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FAST FACTORIZATION BY SIMILARITY OF FUZZY CONCEPT LATTICES WITH HEDGES

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ABSTRACT

The paper presents results on factorization by similarity of fuzzy concept lattices with hedges. A fuzzy concept lattice is a hierarchically ordered collection of clusters extracted from tabular data. The basic idea of factorization by similarity is to have, instead of a possibly large original fuzzy concept lattice, its factor lattice. The factor lattice contains less clusters than the original concept lattice but, at the same time, represents a reasonable approximation of the original concept lattice and provides us with a granular view on the original concept lattice. The factor lattice results by factorization of the original fuzzy concept lattice by a similarity relation. The similarity relation is specified by a user by means of a single parameter, called a similarity threshold. Smaller similarity thresholds lead to smaller factor lattices, i.e. to more comprehensible but less accurate approximations of the original concept lattice. Therefore, factorization by similarity provides a trade-off between comprehensibility and precision.
We first describe the notion of factorization. Second, we present a way to compute the factor lattice directly from input data, i.e. without the need to compute the possibly large original concept lattice. Third, we provide an illustrative example to demonstrate our method.

Keywords: Data analysis; Formal concept analysis; Concept lattice; Fuzzy attribute; Similarity; Factorization.

1. Introduction and motivation

The present paper is a continuation of our previous papers on formal concept analysis (FCA) of data with fuzzy attributes. In particular, it is a continuation of two ways to reduce the size of fuzzy concept lattices.

The first way, see [2, 6, 7], consists in considering, instead of a possibly large fuzzy concept lattice $\mathcal{B}(X, Y, I)$ associated to the input data $\langle X, Y, I \rangle$, a factor lattice $\mathcal{B}(X, Y, I)/{}^a{\approx}$. Note that $\langle X, Y, I \rangle$, which is sometimes called a formal fuzzy context, consists of a finite set X of objects, a finite set Y of attributes, and a fuzzy relation I between X and Y indicating for each $x \in X$ and $y \in Y$ a degree $I(x, y)$ to which object x has attribute y. In addition to that, $\mathcal{B}(X, Y, I)$ is a fuzzy concept lattice in the sense of [4, 19]. Finally, $\mathcal{B}(X, Y, I)/{}^a{\approx}$ is a factor lattice of $\mathcal{B}(X, Y, I)$ by a compatible tolerance relation ${}^a{\approx}$ on $\mathcal{B}(X, Y, I)$ (see e.g. [13] for the notion of a factor lattice by a tolerance). The relation ${}^a{\approx}$ is, in fact, an a-cut of $\approx$ where a is a user-specified threshold (a particular truth degree, e.g. $a = 0.5$) and $\approx$ is a naturally defined fuzzy equivalence relation on $\mathcal{B}(X, Y, I)$ (see later). In [6, 7], two methods to compute the factor lattice $\mathcal{B}(X, Y, I)/{}^a{\approx}$ directly from data, i.e. without the need to compute the whole $\mathcal{B}(X, Y, I)$, have been described.

The second way, see [8, 9, 11], consists in introducing two additional parameters into FCA of data with fuzzy attributes. These parameters, called hedges, are particular unary functions ^{*X} and ^{*Y} on the scale of truth degrees. The hedges are used to modify the basic operators associated to $\langle X, Y, I \rangle$, i.e., the extent and intent forming operators ${}^{\Uparrow}$ and ${}^{\Downarrow}$. Then, instead of $\mathcal{B}(X, Y, I)$, one considers $\mathcal{B}(X^{*X}, Y^{*Y}, I)$ which is defined to be the set of all fixpoints of the modified operators. The basic idea is that stronger hedges ^{*X} and ^{*Y} lead to smaller $\mathcal{B}(X^{*X}, Y^{*Y}, I)$. An interesting point here is that the approach via hedges subsumes some of the earlier approaches to FCA of data with fuzzy attributes. First, if both ^{*X} and ^{*Y} are identities, $\mathcal{B}(X^{*X}, Y^{*Y}, I)$ coincides with $\mathcal{B}(X, Y, I)$. Second, if one of the hedges is identity and the other one is globalization (see later), the resulting $\mathcal{B}(X^{*X}, Y^{*Y}, I)$ is in fact the fuzzy concept lattice considered independently in [10, 12, 18].

The main aim of this paper is to look to what extent the idea of factorization by similarity given by a user-specified threshold can be applied to fuzzy concept lattices with hedges. We present a method of factorization, a direct way to compute the factor lattice from input data, and illustrative examples. The paper is organized as follows. Section 2 presents preliminaries. Section 3 presents the results. An illustrative example is contained in Section 4. Section 5 presents a summary and an outline of a future research.

2. Preliminaries

2.1. Fuzzy sets and fuzzy logic

In this section, we recall necessary notions from fuzzy sets and fuzzy logic. We refer to [4, 15, 17] for further details. The concept of a fuzzy set generalizes that of an ordinary set in that an element may belong to a fuzzy set in an intermediate truth degree not necessarily being 0 or 1. As a structure of truth degrees, equipped with operations for logical connectives, we use complete residuated lattices, i.e. structures $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, where $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); and $\otimes$ and $\rightarrow$ satisfy so-called adjointness property, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$. Elements a of L are called truth degrees, $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication".

The most common set L of truth degrees is the real interval $[0, 1]$; with $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$. Three most important pairs of fuzzy conjunction and fuzzy implication are: Łukasiewicz, with $a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$; minimum, with $a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else; and product, with $a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ else. Very often, we need a finite chain $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ ($a_0 < \cdots < a_n$); with corresponding Łukasiewicz ($a_k \otimes a_l = a_{\max(k+l-n,0)}$, $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$) or minimum ($a_k \otimes a_l = a_{\min(k,l)}$, $a_k \rightarrow a_l = a_n$ for $a_k \leq a_l$ and $a_k \rightarrow a_l = a_l$ otherwise) connectives. Note that complete residuated lattices are basic structures of truth degrees used in fuzzy logic, see [14, 15]. Residuated lattices cover many particular structures, i.e. sets of truth degrees and fuzzy logical connectives, used in applications of fuzzy logic.

For a complete residuated lattice $\mathbf{L}$, a (truth-stressing) hedge is a unary function * satisfying (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, (iv) $a^{**} = a^*$, for all $a, b \in L$. A hedge * is a (truth function of) logical connective "very true" [16]. The largest hedge, by pointwise ordering, is identity, i.e., $a^* = a$. The least hedge is globalization which is defined by $a^* = 1$ for $a = 1$ and $a^* = 0$ for $a < 1$. Note that for $L = \{0, 1\}$, there exists exactly one complete residuated lattice $\mathbf{L}$, which is in fact the well-known two-element Boolean algebra with $L = \{0, 1\}$, and exactly one hedge, namely, the identity on $\{0, 1\}$.

A fuzzy set A in a universe set U is a mapping $A : U \to L$ with $A(u)$ being interpreted as a degree to which u belongs to A. To make $\mathbf{L}$ explicit, fuzzy sets are also called $\mathbf{L}$-sets. By $\mathbf{L}^U$ or L^U we denote the set of all fuzzy sets in universe U, i.e. $L^U = \{A \mid A \text{ is a mapping of } U \text{ to } L\}$. If $U = \{u_1, \ldots, u_n\}$ then A is denoted by $A = \{ {}^{a_1}\!/u_1, \ldots, {}^{a_n}\!/u_n\}$ meaning that $A(u_i)$ equals a_i. For brevity, we omit elements of U whose membership degree is zero. A binary fuzzy relation I between sets X and Y is a fuzzy set in universe $U = X \times Y$, i.e. a mapping $I : X \times Y \to L$ assigning to each $x \in X$ and $y \in Y$ a degree $I(x, y)$ to which x is related to y.

For $A \in L^U$ and $a \in L$, a set ${}^aA = \{u \in U \mid A(u) \geq a\}$ is called an a-cut of A (the ordinary set of elements from U which belong to A to degree at least a); a fuzzy set $a \to A$ in U defined by $(a \to A)(u) = a \to A(u)$ is called an a-shift of A; a fuzzy set $a \otimes A$ in U defined by $(a \otimes A)(u) = a \otimes A(u)$ is called an a-multiple of A.

Given $A, B \in \mathbf{L}^U$, we define a subsethood degree $S(A, B) = \bigwedge_{u \in U} \big(A(u) \to B(u)\big)$, which generalizes the classical subsethood relation $\subseteq$. $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$ (A is fully contained in B). As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$.

2.2. *Fuzzy concept lattices with hedges*

A data table with fuzzy attributes can be identified with a triplet $\langle X, Y, I\rangle$ where X is a non-empty set of objects (table rows), Y is a non-empty set of attributes (table columns), and I is a (binary) fuzzy relation between X and Y, i.e. $I : X \times Y \to L$. In formal concept analysis, the triplet $\langle X, Y, I\rangle$ is called a formal fuzzy context. For $x \in X$ and $y \in Y$, a degree $I(x, y) \in L$ is interpreted as a degree to which object x has attribute y (table entry corresponding to row x and column y). For $L = \{0, 1\}$, formal fuzzy contexts can be identified in an obvious way with ordinary formal contexts.

Let ^{*_X} and ^{*_Y} be hedges. For fuzzy sets $A \in L^X$ and $B \in L^Y$ we define fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$ to make I explicit) by

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x \in X} (A^{*_X}(x) \to I(x, y)), \quad (1)$$
$$B^{\downarrow}(x) = \textstyle\bigwedge_{y \in Y} (B^{*_Y}(y) \to I(x, y)). \quad (2)$$

Using basic rules of predicate fuzzy logic one can see that $A^{\uparrow}$ is a fuzzy set of all attributes common to all objects for which it is true that they are from A, and $B^{\downarrow}$ is a fuzzy set of all objects sharing all attributes for which it is true that they are from B. The set

$$\mathcal{B}\,(X^{*_X}, Y^{*_Y}, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$$

of all fixpoints of $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ is called a fuzzy concept lattice (with hedges ^{*_X} and ^{*_Y}) associated to $\langle X, Y, I\rangle$; elements $\langle A, B\rangle \in \mathcal{B}\,(X^{*_X}, Y^{*_Y}, I)$ will be called formal concepts of $\langle X, Y, I\rangle$; A and B are called the extent and intent of $\langle A, B\rangle$, respectively. For the sake of brevity, we will sometimes write also $\mathcal{B}\,(X^*, Y^*, I)$ instead of $\mathcal{B}\,(X^{*_X}, Y^{*_Y}, I)$. Under a partial order $\leq$ defined on $\mathcal{B}\,(X^{*_X}, Y^{*_Y}, I)$ by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \ \text{ iff } \ A_1 \subseteq A_2,$$

$\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ happens to be a complete lattice and the following theorem (so-called *main theorem of fuzzy concept lattices with hedges* is the characterization of the structure of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. Note that $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is the basic structure used for formal concept analysis of the data table represented by $\langle X, Y, I\rangle$.

Theorem 1 *$\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is under $\leq$ a complete lattice where the infima and suprema are given by*

$$\bigwedge_{j\in J}\langle A_j, B_j\rangle \;=\; \left\langle (\bigcap_{j\in J} A_j)^{\uparrow\downarrow}, (\bigcup_{j\in J} B_j^{*_Y})^{\downarrow\uparrow}\right\rangle, \tag{3}$$

$$\bigvee_{j\in J}\langle A_j, B_j\rangle \;=\; \left\langle (\bigcup_{j\in J} A_j^{*_X})^{\uparrow\downarrow}, (\bigcap_{j\in J} B_j)^{\downarrow\uparrow}\right\rangle. \tag{4}$$

Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K, \leq\rangle$ is isomorphic to some $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ iff there are mappings $\gamma : X \times *_X(L) \to K$, $\mu : Y \times *_Y(L) \to K$ such that

(i) $\gamma(X \times *_X(L))$ is $\bigwedge$-dense in K, $\mu(Y \times *_Y(L))$ is $\bigvee$-dense in K and

(ii) $\gamma(x, a) \leq \mu(y, b)$ iff $a \otimes b \leq I(x, y)$.

Remark 1 Operators $^{\uparrow}$ and $^{\downarrow}$ were introduced in [11] as a parameterized version of operators $A^{\Uparrow}(y) = \bigwedge_{x\in X}(A(x) \to I(x, y))$ and $B^{\Downarrow}(x) = \bigwedge_{y\in Y}(B(y) \to I(x, y))$ which were studied before, see [1, 5, 19]. Clearly, if both ^{*_X} are ^{*_Y} are identities on L, $^{\uparrow}$ and $^{\downarrow}$ coincide with $^{\Uparrow}$ and $^{\Downarrow}$, respectively. If ^{*_X} or ^{*_Y} is the identity on L, we omit ^{*_X} or ^{*_Y} in $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, e.g. we write just $\mathcal{B}(X^{*_X}, Y, I)$ if $^{*_Y} = \mathrm{id}_L$.

3. Factorization of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ by similarity

3.1. The case of $\mathcal{B}(X, Y, I)$

We need to recall the parameterized method of factorization introduced in [2]. Given $\langle X, Y, I\rangle$, introduce a binary fuzzy relation $\approx_{\mathrm{Ext}}$ on $\mathcal{B}(X, Y, I)$ by

$$(\langle A_1, B_1\rangle \approx_{\mathrm{Ext}} \langle A_2, B_2\rangle) \;=\; \bigwedge_{x\in X}(A_1(x) \leftrightarrow A_2(x)) \tag{5}$$

for $\langle A_i, B_i\rangle \in \mathcal{B}(X, Y, I)$, $i = 1, 2$. Here, $\leftrightarrow$ is a so-called biresiduum (i.e., a truth function of equivalence connective) defined by

$$a \leftrightarrow b = (a \to b) \wedge (b \to a).$$

$(\langle A_1, B_1\rangle \approx_{\mathrm{Ext}} \langle A_2, B_2\rangle)$, called a degree of similarity of $\langle A_1, B_1\rangle$ and $\langle A_2, B_2\rangle$, is just the truth degree of "for each object x: x is covered by A_1 iff x is covered by A_2". One can also consider a fuzzy relation $\approx_{\mathrm{Int}}$ defined by

$$(\langle A_1, B_1\rangle \approx_{\mathrm{Int}} \langle A_2, B_2\rangle) \;=\; \bigwedge_{y\in Y}(B_1(y) \leftrightarrow B_2(y)). \tag{6}$$

It can be shown [4] that measuring similarity of formal concepts via intents B_i coincides with measuring similarity via extents A_i, i.e. $\approx_{\mathrm{Ext}}$ coincides with $\approx_{\mathrm{Int}}$, corresponding naturally to the duality of extent/intent view. As a result, we write also just $\approx$ instead of $\approx_{\mathrm{Ext}}$ and $\approx_{\mathrm{Int}}$. Note also that $\approx$ is a fuzzy equivalence relation on $\mathcal{B}(X,Y,I)$.

Given a truth degree $a \in L$ (a similarity threshold specified by a user), consider the thresholded relation ${}^a\!\approx$ on $\mathcal{B}(X,Y,I)$ defined by

$$\langle\langle A_1,B_1\rangle,\langle A_2,B_2\rangle\rangle \in {}^a\!\approx \quad \text{iff} \quad (\langle A_1,B_1\rangle \approx \langle A_2,B_2\rangle) \geq a.$$

That is, ${}^a\!\approx$ is the relation "being similar to degree at least a" and we thereby call it simply similarity (relation). ${}^a\!\approx$ is a reflexive and symmetric binary relation (i.e., a tolerance relation) on $\mathcal{B}(X,Y,I)$. However, ${}^a\!\approx$ need not be transitive (it is transitive if, for instance, $a \otimes b = a \wedge b$ holds true in $\mathbf{L}$). ${}^a\!\approx$ is said to be compatible if it is preserved under arbitrary suprema and infima in $\mathcal{B}(X,Y,I)$, i.e. if $\langle c_j, c'_j\rangle \in {}^a\!\approx$ for $j \in J$ implies both $\langle \bigwedge_{j\in J} c_j, \bigwedge_{j\in J} c'_j\rangle \in {}^a\!\approx$ and $\langle \bigvee_{j\in J} c_j, \bigvee_{j\in J} c'_j\rangle \in {}^a\!\approx$ for any $c_j, c'_j \in \mathcal{B}(X,Y,I)$, $j \in J$. We call $\approx$ compatible if ${}^a\!\approx$ is compatible for each $a \in L$.

Call a subset B of $\mathcal{B}(X,Y,I)$ an ${}^a\!\approx$-block if it is a maximal subset of $\mathcal{B}(X,Y,I)$ such that any two formal concepts from B are similar to degree at least a, i.e., for any $c_1, c_2 \in B$ we have $\langle c_1, c_2\rangle \in {}^a\!\approx$. Note that the notion of an ${}^a\!\approx$-block generalizes that of an equivalence class: if ${}^a\!\approx$ is an equivalence relation then ${}^a\!\approx$-blocks are exactly the equivalence classes of ${}^a\!\approx$. Denote by $\mathcal{B}(X,Y,I)/{}^a\!\approx$ the collection of all ${}^a\!\approx$-blocks. It follows from the results on tolerances on complete lattices [13] that if ${}^a\!\approx$ is compatible, then ${}^a\!\approx$-blocks are special intervals in the concept lattice $\mathcal{B}(X,Y,I)$. For a formal concept $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$, denote by $\langle A,B\rangle_a$ and $\langle A,B\rangle^a$ the infimum and the supremum of the set of all formal concepts which are similar to $\langle A,B\rangle$ to degree at least a, that is,

$$\langle A,B\rangle_a = \bigwedge\{\langle A',B'\rangle \mid \langle\langle A,B\rangle,\langle A',B'\rangle\rangle \in {}^a\!\approx\}, \tag{7}$$
$$\langle A,B\rangle^a = \bigvee\{\langle A',B'\rangle \mid \langle\langle A,B\rangle,\langle A',B'\rangle\rangle \in {}^a\!\approx\}. \tag{8}$$

Operators $\ldots_a$ and $\ldots^a$ are important in description of ${}^a\!\approx$-blocks [13]:

Lemma 1 *${}^a\!\approx$-blocks are exactly intervals of $\mathcal{B}(X,Y,I)$ of the form $[\langle A,B\rangle_a, (\langle A,B\rangle_a)^a]$, i.e.,*

$$\mathcal{B}(X,Y,I)/{}^a\!\approx \;=\; \{[\langle A,B\rangle_a, (\langle A,B\rangle_a)^a] \mid \langle A,B\rangle \in \mathcal{B}(X,Y,I)\}.$$

Note that an interval with lower bound $\langle A_1,B_1\rangle$ and upper bound $\langle A_2,B_2\rangle$ is the subset $[\langle A_1,B_1\rangle,\langle A_2,B_2\rangle] = \{\langle A,B\rangle \in \mathcal{B}(X,Y,I) \mid \langle A_1,B_1\rangle \leq \langle A,B\rangle \leq \langle A_2,B_2\rangle\}$.

Now, define a partial order $\preceq$ on blocks of $\mathcal{B}(X,Y,I)/{}^a\!\approx$ by

$$[c_1,c_2] \preceq [d_1,d_2] \quad \text{iff} \quad c_1 \leq d_1 \quad (\text{iff} \quad c_2 \leq d_2) \tag{9}$$

for any $[c_1,c_2],[d_1,d_2] \in \mathcal{B}(X,Y,I)/{}^a\!\approx$. Then we have [2]:

Theorem 2 *$\mathcal{B}(X,Y,I)/{}^a\!\approx$ equipped with $\preceq$ is a partially ordered set which is a complete lattice, the so-called factor lattice of $\mathcal{B}(X,Y,I)$ by similarity $\approx$ and threshold a.*

Elements of $\mathcal{B}(X,Y,I)/^a{\approx}$ can be seen as similarity-based granules of formal concepts/clusters from $\mathcal{B}(X,Y,I)$. $\mathcal{B}(X,Y,I)/^a{\approx}$ thus provides a granular view on the possibly large $\mathcal{B}(X,Y,I)$. For further details and properties of $\mathcal{B}(X,Y,I)/^a{\approx}$ we refer to [2].

3.2. The case of $\mathcal{B}(X^{*_X},Y^{*_Y},I)$

We now turn our attention to factorization by similarity of $\mathcal{B}(X^{*_X},Y^{*_Y},I)$. Note first that one cannot directly apply the approach which works for $\mathcal{B}(X,Y,I)$. Namely, due to employing hedges, some important properties are no longer available (for instance, the composite mappings ${}^{\uparrow\downarrow}$ and ${}^{\downarrow\uparrow}$ are not fuzzy closure operators in general). Nevertheless, we propose a feasible approach to factorization of concept lattices with hedges.

Remark 2 With $\approx_{\mathrm{Ext}}$ and $\approx_{\mathrm{Int}}$ defined by (5) and (6), compatibility is lost in general. This is still true even if one of the hedges is identity. Consider e.g. ${}^{*_X} = \mathrm{id}_L$. Then, $\approx_{\mathrm{Ext}}$ is compatible with $\bigwedge$. Indeed, one can easily verify that if A_j are extents, i.e. $A_j = A_j^{\Uparrow\downarrow}$, see [11], then $\bigwedge_{j\in J} A_j = \bigcap_{j\in J} A_j$, which immediately yields compatibility of $\approx_{\mathrm{Ext}}$ with $\bigwedge$. However, $\approx_{\mathrm{Ext}}$ need not be compatible with $\bigvee$ as shown by the following example (a dual situation applies to $\approx_{\mathrm{Int}}$).

Take a Łukasiewicz structure on $[0,1]$, let ^{*_X} be identity and ^{*_Y} be globalization, $a = 0.5$, and consider the following data table and fuzzy sets.

I	y_1	y_2	y_3
x_1	1	0.5	0
x_2	0	0	1
x_3	0.5	1	0

$$A_1 = \{\,{}^{0.5}/x_1,\ {}^{0.5}/x_3\}\qquad B_1 = \{\,{}^{1}/y_1,\ {}^{1}/y_2,\ {}^{0.5}/y_3\}$$
$$A_2 = \{\,{}^{0.5}/x_1,\ {}^{1}/x_3\}\qquad B_2 = \{\,{}^{0.5}/y_1,\ {}^{1}/y_2\}$$
$$A_3 = \{\,{}^{1}/x_1,\ {}^{0.5}/x_3\}\qquad B_3 = \{\,{}^{1}/y_1,\ {}^{0.5}/y_2\}$$

One can check that $\langle A_i, B_i\rangle \in \mathcal{B}(X^{*_X},Y^{*_Y},I)$, $i = 1,2,3$, i.e. the fuzzy sets describe concepts, and $\langle\langle A_1,B_1\rangle,\langle A_2,B_2\rangle\rangle \in {}^a{\approx_{\mathrm{Ext}}}$, $\langle\langle A_1,B_1\rangle,\langle A_3,B_3\rangle\rangle \in {}^a{\approx_{\mathrm{Ext}}}$. Then $\approx_{\mathrm{Ext}}$ is compatible with $\bigwedge$, but it is not compatible with $\bigvee$: $\langle\langle A_1,B_1\rangle \wedge \langle A_1,B_1\rangle, \langle A_2,B_2\rangle\wedge\langle A_3,B_3\rangle\rangle \in {}^a{\approx_{\mathrm{Ext}}}$, but $\langle\langle A_1,B_1\rangle\vee\langle A_1,B_1\rangle, \langle A_2,B_2\rangle\vee\langle A_3,B_3\rangle\rangle \notin {}^a{\approx_{\mathrm{Ext}}}$.

Our approach to factorization by similarity, which works for arbitrary hedges ^{*_X} and ^{*_Y}, consists in considering fixpoints of ^{*_X} and ^{*_Y}. We show that if a similarity threshold a is common fixpoint of ^{*_X} and ^{*_Y}, we can introduce a notion of a factor lattice $\mathcal{B}(X^{*_X},Y^{*_Y},I)/^a{\approx}$. Furthermore, we present a method to compute $\mathcal{B}(X^{*_X},Y^{*_Y},I)/^a{\approx}$ directly from input data without the need to compute the possibly large $\mathcal{B}(X^{*_X},Y^{*_Y},I)$. Our method is inspired by and generalizes the method presented in [2] and [6]. The method of [6] becomes a particular case of the method presented in this paper with ^{*_X} and ^{*_Y} being identity mappings on L. Throughout the section, $\approx_{\mathrm{Ext}}$ and $\approx_{\mathrm{Int}}$ denote the fuzzy relations on $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ defined by (5) and (6).

Recall that a fixpoint of a mapping $f : L \to L$ is any $a \in L$ such that $f(a) = a$. We are interested in the set $\mathrm{fix}({}^{*_X},{}^{*_Y})$ of common fixpoints of ^{*_X} and ^{*_Y}, i.e. in a set

$$\mathrm{fix}({}^{*_X},{}^{*_Y}) \;=\; \{a \in L \mid a^{*_X} = a \text{ and } a^{*_Y} = a\}.$$

We need the following auxiliary assertions.

Lemma 2 *If* $a \in \mathrm{fix}({}^{*_X}, {}^{*_Y})$ *then for any* $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ *the following holds:*

$$a \leq (\langle A_1, B_1\rangle \approx_{\mathrm{Ext}} \langle A_2, B_2\rangle) \textit{ implies } a \leq (\langle A_1, B_1\rangle \approx_{\mathrm{Ext}} \langle A_2, B_2\rangle)^{*_X},$$
$$a \leq (\langle A_1, B_1\rangle \approx_{\mathrm{Int}} \langle A_2, B_2\rangle) \textit{ implies } a \leq (\langle A_1, B_1\rangle \approx_{\mathrm{Int}} \langle A_2, B_2\rangle)^{*_Y}.$$

Proof. Follows from the fact that if a is a fixpoint of a hedge $*$ then $a \leq b$ implies $a = a^* \leq b^*$. □

With $(A_1 \approx A_2) = \bigwedge_{x\in X}(A_1(x) \leftrightarrow A_2(x))$ and $(B_1 \approx B_2) = \bigwedge_{y\in Y}(B_1(y) \leftrightarrow B_2(y))$, we have

Lemma 3 *Let* $A_1, A_2 \in L^X$, $B_1, B_2 \in L^Y$. *Then* $(A_1 \approx A_2)^{*_X} \leq (A_1^{*_X} \approx A_2^{*_X})$ *and* $(B_1 \approx B_2)^{*_Y} \leq (B_1^{*_Y} \approx B_2^{*_Y})$.

Proof. Denote ^{*_X} by *. We have $(A_1 \approx A_2)^* \leq (A_1^* \approx A_2^*) = \bigwedge_{x\in X}(A_1(x)^* \leftrightarrow A_2(x)^*)$ iff $(A_1 \approx A_2)^* \leq (A_1(x)^* \leftrightarrow A_2(x)^*)$ for each $x \in X$. Since $(A_1 \approx A_2)^* \leq (A_1(x) \leftrightarrow A_2(x))^*$ for all $x \in X$ it suffices to show $(A_1(x) \leftrightarrow A_2(x))^* \leq (A_1(x)^* \leftrightarrow A_2(x)^*)$, which is true. Indeed,

$$\begin{aligned} & (A_1(x) \leftrightarrow A_2(x))^* \leq (A_1(x) \rightarrow A_2(x))^* \wedge (A_2(x) \rightarrow A_1(x))^* \leq \\ \leq\ & (A_1(x)^* \rightarrow A_2(x)^*) \wedge (A_2(x)^* \rightarrow A_1(x)^*) = (A_1(x)^* \leftrightarrow A_2(x)^*). \end{aligned}$$

The proof for B_i's is similar. □

Lemma 4 *For* $A_1, A_2 \in L^X$ *and* $B_1, B_2 \in L^Y$ *we have* $(A_1 \approx A_2)^{*_X} \leq (A_1^{\uparrow} \approx A_2^{\uparrow})$ *and* $(B_1 \approx B_2)^{*_Y} \leq (B_1^{\downarrow} \approx B_2^{\downarrow})$.

Proof. Follows directly from Lemma 3, and from $(A_1 \approx A_2) \leq (A_1^{\Uparrow} \approx A_2^{\Uparrow})$, and $(B_1 \approx B_2) \leq (B_1^{\Downarrow} \approx B_2^{\Downarrow})$ [2]. □

As shown in [2], if both ^{*_X} and ^{*_Y} are identities, then $\approx_{\mathrm{Ext}}$ coincides with $\approx_{\mathrm{Int}}$. With arbitrary hedges ^{*_X} and ^{*_Y}, the situation is different. Namely, $\approx_{\mathrm{Ext}}$ may differ from $\approx_{\mathrm{Int}}$. Nevertheless, as shown by the following theorem, if a is a common fixpoint of ^{*_X} and ^{*_Y}, ${}^a{\approx_{\mathrm{Ext}}}$ equals ${}^a{\approx_{\mathrm{Int}}}$.

Lemma 5 *Let* $a \in \mathrm{fix}({}^{*_X}, {}^{*_Y})$. *Then* ${}^a{\approx_{\mathrm{Ext}}} = {}^a{\approx_{\mathrm{Int}}}$.

Proof. Let $\langle\langle A_1, B_1\rangle, \langle A_2, B_2\rangle\rangle \in {}^a{\approx_{\mathrm{Ext}}}$, i.e. $a \leq (\langle A_1, B_1\rangle \approx_{\mathrm{Ext}} \langle A_2, B_2\rangle)$. Using Lemma 2 and Lemma 4, we have

$$\begin{aligned} a\ &\leq\ (\langle A_1, B_1\rangle \approx_{\mathrm{Ext}} \langle A_2, B_2\rangle)^{*_X} = (A_1 \approx A_2)^{*_X} \leq (A_1^{*_X} \approx A_2^{*_X}) \leq \\ &\leq\ (A_1^{\uparrow} \approx A_2^{\uparrow}) = (B_1 \approx B_2) = (\langle A_1, B_1\rangle \approx_{\mathrm{Int}} \langle A_2, B_2\rangle), \end{aligned}$$

i.e. $\langle\langle A_1, B_1\rangle, \langle A_2, B_2\rangle\rangle \in {}^a{\approx_{\mathrm{Int}}}$. In a similar way, if $\langle\langle A_1, B_1\rangle, \langle A_2, B_2\rangle\rangle \in {}^a{\approx_{\mathrm{Int}}}$ then $\langle\langle A_1, B_1\rangle, \langle A_2, B_2\rangle\rangle \in {}^a{\approx_{\mathrm{Ext}}}$, proving ${}^a{\approx_{\mathrm{Ext}}} = {}^a{\approx_{\mathrm{Int}}}$. □

We can therefore write ${}^a{\approx}$ instead of ${}^a{\approx_{\mathrm{Ext}}}$ and ${}^a{\approx_{\mathrm{Int}}}$. With the above notation, the following theorem shows a way to factorize $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$.

Lemma 6 *Let $a \in \mathrm{fix}({}^{*_X}, {}^{*_Y})$. Then ${}^a\!\approx$ is a compatible tolerance on $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. That is, for any $c_j, c'_j \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, $j \in J$, if $\langle c_j, c'_j\rangle \in {}^a\!\approx$ then $\langle \bigwedge_{j\in J} c_j, \bigwedge_{j\in J} c'_j\rangle \in {}^a\!\approx$ and $\langle \bigvee_{j\in J} c_j, \bigvee_{j\in J} c'_j\rangle \in {}^a\!\approx$.*

Proof. Follow the proof of compatibility of ${}^a\!\approx$ in [2] and apply Lemma 2 and Lemma 4 twice at the end of that proof. □

Denote by $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^a\!\approx$ the collection of all ${}^a\!\approx$-blocks, i.e. maximal subsets B of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^a\!\approx$ such that for any $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in B$ we have $\langle\langle A_1, B_1\rangle, \langle A_2, B_2\rangle\rangle \in {}^a\!\approx$. Then we have

Theorem 3 (factor lattice by similarity) *Let $a \in \mathrm{fix}({}^{*_X}, {}^{*_Y})$. Then*

$$\mathcal{B}(X^{*_X}, Y^{*_Y}, I) = \{[\langle A, B\rangle_a, (\langle A, B\rangle_a)^a] \mid \langle A, B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)\},$$

where $\langle A, B\rangle_a$ and $\langle A, B\rangle^a$ are given by (7) *and* (8). *Moreover, $\preceq$ defined by* (9) *makes $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ a complete lattice.*

Proof. The assertion is a direct consequence of Lemma 6 and general results on factorization of complete lattices by compatible tolerance relations. □

The factor lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^a\!\approx$ from Theorem 3 is called a *factor lattice of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ by similarity $\approx$ and threshold a.*

Therefore, if a is a fixpoint of both ^{*_X} and ^{*_Y}, we can consider the factor lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^a\!\approx$ of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. We now turn our attention to the problem of how to compute the factor lattice. One way is to follow the definition and to split the computation of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^a\!\approx$ into two steps: (1) compute the possibly large fuzzy concept lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ and (2) compute the ${}^a\!\approx$-blocks, i.e. the elements of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^a\!\approx$. Although there are efficient algorithms for both (1) and (2), computing $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^a\!\approx$ this way is time demanding. In what follows, we present a way to obtain $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^a\!\approx$ directly, without the need to compute $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ first and then to compute the blocks of ${}^a\!\approx$. Basically, we follow and appropriately modify the method proposed in [6]. We need the following lemma.

Lemma 7 *Let $a \in \mathrm{fix}({}^{*_X}, {}^{*_Y})$. For $\langle A, B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, we have*

(a) $\langle A, B\rangle_a = \langle (a \otimes A)^{\uparrow\downarrow}, (a \to B)^{\downarrow\uparrow}\rangle$,

(b) $\langle A, B\rangle^a = \langle (a \to A)^{\uparrow\downarrow}, (a \otimes B)^{\downarrow\uparrow}\rangle$.

Proof. Due to duality we verify (a) only. We need to prove that $(a \otimes A)^{\uparrow\downarrow}$ is an extent of the least formal concept from $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ which is similar to $\langle A, B\rangle$ to degree at least a and that $(a \to B)^{\downarrow\uparrow}$ is the corresponding intent. That is, we need to prove that (1) $(a \otimes A)^{\uparrow\downarrow}$ is an extent of a formal concept $\langle (a \otimes A)^{\uparrow\downarrow}, D\rangle$ which is similar to $\langle A, B\rangle$ to degree at least a, (2) if $\langle C, F\rangle$ is a formal concept similar to $\langle A, B\rangle$ to degree at least a then $\langle (a \otimes A)^{\uparrow\downarrow}, D\rangle \leq \langle C, F\rangle$, (3) $(a \to B)^{\downarrow\uparrow}$ is an intent of a formal concept $\langle E, (a \to B)^{\downarrow\uparrow}\rangle$ which is similar to $\langle A, B\rangle$ to degree at least a, (4) if $\langle C, F\rangle$ is a formal concept similar to $\langle A, B\rangle$ to degree at least a then $\langle E, (a \to B)^{\downarrow\uparrow}\rangle \leq \langle C, F\rangle$. Clearly, (1)–(4) imply $E = (a \otimes A)^{\uparrow\downarrow}$ and $D = (a \to B)^{\downarrow\uparrow}$. We thus have to verify (1)–(4).

(1): One can easily check that $a \leq ((a \otimes A) \approx A)$. Since a is a fixpoint of ^{*X}, Lemma 4 yields $a = a^{*X} \leq ((a \otimes A) \approx A)^{*X} \leq ((a \otimes A)^{\uparrow} \approx A^{\uparrow})$. Taking into account that a is a fixpoint of ^{*Y} we further obtain $a \leq ((a \otimes A)^{\uparrow} \approx A^{\uparrow})^{*Y} \leq ((a \otimes A)^{\uparrow\downarrow} \approx A^{\uparrow\downarrow}) = ((a \otimes A)^{\uparrow\downarrow} \approx A)$ since A is an extent.

(2): If $a \leq (A \approx C)$ then using adjointness, we get $a \otimes A \subseteq C$ from which we have $(a \otimes A)^{\uparrow\downarrow} \subseteq C^{\uparrow\downarrow} = C$, proving (2).

(3): Since $a \leq ((a \to B) \approx B)$, we have $a = a^{*Y} \leq ((a \to B) \approx B)^{*Y} \leq ((a \to B)^{\downarrow} \approx B^{\downarrow})$ and then $a \leq ((a \to B)^{\downarrow} \approx B^{\downarrow})^{*X} \leq ((a \to B)^{\downarrow\uparrow} \approx B^{\downarrow\uparrow}) = ((a \to B)^{\downarrow\uparrow} \approx B)$ since B is an intent.

(4): We need to show $F \subseteq (a \to B)^{\downarrow\uparrow}$. Since $a \leq (B \approx F)$, applying adjointness twice gives $F \subseteq a \to B$ from which we have $F = F^{\downarrow\uparrow} \subseteq (a \to B)^{\downarrow\uparrow}$. The proof is complete. □

Remark 3 Thus we have $(\langle A, B\rangle_a)^a = \left\langle (a \to (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow}, (a \otimes (a \to B)^{\uparrow\downarrow})^{\downarrow\uparrow}\right\rangle$.

Recall now that an $\mathbf{L}_{\{1\}}$-closure operator in U is a mapping $C : \mathbf{L}^U \to \mathbf{L}^U$ satisfying

(i) $A \subseteq C(A)$,

(ii) $A_1 \subseteq A_2$ implies $C(A_1) \subseteq C(A_2)$,

(iii) $C(A) = C(C(A))$.

An important role in our investigation is played by operators $C : \mathbf{L}^U \to \mathbf{L}^U$ satisfying the above conditions (ii) and (iii), and, in addition to that, condition

(i*) $A^* \subseteq C(A)$

where * is a hedge. Note that (i*) is weaker than (i). In what follows, we call an operator C satisfying conditions (i*), (ii), and (iii), an *almost* $\mathbf{L}_{\{1\}}$*-closure operator in* U. Recall that a fixpoint of C is any fuzzy set A in U such that $A = C(A)$ and denote by $\mathrm{fix}(C)$ the set of all fixpoints of C, i.e.

$$\mathrm{fix}(C) = \{A \in \mathbf{L}^U \mid C(A) = A\}.$$

Our way to obtain the factor lattice $\mathcal{B}(X^{*X}, Y^{*Y}, I)/{}^a\!\approx$ directly from input data $\langle X, Y, I\rangle$ is based on the next theorem. Denoting the set of all extents of suprema of ${}^a\!\approx$-blocks by $\mathrm{ESB}(a)$, i.e.

$$\mathrm{ESB}(a) = \{A \in L^X \mid \langle A, B\rangle \in \mathcal{B}(X^{*X}, Y^{*Y}, I) \text{ and } [\langle A, B\rangle_a, \langle A, B\rangle] \in \mathcal{B}(X, Y, I)/{}^a\!\approx\},$$

we have

Theorem 4 *Let* $a \in \mathrm{fix}({}^{*X}, {}^{*Y})$. *Then the mapping* $C_a : A \mapsto (a \to (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow}$ *is an almost* $\mathbf{L}_{\{1\}}$*-closure operator in* X *such that*

$$\mathrm{fix}(C_a) = \mathrm{ESB}(a),$$

i.e., the fixpoints of C_a *are just the extents of suprema of* ${}^a\!\approx$*-blocks.*

Proof. First, we verify that C_a is an almost $\mathbf{L}_{\{1\}}$-closure operator. To verify $A^{*_X} \subseteq C_a(A)$ it is sufficient to check $A^{*_X} \subseteq (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{*_X}$. Namely, $A^{*_X} \subseteq (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{*_X}$ together with $(a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{*_X} \subseteq (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow}$ yields $A^{*_X} \subseteq (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow} = C_a(A)$, as desired. Therefore, to verify $A^{*_X} \subseteq C_a(A)$, it remains to check $A^{*_X} \subseteq (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{*_X}$: Since $c^{*_X} \leq d^{*_X}$ iff $c^{*_X} \leq d$, it is sufficient to show $A^{*_X} \subseteq a \rightarrow (a \otimes A)^{\uparrow\downarrow}$ which is equivalent to $a \otimes A^{*_X} \subseteq (a \otimes A)^{\uparrow\downarrow}$. Since a is a fixpoint of ^{*_X}, i.e. $a^{*_X} = a$, and since $c^{*_X} \otimes d^{*_X} \leq (c \otimes d)^{*_X}$ (for any $c, d \in L$), we have $a \otimes A^{*_X} = a^{*_X} \otimes A^{*_X} \subseteq (a \otimes A)^{*_X} \subseteq (a \otimes A)^{\uparrow\downarrow}$. We showed $A^{*_X} \subseteq C_a(A)$.
We verify that $A_1 \subseteq A_2$ implies $C_a(A_1) \subseteq C_a(A_2)$: Since for $D_1, D_2 \in L^X$, $D_1 \subseteq D_2$ implies $(a \otimes D_1) \subseteq (a \otimes D_2)$, $D_1 \subseteq D_2$ implies $(a \rightarrow D_1) \subseteq (a \rightarrow D_2)$, and $D_1 \subseteq D_2$ implies $D_1^{\uparrow\downarrow} \subseteq D_2^{\uparrow\downarrow}$, one can easily see that $A_1 \subseteq A_2$ implies $C_a(A_1) \subseteq C_a(A_2)$.
To verify $C_a(A) = C_a(C_a(A))$, we will need the following claim.

Claim 1 *If a is a fixpoint of ^{*_X} and ^{*_Y} then* (1) $a \otimes (a \rightarrow A)^{\uparrow\downarrow} \subseteq A^{\uparrow\downarrow}$; (2) $A^{\uparrow\downarrow} \subseteq a \rightarrow (a \otimes A)^{\uparrow\downarrow}$ *for any $A \in L^X$.*

Proof of Claim 1. Using adjointness, $a \otimes (a \rightarrow A)^{\uparrow\downarrow} \subseteq A^{\uparrow\downarrow}$ is equivalent to $a \leq S((a \rightarrow A)^{\uparrow\downarrow}, A^{\uparrow\downarrow})$ and $A^{\uparrow\downarrow} \subseteq a \rightarrow (a \otimes A)^{\uparrow\downarrow}$ is equivalent to $a \leq S(A^{\uparrow\downarrow}, (a \otimes A)^{\uparrow\downarrow})$. Since $S((a \rightarrow A)^{\uparrow\downarrow}, A^{\uparrow\downarrow}) \geq (a \rightarrow A)^{\uparrow\downarrow} \approx A^{\uparrow\downarrow}$ and $S(A^{\uparrow\downarrow}, (a \otimes A)^{\uparrow\downarrow}) \geq A^{\uparrow\downarrow} \approx (a \otimes A)^{\uparrow\downarrow}$, it suffices to show that $a \leq (a \rightarrow A)^{\uparrow\downarrow} \approx A^{\uparrow\downarrow}$ and $a \leq A^{\uparrow\downarrow} \approx (a \otimes A)^{\uparrow\downarrow}$. The inequalities can be shown the similar way as (3) and (1), respectively, in the proof of the Lemma 7.

Now, taking $C_a(A)$ for A in (2) of Claim 1 we get $a \rightarrow (a \otimes C_a(A))^{\uparrow\downarrow} \supseteq (C_a(A))^{\uparrow\downarrow}$, which implies $C_a(C_a(A)) = (a \rightarrow (a \otimes C_a(A))^{\uparrow\downarrow})^{\uparrow\downarrow} \supseteq (C_a(A))^{\uparrow\downarrow} = C_a(A)$, proving $C_a(A) \subseteq C_a(C_a(A))$, and, taking $(a \otimes A)^{\uparrow\downarrow}$ for A in (1) of Claim 1 we get $a \otimes (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow} \subseteq (a \otimes A)^{\uparrow\downarrow\uparrow\downarrow} = (a \otimes A)^{\uparrow\downarrow}$, which implies subsequently $(a \otimes (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow})^{\uparrow\downarrow} \subseteq ((a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow} = (a \otimes A)^{\uparrow\downarrow}$, $a \rightarrow (a \otimes (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow})^{\uparrow\downarrow} \subseteq a \rightarrow (a \otimes A)^{\uparrow\downarrow}$ and $C_a(C_a(A)) = (a \rightarrow (a \otimes (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow})^{\uparrow\downarrow})^{\uparrow\downarrow} \subseteq (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow} = C_a(A)$, proving $C_a(C_a(A)) \subseteq C_a(A)$, i.e., $C_a(A) = C_a(C_a(A))$. We proved that C_a is an almost $\mathbf{L}_{\{1\}}$-closure operator.

Second, we verify $\text{fix}(C_a) = \text{ESB}(a)$, i.e. that the set of fixed points of C_a equals the set of extents of suprema of ${}^a\!\approx$-blocks. Let A be a fixpoint of C_a. By Lemma 1, the interval $[\langle A, A^{\uparrow}\rangle_a, (\langle A, A^{\uparrow}\rangle_a)^a]$ is an ${}^a\!\approx$-block, and by Lemma 7, $(\langle A, A^{\uparrow}\rangle_a)^a = \langle (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow}, \ldots\rangle$. Since $A = C_a(A) = (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow}$, A is the extent of a supremum of an ${}^a\!\approx$-block. Conversely, let A be a supremum of an ${}^a\!\approx$-block. Then $[\langle A, A^{\uparrow}\rangle_a, \langle A, A^{\uparrow}\rangle]$ is an ${}^a\!\approx$-block and so $(\langle A, A^{\uparrow}\rangle_a)^a = \langle A, A^{\uparrow}\rangle$. Lemma 7 now gives $A = (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow}$, i.e. $A = C_a(A)$ verifying that A is a fixpoint of C_a. The proof is complete. □

Remark 4 Note that $C_a : A \mapsto (a \rightarrow (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow}$ does neither satisfy $A \subseteq C_a(A)$ nor $S(A_1, A_2) \leq S(C_a(A_1), C_a(A_2))$ which is a monotony condition stronger than the above condition (ii). Namely, consider the structure of truth degrees, the hedges and the data table from Remark 2, with the exception that both ^{*_X} and ^{*_Y} be globalizations now. Take $A_1 = \{ {}^{0.5}\!/x_1, {}^{1}\!/x_2, {}^{0.5}\!/x_3\}$ and $A_2 = \{ {}^{1}\!/x_1, {}^{0.5}\!/x_2\}$. One can check that given $a = 1$ (fixpoint of both ^{*_X} and ^{*_Y}), $C_a(A_1) = A_1^{\uparrow\downarrow} =$

$\{\,{}^{1}/x_2\}$ and $C_a(A_2) = A_2^{\uparrow\downarrow} = \{\,{}^{1}/x_1,\ {}^{0.5}/x_3\}$, hence $A_1 \nsubseteq C_a(A_1)$, $A_2 \nsubseteq C_a(A_2)$ and $0.5 = \mathrm{S}(A_1, A_2) \nleq \mathrm{S}(C_a(A_1), C_a(A_2)) = 0$.

Remark 5 It is important to realize that the ${}^{a}\!\approx$-blocks are of the form

$$[\langle A, B\rangle_a, (\langle A, B\rangle_a)^a].$$

Moreover, because $\langle A, B\rangle_a = ((\langle A, B\rangle_a)^a)_a$, the blocks are uniquely given by their suprema

$$(\langle A, B\rangle_a)^a = \big\langle (a \to (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow}, (a \otimes (a \to B)^{\uparrow\downarrow})^{\downarrow\uparrow}\big\rangle.$$

Furthermore, since each $(\langle A, B\rangle_a)^a$ is given by its extent, i.e. by $A = (a \to (a \otimes A)^{\uparrow\downarrow})^{\uparrow\downarrow}$, we can conclude that the ${}^{a}\!\approx$-blocks, i.e. the elements of the factor lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^{a}\!\approx$, are uniquely given by the extents of their suprema. Therefore, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^{a}\!\approx$ is uniquely given by $\mathrm{ESB}(a)$.

According to Remark 5, in order to compute the factor lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^{a}\!\approx$, it is sufficient to compute $\mathrm{ESB}(a)$. According to Theorem 4, $\mathrm{ESB}(a)$ is the set of fixpoints of C_a which is an almost $\mathbf{L}_{\{1\}}$-closure operator. Now, computing sets of fixpoints of $\mathbf{L}_{\{1\}}$-closure operators can be performed by an extension of Ganter's NextClosure algorithm, see e.g. [3]. For computing sets of fixpoints of almost $\mathbf{L}_{\{1\}}$-closure operators, one can modify this extension the same way as in [10]. Details will be presented in a forthcoming paper.

4. Illustrative example

We present an illustrative example demonstrating the notion of a factor concept lattice. Take a finite Łukasiewicz chain $\mathbf{L}$ with $L = \{0, 0.25, 0.5, 0.75, 1\}$ as a structure of truth degrees. Consider an input data table $\langle X, Y, I\rangle$ depicted in Fig. 1 (left) which describes properties of planets of our solar system. The set X of objects consists of objects "Mercury", "Venus", …, set Y contains four (fuzzy) attributes: size of the planet (small / large) and distance from Sun (far / near). Let $*_X$ be identity and $*_Y$ be a hedge defined as follows: for $a \in L$, $a^{*_Y} = 0.5$ if $a = 0.75$ and $a^{*_Y} = a$ otherwise.

The whole fuzzy concept lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ has 94 formal concepts. There are four common fixpoints of ^{*_X} and ^{*_Y}, namely,

$$\mathrm{fix}({}^{*_X}, {}^{*_Y}) = \{0, 0.25, 0.5, 1\}.$$

We show and compare the factor concept lattices $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/{}^{a}\!\approx$ for thresholds $a = 0.25$ and $a = 0.5$. Note that for $a = 0$, the factor concept lattice degenerates into a one-element lattice, while for $a = 1$, the factor concept lattice is isomorphic to the original concept lattice. The factor lattices $\mathcal{B}(X, Y^{*_Y}, I)/{}^{0.25}\!\approx$ and $\mathcal{B}(X, Y^{*_Y}, I)/{}^{0.5}\!\approx$ happen to be isomorphic lattices and have 12 similarity blocks. The lattice of blocks is depicted in Fig. 1 (right). The suprema of ${}^{0.25}\!\approx$-blocks and ${}^{0.5}\!\approx$-blocks are listed in Table 1.

	size		distance	
	small (s)	large (l)	far (f)	near (n)
Mercury	1	0	0	1
Venus	0.75	0	0	1
Earth	0.75	0	0	0.75
Mars	1	0	0.5	0.75
Jupiter	0	1	0.75	0.5
Saturn	0	1	0.75	0.5
Uranus	0.25	0.5	1	0.25
Neptune	0.25	0.5	1	0
Pluto	1	0	1	0

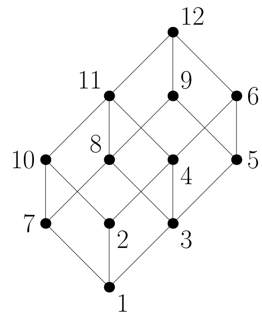

Figure 1: Data table with fuzzy attributes and factor lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/^a\!\approx$ of the corresponding fuzzy concept lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ for $a = 0.25$ and $a = 0.5$.

Table 1: Suprema of $^a\!\approx$-blocks of factor lattice $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)/^a\!\approx$ for $a = 0.25$ (top) and $a = 0.5$ (bottom).

	extent									intent			
1	.75	.75	.75	.75	.75	.75	1	.75	.75	.25	.25	.25	.25
2	.75	.75	.75	.75	.75	.75	1	1	.75	.25	.25	.25	0
3	.75	.75	.75	1	.75	.75	1	.75	.75	.25	0	.25	.25
4	.75	.75	.75	1	.75	.75	1	1	1	.25	0	.25	0
5	1	1	1	1	.75	.75	1	.75	.75	.25	0	0	.25
6	1	1	1	1	.75	.75	1	1	1	.25	0	0	0
7	.75	.75	.75	.75	1	1	1	.75	.75	0	.25	.25	.25
8	.75	.75	.75	1	1	1	1	.75	.75	0	0	.25	.25
9	1	1	1	1	1	1	1	.75	.75	0	0	0	.25
10	.75	.75	.75	.75	1	1	1	1	.75	0	.25	.25	0
11	.75	.75	.75	1	1	1	1	1	1	0	0	.25	0
12	1	1	1	1	1	1	1	1	1	0	0	0	0

	extent									intent			
1	.5	.5	.5	.5	.5	.5	.75	.5	.5	.5	.5	.5	.5
2	.5	.5	.5	.5	.5	.5	.75	.75	.5	.5	.5	.5	.25
3	.5	.5	.5	1	.5	.5	.75	.5	.5	.5	0	.5	.5
4	.5	.5	.5	1	.5	.5	.75	.75	1	.5	0	.5	0
5	1	1	1	1	.5	.5	.75	.5	.5	.5	0	0	.5
6	1	1	1	1	.5	.5	.75	.75	1	.5	0	0	0
7	.5	.5	.5	.5	1	1	.75	.5	.5	0	.5	.5	.5
8	.5	.5	.5	1	1	1	.75	.5	.5	0	0	.5	.5
9	1	1	1	1	1	1	.75	.5	.5	0	0	0	.5
10	.5	.5	.5	.5	1	1	1	1	.5	0	.5	.5	0
11	.5	.5	.5	1	1	1	1	1	1	0	0	.5	0
12	1	1	1	1	1	1	1	1	1	0	0	0	0

5. Conclusions and future research

We presented a method of factorization of fuzzy concept lattices with hedges. A factor lattice represents an approximate version of the original fuzzy concept lattice. The size of the factor lattice is controlled by a user-specified threshold. The factor lattice can be computed directly from input data, without first computing the possibly large original fuzzy concept lattice. Future research will focus on exploring so-called shifted attributes to compute the factor concept lattice directly from input data. From a wider point of view, our research will focus on reducing the size of concept lattices according to user-specified requests and preferences.

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