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Triadic fuzzy Galois connections as ordinary connections ☆

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Abstract

We present results on triadic Galois connections and trilattices associated with ternary fuzzy relations. These structures are fundamental in the analysis of three-way relational data. We provide an axiomatization of triadic fuzzy Galois connections and establish two ways of representing these structures by their ordinary counterparts—one via a Cartesian representation and the other via a-cuts. The results allow us to easily transfer some of the known results from the ordinary case to the fuzzy case. This transfer is illustrated by an alternative proof of the basic theorem of fuzzy concept trilattices via reduction to the ordinary basic theorem. In addition, we provide a hint to generalize the presented results to n-ary structures which are of increasing importance in data analysis of n-way data.

Keywords: Galois connection; Trilattice; Fuzzy logic; Ternary relation; Formal concept analysis

1. Introduction

Many results about ordinary relations have been generalized to the setting of fuzzy relations in the past. Needless to say, a fundamental question, important both from practical and theoretical point of view, is that of the relationship of the generalized notions and results to the ordinary ones.

In this paper we deal with basic structures associated with ternary relations, particularly with Galois connections and the trilattices of their fixpoints, i.e. the structures of maximal Cartesian subrelations, which appear as fundamental structures in relational analysis of three-way data. Namely, such structures appear in triadic concept analysis [18,22], triadic association rules [14], or in factor analysis of triadic data [5,7]. We focus on ternary relations but provide hints to generalize the results in a straightforward way to general n-ary relations. Note that general n-ary relations are becoming increasingly important for their role in the analysis of n-way data [10,16,17].

Our paper is organized as follows. We first provide preliminaries in Section 2. In Section 3, we introduce the Galois connections induced by ternary fuzzy relations and provide their axiomatization. In Section 4, we describe a

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representation of these Galois connections by means of Galois connections induced by ordinary relations. In particular, we provide two kinds of representation, one based on a Cartesian representation of fuzzy sets by ordinary sets and the other based on a-cuts. The representation theorems allow us to transfer almost automatically the results from the ordinary to the fuzzy setting which we illustrate using the basic theorem on concept trilattices as an example. In Section 5, we present conclusions and outline some topics for future research.

2. Preliminaries

2.1. Fuzzy logic and fuzzy sets

We assume that the scale L of truth degrees forms a complete residuated lattice [21], i.e. an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$$

such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy the so-called adjointness property:

$$a \otimes b \leqslant c \quad \text{iff} \quad a \leqslant b \rightarrow c \tag{1}$$

for each $a, b, c \in L$. In fuzzy logic, elements a of L are called truth degrees and $\otimes$ and $\rightarrow$ are considered as the (truth functions of) many-valued conjunction and implication.

Examples of complete residuated lattices, particularly those with L being [0, 1] or a finite subchain of [0, 1] which are based on t-norms and their residua, are well known and we refer to [3,12,13] for details. A special case of a complete residuated lattice is the two-element Boolean algebra $\langle\{0, 1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of classical logic. This is important because for the particular case $\mathbf{L} = \mathbf{2}$, the notions and results become the ones regarding ordinary sets and relations.

Given a complete residuated lattice $\mathbf{L}$, we define the usual notions: an $\mathbf{L}$-set (fuzzy set, graded set) A in a universe U is a mapping $A : U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Let L^U denote the collection of all $\mathbf{L}$-sets in U. The basic operations with $\mathbf{L}$-sets are based on the residuated lattice operations and are defined componentwise. Clearly, $\mathbf{2}$-sets and operations with $\mathbf{2}$-sets can be identified with ordinary sets and operations with ordinary sets, respectively. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$; similarly for ternary relations. For $a \in L$ and $u \in U$, we denote by $\{a/u\}$ the $\mathbf{L}$-set A in U for which $A(x) = a$ if $x = u$ and $A(x) = 0$ if $x \neq u$. Given $A, B \in L^U$, we define the degree $S(A, B)$ of inclusion of A in B by

$$S(A, B) = \bigwedge_{u \in U} \big(A(u) \rightarrow B(u)\big) \tag{2}$$

and the degree of equality of A and B by

$$A \approx B = \bigwedge_{u \in U} \big(A(u) \leftrightarrow B(u)\big) \tag{3}$$

Note that (2) generalizes the ordinary subsethood relation $\subseteq$. Described verbally, $S(A, B)$ represents the degree to which every element of A is an element of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leqslant B(u)$ for each $u \in U$. Likewise, (3) generalizes the ordinary equality relation $=$. Note also that $A \approx B$ represents the degree to which every element belongs to A iff it belongs to B. Clearly, $A = B$ iff $A \approx B = 1$. For details on fuzzy sets we refer to [3,12,13,15].

2.2. Triadic concept analysis and Galois connections

We now provide the basic notions regarding the structures related to ternary relations, mainly in terms of triadic concept analysis [18,22] which is the main intended application area of our results (see [20] for the polyadic case). A *triadic context* is a quadruple $\langle X, Y, Z, I\rangle$ where X, Y, and Z are non-empty sets, and I is a ternary relation between X, Y, and Z, i.e. $I \subseteq X \times Y \times Z$. The sets X, Y, and Z are interpreted as the sets of objects, attributes, and conditions, respectively; I is interpreted as the incidence relation ("to have-under relation"). That is, if $x \in X$, $y \in Y$, and $z \in Z$, then $\langle x, y, z\rangle \in I$ is interpreted as: object x has attribute y under condition z, in which case we say that x, y, z (or y, x, z, or the result of listing x, y, z in any other sequence) are related by I. For convenience, a triadic context is denoted by $\langle X_1, X_2, X_3, I\rangle$.

Let $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ be a triadic context. For $\{i, j, k\} = \{1, 2, 3\}$ and $C_k \subseteq X_k$, we define a dyadic context [11]

$$\mathbf{K}^{ij}_{C_k} = \langle X_i, X_j, I^{ij}_{C_k}\rangle$$

by

$$\langle x_i, x_j\rangle \in I^{ij}_{C_k} \quad \text{iff} \quad \text{for each } x_k \in C_k\text{: } x_i, x_j, x_k \text{ are related by } I.$$

The dyadic concept-forming operators [11] induced by $\mathbf{K}^{ij}_{C_k}$ are denoted by ${}^{(i,j,C_k)}$. That is, for $C_i \subseteq X_i$ and $C_j \subseteq X_j$ we have

$$C_i^{(i,j,C_k)} = \{x_j \in X_j\colon \langle x_i, x_j\rangle \in I^{ij}_{C_k} \text{ for each } x_i \in C_i\},$$
$$C_j^{(i,j,C_k)} = \{x_i \in X_i\colon \langle x_i, x_j\rangle \in I^{ij}_{C_k} \text{ for each } x_j \in C_j\}.$$

A *triadic concept* of $\langle X_1, X_2, X_3, I\rangle$ is a triplet $\langle C_1, C_2, C_3\rangle$ of $C_1 \subseteq X_1$, $C_2 \subseteq X_2$, and $C_3 \subseteq X_3$, satisfying the following three conditions: C_1 is the set of all $x_1 \in X_1$ such that x_1, x_2, x_3 are related by I for every $x_2 \in X_2$ and $x_3 \in X_3$; C_2 is the set of all $x_2 \in X_2$ such that x_1, x_2, x_3 are related by I for every $x_1 \in X_1$ and $x_3 \in X_3$; and C_3 is the set of all $x_3 \in X_3$ such that x_1, x_2, x_3 are related by I for every $x_1 \in X_1$ and $x_2 \in X_2$. In terms of the above operators, these three conditions are equivalent to

$$C_1 = C_2^{(1,2,C_3)}, \qquad C_2 = C_3^{(2,3,C_1)}, \quad \text{and} \quad C_3 = C_1^{(3,1,C_2)},$$

or, since $C_i^{(i,j,C_k)} = C_i^{(j,i,C_k)}$ and $C_i^{(i,j,C_k)} = C_k^{(k,j,C_i)}$, to any of the other (sixty-three) possible ways of expressing the three conditions. Clearly, the three conditions may also be expressed by saying that for every assignment $\{i, j, k\} = \{1, 2, 3\}$ we have $C_i = C_j^{(i,j,C_k)}$, or that for some such assignment we have $C_i = C_j^{(i,j,C_k)}$, $C_j = C_k^{(j,k,C_i)}$, and $C_k = C_i^{(k,i,C_j)}$. C_1, C_2, and C_3 are called the *extent*, *intent*, and *modus* of $\langle C_1, C_2, C_3\rangle$. Geometrically, triadic concepts are just the maximal cuboids contained in I, i.e. maximal subrelations of I that result as Cartesian products of sets of objects, attributes, and modi. The set of all triadic concepts of $\langle X_1, X_2, X_3, I\rangle$ is denoted by $\mathcal{T}(X_1, X_2, X_3, I)$ and is called the concept trilattice (see Section 4) of $\langle X_1, X_2, X_3, I\rangle$. Note that complete trilattices are the appropriate generalizations of complete lattices (i.e. dyadic lattices) that result as naturally structured sets of fixpoints of the connections induced by ternary relations.

For $\{i, j, k\} = \{1, 2, 3\}$ with $j < k$, a triadic context $\langle X_1, X_2, X_3, I\rangle$ induces the operator

$$(i)_I : 2^{X_j} \times 2^{X_k} \to 2^{X_i}$$

defined by

$$(C_j, C_k)^{(i)_I} = C_j^{(j,i,C_k)}$$

for any $C_j \subseteq X_j$ and $C_k \subseteq X_k$. The triplet $\langle{}^{(1)_I}, {}^{(2)_I}, {}^{(3)_I}\rangle$, denoted also simply by $\langle{}^{(1)}, {}^{(2)}, {}^{(3)}\rangle$, forms an (ordinary) triadic Galois connection [9], see also Section 3. That is, for every $C_i \subseteq X_i$, $C_j \subseteq X_j$, and $C_k \subseteq X_k$, one has

$$C_3 \subseteq (C_1, C_2)^{(3)} \quad \text{iff} \quad C_1 \subseteq (C_2, C_3)^{(1)} \quad \text{iff} \quad C_2 \subseteq (C_1, C_3)^{(2)}. \tag{4}$$

Conversely, every triplet $\langle{}^{(1)_I}, {}^{(2)_I}, {}^{(3)_I}\rangle$ satisfying (4) is induced by some triadic context [9].

3. Triadic fuzzy Galois connections

3.1. Triadic fuzzy contexts and their Galois connections

We first recall the basic notions of triadic concept analysis as generalized in [6] for data with fuzzy attributes, i.e. for ternary fuzzy relations. These notions have been utilized for factor analysis of three-way data in [7].

A *triadic* **L**-*context* (triadic fuzzy context, or just triadic context) is a quadruple $\langle X, Y, Z, I\rangle$ where X, Y, and Z are non-empty sets, and I is a ternary fuzzy relation between X, Y, and Z, i.e. $I : X \times Y \times Z \to L$. Again, $x \in X$, $y \in Y$, and $z \in Z$ are interpreted as objects, attributes, and conditions, respectively, and the degree $I(x, y, z) \in L$ is interpreted as the degree to which object x has attribute y under condition z. In this case, we also say that $I(x, y, z)$ is the degree to which x, y, z (or y, x, z or z, x, y, etc.) are related and, for convenience, denote $I(x, y, z)$ also by $I\{x, y, z\}$ or $I\{y, x, z\}$ or $I\{z, x, y\}$, etc. As in the ordinary case, we denote a triadic fuzzy context by $\langle X_1, X_2, X_3, I\rangle$. Except for mathematical arguments, the motivation for considering triadic fuzzy contexts is that in several situations, the relationship between objects, attributes, and modi naturally comes in degrees. For example, a degree to which object x has feature y under condition z may be interpreted as the degree to which customer z considers product x as having feature y (e.g., the degree 3/4 means that customer z considers food product x as having a good taste), see [6].

For every $\{i, j, k\} = \{1, 2, 3\}$ and a fuzzy set $A_k \in L^{X_k}$, a triadic **L**-context $\mathbf{K} = \langle X_1, X_2, X_3, I\rangle$ induces a dyadic **L**-context

$$\mathbf{K}^{ij}_{A_k} = \langle X_i, X_j, I^{ij}_{A_k}\rangle$$

in which the fuzzy relation $I^{ij}_{A_k}$ between X_i and X_j is defined by

$$I^{ij}_{A_k}(x_i, x_j) = \bigwedge_{x_k \in X_k} \big(A_k(x_k) \to I\{x_i, x_j, x_k\}\big) \tag{5}$$

for every $x_i \in X_i$ and $x_j \in X_j$. Dyadic **L**-contexts and the associated structures including concept-forming operators and concept lattices were studied in a series of papers, see e.g. [3,4,19]. The concept-forming operators induced by $\mathbf{K}^{ij}_{A_k}$ are denoted by ${}^{(i,j,A_k)}$. That is, for a fuzzy set A_i in X_i, we define a fuzzy set $A_i^{(i,j,A_k)}$ in X_j by

$$A_i^{(i,j,A_k)}(x_j) = \bigwedge_{x_i \in X_i} A_i(x_i) \to I^{ij}_{A_k}(x_i, x_j). \tag{6}$$

Similarly, for a fuzzy set A_j in X_j, we define a fuzzy set $A_j^{(i,j,A_k)}$ in X_i by

$$A_j^{(i,j,A_k)}(x_i) = \bigwedge_{x_j \in X_j} A_j(x_j) \to I^{ij}_{A_k}(x_i, x_j).$$

Generalizing the ordinary case, a *triadic* **L**-*concept* (triadic fuzzy concept) of $\langle X_1, X_2, X_3, I\rangle$ is a triplet $\langle A_1, A_2, A_3\rangle$ consisting of fuzzy sets $A_1 \in L^{X_1}$, $A_2 \in L^{X_2}$, and $A_3 \in L^{X_3}$, for which

$$A_1 = A_2^{(1,2,A_3)}, \qquad A_2 = A_3^{(2,3,A_1)}, \quad \text{and} \quad A_3 = A_1^{(3,1,A_2)}.$$

As in the ordinary case, due to $A_i^{(i,j,A_k)} = A_i^{(j,i,A_k)}$ (obvious) and $A_i^{(i,j,A_k)} = A_k^{(k,j,A_i)}$ (see [6, Lemma 3.1 (b)]), the definition of a triadic fuzzy concept may equivalently be rephrased, e.g. by requiring that for every assignment $\{i, j, k\} = \{1, 2, 3\}$ we have $A_i = A_j^{(i,j,C_k)}$. In this case, A_1, A_2, and A_3 are called the *extent*, *intent*, and *modus* of $\langle A_1, A_2, A_3\rangle$. The set of all triadic concepts of $\langle X_1, X_2, X_3, I\rangle$ is denoted by $\mathcal{T}(X_1, X_2, X_3, I)$ or simply $\mathcal{T}(I)$ and is called the **L**-*concept trilattice* (fuzzy concept trilattice) of $\langle X_1, X_2, X_3, I\rangle$.

Remark 1. Clearly, the notions introduced in this section generalize the corresponding ordinary notions reviewed in Section 2. Namely, putting $L = \{0, 1\}$, the notions of a triadic **L**-context, the induced operators and so on may be identified with the ordinary notions.

Example 1. A triadic **L**-context $\langle X, Y, Z, I\rangle$ is usually depicted using $|Z|$ tables, one table for every condition $z \in Z$. The tables have $|X|$ rows and $|Y|$ columns which correspond to objects and attributes, respectively. An entry of the table that corresponds to z, on the row corresponding to x and column corresponding to y is just the degree $I(x, y, z)$.

Let **L** be a three-element Gödel chain, that is a residuated lattice with elements $\{0, \frac{1}{2}, 1\}$, the operation $\otimes$ defined by $a \otimes b = \min(a, b)$, and $\to$ given by $a \to b = 1$ for all $a \leqslant b$ and $a \to b = b$ otherwise.

The following two tables depict a triadic **L**-context with the set of objects $X = x_1, x_2, x_3$, the set of attributes $Y = \{y_1, y_2, y_3\}$, and the set of conditions $Z = \{z_1, z_2\}$. The table on the left-hand side corresponds to the condition z_1, while the one on the right-hand side corresponds to the condition z_2.

z_1	y_1	y_2	y_3
x_1	0	1	1
x_2	0	0	$\frac{1}{2}$
x_3	$\frac{1}{2}$	$\frac{1}{2}$	1

z_2	y_1	y_2	y_3
x_1	1	0	0
x_2	1	1	0
x_3	1	0	$\frac{1}{2}$

Consider an **L**-set of objects A given by $A(x_1) = 0$, $A(x_2) = 0$, $A(x_3) = \frac{1}{2}$, which we denote by $A = \{0/x_1, 0/x_2, \frac{1}{2}/x_3\}$, and an **L**-set of conditions $C = \{1/z_1, \frac{1}{2}/z_2\}$. We now compute $A^{(1,2,C)}$ (note that we assume that X corresponds to X_1, Y to X_2, and Z to X_3, as we did in the above section). Using (5) with $i = 1$, $j = 2$, $k = 3$ one obtains a binary fuzzy relation I_C^{12} between X and Y, the relation is depicted by the following table.

	y_1	y_2	y_3
x_1	0	0	0
x_2	0	0	0
x_3	$\frac{1}{2}$	0	1

Using (6), one computes that $A^{(1,2,C)} = \{1/y_1, 0/y_2, 1/y_3\}$. Moreover, one can easily verify that

$$\left\langle \left\{ \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3} \right\}, \left\{ \frac{0}{y_1}, \frac{0}{y_2}, \frac{\frac{1}{2}}{y_3} \right\}, \left\{ \frac{1}{z_1}, \frac{0}{z_2} \right\} \right\rangle$$

is a triadic **L**-concept.

3.2. Axiomatizing Galois connections of triadic fuzzy contexts

A triadic **L**-context $\langle X_1, X_2, X_3, I\rangle$ induces three operators

$$^{(i)_I} : L^{X_j} \times L^{X_k} \to L^{X_i}$$

for $\{i, j, k\} = \{1, 2, 3\}$ with $j < k$, defined by

$$(A_j, A_k)^{(i)_I} = A_j^{(j,i,A_k)} \tag{7}$$

for any $A_j \in L^{X_j}$ and $A_k \in L^{X_k}$. Note that since, as mentioned above, $A_j^{(j,i,A_k)} = A_k^{(k,i,A_j)}$, we may take the liberty of writing $(A_k, A_j)^{(i)_I}$ instead of $(A_j, A_k)^{(i)_I}$, i.e. consider also $^{(i)_I} : L^{X_k} \times L^{X_j} \to L^{X_i}$. The triplet $\langle ^{(1)_I}, ^{(2)_I}, ^{(3)_I}\rangle$, denoted also just by $\langle ^{(1)}, ^{(2)}, ^{(3)}\rangle$, is axiomatized below. In fact, we provide an axiomatization of a wider class of operators for reasons that become apparent later.

Recall that an order filter in a partially ordered set $\langle L, \leqslant\rangle$ (known also as upward closed subset or upper set) is any subset $K \subseteq L$ for which $a \in K$ and $a \leqslant b$ imply $b \in K$ for any $a, b \in L$.

Definition 1. Let K be an order filter in $\langle L, \leqslant\rangle$. A *triadic* $\mathbf{L}_K$*-Galois connection* between sets X_1, X_2, and X_3 is a triplet $\langle ^{(1)}, ^{(2)}, ^{(3)}\rangle$ of mappings $^{(1)} : L^{X_2} \times L^{X_3} \to L^{X_1}$, $^{(2)} : L^{X_1} \times L^{X_3} \to L^{X_2}$, and $^{(3)} : L^{X_1} \times L^{X_2} \to L^{X_3}$, satisfying for every $A_1 \in L^{X_1}$, $A_2 \in L^{X_2}$, and $A_3 \in L^{X_3}$, that if $S(A_3, (A_1, A_2)^{(3)}) \in K$ or $S(A_1, (A_2, A_3)^{(1)}) \in K$ or $S(A_2, (A_1, A_3)^{(2)}) \in K$, then

$$S\big(A_3, (A_1, A_2)^{(3)}\big) = S\big(A_1, (A_2, A_3)^{(1)}\big) = S\big(A_2, (A_1, A_3)^{(2)}\big). \tag{8}$$

Remark 2. (a) One can easily see that for $L = \{0, 1\}$, triadic $\mathbf{L}_K$-Galois connections become ordinary triadic Galois connections (observe that in this case, there are only two filters, namely $K = L$ and $K = \{1\}$ and both lead to the same notion).

(b) In accordance with [1], we use the term **L**-Galois connections for $\mathbf{L}_L$-Galois connections.

The following theorem provides an alternative characterization of $\mathbf{L}_K$-Galois connections in terms of extensivity and antitony.

Theorem 1. *A triplet* $\langle ^{(1)}, ^{(2)}, ^{(3)} \rangle$ *is a triadic* $\mathbf{L}_K$*-Galois connection iff for any* $\{i, j, k\} = \{1, 2, 3\}$*, the following conditions hold for all* $A_i, A_i' \in L^{X_i}$, $A_j \in L^{X_j}$, $A_k \in L^{X_k}$:

(a) $A_i \subseteq (A_j, (A_i, A_j)^{(k)})^{(i)}$ *(extensivity)*,
(b) *if* $S(A_i, A_i') \in K$ *then* $S(A_i, A_i') \leqslant S((A_i', A_j)^{(k)}, (A_i, A_j)^{(k)})$ *(antitony)*.

Proof. "⇒" Assume that (8) holds for $\langle ^{(1)}, ^{(2)}, ^{(3)} \rangle$. Then

$$S\big(A_i, \big(A_j, (A_i, A_j)^{(k)}\big)^{(i)}\big) = S\big((A_i, A_j)^{(k)}, (A_i, A_j)^{(k)}\big) = 1 \in K,$$

proving (a) on account of "$A \subseteq B$ iff $S(A, B) = 1$".

By (a) and (8), we have

$$S\big(A_i, A_i'\big) \leqslant S\big(A_i, \big(A_j, \big(A_i', A_j\big)^{(k)}\big)\big) = S\big(\big(A_i', A_j\big)^{(k)}, (A_i, A_j)^{(k)}\big)$$

proving (b).

"⇐": Assume (a) and (b), and let $S(A_3, (A_1, A_2)^{(3)}) \in K$. Due to (b),

$$S\big(A_3, (A_1, A_2)^{(3)}\big) \leqslant S\big(\big((A_1, A_2)^{(3)}, A_1\big)^{(2)}, (A_3, A_1)^{(2)}\big).$$

Due to (a), $A_2 \subseteq ((A_1, A_2)^{(3)}, A_1)^{(2)}$. Since $(A_3, A_1)^{(2)} = (A_1, A_3)^{(2)}$, we get

$$S\big(\big((A_1, A_2)^{(3)}, A_1\big)^{(2)}, (A_3, A_1)^{(2)}\big) \leqslant S\big(A_2, (A_1, A_3)^{(2)}\big).$$

Putting the displayed inequalities together, we obtain

$$S\big(A_3, (A_1, A_2)^{(3)}\big) \leqslant S\big(A_2, (A_1, A_3)^{(2)}\big).$$

Since $S(A_3, (A_1, A_2)^{(3)}) \in K$ and since K is a filter, we get $S(A_2, (A_1, A_3)^{(2)}) \in K$. Applying now the above reasoning to $S(A_2, (A_1, A_3)^{(2)})$ yields

$$S\big(A_2, (A_1, A_3)^{(2)}\big) \leqslant S\big(A_3, (A_1, A_2)^{(3)}\big),$$

establishing $S(A_3, (A_1, A_2)^{(3)}) = S(A_2, (A_1, A_3)^{(2)})$. The other equality of (8) is proven symmetrically. □

Next, we provide some properties that are needed to show a bijective correspondence between ternary **L**-relations and triadic **L**-Galois connections.

Lemma 1. *For* $\{i, j, k\} = \{1, 2, 3\}$*, index sets* P, Q*, and fuzzy sets* $A_{ip} \in L^{X_i}$*, and* $A_{jp} \in L^{X_j}$ *the following equality holds*:

$$\Big(\bigvee_{p \in P} A_{ip}, \bigvee_{q \in Q} A_{jq}\Big)^{(k)} = \bigwedge_{p \in P, q \in Q} (A_{ip}, A_{jq})^{(k)} \tag{9}$$

Proof. We prove $(\bigvee_{p \in P} A_{ip}, A_j)^{(k)} = \bigwedge_{p \in P} (A_{ip}, A_j)^{(k)}$ by proving that for every $A_k \in L^{X_k}$,

$$A_k \subseteq \Big(\bigvee_{p \in P} A_{ip}, A_j\Big)^{(k)} \quad \text{iff} \quad A_k \subseteq \bigwedge_{p \in P} (A_{ip}, A_j)^{(k)}.$$

$A_k \subseteq (\bigvee_{p \in P} A_{ip}, A_j)^{(k)}$ iff (due to (8)) $\bigvee_{p \in P} A_{ip} \subseteq (A_j, A_k)^{(i)}$ iff for each $p \in P$, $A_{ip} \subseteq (A_j, A_k)^{(i)}$ iff for each $p \in P$, $A_k \subseteq (A_{ip}, A_j)^{(k)}$. The same property for j is proven analogously. The assertion now follows. □

Lemma 2. *Let* $\langle ^{(1)}, ^{(2)}, ^{(3)} \rangle$ *be a triadic* **L***-Galois connection. For* $\{i, j, k\} = \{1, 2, 3\}$ *and* $A_i \in L^{X_i}$*, let the mappings* $^{\uparrow_{A_i}} : L^{X_j} \to L^{X_k}$ *and* $^{\downarrow_{A_i}} : L^{X_k} \to L^{X_j}$ *be defined by*

$$A_k^{\uparrow_{A_i}} = (A_k, A_i)^{(j)},$$
$$A_j^{\downarrow_{A_i}} = (A_j, A_i)^{(k)}.$$

Then $\langle {}^{\uparrow_{A_i}}, {}^{\downarrow_{A_i}} \rangle$ *forms a dyadic* **L**-*Galois connection between* X_j *and* X_k [1].

Proof. $S(A_j, A_k^{\downarrow_{A_i}}) = S(A_j, (A_k, A_i)^{(j)}) = S(A_k, (A_j, A_i)^{(k)}) = S(A_k, A_j^{\uparrow_{A_i}})$, verifying the defining condition for dyadic **L**-Galois connections. □

Lemma 3. *For* $\{i, j, k\} = \{1, 2, 3\}$ *and any* $x_i \in X_i$, $x_j \in X_j$, *it holds*

(a) $a \to (\{1/x_i\}, \{1/x_j\})^{(k)} = (\{a/x_i\}, \{1/x_j\})^{(k)}$,
(b) $\bigwedge_{x_i \in X_i} A_i(x_i) \to (\{1/x_i\}, \{1/x_j\})^{(k)} = (A_i, \{1/x_j\})^{(k)}$.

Proof. (a) By Lemma 2 we get that

$$a \to \left(\left\{\frac{1}{x_i}\right\}, \left\{\frac{1}{x_j}\right\}\right)^{(k)} = a \to \left\{\frac{1}{x_i}\right\}^{\uparrow_{\{\frac{1}{x_j}\}}}.$$

[1] implies that

$$a \to \left\{\frac{1}{x_i}\right\}^{\uparrow_{\{\frac{1}{x_j}\}}} = \left\{\frac{a}{x_i}\right\}^{\uparrow_{\{\frac{1}{x_j}\}}}.$$

Finally, by Lemma 2 we have

$$\left\{\frac{a}{x_i}\right\}^{\uparrow_{\{\frac{1}{x_j}\}}} = \left(\left\{\frac{a}{x_i}\right\}, \left\{\frac{1}{x_j}\right\}\right)^{(k)}.$$

(b) Using (a) and Lemma 1 we get

$$\begin{aligned}\bigwedge_{x_i \in X_i} A_i(x_i) &\to \left(\left\{\frac{1}{x_i}\right\}, \left\{\frac{1}{x_j}\right\}\right)^{(k)} \\ &= \bigwedge_{x_i \in X_i} \left(\left\{\frac{A_i(x_i)}{x_i}\right\}, \left\{\frac{1}{x_j}\right\}\right)^{(k)} \\ &= \left(\bigvee_{x_i \in X_i} \left\{\frac{A_i(x_i)}{x_i}\right\}, \left\{\frac{1}{x_j}\right\}\right)^{(k)} \\ &= \left(A_i, \left\{\frac{1}{x_j}\right\}\right)^{(k)} \end{aligned}$$ □

The next theorem shows that triadic **L**-Galois connections are just the mappings obtained from ternary fuzzy relations by (7).

Theorem 2. *Let* $I \in L^{X_1 \times X_2 \times X_3}$. *Let* $\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle$ *be a triadic* **L**-*Galois connection between* X_1, X_2, *and* X_3 *and define a ternary* **L**-*relation* $I_{\langle (1), (2), (3) \rangle}$ *between* X_1, X_2, *and* X_3 *by*

$$I_{\langle (1), (2), (3) \rangle}(x_1, x_2, x_3) = \left(\left\{\frac{1}{x_1}\right\}, \left\{\frac{1}{x_2}\right\}\right)^{(3)}(x_3) = \left(\left\{\frac{1}{x_1}\right\}, \left\{\frac{1}{x_3}\right\}\right)^{(2)}(x_2) = \left(\left\{\frac{1}{x_2}\right\}, \left\{\frac{1}{x_3}\right\}\right)^{(1)}(x_1).$$

Then

(a) $\langle {}^{(1)_I}, {}^{(2)_I}, {}^{(3)_I} \rangle$ *forms a triadic* **L**-*Galois connection*;
(b) $I = I_{\langle (1)_I, (2)_I, (3)_I \rangle}$;

(c) $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle = \langle^{(1)}I_{\langle^{(1)},^{(2)},^{(3)}\rangle}, {}^{(2)}I_{\langle^{(1)},^{(2)},^{(3)}\rangle}, {}^{(3)}I_{\langle^{(1)},^{(2)},^{(3)}\rangle}\rangle$.

Proof. Observe first that the definition of $I_{\langle^{(1)},^{(2)},^{(3)}\rangle}$ is correct. Indeed, let $^{\uparrow\{1/x_i\}}$ and $^{\downarrow\{1/x_i\}}$ be the dyadic **L**-Galois connection defined in Lemma 2. Then, due to [1, Lemma 4] we have $(\{1/x_i\}, \{1/x_j\})^{(k)}(x_k) = \{1/x_j\}^{\downarrow\{1/x_i\}}(x_k) = \{1/x_k\}^{\uparrow\{1/x_i\}}(x_j) = (\{1/x_i\}, \{1/x_k\})^{(j)}(x_j)$ as well as the above-mentioned $(\{1/x_i\}, \{1/x_j\})^{(k)}(x_k) = (\{1/x_j\}, \{1/x_i\})^{(k)}(x_k)$, the correctness follows.

(a) Using $a \to \bigwedge_{j\in J} b_j = \bigwedge_{j\in J}(a \to b_j)$ and $(a \otimes b) \to c = a \to (b \to c)$ we get

$$
\begin{aligned}
&S\big(A_i, (A_j, A_k)^{(i_I)}\big)\\
&\quad= \bigwedge_{x_i\in X_i} A_i(x_i) \to \bigwedge_{\substack{x_j\in X_j\\ x_k\in X_k}} \big(A_j(x_j) \otimes A_k(k_k) \to I\{x_i, x_j, x_k\}\big)\\
&\quad= \bigwedge_{x_j\in X_j} A_j(x_j) \to \bigwedge_{\substack{x_i\in X_i\\ x_k\in X_k}} \big(A_i(x_i) \otimes A_k(k_k) \to I\{x_i, x_j, x_k\}\big)\\
&\quad= S\big(A_j, (A_i, A_k)^{(j_I)}\big),
\end{aligned}
$$

checking property (8) for $\langle^{(1)}I, ^{(2)}I, ^{(3)}I\rangle$.

(b) For every $x_1 \in X_1, x_2 \in X_2$, and $x_3 \in X_3$ we have

$$
\begin{aligned}
I_{\langle^{(1)}I,^{(2)}I,^{(3)}I\rangle}(x_1, x_2, x_3) &= \left(\left\{\frac{1}{x_1}\right\}, \left\{\frac{1}{x_2}\right\}\right)^{(3)_I}(x_3)\\
&= \left\{\frac{1}{x_1}\right\}^{(1,3,\{\frac{1}{x_2}\})}(x_3) = 1 \otimes 1 \to I(x_1, x_2, x_3) = I(x_1, x_2, x_3).
\end{aligned}
$$

(c) Using the properties of residuated lattices and Lemma 3 (b) we get

$$
\begin{aligned}
&(A_i, A_j)^{(k)I_{\langle(1),(2),(3)\rangle}}(x_k)\\
&\quad= \bigwedge_{x_i\in X_i} A_i(x_i) \to \Big(\bigwedge_{x_j\in X_j} A_j(x_j) \to I_{\langle(1),(2),(3)\rangle}\{x_i, x_j, x_k\}\Big)\\
&\quad= \bigwedge_{x_i\in X_i} A_i(x_i) \to \left(\bigwedge_{x_j\in X_j} A_j(x_j) \to \left(\left\{\frac{1}{x_i}\right\}, \left\{\frac{1}{x_j}\right\}\right)^{(k)}(x_k)\right)\\
&\quad= \bigwedge_{x_i\in X_i} A_i(x_i) \to \left(\left\{\frac{1}{x_i}\right\}, A_j\right)^{(k)}(x_k)\\
&\quad= (A_i, A_j)^{(k)}(x_k) \qquad \square
\end{aligned}
$$

Therefore, the notion of a triadic **L**-Galois connection (Definition 1 for $K = L$) provides us with an axiomatization of the mappings induced by ternary fuzzy relations by (7).

4. Representation of triadic fuzzy Galois connections by ordinary connections

In this section, we provide two kinds of representation of triadic fuzzy Galois connections using ordinary triadic Galois connections. In Section 4.1, we present a representation which is based on looking at a fuzzy set A in U as the area below the membership function, i.e. a subset of the Cartesian product $U \times L$ of U and the set L of truth degrees. In Section 4.2, we present another representation, a cut-like one, using which a triadic fuzzy Galois connection is represented as a nested system of ordinary triadic connections. In Section 4.3, we present an application of the Cartesian representation in proving in a simple way by reduction the basic theorem about fuzzy concept trilattices.

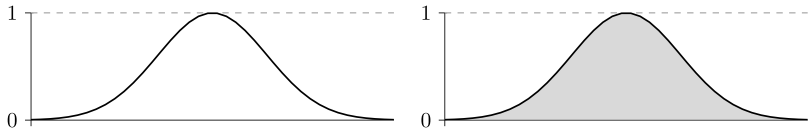

Fig. 1. Fuzzy set A (left) and its Cartesian representation $\lfloor A \rfloor$ (right).

4.1. Cartesian representation

We utilize a representation of fuzzy sets studied in [2] and further developed in [3,8]. The representation is based on the following mappings (note that these mappings were independently introduced in [19]). For a fuzzy set $A \in L^U$ put

$$\lfloor A \rfloor = \{\langle u, a\rangle \in U \times L \mid a \leqslant A(u)\}.$$

For an ordinary set $B \subseteq U \times L$, define a fuzzy set $\lceil B \rceil$ in U by

$$\lceil B \rceil(u) = \bigvee_{\langle u,a\rangle \in B} a.$$

The set $\lfloor A \rfloor$ may be thought of as the area below A, see Fig. 1, while $\lceil B \rceil$ may be thought of as an upper envelope of B. In what follows, we use the properties of $\lfloor\ \rfloor$ and $\lceil\ \rceil$ which may be found in [3]. Most important are the monotony of $\lfloor\ \rfloor$ and $\lceil\ \rceil$ w.r.t. inclusion of fuzzy sets and ordinary sets, respectively; $A = \lceil \lfloor A \rfloor \rceil$ for any $A \in L^U$; $B \subseteq \lfloor \lceil B \rceil \rfloor$ and $\lceil B \rceil = \lceil \lfloor \lceil B \rceil \rfloor \rceil$ for any $B \subseteq U \times L$.

Definition 2. An (ordinary) triadic Galois connection $\langle^{\langle 1\rangle}, ^{\langle 2\rangle}, ^{\langle 3\rangle}\rangle$ between $X_1 \times L$, $X_2 \times L$, $X_3 \times L$ is called *commutative with respect to* $\lfloor\lceil\rceil\rfloor$ iff

$$\big(\lfloor\lceil A_i \rceil\rfloor, \lfloor\lceil A_j \rceil\rfloor\big)^{\langle k\rangle} = \big\lfloor\big\lceil (A_i, A_j)^{\langle k\rangle}\big\rceil\big\rfloor \tag{10}$$

holds for any $\{i, j, k\} = \{1, 2, 3\}$ and any sets $A_1 \subseteq X_1 \times L$, $A_2 \subseteq X_2 \times L$, and $A_3 \subseteq X_3 \times L$.

The following definition shows how triplets of mappings on fuzzy sets in X_is may be defined from triplets of mappings on subsets of $X_i \times L$s and vice versa.

Definition 3. Let $\{i, j, k\} = \{1, 2, 3\}$, $j < k$. For a triadic Galois connection $\langle^{\langle 1\rangle}, ^{\langle 2\rangle}, ^{\langle 3\rangle}\rangle$ between $X_1 \times L$, $X_2 \times L$, $X_3 \times L$, and fuzzy sets $A_j \in L^{X_j}$ and $A_k \in L^{X_k}$, we define mappings $(\langle i\rangle) : L^{X_j} \times L^{X_k} \to L^{X_i}$ by

$$(A_j, A_k)^{(\langle i\rangle)} = \big\lceil\big(\lfloor A_j \rfloor, \lfloor A_k \rfloor\big)^{\langle i\rangle}\big\rceil. \tag{11}$$

Let $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$ be a triadic **L**-Galois connection between X_1, X_2, and X_3. For sets $A_j \subseteq X_j \times L$ and $A_k \subseteq X_k \times L$, we define mappings $\langle(i)\rangle : 2^{X_j \times L} \times 2^{X_k \times L} \to 2^{X_i \times L}$ by

$$(A_j, A_k)^{\langle(i)\rangle} = \big\lfloor\big(\lceil A_j \rceil, \lceil A_k \rceil\big)^{(i)}\big\rfloor \tag{12}$$

The following theorem provides the first way to represent triadic fuzzy Galois connections using ordinary connections.

Theorem 3. *Let $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$ be a triadic $\mathbf{L}_{\{1\}}$-Galois connection between X_1, X_2, and X_3, let $\langle^{\langle 1\rangle}, ^{\langle 2\rangle}, ^{\langle 3\rangle}\rangle$ be a triadic Galois connection between $X_1 \times L$, $X_2 \times L$, and $X_3 \times L$ commutative w.r.t. $\lfloor\lceil\rceil\rfloor$. Then the following hold*:

(a) $\langle^{\langle(1)\rangle}, ^{\langle(2)\rangle}, ^{\langle(3)\rangle}\rangle$ *is a triadic Galois connection commutative with respect to* $\lfloor\lceil\rceil\rfloor$.
(b) $\langle^{(\langle 1\rangle)}, ^{(\langle 2\rangle)}, ^{(\langle 3\rangle)}\rangle$ *is a triadic* $\mathbf{L}_{\{1\}}$*-Galois connection.*
(c) *Definition 3 describes a one-to-one mapping between the set of all triadic $\mathbf{L}_{\{1\}}$-Galois connections between X_1, X_2, and X_3 and the set of all triadic Galois connections between $X_1 \times L$, $X_2 \times L$, and $X_3 \times L$ that are commutative with respect to* $\lfloor\lceil\rceil\rfloor$.

Proof. Let $\{i, j, k\} = \{1, 2, 3\}$.

(a) Let $A_i \subseteq X_i \times L$, $A_j \subseteq X_j \times L$, $A_k \subseteq X_k \times L$, and let $A_k \subseteq (A_i, A_j)^{\langle(k)\rangle}$. Then $\lfloor\lceil A_k\rceil\rfloor \subseteq \lfloor\lceil (A_i, A_j)^{\langle(k)\rangle}\rceil\rfloor$ and since

$$\lfloor\lceil (A_i, A_j)^{\langle(k)\rangle}\rceil\rfloor = \lfloor\lceil\lfloor(\lceil A_i\rceil, \lceil A_j\rceil)^{(k)}\rfloor\rceil\rfloor = \lfloor(\lceil A_i\rceil, \lceil A_j\rceil)^{(k)}\rfloor,$$

we get $\lfloor\lceil A_k\rceil\rfloor \subseteq \lfloor(\lceil A_i\rceil, \lceil A_j\rceil)^{(k)}\rfloor$. Now, observe that ($\rightsquigarrow$ stands for "implies")

$$\begin{aligned}\lfloor\lceil A_k\rceil\rfloor \subseteq \lfloor(\lceil A_i\rceil, \lceil A_j\rceil)^{(k)}\rfloor &\rightsquigarrow S(\lceil A_k\rceil, (\lceil A_i\rceil, \lceil A_j\rceil)^{(k)}) = 1\\ &\rightsquigarrow S(\lceil A_i\rceil, (\lceil A_j\rceil, \lceil A_k\rceil)^{(i)}) = 1\\ &\rightsquigarrow \lfloor\lceil A_i\rceil\rfloor \subseteq \lfloor(\lceil A_j\rceil, \lceil A_k\rceil)^{(i)}\rfloor\\ &\rightsquigarrow A_i \subseteq (A_j, A_k)^{\langle(i)\rangle}.\end{aligned}$$

Clearly, due to symmetry in i, j, k, we established that $\langle^{\langle(1)\rangle}, ^{\langle(2)\rangle}, ^{\langle(3)\rangle}\rangle$ is a triadic Galois connection, i.e. satisfies (4).

To prove commutativity with respect to $\lfloor\lceil\rceil\rfloor$ observe that

$$\begin{aligned}(\lfloor\lceil A_i\rceil\rfloor, \lfloor\lceil A_j\rceil\rfloor)^{\langle(k)\rangle} &= \lfloor(\lceil\lfloor\lceil A_i\rceil\rfloor\rceil, \lceil\lfloor\lceil A_j\rceil\rfloor\rceil)^{(k)}\rfloor\\ &= \lfloor(\lceil A_i\rceil, \lceil A_j\rceil)^{(k)}\rfloor\\ &= (A_1, A_2)^{\langle(k)\rangle}\\ &= \lfloor\lceil(A_1, A_2)^{\langle(k)\rangle}\rceil\rfloor.\end{aligned}$$

(b) Let $A_i \in L^{X_i}$, $A_j \in L^{X_j}$, $A_k \in L^{X_k}$, and let $A_k \subseteq (A_i, A_j)^{(\langle k\rangle)}$. The last assumption, definition of $^{(\langle k\rangle)}$, and commutativity of $^{\langle k\rangle}$ imply

$$\begin{aligned}\lfloor A_k\rfloor \subseteq \lfloor(A_i, A_j)^{(\langle k\rangle)}\rfloor &= \lfloor\lceil(\lfloor A_i\rfloor, \lfloor A_j\rfloor)^{\langle k\rangle}\rceil\rfloor = (\lfloor\lceil\lfloor A_i\rfloor\rceil\rfloor, \lfloor\lceil\lfloor A_j\rfloor\rceil\rfloor)^{\langle k\rangle}\\ &= (\lfloor A_i\rfloor, \lfloor A_j\rfloor)^{\langle k\rangle}.\end{aligned}$$

As $\langle^{\langle 1\rangle}, ^{\langle 2\rangle}, ^{\langle 3\rangle}\rangle$ is a triadic Galois connection, $\lfloor A_k\rfloor \subseteq (\lfloor A_i\rfloor, \lfloor A_j\rfloor)^{\langle k\rangle}$ implies

$$\begin{aligned}\lfloor A_i\rfloor \subseteq (\lfloor A_j\rfloor, \lfloor A_k\rfloor)^{\langle i\rangle} &= (\lfloor\lceil\lfloor A_j\rfloor\rceil\rfloor, \lfloor\lceil\lfloor A_k\rfloor\rceil\rfloor)^{\langle i\rangle}\\ &= \lfloor\lceil(\lfloor A_j\rfloor, \lfloor A_k\rfloor)^{\langle i\rangle}\rceil\rfloor\\ &= \lfloor(A_j, A_k)^{(\langle i\rangle)}\rfloor\end{aligned}$$

and thus $A_i \subseteq (A_j, A_k)^{(\langle i\rangle)}$.

(c) We prove that $\langle^{\langle(\langle 1\rangle)\rangle}, ^{\langle(\langle 2\rangle)\rangle}, ^{\langle(\langle 3\rangle)\rangle}\rangle = \langle^{\langle 1\rangle}, ^{\langle 2\rangle}, ^{\langle 3\rangle}\rangle$. Let $A_i \subseteq X_i \times L$, $A_j \subseteq X_j \times L$. First observe that since $A_i \subseteq \lfloor\lceil A_i\rceil\rfloor$ and $A_j \subseteq \lfloor\lceil A_j\rceil\rfloor$, we have

$$(\lfloor\lceil A_i\rceil\rfloor, \lfloor\lceil A_j\rceil\rfloor)^{\langle k\rangle} \subseteq (A_i, A_j)^{\langle k\rangle} \subseteq \lfloor\lceil(A_i, A_j)^{\langle k\rangle}\rceil\rfloor,$$

whence commutativity with respect to $\lfloor\lceil\rceil\rfloor$ yields

$$(\lfloor\lceil A_i\rceil\rfloor, \lfloor\lceil A_j\rceil\rfloor)^{\langle k\rangle} = (A_i, A_j)^{\langle k\rangle}.$$

Now, by using (11) and (12) we get

$$(A_i, A_j)^{\langle(\langle k\rangle)\rangle} = (\lfloor\lceil A_i\rceil\rfloor, \lfloor\lceil A_j\rceil\rfloor)^{\langle k\rangle} = (A_i, A_j)^{\langle k\rangle}.$$

It remains to prove $\langle^{(\langle(1)\rangle)}, ^{(\langle(2)\rangle)}, ^{(\langle(3)\rangle)}\rangle = \langle^{(1)}, ^{(2)}, ^{(3)}\rangle$. For $A_i \in L^{X_i}$ and $A_j \in L^{X_j}$, (11) and (12) imply

$$(A_i, A_j)^{(\langle(k)\rangle)} = \lceil\lfloor(\lceil\lfloor A_i\rfloor\rceil, \lceil\lfloor A_j\rfloor\rceil)^{(k)}\rfloor\rceil = (A_i, A_j)^{(k)},$$

finishing the proof. □

4.2. Cut-like representation

Recall that for a fuzzy set $A \in L^U$ and a degree $a \in L$, the a-cut ${}^aA$ of A is the ordinary subset of U defined by

$${}^aA = \{u \in U \mid a \leqslant A(u)\}.$$

It is well known that each fuzzy set is uniquely represented by the system of its a-cuts. Depending on the properties of the scale of truth degrees, one may introduce an appropriate notion of a nested system of subsets of U in such a way that nested systems become just the system of a-cuts of fuzzy sets, see e.g. [3].

One may easily verify that straightforward conditions such as $({}^aA_1, {}^aA_2)^{(3)} = {}^a(A_1, A_2)^{(3)}$ do not hold for triadic fuzzy Galois connections. Nevertheless, a cut-like representation of triadic fuzzy Galois connections is possible, as shown in the rest of this section. The representation is based on the following notion.

Definition 4. A system $\{\langle {}^{(1_a)}, {}^{(2_a)}, {}^{(3_a)} \rangle \mid a \in L\}$ of (ordinary) triadic Galois connections is called **L**-*nested* iff for every assignment $\{i, j, k\} = \{1, 2, 3\}$ we have

1. for each $a, b \in L$ such that $a \leqslant b$, and $A_i \in L^{X_i}$, $A_j \in L^{X_j}$ it holds $(A_i, A_j)^{(k_a)} \supseteq (A_i, A_j)^{(k_b)}$
2. for all $x_i \in X_i$, $x_j \in X_j$, $x_k \in X_k$ the set $\{a \in L \mid x_i \in (\{x_j\}, \{x_k\})^{(i_a)}\}$ has a greatest element.

We need the following lemmas.

Lemma 4. *For $\{i, j, k\} = \{1, 2, 3\}$, let $I \in L^{X_1 \times X_2 \times X_3}$ be an* **L**-*relation, $\langle {}^{(1)}, {}^{(2)}, {}^{(3)} \rangle$ be the triadic* **L**-*Galois connection induced by I and for $a \in L$ let $\langle {}^{(1_a)}, {}^{(2_a)}, {}^{(3_a)} \rangle$ be the triadic Galois connections induced by the cuts ${}^aI$. Then*

(a) *for every $A_i \in 2^{X_i}$, $A_j \in 2^{X_j}$, and $a \in L$ we have*

$${}^a(A_i, A_j)^{(k)} = (A_i, A_j)^{(k_a)},$$

(b) *for all fuzzy sets $A_i \in L^{X_i}$, $A_j \in L^{X_j}$, and $b, c \in L$ we have*

$${}^a(A_i, A_j)^{(k)} = \bigcap_{b,c \in L} \left({}^bA_i, {}^cA_j\right)^{(k_{a \otimes b \otimes c})}.$$

Proof. (a) Let $A_i \in 2^{X_i}$, $A_j \in 2^{X_j}$, and $a \in L$. Then for any $x_k \in X_k$ we have

$$x_k \in {}^a(A_i, A_j)^{(k)} \quad \text{iff} \quad \bigwedge_{\substack{x_i \in X_i \\ x_j \in X_j}} A_i(x_i) \otimes A_j(x_j) \to I\{x_i, x_j, x_k\} \geqslant a.$$

Since A_i and A_j are ordinary sets the following holds

$$\bigwedge_{\substack{x_i \in X_i \\ x_j \in X_j}} A_i(x_i) \otimes A_j(x_j) \to I\{x_i, x_j, x_k\} = \bigwedge_{\substack{x_i \in A_i \\ x_j \in A_j}} 1 \otimes 1 \to I\{x_i, x_j, x_k\} = \bigwedge_{\substack{x_i \in A_i \\ x_j \in A_j}} I\{x_i, x_j, x_k\}.$$

To see the claim, observe that $x_k \in (A_i, A_k)^{(k_a)}$ iff $A_i \times A_j \times \{x_k\} \subseteq {}^aI$ iff $a \leqslant I\{x_i, x_j, x_k\}$ for all $x_i \in A_i, x_j \in A_j$ iff

$$a \leqslant \bigwedge_{\substack{x_i \in A_i \\ x_j \in A_j}} I\{x_i, x_j, x_k\}.$$

(b) Let $A_i \in L^{X_i}$, $A_j \in L^{X_j}$. Assume that $x_k \in {}^a(A_i, A_j)^k$. Then

$$\bigwedge_{\substack{x_i \in A_i \\ x_j \in A_j}} A(x_i) \otimes A(x_j) \to I\{x_i, x_j, x_k\} \geqslant a$$

and thus $a \leqslant A(x_i) \otimes A(x_j) \to I\{x_i, x_j, x_k\}$ for all $x_i \in X_i, x_j \in X_j$. By adjunction we get $a \otimes A(x_i) \otimes A(x_j) \leqslant I\{x_i, x_j, x_k\}$.

For arbitrary $b, c \in L$ and any $x_i \in {}^bA_i, x_j \in {}^cA_j$ we have

$$a \otimes b \otimes c \leqslant a \otimes A_i(x_i) \otimes A_j(x_j) \leqslant I\{x_i, x_j, x_k\}.$$

This implies that ${}^bA_i \times {}^cA_j \times \{x_k\} \subseteq {}^{a\otimes b\otimes c}I$ and thus $x_k \in ({}^bA_i, {}^cA_j)^{(k_{a\otimes b\otimes c})}$, which proves ${}^a(A_i, A_j)^k \subseteq \bigcap_{b,c\in L}({}^bA_i, {}^cA_j)^{(k_{a\otimes b\otimes c})}$.

To prove the converse, let $x_k \in ({}^bA_i, {}^cA_j)^{(k_{a\otimes b\otimes c})}$ for all $b, c \in L$. To show $x_k \in {}^a(A_i, A_j)^{(k)}$, we need to prove that for every $x_i \in X_i$ and $x_j \in X_j$ we have $a \leqslant A(x_i) \otimes A(x_j) \to I\{x_i, x_j, x_k\}$, that is, due to adjunction, $a \otimes A(x_i) \otimes A(x_j) \leqslant I\{x_i, x_j, x_k\}$. For $b = A_i(x_i)$ and $c = A_j(x_j)$, the assumption $x_k \in ({}^bA_i, {}^cA_j)^{(k_{a\otimes b\otimes c})}$ implies ${}^bA_i \times {}^cA_j \times \{x_k\} \subseteq {}^{a\otimes b\otimes c}I$ and therefore $a \otimes b \otimes c \leqslant I\{y_i, y_j, x_k\}$ for all $y_i \in {}^bA_i, y_j \in {}^cA_j$. Since $x_i \in {}^bA_i, x_j \in {}^cA_j$, we get $a \otimes A(x_i) \otimes A(x_j) = a \otimes b \otimes c \leqslant I\{x_i, x_j, x_k\}$, the required inequality. □

Lemma 5. *Let $\langle ^{(1)_1}, ^{(2)_1}, ^{(3)_1}\rangle$ and $\langle ^{(1)_2}, ^{(2)_2}, ^{(3)_2}\rangle$ be triadic* **L***-Galois connections, let I_1 and I_2 be the corresponding* **L***-relations between X_1, X_2, and X_3. Then $I_1 \subseteq I_2$ iff for each $\{i, j, k\} = \{1, 2, 3\}$ and every $A_i \in L^{X_i}$, $A_j \in L^{X_j}$ it holds $(A_i, A_j)^{(k)_1} \subseteq (A_i, A_j)^{(k)_2}$.*

Proof. "⇒": The claim follows from Theorem 2, the definition of $^{(k)}$, and antitony of $\to$ in the second argument.

"⇐": For any $x_i \in X_i, x_j \in X_j, x_k \in X_k$ it holds $I_1\{x_i, x_j, x_k\} = 1 \otimes 1 \to I_1\{x_i, x_j, x_k\} = ({}^1/\{x_i\}, {}^1/\{x_j\})^{(k)_1}(x_k) \leqslant ({}^1/\{x_i\}, {}^1/\{x_j\})^{(k)_2}(x_k) = 1 \otimes 1 \to I_2\{x_i, x_j, x_k\} = I_2\{x_i, x_j, x_k\}$. □

The next theorem provides the cut-like representation of triadic fuzzy Galois connections.

Theorem 4. *For a triadic* **L***-Galois connection $\langle ^{(1)}, ^{(2)}, ^{(3)}\rangle$ between X_1, X_2, and X_3, let $\mathcal{C}_{\langle ^{(1)}, ^{(2)}, ^{(3)}\rangle} = \{\langle ^{(1_a)}, ^{(2_a)}, ^{(3_a)}\rangle \mid a \in L\}$.*

For an **L***-nested system $\mathcal{C} = \{\langle ^{(1_a)}, ^{(2_a)}, ^{(3_a)}\rangle \mid a \in L\}$ of triadic Galois connections between X_1, X_2, and X_3, denote by $\langle ^{(1)\mathcal{C}}, ^{(2)\mathcal{C}}, ^{(3)\mathcal{C}}\rangle$ the mappings defined for $\{i, j, k\} = \{1, 2, 3\}$, and $A_i \in L^{X_i}$, $A_j \in L^{X_j}$ by*

$$(A_i, A_j)^{(k)\mathcal{C}}(x_k) = \bigvee\left\{a \,\middle|\, x_k \in \bigcap_{b,c\in L}\left({}^bA_i, {}^cA_j\right)^{(k_{a\otimes b\otimes c})}\right\}.$$

Then

(a) *$\mathcal{C}_{\langle ^{(1)}, ^{(2)}, ^{(3)}\rangle}$ is an* **L***-nested system of triadic Galois connections*;
(b) *$\langle ^{(1)\mathcal{C}}, ^{(2)\mathcal{C}}, ^{(3)\mathcal{C}}\rangle$ is a triadic* **L***-Galois connection*;
(c) *$\langle ^{(1)}, ^{(2)}, ^{(3)}\rangle = \langle ^{(1)\mathcal{C}_{\langle (1),(2),(3)\rangle}}, ^{(2)\mathcal{C}_{\langle (1),(2),(3)\rangle}}, ^{(3)\mathcal{C}_{\langle (1),(2),(3)\rangle}}\rangle$, and $\mathcal{C} = \mathcal{C}_{\langle ^{(1)\mathcal{C}}, ^{(2)\mathcal{C}}, ^{(3)\mathcal{C}}\rangle}$, i.e. the mappings between the sets of all triadic* **L***-Galois connections and all nested systems of triadic Galois connections are mutually inverse bijections.*

Proof. (a) It suffices to check the conditions of Definition 4. To check the first condition, see that if $a \leqslant b$ then ${}^aI \supseteq {}^bI$ and by Lemma 5 it holds $(A_i, A_j)^{(k_a)} \geqslant (A_i, A_j)^{(k_b)}$ for all $A_i \subseteq X_i, A_j \subseteq X_j$. The second condition: Since $x_k \in (\{x_i\}, \{x_j\})^{(k_a)}$ iff $\langle x_i, x_j, x_k\rangle \in {}^aI$ iff $I\{x_i, x_j, x_k\} \geqslant a$, the greatest element a such that $x_k \in (\{x_i\}, \{x_j\})^{(k_a)}$ is clearly $I\{x_i, x_j, x_k\}$.

(b) Let $x_k \in \bigcap_{b,c\in L}({}^bA_i, {}^cA_j)^{(k_{a\otimes b\otimes c})}$. First, define $I \in L^{X_1\times X_2\times X_3}$ by

$$I\{x_i, x_j, x_k\} = \bigvee\{a \mid x_i, x_j, x_k \text{ are related by } I_{\langle (1_a), (2_a), (3_a)\rangle}\},$$

where $I_{\langle (1_a), (2_a), (3_a)\rangle}$s are ordinary relations induced by triadic Galois connections in $\mathcal{C}$ (cf. Theorem 2 for $\mathbf{L} = \mathbf{2}$). The **L**-nestedness of $\mathcal{C}$ ensures that for every $a \in L$ we have $I_{\langle (1_a), (2_a), (3_a)\rangle} = {}^aI$. Indeed, (1) of Definition 4 and Lemma 5 yield that $I_{\langle (1)_a, (2)_a, (3)_a\rangle} \supseteq I_{\langle (1)_b, (2)_b, (3)_b\rangle}$ whenever $a \leqslant b$. By (2) of Definition 4, $\bigvee\{a \mid x_i, x_j, x_k \text{ are related by} \in I_{\langle (1_a), (2_a), (3_a)\rangle}\}$ has a greatest element. It therefore follows from the properties of a-cuts of fuzzy sets that $I_{\langle (1_a), (2_a), (3_a)\rangle} = {}^aI$.

Now, by Lemma 4, $x_k \in \bigcap_{b,c\in L}({}^{b}A_i, {}^{c}A_j)^{(k_{a\otimes b\otimes c})}$ is equivalent to $x_k \in {}^{a}(A_i, A_j)^{(k)}$, where $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$ is the triadic **L**-Galois connection induced by I. Therefore,

$$(A_i, A_j)^{(k)\mathcal{C}}(x_k) = \bigvee\left\{a \,\middle|\, x_k \in \bigcap_{b,c\in L}\left({}^{b}A_i, {}^{c}A_j\right)^{(k_{a\otimes b\otimes c})}\right\}$$

$$= \bigvee\left\{a \,\middle|\, x_k \in {}^{a}(A_i, A_j)^{(k)}\right\} = (A_i, A_j)^{(k)}(x_k),$$

showing that ${}^{(k)\mathcal{C}} = {}^{(k)}$. As a result, $\langle^{(1)\mathcal{C}}, ^{(2)\mathcal{C}}, ^{(3)\mathcal{C}}\rangle$ is an **L**-Galois connection.

(c) Follows easily from the proofs of (a) and (b). □

4.3. Application of the Cartesian representation

Recall that the so-called main (or basic) theorem of triadic concept analysis characterizes concept trilattices by their the structural properties [22]. This theorem was generalized for fuzzy concept trilattices in [6]. As an application of the representation provided in Section 4.1, we present a simple proof of the main theorem of fuzzy concept trilattices by a certain reduction utilizing the theorem for ordinary concept trilattices from [22]. For this purpose, we need to recall the basic notions regarding trilattices which are best seen as triadic extensions of lattices. Details may be found in [22].

Let V be a non-empty set, $\lesssim_1$, $\lesssim_2$, and $\lesssim_3$ be quasiorder relations on V. A tuple $\langle V, \lesssim_1, \lesssim_2, \lesssim_3\rangle$ is called a *triordered set* if and only if the following two conditions hold for every $\{i, j, k\} = \{1, 2, 3\}$:

1. $v \lesssim_i w$ and $v \lesssim_j w$ imply $w \lesssim_k v$ for every $v, w \in V$;
2. $\sim_i \cap \sim_j \cap \sim_k$, where $\sim_l = \lesssim_l \cap \lesssim_l^{-1}$, is the identity relation on V.

Let $V_i, V_k \subseteq V$. An element $v \in V$ is called an *ik-bound* of $\langle V_i, V_k\rangle$ if $v_i \lesssim_i v$ and $v_k \lesssim_k v$ for every $v_i \in V_i$ and $v_k \in V_k$. An ik-bound v is called an *ik-limit* of $\langle V_i, V_k\rangle$ if $u \lesssim_j v$ for every ik-bound u of $\langle V_i, V_k\rangle$. In every triordered set $(V, \lesssim_1, \lesssim_2, \lesssim_3)$ there is at most one ik-limit v of $\langle V_i, V_k\rangle$ satisfying $v \lesssim_k u$ for every ik-limit u of $\langle V_i, V_k\rangle$. If such v exists, we call v an *ik-join* of $\langle V_i, V_k\rangle$ and denote it by $\nabla_{ik}(V_i, V_k)$. A triordered set $(V, \lesssim_1, \lesssim_2, \lesssim_3)$ in which the ik-join exists for all $i \neq k$ $(i, k \in \{1, 2, 3\})$ and all pairs $\langle V_i, V_k\rangle$ of subsets of V is called a *complete trilattice*. Let $\mathbf{V} = \langle V, \lesssim_1, \lesssim_2, \lesssim_3\rangle$ be a triordered set. For $l = 1, 2, 3$, an *order filter* in the quasiordered set $\langle V, \lesssim_l\rangle$ is a subset $F \subseteq V$ for which $v \in F$ whenever $u \in F$ and $u \lesssim_l v$ for every $u, v \in V$. The set of all order filters of $\langle V, \lesssim_l\rangle$ is denoted by $\mathcal{F}_l(\mathbf{V})$. A *principal filter* of $\langle V, \lesssim_l\rangle$ generated by $v \in V$ is the order filter $[v)_l = \{u \in V \mid v \lesssim_l u\}$. A subset $\mathcal{X} \subseteq \mathcal{F}_l(\mathbf{V})$ is called *l-dense* with respect to **V** if each principal filter of $\langle V, \lesssim_l\rangle$ is the intersection of some order filters from $\mathcal{X}$.

For a triadic (fuzzy or ordinary) Galois connection $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$ between X_1, X_2, and X_3 we denote by $\mathcal{T}(X_1, X_2, X_3, \langle^{(1)}, ^{(2)}, ^{(3)}\rangle)$ or just $\mathcal{T}(\langle^{(1)}, ^{(2)}, ^{(3)}\rangle)$ the set of its fixpoints, i.e.

$$\mathcal{T}\left(X_1, X_2, X_3, \left\langle^{(1)}, ^{(2)}, ^{(3)}\right\rangle\right) = \left\{\langle A_1, A_2, A_3\rangle \,\middle|\, (A_i, A_j)^{(k)} = A_k \text{ for every } \{i, j, k\} = \{1, 2, 3\}\right\}.$$

We consider this set equipped with the quasiorders $\lesssim_l$ defined by

$$\langle A_1, A_2, A_3\rangle \lesssim_l \langle B_1, B_2, B_3\rangle \quad \text{if and only if} \quad A_l \subseteq B_l$$

and do not mention these quasiorders explicitly in what follows. Recall that for an ordinary triadic Galois connection $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$ we may consider the corresponding ternary relation I. Therefore, $\mathcal{T}(X_1, X_2, X_3, \langle^{(1)}, ^{(2)}, ^{(3)}\rangle)$ is a complete trilattice because it is just the trilattice $\mathcal{T}(X_1, X_2, X_3, I)$ to which the basic theorem from [22] applies. Using this fact, we obtain:

Theorem 5. *For a triadic $\mathbf{L}_K$-Galois connection $\langle^{(1)}, ^{(2)}, ^{(3)}\rangle$, the structure $\mathcal{T}(X_1, X_2, X_3, \langle^{(1)}, ^{(2)}, ^{(3)}\rangle)$ is a complete trilattice which is isomorphic to $\mathcal{T}(X_1 \times L, X_2 \times L, X_3 \times L, \langle^{\langle(1)\rangle}, ^{\langle(2)\rangle}, ^{\langle(3)\rangle}\rangle)$. Moreover,*

$$\mathcal{T}\left(X_1 \times L, X_2 \times L, X_3 \times L, \left\langle^{\langle(1)\rangle}, ^{\langle(2)\rangle}, ^{\langle(3)\rangle}\right\rangle\right) = \mathcal{T}\left(X_1 \times L, X_2 \times L, X_3 \times L, I^{\times}\right),$$

where

$$\langle\langle x_1, a\rangle, \langle x_2, b\rangle, \langle x_3, c\rangle\rangle \in I^\times \quad \textit{iff} \quad c \leqslant \left(\left\{\frac{a}{x_1}\right\}, \left\{\frac{b}{x_2}\right\}\right)^{(3)}(x_3).$$

Proof. Denote $\mathcal{T}(\langle^{(1)}, {}^{(2)}, {}^{(3)}\rangle)$ and $\mathcal{T}(\langle^{\langle(1)\rangle}, {}^{\langle(2)\rangle}, {}^{\langle(3)\rangle}\rangle)$ by $\mathcal{T}$ and $\mathcal{T}^\times$, respectively. We consider the mappings $h : \mathcal{T} \to \mathcal{T}^\times$ and $g : \mathcal{T}^\times \to \mathcal{T}$ defined by

$$h(\langle A_1, A_2, A_3\rangle) = \langle\lfloor A_1\rfloor, \lfloor A_2\rfloor, \lfloor A_2\rfloor\rangle,$$
$$g(\langle A_1, A_2, A_3\rangle) = \langle\lceil A_1\rceil, \lceil A_2\rceil, \lceil A_2\rceil\rangle.$$

First we show that the mappings are defined correctly. Let $\langle A_1, A_2, A_3\rangle \in \mathcal{T}$. Then by Definition 3 for any assignment $\{i, j, k\} = \{1, 2, 3\}$,

$$(\lfloor A_i\rfloor, \lfloor A_j\rfloor)^{\langle(k)\rangle} = \lfloor(\lceil\lfloor A_i\rfloor\rceil, \lceil\lfloor A_j\rfloor\rceil)^{(k)}\rfloor = \lfloor(A_i, A_j)^{(k)}\rfloor = \lfloor A_k\rfloor.$$

For $\langle A_1, A_2, A_3\rangle \in \mathcal{T}^\times$ we obtain due to Theorem 3 and the definition of commutativity w.r.t. $\lfloor\lceil\,\rceil\rfloor$

$$\begin{aligned}(\lceil A_i\rceil, \lceil A_j\rceil)^{(k)} &= \lceil(\lfloor\lceil A_i\rceil\rfloor, \lfloor\lceil A_j\rceil\rfloor)^{\langle(k)\rangle}\rceil = \lceil\lfloor\lceil(A_i, A_j)^{\langle(k)\rangle}\rceil\rfloor\rceil \\ &= \lceil\lfloor\lceil A_k\rceil\rfloor\rceil = \lceil A_k\rceil.\end{aligned}$$

Clearly, both g and h are order preserving. To show that g and h are mutually inverse, we need to verify that $\langle A_1, A_2, A_3\rangle = g(h(A_1, A_2, A_3))$ for each $\langle A_1, A_2, A_3\rangle \in \mathcal{T}$ and $\langle A_1, A_2, A_3\rangle = h(g(A_1, A_2, A_3))$ for each $\langle A_1, A_2, A_3\rangle \in \mathcal{T}^\times$. $\langle A_1, A_2, A_3\rangle = g(h(A_1, A_2, A_3))$ is trivial because for every fuzzy set A_i we always have $A_i = \lceil\lfloor A_i\rfloor\rceil$.

If $\langle A_1, A_2, A_3\rangle \in \mathcal{T}^\times$, then by definition $(A_j, A_k)^{\langle(i)\rangle} = A_i$. As shown above, $(\lceil A_j\rceil, \lceil A_k\rceil)^{(i)} = \lceil A_i\rceil$. Therefore, we get $(A_j, A_k)^{\langle(i)\rangle} = \lfloor(\lceil A_j\rceil, \lceil A_k\rceil)^{(i)}\rfloor = \lfloor\lceil A_i\rceil\rfloor$. Putting together, we obtain $A_i = \lfloor\lceil A_i\rceil\rfloor$, proving $\langle A_1, A_2, A_3\rangle = h(g(A_1, A_2, A_3))$.

To see that $\mathcal{T}(X_1, X_2, X_3, \langle^{\langle(1)\rangle}, {}^{\langle(2)\rangle}, {}^{\langle(3)\rangle}\rangle) = \mathcal{T}(X_1, X_2, X_3, I^\times)$ it suffices to show that $I^\times$ is precisely the relation $I_{\langle^{\langle(1)\rangle}, {}^{\langle(2)\rangle}, {}^{\langle(3)\rangle}\rangle}$ of Theorem 2 corresponding to $\langle^{\langle(1)\rangle}, {}^{\langle(2)\rangle}, {}^{\langle(3)\rangle}\rangle$. That is, we need to check that $c \leqslant (\{a/x_1\}, \{b/x_2\})^{(3)}(x_3)$ iff $\langle x_3, c\rangle \in (\{\langle x_1, a\rangle\}, \{\langle x_2, b\rangle\})^{\langle(3)\rangle}$ which is indeed true due to Definition 3. □

Note that the condition $c \leqslant (\{a/x_1\}, \{b/x_2\})^{(3)}(x_3)$ in the definition of $I^\times$ may equivalently be replaced by its symmetric counterparts, namely $a \leqslant (\{b/x_2\}, \{c/x_3\})^{(1)}(x_1)$ and $b \leqslant (\{a/x_1\}, \{c/x_3\})^{(2)}(x_2)$. This easily follows from the previous results and definitions.

The following theorem shows an important fact that every fuzzy concept trilattice is isomorphic to a certain concept trilattice.

Theorem 6. *Any* **L***-concept trilattice* $\mathcal{T}(X_1, X_2, X_3, I)$ *is isomorphic to the* (*ordinary*) *concept trilattice* $\mathcal{T}(X_1 \times L, X_2 \times L, X_3 \times L, I^\times)$*, where*

$$\langle\langle x_1, a\rangle, \langle x_2, b\rangle, \langle x_3, c\rangle\rangle \in I^\times \quad \textit{iff} \quad a \otimes b \otimes c \leqslant I(x_1, x_2, x_3).$$

Proof. Let $\langle^{(1)}, {}^{(2)}, {}^{(3)}\rangle$ be the triadic **L**-Galois connection induced by I by Theorem 2. We have $(\{a/x_1\}, \{b/x_2\})^{(3)}(x_3) = \{a/x_1\}(x_1) \otimes \{b/x_2\}(x_2) \to I(x_1, x_2, x_3)$ from which we get by adjointness that

$$a \otimes b \otimes c \leqslant I(x_1, x_2, x_3) \quad \text{iff} \quad c \leqslant \left(\left\{\frac{a}{x_1}\right\}, \left\{\frac{b}{x_2}\right\}\right)^{(3)}(x_3).$$

The claim now follows from Theorem 5. □

For $\{i, j, k\} = \{1, 2, 3\}$ let $C_i \in L^{X_i}$ and $C_k \in L^{X_k}$. We denote by $\mathfrak{b}_{ik}$ the mapping that assigns to every pair of fuzzy sets C_i and C_k the triplet $\langle A_1, A_2, A_3\rangle$ defined by $A_j = (C_i, C_k)^{(j)}$, $A_i = (A_j, C_k)^{(i)}$, and $A_k = (A_i, A_j)^{(k)}$. Note that $\mathfrak{b}_{ik}(C_i, C_k)$ is in fact a triadic fuzzy concept with convenient properties, see [6].

The rest of the chapter is devoted to the basic theorem describing the structure of fuzzy concept trilattices. We prove it by reduction using the basic theorem on ordinary concept trilattices [22] which we include for completeness.

Theorem 7. *Let* $\mathbf{K} = \langle X_1, X_2, X_3, I^{\mathrm{w}}\rangle$ *be an ordinary triadic context. Then*

(1) $\mathcal{T}(I)$ *is a complete trilattice for which the* ik*-joins can be described as follows*:

$$\nabla_{ik}(\mathcal{X}_i, \mathcal{X}_k) = \mathfrak{b}^{\mathrm{w}}_{ik}\Big(\bigcup\big\{A_i \mid \langle A_1, A_2, A_3\rangle \in \mathcal{X}_i\big\}, \bigcup\big\{A_k \mid \langle A_1, A_2, A_3\rangle \in \mathcal{X}_k\big\}\Big),$$

(2) *A complete trilattice* $\mathbf{V} = \langle V, \lesssim_1, \lesssim_2, \lesssim_3\rangle$ *is isomorphic to* $\mathcal{T}(I^{\mathrm{w}})$ *if and only if there exist mappings* $\tilde{\kappa}^w_l : X_l \to \mathcal{F}_l(\mathbf{V})$ $(l = 1, 2, 3)$ *such that*:
(i)$^{\mathrm{w}}$ $\tilde{\kappa}^{\mathrm{w}}_l$ *is* l*-dense with respect to* V
(ii)$^{\mathrm{w}}$ $A_1 \times A_2 \times A_3 \subseteq I^{\mathrm{w}}$ *iff* $\bigcap_{l=1}^{3}\bigcap_{a_l \in A_l}\tilde{\kappa}^w_l(a_i) \neq \emptyset$ *for all* $A_i \in X_i$, $A_j \in X_j$, $A_k \in X_k$, $\{i, j, k\} = \{1, 2, 3\}$.

The mapping $\mathfrak{b}^{\mathrm{w}}_{ik}$ from the previous theorem is a mapping defined in the same way as the mapping $\mathfrak{b}_{ik}$ above for ordinary triadic Galois connection induced by I^{w}. We are now ready to prove the basic theorem in fuzzy setting.

Theorem 8. (1) $\mathcal{T}(X_1, X_2, X_3, \langle^{(1)}, ^{(2)}, ^{(3)}\rangle)$ *is a complete trilattice for which the* ik*-joins are defined for every* $i, k \in \{1, 2, 3\}$, $i \neq k$, *by*:

$$\nabla_{ik}(\mathcal{X}_i, \mathcal{X}_k) = \mathfrak{b}_{ik}\Big(\bigcup\big\{A_i \mid \langle A_1, A_2, A_3\rangle \in \mathcal{X}_i\big\}, \bigcup\big\{A_k \mid \langle A_1, A_2, A_3\rangle \in \mathcal{X}_k\big\}\Big).$$

(2) *A complete trilattice* $\mathbf{V} = \langle V, \lesssim_1, \lesssim_2, \lesssim_3\rangle$ *is isomorphic to a concept trilattice* $\mathcal{T}(X_1, X_2, X_3, \langle^{(1)}, ^{(2)}, ^{(3)}\rangle)$ *if and only if there are mappings* $\tilde{\kappa}_l : X_l \times L \to \mathcal{F}_l(\mathbf{V})$, $l = 1, 2, 3$, *such that*

(i) $\tilde{\kappa}_l(X_l \times L)$ *is* l*-dense with respect to* $\mathbf{V}$;
(ii) $A_i \subseteq (A_j, A_k)^{(i)}$ *iff* $\bigcap_{l=1}^{3}\bigcap_{x_l \in X_l}\tilde{\kappa}_l(x_l, A_l(x_l)) \neq \emptyset$, *for every* $A_i \in L^{X_i}$, $A_j \in L^{X_j}$, $A_k \in L^{X_k}$, $\{i, j, k\} = \{1, 2, 3\}$;
(iii) $a \leqslant b$ *implies* $\tilde{\kappa}_l(x_l, b) \subseteq \tilde{\kappa}_l(x_l, a)$ *for every* $a, b \in L$, $x_l \in X_l$, $l = 1, 2, 3$.

Proof. The fact that $\mathcal{T}(X_1, X_2, X_3, \langle^{(1)}, ^{(2)}, ^{(3)}\rangle)$ is a complete trilattice was proved in Theorem 6. Moreover, due to Theorem 6, the form of the isomorphism in Theorem 6 and its inverse, which is shown in the proof of Theorem 5, and Theorem 7, we have $\nabla_{ik}(\mathcal{X}_i, \mathcal{X}_k) = \langle\lceil B_1\rceil, \lceil B_2\rceil, \lceil B_3\rceil\rangle$, where

$$\langle B_1, B_2, B_3\rangle = \mathfrak{b}^{\mathrm{w}}_{ik}\Big(\bigcup\big\{\lfloor A_i\rfloor \mid \langle A_1, A_2, A_3\rangle \in \mathcal{X}_i\big\}, \bigcup\big\{\lfloor A_k\rfloor \mid \langle A_1, A_2, A_3\rangle \in \mathcal{X}_k\big\}\Big).$$

Now, using $\bigcup_t \lfloor A_t\rfloor = \lfloor\bigcup_t A_t\rfloor$ for any system of fuzzy sets $\{A_t\}$, and Theorem 3 we can transform the previous equality into

$$\big\langle\lceil B_1\rceil, \lceil B_2\rceil, \lceil B_3\rceil\big\rangle = \mathfrak{b}_{ik}\Big(\bigcup\big\{A_i \mid \langle A_1, A_2, A_3\rangle \in \mathcal{X}_i\big\}, \bigcup\big\{A_k \mid \langle A_1, A_2, A_3\rangle \in \mathcal{X}_k\big\}\Big),$$

which concludes the proof.

(2) First observe that if we consider the ordinary triadic context $I^{\times}$ the mappings $\tilde{\kappa}^w_l$ of Theorem 7 are exactly in the same form as mappings $\tilde{\kappa}_l$.

"⇒": Assume that there are mappings $\tilde{\kappa}_i$ such that (i), (ii), (iii) hold. It suffices to show, that for these mappings (i)$^{\mathrm{w}}$ and (ii)$^{\mathrm{w}}$ hold, because in such case V is isomorphic to $\mathcal{T}(I^{\times})$ and by Theorem 6 it is also isomorphic to $\mathcal{T}(\langle^{(1)}, ^{(2)}, ^{(3)}\rangle)$.

It is easy to see that (i)$^{\mathrm{w}}$ holds iff (i) holds. To prove (ii)$^{\mathrm{w}}$ observe that for all $A_i \in L^{X_i}$, $A_j \in L^{X_j}$, $A_k \in L^{X_k}$ we have

$$A_i \subseteq (A_j, A_k)^{(i)} \quad \text{iff} \quad A_1 \otimes A_2 \otimes A_3 \subseteq I \quad \text{iff} \quad \lfloor A_1\rfloor \times \lfloor A_2\rfloor \times \lfloor A_3\rfloor \subseteq I^{\times}.$$

Moreover, by $A(x) = \bigvee\{a \mid (x, a) \in \lfloor A\rfloor\}$ for any fuzzy set A and by (iii) we obtain $\bigcap_{l=1}^{3}\bigcap_{x_l \in X_l}\tilde{\kappa}_l(x_l, A_l(x_l)) \neq \emptyset$ iff $\bigcap_{l=1}^{3}\bigcap_{x_l \in X_l}\tilde{\kappa}_l(x_l, \bigvee\{a \mid (x_l, a) \in \lfloor A_l\rfloor\}) \neq \emptyset$ iff $\bigcap_{l=1}^{3}\bigcap_{x_l \in X_l}\bigcap_{a \leqslant A_l(x_l)}\tilde{\kappa}_l(x_l, a) \neq \emptyset$ iff $\bigcap_{l=1}^{3}\bigcap_{(x_l, a) \in \lfloor A_l\rfloor}\tilde{\kappa}_l(x_l, a) \neq \emptyset$. Hence due to (ii) we have $\lfloor A_1\rfloor \times \lfloor A_2\rfloor \times \lfloor A_3\rfloor \subseteq I^{\times}$ iff $\bigcap_{l=1}^{3}\bigcap_{(x_l, a) \in \lfloor A_l\rfloor}\tilde{\kappa}_l(x_l, a) \neq \emptyset$ which is just (ii)$^{\mathrm{w}}$.

"⇐": Assume that V is isomorphic to $\mathcal{T}(\langle^{(1)}, {}^{(2)}, {}^{(3)}\rangle)$. By Theorem 6 it is also isomorphic to $\mathcal{T}(I^\times)$. Denote by $\varphi^{\mathrm{w}} : \mathcal{T}(I^\times) \to V$ the isomorphism between V and $\mathcal{T}(I^\times)$. Let κ_l be mappings $\kappa_l : (X_l \times L) \to \mathcal{F}_l(\mathcal{T}(I^\times))$ defined by

$$\kappa_l(x_l, a) = \left\{ \langle B_1, B_2, B_3 \rangle \in \mathcal{T}\left(I^\times\right) \;\middle|\; \langle x_l, a \rangle \in B_l \right\}, \tag{13}$$

and define

$$\tilde{\kappa}_l^{\mathrm{w}}(x_l, a) = \varphi^{\mathrm{w}}\left(\kappa_l(x_l, a)\right). \tag{14}$$

Then $\tilde{\kappa}_l^{\mathrm{w}}$ fulfill (i)$^{\mathrm{w}}$ and (ii)$^{\mathrm{w}}$, see [22]. It remains to prove that they comply with (i), (ii), and (iii).

We immediately obtain that (i) holds. For all $\langle A_1, A_2, A_3 \rangle \in \mathcal{T}(I^\times)$ we have that if $(x_l, a) \in A_l$ then $(x_l, b) \in A_l$ for all $b \leqslant a$. By (13) we get that $b \leqslant a$ implies $\tilde{\kappa}_l(x_l, a) \subseteq \tilde{\kappa}_l(x_l, b)$. Hence (iii) holds. To see that (ii) holds, observe that $A_i \subseteq (A_j, A_k)^{(i)}$ iff $A_1 \otimes A_2 \otimes A_3 \in I$ iff $\lfloor A_1 \rfloor \times \lfloor A_2 \rfloor \times \lfloor A_3 \rfloor \subseteq I^\times$ iff $\bigcap_{l=1}^{3} \bigcap_{(x_l, a) \in \lfloor A_l \rfloor} \tilde{\kappa}_l^{\mathrm{w}}(x_l, a) \neq \emptyset$ iff $\bigcap_{l=1}^{3} \bigcap_{x_l \in X_l} \tilde{\kappa}_l^{w}(x_l, A_l(x_l)) \neq \emptyset$. □

5. Conclusions and further issues

We provided an axiomatic characterization of triadic fuzzy Galois connections and two ways to represent them by ordinary triadic connections. These connections appear in data analysis of three-way relational data. Most importantly, their fixpoints are maximal cuboids contained in the data (maximal Cartesian subrelations of the relation representing the data).

The results establish important connections between the ordinary and fuzzy cases that enable us to easily carry over results (theorems, algorithms) for triadic fuzzy data from those for ordinary triadic data. As an example, we presented a simple proof of the basic theorem describing the structure of these fixpoints (so-called fuzzy concept trilattices) by reduction using the proof from the ordinary case.

The following topics are left for future research:

- Identify, formally if possible, the types of results that may be automatically carried over from the ordinary case to fuzzy case.
- Develop other possible types of reduction. Extend the applicability of the presented representation to a wider class of relational methods (see [3] for a general cut-like semantics for predicate fuzzy logic).
- Study the computational efficiency of the representation results with the aim of obtaining algorithms for fuzzy relations from those for ordinary relations.

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