set $A \otimes B \otimes C$ in $X_1 \times X_2 \times X_3$ defined by

$$(A \otimes B \otimes C)(x_1, x_2, x_3) = A(x_1) \otimes B(x_2) \otimes C(x_3).$$

Theorem 3.5:

(a) *If* $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K})$ *then* $A_1 \otimes A_2 \otimes A_3 \subseteq I$. *Moreover,* $\langle A_1, A_2, A_3\rangle$ *is maximal with respect to pointwise set inclusion, i.e. there does not exist* $\langle B_1, B_2, B_3\rangle \in \langle \mathbf{L}^{X_1}, \mathbf{L}^{X_2}, \mathbf{L}^{X_3}\rangle$ *other than* $\langle A_1, A_2, A_3\rangle$ *such that* $A_i \subseteq B_i$ *for every* $i = 1, 2, 3$.

(b) *If* $A_1 \otimes A_2 \otimes A_3 \subseteq I$ *then there exists* $\langle B_1, B_2, B_3\rangle \in \mathcal{T}(\mathbf{K})$ *such that* $A_i \subseteq B_i$ *for every* $i = 1, 2, 3$.

Proof: (a) Let $\{i, j, k\} = \{1, 2, 3\}$. From $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K})$ it follows that $A_k(x_k) = A_i^{(i,j,k)} = \bigwedge_{(x_i,x_j)\in X_i\times X_j} A_i(x_i) \otimes A_j(x_j) \to I\{x_i, x_j, x_k\}$. Furthermore,

$$\begin{aligned} & A_i(x_i) \otimes A_j(x_j) \otimes A_k(x_k) = \\ &= A_i(x_i) \otimes A_j(x_j) \otimes \textstyle\bigwedge_{x_i\in X_i, x_j\in X_j} (A_i(x_i) \otimes A_j(x_j) \to I\{x_i, x_j, x_k\}) \leq \\ &\leq A_i(x_i) \otimes A_j(x_j) \otimes (A_i(x_i) \otimes A_j(x_j) \to I\{x_i, x_j, x_k\}) \leq \\ &\leq I\{x_i, x_j, x_k\}. \end{aligned}$$

Let $\langle A_1, A_2, A_3\rangle$ and $\langle B_1, B_2, B_3\rangle$ be triadic fuzzy concepts with $A_i \subseteq B_i$ for every $i = 1, 2, 3$. Applying Corollary 3.3 to $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$ we get $B_3 \subseteq A_3$; in a similar manner, $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$, hence $\langle A_1, A_2, A_3\rangle = \langle B_1, B_2, B_3\rangle$, proving maximality of $\langle A_1, A_2, A_3\rangle$.

(b) Let $\{i, j, k\} = \{1, 2, 3\}$ and $\mathfrak{b}_{ik}(A_i, A_k) = \langle B_1, B_2, B_3\rangle$. Due to Theorem 3.4, $A_i \subseteq B_i$ and $A_k \subseteq B_k$. Moreover

$$\begin{aligned} & B_j(x_j) = A_i^{(i,j,A_k)}(x_j) = \\ &= \textstyle\bigwedge_{x_i\in X_i, x_k\in X_k} (A_i(x_i) \otimes A_j(x_k) \to I\{x_i, x_j, x_k)\} \geq \\ &\geq \textstyle\bigwedge_{x_i\in X_i, x_k\in X_k} (A_i(x_i) \otimes A_k(x_k) \to A_i(x_i) \otimes A_k(x_k) \otimes A_j(x_j)) \geq \\ &\geq A_j(x_j), \end{aligned}$$

thus $A_j \subseteq B_j$, finishing the proof. □

Table 1. Triadic fuzzy context (t = taste, a = aroma, l = look, p = price)

	Fry				Bender				Leela				Zoidberg			
	t	a	l	p	t	a	l	p	t	a	l	p	t	a	l	p
steak	1	1	1	0	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	1	1	0
salad	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	0	1	1	1	1	1	1	1	$\frac{1}{2}$
veget.	0	0	$\frac{1}{2}$	1	0	0	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	1	1	0
wings	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	1	0	0	0	$\frac{1}{2}$	1	1	1	$\frac{1}{2}$

4. Illustrative Example

In this section, we present a small illustrative example demonstrating that triadic fuzzy concepts represent interesting patterns in data.

One of typical examples of data that are easily transformed into triadic fuzzy context are customer surveys. In such data, we take products as objects, product features as attributes, and customers participating in the survey as conditions. The relation between objects, attributes, and products then captures the degree to which customers regard particular features of products as being of good quality. In our example, we take a customer survey in a hypothetical restaurant as the data we use.

We consider a triadic fuzzy context consisting of a set of four objects, each of which represents a dish in a restaurant (beef steak, cheese salad, vegetable plate and fried chicken wings); a set of four attributes capturing features of the dishes (taste, aroma, look, and price); and a set of four customers who evaluate the dishes (Fry, Bender, Leela, Zoidberg). The context is depicted in Table 1.

We use a three element set $\{0, \frac{1}{2}, 1\}$ as a scale of truth degrees with the degrees representing "bad", "neutral" and "excellent". A degree to which a dish x, its feature y and a customer z are related is then interpreted as a degree to which according to customer z, x has feature y. For example, the degree 1 in which beef steak, taste and Fry are related is interpreted as Fry considering the beef steak as having excellent taste.

The corresponding fuzzy concept trilattice consists of 112 triadic concepts, therefore we do not comment on the interpretation of all of them. Instead, we present a list of five interesting ones in Table 2 to illustrate that triadic concepts are easily interpretable. Namely:

- Concept No. 1 represents a group of customers who find taste and aroma of beaf steak and fried chicken wings excellent and their look at least neutral. We can say that it describes customers who like meat dishes for their taste and aroma.
- Concept No. 2 represents customers who like cheese salad for its excellent taste, aroma and look, and partly for its price.
- Concept No. 3 can be interpreted as "customers who have no preferences in food."
- Concept No. 4 represents customers who like beef steak and partly fried chicken wings for their excellent taste and look and at least neutral aroma.
- Concept No. 5 shows that there is no customer who finds all properties, including price, of all dishes excellent.

Let us remark, that the selected concepts are potentially helpful for the management of our imaginary restaurant, because they illustrate the trends in customer behaviour and the reasons for their occurrence. For example, concept No. 1 indicates that there is a numerous group of customers who like meat dishes. Moreover, one can see that it is because the customers like the excellent taste and aroma of

Table 2. Five interesting triadic concepts. The concepts are represented by columns 1, 2, . . . , 5.

	1	2	3	4	5
steak	1	0	1	1	1
salad	0	1	1	0	1
veget.	0	0	1	0	1
wings	1	0	1	$\frac{1}{2}$	1
taste	1	1	1	1	1
aroma	1	1	1	$\frac{1}{2}$	1
look	$\frac{1}{2}$	1	1	1	1
price	0	$\frac{1}{2}$	0	0	1
Fry	1	0	0	1	0
Bender	1	0	0	1	0
Leila	0	1	0	0	0
Zoidberg	1	1	1	1	0

the dishes.

5. Basic Theorem: The Structure of Triadic Fuzzy Concepts

Let V be a non-empty set, and for $i \in \{1,2,3\}$ let $\lesssim_i$ be quasiorder relations on V. As is well-known, $\lesssim_i \cap \gtrsim_i$ is an equivalence relation on V, denoted by $\sim_i$ in what follows. Furthermore, $\lesssim_i$ induces a partial order relation $\leq_i$ on the factor set $V/\sim_i = \{[v]_{\sim_i} \,|\, v \in V\}$ by putting $[v]_{\sim_i} \leq_i [w]_{\sim_i}$ if and only if $v \lesssim_i w$, for each $i \in \{1,2,3\}$. Here, $[v]_{\sim_i} = \{w \in V \,|\, v \sim_i w\}$ is the equivalence class of v. $(V, \lesssim_1, \lesssim_2, \lesssim_3)$ is called a *triordered set* (Wille 1995) if and only if the following two conditions hold for every $\{i,j,k\} = \{1,2,3\}$:

(to1) $v \lesssim_i w$ and $v \lesssim_j w$ implies $w \lesssim_k v$ for every $v, w \in V$;
(to2) $\sim_i \cap \sim_j \cap \sim_k$ is an identity relation on V.

It is easy to see that if $(V, \lesssim_1, \lesssim_2, \lesssim_3)$ is a triordered set, $\sim_i \cap \sim_j$ is an identity on V for every $i \neq j$ (Wille 1995).

We recall the basic notions and results regarding triordered sets (Wille 1995). Let $V_i, V_k \subseteq V$. An element $v \in V$ is called an *ik-bound* of if $v_i \lesssim_i v$ and $v_k \lesssim_k v$ for every $v_i \in V_i$ and $v_k \in V_k$. An ik-bound v is called an *ik-limit* of $\langle V_i, V_k\rangle$ if $u \lesssim_j v$ for every ik-bound u of $\langle V_i, V_k\rangle$. In every triordered set $(V, \lesssim_1, \lesssim_2, \lesssim_3)$ there is at most one ik-limit v of $\langle V_i, V_k\rangle$ satisfying $v \lesssim_k u$ for every ik-limit u of $\langle V_i, V_k\rangle$. If such v exists, we call v an *ik-join* of $\langle V_i, V_k\rangle$ and denote it $\nabla_{ik}(V_i, V_k)$. A triordered set $(V, \lesssim_1, \lesssim_2, \lesssim_3)$ in which the ik-join exists for all $i \neq k$ ($i, k \in \{1,2,3\}$) and all pairs $\langle V_i, V_k\rangle$ of subsets of V is called a *complete trilattice*.

Let $\mathbf{V} = \langle V, \lesssim_1, \lesssim_2, \lesssim_3\rangle$ be a triordered set. An *order filter* in the quasiordered set $\langle V, \lesssim_i\rangle$ is a subset $F \subseteq V$ for which $v \in F$ whenever $u \in F$ and $u \lesssim_i v$ for every $u, v \in V$. The set of all order filters of $\langle V, \lesssim_i\rangle$ is denoted by $\mathcal{F}_i(\mathbf{V})$. A *principal filter* of $\langle V, \lesssim_i\rangle$ generated by $v \in V$ is the order filter $[v)_i = \{u \in V \,|\, v \lesssim_i u\}$. A subset $\mathcal{X} \subseteq \mathcal{F}_i(\mathbf{V})$ is called *i-dense* with respect to $\mathbf{V}$ if each principal filter of $\langle V, \lesssim_i\rangle$ is the intersection of some order filters from $\mathcal{X}$.

Let now $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ be a triadic fuzzy context. It is easy to see (Theorem 5.1 (1)) that $\mathcal{T}(\mathbf{K})$ is a triordered set. Consider the mapping $\kappa_i : X_i \times L \to \mathcal{T}(\mathbf{K})$ defined by

$$\kappa_i(x_i, a) = \{\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K}) \,|\, A_i(x_i) \geq a\}$$

for every $i \in \{1,2,3\}$, $x_i \in X_i$, and $a \in L$. It is immediate that $\kappa_i(x_i, a)$ is an order filter in $\langle \mathcal{T}(\mathbf{K}), \lesssim_i\rangle$, i.e. $\kappa_i(x_i, a) \in \mathcal{F}_i(\mathcal{T}(\mathbf{K}))$. It is easy to see that the principal filter $[\langle A_1, A_2, A_3\rangle)_i$ generated by $\langle A_1, A_2, A_3\rangle \in \mathcal{T}(\mathbf{K})$ satisfies

$$[\langle A_1, A_2, A_3\rangle)_i = \bigcap_{x_i \in X_i} \kappa_i(x_i, A_i(x_i)).$$

The set $\kappa_i(X_i \times L)$ is therefore i-dense with respect to $\langle \mathcal{T}(\mathbf{K}), \lesssim_1, \lesssim_2, \lesssim_3\rangle$. Moreover, $a \leq b$ implies $\kappa_i(x_i, b) \subseteq \kappa_i(x_i, a)$.

The following theorem, which generalizes the basic theorem of triadic concept analysis (Wille 1995), describes the structure of trilattices of triadic **L**-contexts.

Theorem 5.1: *Let $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ be a triadic fuzzy context.*

(1) *$\mathcal{T}(\mathbf{K})$ is a complete trilattice for which the ik-joins are defined for every $i, k \in \{1,2,3\}$, $i \neq k$, by:*

$$\nabla_{ik}(\mathcal{X}_i, \mathcal{X}_k) = \mathfrak{b}_{ik}\left(\bigcup\{A_i | \langle A_1, A_2, A_3\rangle \in \mathcal{X}_i\}, \bigcup\{A_k | \langle A_1, A_2, A_3\rangle \in \mathcal{X}_k\}\right).$$

(2) *A complete trilattice $\mathbf{V} = \langle V, \lesssim_1, \lesssim_2, \lesssim_3\rangle$ is isomorphic to $\mathcal{T}(\mathbf{K})$ if and only if there are mappings $\tilde{\kappa}_i : X_i \times L \to \mathcal{F}_i(\mathbf{V})$, $i = 1,2,3$, such that*

(i) *$\tilde{\kappa}_i(X_i \times L)$ is i-dense with respect to $\mathbf{V}$;*
(ii) *$A_1 \otimes A_2 \otimes A_3 \subseteq I$ iff $\bigcap_{i=1}^3 \bigcap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i(x_i)) \neq \emptyset$, for every $A_i \in L^{X_i}$, $i = 1,2,3$;*
(iii) *$a \leq b$ implies $\tilde{\kappa}_i(x_i, b) \subseteq \tilde{\kappa}_i(x_i, a)$ for every $a, b \in L$, $x_i \in X_i$, $i = 1,2,3$.*

Proof: (1): Corollary 3.3 implies that $\mathcal{T}(\mathbf{K})$ is a triordered set. Moreover, $\mathcal{T}(\mathbf{K})$ is a complete trilattice due to Theorem 3.4.

(2): "$\Rightarrow$": Let φ be an isomorphism between $\mathbf{V}$ and $\mathcal{T}(\mathbf{K})$. Define a mapping $\tilde{\kappa}_i : X_i \times L \to V$ by $\tilde{\kappa}_i(x_i, b) = \varphi(\kappa(x_i, b))$. As observed above, $\kappa_i(X_i \times L)$ is i-dense w.r.t. $\mathcal{T}(\mathbf{K})$ and $a \leq b$ implies $\kappa_i(x_i, b) \subseteq \kappa_i(x_i, a)$. Therefore, $\tilde{\kappa}_i$ satisfies (i) and (iii). If $A_1 \otimes A_2 \otimes A_3 \subseteq I$, Theorem 3.5 (b) yields a concept $\langle B_1, B_2, B_2\rangle \in \mathcal{T}(\mathbf{K})$ for which $A_i \subseteq B_i$ for all $i \in \{1,2,3\}$. Clearly, $\langle B_1, B_2, B_2\rangle \in \bigcap_{i=1}^3 \bigcap_{x_i \in X_i} \kappa_i(x_i, A_i(x_i))$. Conversely, if $\langle B_1, B_2, B_2\rangle$ is an element of $\bigcap_{i=1}^3 \bigcap_{x_i \in X_i} \kappa_i(x_i, A_i(x_i))$, one has $A_i \subseteq B_i$ for all $i \in \{1,2,3\}$. Monotony of $\otimes$ and Theorem 3.5 (a) thus yield $A_1 \otimes A_2 \otimes A_3 \subseteq B_1 \otimes B_2 \otimes B_3 \subseteq I$. Therefore, $A_1 \otimes A_2 \otimes A_3 \subseteq I$ iff $\bigcap_{i=1}^3 \bigcap_{x_i \in X_i} \kappa_i(x_i, A_i(x_i)) \neq \emptyset$ iff $\bigcap_{i=1}^3 \bigcap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i(x_i)) \neq \emptyset$, proving (ii).

"$\Leftarrow$": Let ψ be a mapping $\psi : V \to L^{X_1} \times L^{X_2} \times L^{X_3}$ defined by $\psi(v) = \langle A_1^v, A_2^v, A_2^v\rangle$ where

$$A_i^v(x_i) = \bigvee L_{i,x_i}^v.$$

where $L_{i,x_i}^v = \{a \in L \,|\, v \in \tilde{\kappa}_i(x_i, a)\}$. Since $\mathbf{V}$ is a triordered set, $[v)_1 \cap [v)_2 \cap [v)_3 = \{v\}$. (i) implies that $[v)_i$ is the intersection of all $\tilde{\kappa}_i(x_i, a)$ that contain v, i.e. $[v)_i = \bigcap_{x_i \in X_i} \bigcap_{a \in L_{i,x_i}^v} \tilde{\kappa}_i(x_i, a)$. Therefore, $\bigcap_{i=1}^3 \bigcap_{x_i \in X_i} \bigcap_{a \in L_{i,x_i}^v} \tilde{\kappa}_i(x_i, a) = \{v\} \neq \emptyset$. In particular, for every collection of $a_{i,x_i} \in L_{i,x_i}^v$ ($i = 1,2,3$ and $x_i \in X_i$), $\bigcap_{i=1}^3 \bigcap_{x_i \in X_i} \tilde{\kappa}_i(x_i, a_{i,x_i}) \neq \emptyset$, and (ii), applied to fuzzy sets A_i defined by $A_i(x_i) = a_{i,x_i}$ for each $x_i \in X_i$, thus yields

$$a_{1,x_1} \otimes a_{2,x_2} \otimes a_{3,x_3} \leq I(x_1, x_2, x_3).$$

Using $a \otimes (\bigvee_j b_j) = \bigvee_j (a \otimes b_j)$ (identity of complete residuated lattices), one therefore gets

$$\begin{aligned} & A_1^v(x_1) \otimes A_2^v(x_2) \otimes A_2^v(x_2) = \\ = & (\bigvee_{a_{1,x_1} \in L^v_{1,x_1}} a_{1,x_1}) \otimes (\bigvee_{a_{2,x_2} \in L^v_{2,x_2}} a_{2,x_2}) \otimes (\bigvee_{a_{3,x_3} \in L^v_{3,x_3}} a_{3,x_3}) = \\ = & \bigvee_{a_{1,x_1} \in L^v_{1,x_1}} \bigvee_{a_{2,x_2} \in L^v_{2,x_2}} \bigvee_{a_{3,x_3} \in L^v_{3,x_3}} a_{1,x_1} \otimes a_{2,x_2} \otimes a_{3,x_3} \leq I(x_1, x_2, x_3), \end{aligned}$$

verifying $A_1^v \otimes A_2^v \otimes A_2^v(x_2) \subseteq I$. (ii) then yields $\bigcap_{i=1}^3 \bigcap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i^v(x_i)) \neq \emptyset$. Due to (iii),

$$\bigcap_{i=1}^3 \bigcap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i^v(x_i)) \subseteq \bigcap_{i=1}^3 \bigcap_{x_i \in X_i} \bigcap_{a \in L^v_{i,x_i}} \tilde{\kappa}_i(x_i, a) = \{v\},$$

whence

$$\bigcap_{i=1}^3 \bigcap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i^v(x_i)) = \{v\}. \tag{6}$$

Using adjointness, one easily verifies that for $\hat{A}_3^v = A_1^{v(1,3,A_2^v)}$ we also have $A_3^v \subseteq \hat{A}_3^v$ and $A_1^v \otimes A_2^v \otimes \hat{A}_3^v \subseteq I$. Due to (ii) and (iii), the latter inclusion and (6) imply $\bigcap_{i=1}^2 \bigcap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i^v(x_i)) \cap \bigcap_{x_3 \in X_3} \tilde{\kappa}_3(x_3, \hat{A}_3^v(x_3)) = \{v\}$. In particular, $\hat{A}_3^v(x_3) \in L^v_{3,x_3}$ which implies $\hat{A}_3^v \subseteq A_3^v$ because as $A_3^v(x_3) = \bigvee L^v_{3,x_3} \geq \hat{A}_3^v(x_3)$. To sum up, $\hat{A}_3^v = A_3^v$. The same way, one proves $\hat{A}_1^v = A_1^v$ and $\hat{A}_2^v = A_2^v$. This shows $\psi(v) \in \mathcal{T}(\mathbf{K})$.

If $v_1 \lesssim_i v_2$ for $v_1, v_2 \in V$ then clearly, $L^{v_1}_{i,x_i} \subseteq L^{v_2}_{i,x_i}$ for every $x_i \in X_i$, whence $A_i^{v_1} \subseteq A_i^{v_2}$, showing that ψ preserves $\lesssim_i$.

Let $\langle A_1, A_2, A_3 \rangle \in \mathcal{T}(\mathbf{K})$. Theorem 3.5 (a) and (ii) imply that there exists $v \in \bigcap_{i=1}^3 \bigcap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i(x_i))$ and thus $v \in \bigcap_{x_i \in X_i} \tilde{\kappa}_i(x_i, A_i(x_i))$. The key observation is that $v \notin \tilde{\kappa}_i(x_i, d)$ for all $d \nleq A(x_i)$ for all $x_i \in X_i$. In order to prove it, assume by contradiction that there are $x_i' \in X_i$ and $d \nleq A_i(x_i')$ such that $v \in \tilde{\kappa}_i(x_i', d)$. Then (ii) implies that for all $x_j \in X_j, x_k \in X_k$ we have $d \otimes A_j(x_j) \otimes A_k(x_k) \leq I(x_i', x_j, x_k)$ and thus $(d \vee A_i(x_i)) \otimes A_j(x_j) \otimes A_k(x_k) \leq I(x_i', x_j, x_k)$, a contradiction to the maximality of $\langle A_1, A_2, A_3 \rangle$ (Theorem 3.5). Therefore $A_i(x_i) = \bigvee \{a \mid v \in \tilde{\kappa}_i(x_i, a)\}$ for each $x_i \in X_i$, $i = 1, 2, 3$. This proves $\psi(v) = \langle A_1, A_2, A_3 \rangle$, whence the surjectivity of ψ.

(6) implies that if $v \neq w$, then $\psi(v) \neq \psi(w)$. Therefore ψ is injective.

If $\psi(v_1) = \langle A_1, A_2, A_3 \rangle \lesssim_i \langle B_1, B_2, B_3 \rangle = \psi(v_2)$ then $A_i \subseteq B_i$. (iii) implies that for each $x_i \in X_i$ we have $\tilde{\kappa}_i(x_i, B_i(x_i)) \subseteq \tilde{\kappa}_i(x_i, A_i(x_i))$ which implies $[v_2)_i \subseteq [v_1)_i$ and therefore $v_1 \lesssim_i v_2$. Thus ψ^-1 preserves $\lesssim_i$.

□

Remark 1 : Let us see that Theorem 5.1 indeed generalizes Wille's basic theorem of triadic concept analysis (Wille 1995). This is clear for (1) because Wille's is a particular case of (1) for $L = \{0, 1\}$. For part (2), we show that for $L = \{0, 1\}$, the existence of mappings $\tilde{\kappa}_i$ satisfying (i), (ii), and (iii), is equivalent to Wille's conditions, i.e. to the existence of mappings $\tilde{\kappa}_i' : X_1 \to \mathcal{F}_i(\mathbf{V})$ satisfying: (i') $\tilde{\kappa}_i'(X_i)$ is i-dense in $\mathbf{V}$ and (ii') $A_1 \times A_2 \times A_3 \subseteq I$ iff $\bigcap_{i=1}^3 \bigcap_{x_i \in A_i} \tilde{\kappa}_i'(x_i) \neq \emptyset$. In doing so, we identify sets and relations (used in the ordinary setting) with their characteristic functions (i.e. with fuzzy sets, used in a fuzzy setting).

Let $\tilde{\kappa}_i$ satisfy (i)–(iii). Define $\tilde{\kappa}'_i(x_i) = \tilde{\kappa}_i(x_i, 1)$. To see that $\tilde{\kappa}'_i$ satisfy (i') and (ii'), it is sufficient to observe that $\tilde{\kappa}_i(x_i, 0) = V$ for every $i = 1, 2, 3$ and $x_i \in X_i$. The fact $\tilde{\kappa}_i(x_i, 0) = V$ is established in the following Claim.

Claim: If $\tilde{\kappa}_i$ sasitfy (i)–(iii) of Theorem 5.1 then $\tilde{\kappa}_i(x_i, 0) = V$ for each $x_i \in X_i$, $i = 1, 2, 3$.

Proof of Claim: Assume $i = 1$. Consider fuzzy sets $0_1 \in \mathbf{L}^{X_1}$, $1_2 \in \mathbf{L}^{X_2}$, and $1_3 \in \mathbf{L}^{X_3}$ defined for any $x_1 \in X_1$, $x_2 \in X_2$, and $x_3 \in X_3$ by $0_1(x_1) = 0$, $1_2(x_2) = 1$, and $1_3(x_3) = 1$. Due to (ii),

$$0 \times 1_2 \times 1_3 \subseteq I \text{ iff } \bigcap_{x_1 \in X_1} \tilde{\kappa}_1(x_1, 0) \cap \bigcap_{x_2 \in X_2} \tilde{\kappa}_2(x_2, 1) \cap \bigcap_{x_3 \in X_3} \tilde{\kappa}_3(x_3, 1) \neq \emptyset$$

Let $v = \nabla_{23}(V, V)$. Then v is an 23-bound of $\langle V, V\rangle$ and, therefore, $w \lesssim_2 v$ and $w \lesssim_3 v$ for each $w \in V$. Applying (to1) we get that $v \lesssim_1 w$ for each $w \in V$.

We now claim that v is the only member of $\cap_{x_2 \in X_2} \tilde{\kappa}_2(x_2, 1) \cap \cap_{x_3 \in X_3} \tilde{\kappa}_3(x_3, 1)$. Indeed, assume by contradiction that there is $w \neq v$ such that $w \in \cap_{x_2 \in X_2} \tilde{\kappa}_2(x_2, 1) \cap \cap_{x_3 \in X_3} \tilde{\kappa}_3(x_3, 1)$. Then $w \in \tilde{\kappa}_2(x_2, 1)$ for each $x_2 \in X_2$ and by (iii) $w \in \tilde{\kappa}_2(x_2, a)$ for each $x_2 \in X_2$ and $a \in L$. (i) implies that $v \sim_2 w$ (if $w <_2 v$ than $[v)_i$ cannot be obtained as an intersection of some subset of $\tilde{\kappa}_2(X_2 \times L)$). Similarly, we get $w \sim_3 v$. Therefore, (to1) and (to2) imply $v = w$, a contradiction.

Moreover, we have

$$\begin{aligned}
&\bigcap_{x_1 \in X_1} \tilde{\kappa}_1(x_1, 0) \cap \bigcap_{x_2 \in X_2} \tilde{\kappa}_2(x_2, 1) \cap \bigcap_{x_3 \in X_3} \tilde{\kappa}_3(x_3, 1) \neq \emptyset \text{ iff} \\
\text{iff } &\bigcap_{x_1 \in X_1} \tilde{\kappa}_1(x_1, 0) \cap \{v\} \neq \emptyset \text{ iff} \\
\text{iff } &v \in \bigcap_{x_1 \in X_1} \tilde{\kappa}_1(x_1, 0) \text{ iff} \\
\text{iff } &v \in \tilde{\kappa}_1(x_1, 0) \text{ for each } x_2 \in X_2.
\end{aligned}$$

Since $v \lesssim_1 w$ for each $w \in V$ and since $\tilde{\kappa}_1(x_1, 0)$ is an order filter, it follows that $\tilde{\kappa}_1(x_1, 0) = V$. The proofs for $i = 2, 3$ are analogous. *QED Claim*

If $\tilde{\kappa}'_i$ satisfy (i') and (ii') then putting $\tilde{\kappa}_i(x_i, 0) = V$ and $\tilde{\kappa}_i(x_i, 1) = \tilde{\kappa}'_i(x_i)$, $\tilde{\kappa}_i$ clearly satisfy (i)–(iii).

It remains an open problem whether condition (iii) follows from (i) and (ii) in Theorem 5.1.

6. Conclusions and Future Work

We presented an approach to triadic concept analysis of data with graded attributes. In particular, we provided motivations, mathematical foundations, and an illustrative example. The paper shows that triadic concept analysis can be cleanly developed for data with graded attributes. Future research shall include the following topics.

– Applications. In (Belohlavek *at al.* 2011), we prove that triadic fuzzy concepts may be used as optimal factors in factorizing three-way ordinal data. This and other types of applications of triadic fuzzy concepts in the analysis of triadic data are to be explored thoroughly. Note that the analysis of three-way data is considered an important recent topic in data analysis (Kolda and Bader 2009).

- Algorithms. For binary data, Jäschke *at al.* (2006) presents an algorithm for computing triadic concepts. Clearly, efficient algorithms for computing all triadic concepts or various parts of the trilattice of triadic concepts for graded data are necessary for applications.
- Foundations. In particular, it is essential to establish links between triadic concept analysis of data with graded attributes to the ordinary case of data with binary attributes. As in the dyadic case (Belohlavek 2002), this may provide theorems for obtaining easily certain type of results from the ordinary to the fuzzy setting. Another interesting direction consists in generalizing the ideas from (Belohlavek 2011a,b) to work out a kind of triadic concept analysis with degrees from possibly different sets for extents, intents, and modi, and with other types of concepts than those corresponding to cuboids.

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