Crisply Generated Fuzzy Concepts

Radim Bělohlávek, Vladimír Sklenář, and Jiří Zacpal

Dept. Computer Science, Palacký University,
Tomkova 40, CZ-779 00 Olomouc, Czech Republic
{radim.belohlavek, vladimir.sklenar, jiri.zacpal}@upol.cz

Abstract. In formal concept analysis of data with fuzzy attributes, both the extent and the intent of a formal (fuzzy) concept may be fuzzy sets. In this paper we focus on so-called crisply generated formal concepts. A concept $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ is crisply generated if $A = D^{\downarrow}$ (and so $B = D^{\downarrow\uparrow}$) for some crisp (i.e., ordinary) set $D \subseteq Y$ of attributes (generator). Considering only crisply generated concepts has two practical consequences. First, the number of crisply generated formal concepts is considerably less than the number of all formal fuzzy concepts. Second, since crisply generated concepts may be identified with a (ordinary, not fuzzy) set of attributes (the largest generator), they might be considered "the important ones" among all formal fuzzy concepts. We present basic properties of the set of all crisply generated concepts, an algorithm for listing all crisply generated concepts, a version of the main theorem of concept lattices for crisply generated concepts, and show that crisply generated concepts are just the fixed points of pairs of mappings resembling Galois connections. Furthermore, we show connections to other papers on formal concept analysis of data with fuzzy attributes. Also, we present examples demonstrating the reduction of the number of formal concepts and the speed-up of our algorithm (compared to listing of all formal concepts and testing whether a concept is crisply generated).

1 Problem Setting and Preliminaries

1.1 Problem Setting

Formal concept analysis (FCA) [12] deals with object-attribute data tables (objects and attributes corresponding to table rows and columns, respectively). In the basic setting, attributes are assumed to be binary, i.e. table entries are 1 or 0 according to whether an attribute applies to an object or not. If the attributes under consideration are fuzzy (like "cheap", "expensive"), each table entry contains a truth degree to which an attribute applies to an object. The degrees can be taken from some appropriate scale containing 0 (does not apply at all) and 1 (fully applies) as bounds. The most popular choice is some subinterval of $[0, 1]$, but in general, degrees need not be numbers. A data table with truth degrees can be considered a many-valued context and can be transformed to a binary data table via so-called conceptual scaling [12]. Alternatively, the table with truth degrees can be approached using the apparatus of FCA generalized

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to fuzzy setting (generalization of FCA from the point of view of fuzzy logic). A general discussion about the relationship between conceptual scaling in the sense of FCA and membership functions in the sense of fuzzy set can be found in [21].

In the present paper, we are interested in FCA of data with fuzzy attributes (FCAf) in the framework of fuzzy logic and fuzzy set theory. Probably the first paper on this was [11]. Later on, FCAf was developed by Pollandt [18] and, independently, by the first author of this paper, e.g. [1, 2, 3, 7]. An important aspect of FCA in general is the possibly large number of formal concepts extracted from data. In this paper, we propose and study what we call crisply generated formal fuzzy concepts. These are particular formal fuzzy concepts which can be considered "more important" than the others (non-crisply generated). Considering only crisply generated concepts, the main practical effect is the reduction of the number of formal concepts extracted from data. In the rest of this section, we present preliminaries on fuzzy logic and FCAf. In Section 2 we present our approach and theoretical results. Section 3 contains examples and experiments studying mainly the reduction of the number of extracted concepts.

1.2 Preliminaries

Fuzzy Sets and Fuzzy Logic. We assume basic familiarity with fuzzy logic and fuzzy sets [16, 13, 6]. An element may belong to a fuzzy set in an intermediate degree not necessarily being 0 or 1. Formally, a fuzzy set A in a universe X is a mapping assigning to each $x \in X$ a truth degree $A(x) \in L$ where L is some partially ordered set of truth degrees containing at least 0 (full falsity) and 1 (full truth). L needs to be equipped with logical connectives, e.g. $\otimes$ (fuzzy conjunction), $\rightarrow$ (fuzzy implication), etc. L together with logical connectives forms a structure $\mathbf{L}$ of truth degrees. We assume that $\mathbf{L}$ forms a so-called complete residuated lattice. Recall that a complete residuated lattice [6, 13, 14] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that (1) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice (with the least element 0, greatest element 1), i.e. a partially ordered set in which arbitrary infima ($\bigwedge$) and suprema ($\bigvee$) exist; (2) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation satisfying $x \otimes (y \otimes z) = (x \otimes y) \otimes z$, $x \otimes y = y \otimes x$, and $x \otimes 1 = x$; (3) $\otimes, \rightarrow$ satisfy $x \otimes y \leq z$ iff $x \leq y \rightarrow z$. In what follows, $\mathbf{L}$ always denotes a fixed complete residuated lattice.

The most applied set of truth degrees is the real interval $[0, 1]$; with $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and with three important pairs of fuzzy conjunction and fuzzy implication: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), minimum ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and product ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ else). Another possibility is to take a finite chain $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ ($a_0 < \cdots < a_n$) equipped with Łukasiewicz structure ($a_k \otimes a_l = a_{\max(k+l-n,0)}$, $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$) or minimum ($a_k \otimes a_l = a_{\min(k,l)}$, $a_k \rightarrow a_l = a_n$ for $a_k \leq a_l$ and $a_k \rightarrow a_l = a_l$ otherwise).

The set of all fuzzy sets (or $\mathbf{L}$-sets) in X is denoted L^X. For a fuzzy set $A \in L^X$, the 1-cut ${}^1A$ of A is an ordinary set ${}^1A = \{x \in X \mid A(x) = 1\}$. A is called crisp if $A(x) \in \{0,1\}$. By $\{\,{}^{a}\!/x\}$ we denote a fuzzy set A for which $A(x) = a$ and $A(y) = 0$ for $y \neq x$. For fuzzy sets A, B in X we put $A \subseteq B$ (A is a subset of B) if for each $x \in X$ we have $A(x) \leq B(x)$. More generally, the degree $S(A, B)$ to which A is a subset of B is defined by $S(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Then, $A \subseteq B$ means $S(A, B) = 1$.

Formal Concept Analysis of Data with Fuzzy Attributes. Let X and Y be sets of objects and attributes, respectively, I be a fuzzy relation between X and Y. That is, $I : X \times Y \to L$ assigns to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ to which object x has attribute y (L is a support set of some complete residuated lattice $\mathbf{L}$). The triplet $\langle X, Y, I\rangle$ is called a formal fuzzy context (corresponds to a data table with fuzzy attributes). For fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$) defined by

$$A^{\uparrow}(y) = \bigwedge_{x \in X} (A(x) \to I(x, y)), \tag{1}$$

$$B^{\downarrow}(x) = \bigwedge_{y \in Y} (B(y) \to I(x, y)) \tag{2}$$

for $y \in Y$ and $x \in X$. Using basic rules of predicate fuzzy logic, one can see that $A^{\uparrow}(y)$ is the truth degree of the proposition "y is shared by all objects from A" and $B^{\downarrow}(x)$ is the truth degree of "x has all attributes from B". Putting

$$\mathcal{B}\,(X, Y, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\},$$

$\mathcal{B}\,(X, Y, I)$ is the set of all pairs $\langle A, B\rangle$ such that (a) A is the collection of all objects that have all the attributes of (the intent) B and (b) B is the collection of all attributes that are shared by all the objects of (the extent) A. Elements of $\mathcal{B}\,(X, Y, I)$ are called formal concepts of $\langle X, Y, I\rangle$ (formal fuzzy concepts, formal $\mathbf{L}$-concepts); $\mathcal{B}\,(X, Y, I)$ is called the concept lattice given by $\langle X, Y, I\rangle$ (fuzzy concept lattice, $\mathbf{L}$-concept lattice). Both the extent A and the intent B of a formal concept $\langle A, B\rangle$ are in general fuzzy sets. This corresponds to the fact that in general, concepts apply to objects and attributes to various intermediate degrees, not only 0 and 1.

Putting

$$\langle A_1, B_1\rangle \leq \langle A_1, B_1\rangle \text{ iff } A_1 \subseteq A_2 (\text{iff } B_1 \supseteq B_2) \tag{3}$$

for $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}\,(X, Y, I)$, $\leq$ models the subconcept-superconcept hierarchy in $\mathcal{B}\,(X, Y, I)$.

The following is a version of the main theorem for fuzzy concept lattices (see [7, 18]).

Theorem 1. *The set $\mathcal{B}\,(X, Y, I)$ is under $\leq$ a complete lattice where infima and suprema are given by*

$$\bigwedge_{j \in J} \langle A_j, B_j\rangle = \langle \bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)^{\downarrow\uparrow}\rangle\ , \tag{4}$$

$$\bigvee_{j \in J} \langle A_j, B_j\rangle = \langle (\bigcup_{j \in J} A_j)^{\uparrow\downarrow}, \bigcap_{j \in J} B_j\rangle\ . \tag{5}$$

Moreover, an arbitrary complete lattice $\mathbf{V} = \langle V, \wedge, \vee \rangle$ *is isomorphic to some* $\mathcal{B}(X,Y,I)$ *iff there are mappings* $\gamma : X \times L \to V$, $\mu : Y \times L \to V$ *such that* $\gamma(X,L)$ *is* $\bigvee$*-dense in* V, $\mu(Y,L)$ *is* $\bigwedge$*-dense in* V; $a \otimes b \leq I(x,y)$ *iff* $\gamma(x,a) \leq \mu(y,b)$.

Note that Theorem 1 can be proved by reduction (see [18, 4]) to the main theorem of ordinary concept lattices [20] or directly in the framework of fuzzy logic [7]. Note also that Theorem 1 is concerned with bivalent order, there is still a more general version [7] dealing with many-valued (fuzzy) order.

Taking $L = \{0,1\}$ (two truth degrees; bivalent case), the notions of formal fuzzy context, formal fuzzy concept, and fuzzy concept lattice coincide with the ordinary notions [12]. In the following we denote $\mathrm{Ext}(I) = \{A \mid \langle A,B\rangle \in \mathcal{B}(X,Y,I)$ for some $B\}$ (extents of concepts) and $\mathrm{Int}(I) = \{B \mid \langle A,B\rangle \in \mathcal{B}(X,Y,I)$ for some $A\}$ (intents of concepts). Recall [3] that $^{\uparrow}$ and $^{\downarrow}$ satisfy $S(A_1,A_2) \leq S(A_2^{\uparrow},A_1^{\uparrow})$; $S(B_1,\ B_2) \leq S(B_2^{\downarrow},B_1^{\downarrow})$; $A \subseteq A^{\uparrow\downarrow}$; $B \subseteq B^{\downarrow\uparrow}$. As a consequence, $A^{\uparrow} = A^{\uparrow\downarrow\uparrow}$ and $B^{\downarrow} = B^{\downarrow\uparrow\downarrow}$.

2 Crisply Generated Formal Concepts

2.1 Motivation and Definition

A formal concept $\langle A,B\rangle$ consists of a fuzzy set A and a fuzzy set B such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$. Due to (1) and (2), and the basic rules of predicate fuzzy logic, this directly captures the verbal definition of a formal concept inspired by Port-Royal logic. Nevertheless, this definition actually allows for formal fuzzy concepts $\langle A,B\rangle$ such that, for example, for any $x \in X$ and $y \in Y$ we have $A(x) = 1/2$ and $B(y) = 1/2$. A verbal description of such a concept is "a concept to which each attribute belongs to degree 1/2". Such a concept, although satisfying the verbally described condition $A^{\uparrow} = B$, $B^{\downarrow} = A$, will probably be considered "not the important one". This is because people expect concepts to be determined by "some attributes", i.e. by an ordinary set of attributes. This leads to the following definition.

Definition 1. *A formal fuzzy concept* $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$ *is called* crisply generated *if there is a crisp set* $B_c \subseteq Y$ *such that* $A = B_c^{\downarrow}$ *(and thus* $B = B_c^{\downarrow\uparrow}$*).*

We say that B_c crisply generates $\langle A,B\rangle$. Let $\mathcal{B}_c(X,Y,I)$ denote the collection of all crisply generated formal concepts in $\langle X,Y,I\rangle$, i.e.

$$\mathcal{B}_c(X,Y,I) = \{\langle A,B\rangle \in \mathcal{B}(X,Y,I) \mid \text{ there is } B_c \subseteq Y : A = B_c^{\downarrow}\}.$$

If $\langle A,B\rangle$ is a crisply generated concept with $A = B_c^{\downarrow}$, it might be actually more informative to write $\langle A,B_c\rangle$ instead of $\langle A,B\rangle$. Doing so, no information is lost since the corresponding fuzzy concept $\langle A,B\rangle$ can be obtained from $\langle A,B_c\rangle$ by taking $B = B_c^{\downarrow\uparrow}$. In general, there may be several crisp B_c's with $A = B_c^{\downarrow}$. To remove this ambiguity, we can always take the greatest B_c:

Lemma 1. *For a crisply generated formal concept* $\langle A,B\rangle$, ${}^1B$ *is the largest crisp set* B_c *for which* $A = B_c^{\downarrow}$.

Proof. Let $\langle A, B\rangle$ be crisply generated by some B_c, i.e. $A = B_c^{\downarrow}$. Since $B_c \subseteq B_c^{\downarrow\uparrow} = B$, we have $B_c = {}^1B_c \subseteq {}^1B$. That is, ${}^1B$ contains any crisp B_c which generates $\langle A, B\rangle$. Moreover, ${}^1B$ itself is a crisp set which generates $\langle A, B\rangle$. Indeed, take some crisp B_c which generates $\langle A, B\rangle$. We know that $B_c \subseteq {}^1B$, from which we get $B = B_c^{\downarrow\uparrow} \subseteq ({}^1B)^{\downarrow\uparrow}$. On the other hand, ${}^1B \subseteq B$ gives $({}^1B)^{\downarrow\uparrow} \subseteq B^{\downarrow\uparrow} = B$ which shows $({}^1B)^{\downarrow\uparrow} = B$.

Crisply generated formal concepts can be alternatively defined as maximal rectangles $\langle A, B\rangle$ contained in I for which A is the projection of ${}^1B$. Call a fuzzy relation $I' \in L^{X\times Y}$ a *rectangular relation* if there are $A \in L^X$, $B \in L^Y$ such that $I'(x, y) = A(x) \otimes B(y)$, written $I' = A \otimes B$ (call then $\langle A, B\rangle$ a *rectangle*). $\langle A, B\rangle$ is said to be *contained* in $I'' \in L^{X\times Y}$ if $A \otimes B \subseteq I''$. We put $\langle A_1, B_1\rangle \trianglelefteq \langle A_2, B_2\rangle$ if for each $x \in X$, $y \in Y$ we have $A_1(x) \leq A_2(x)$ and $B_1(y) \leq B_2(y)$. By an I-projection of a subset $C \subseteq Y$ on X we mean a fuzzy set A in X defined by $A(x) = \bigwedge_{y\in C} I(x, y)$.

Lemma 2. *$\langle A, B\rangle$ is a crisply generated concept iff $\langle A, B\rangle$ is a maximal (w.r.t. $\trianglelefteq$) rectangle contained in I such that A is the projection of ${}^1B$.*

Proof. The assertion follows from [6–Theorem 5.7], the fact that $\langle A, B\rangle$ is crisply generated iff $A = ({}^1B)^{\downarrow}$ (see Lemma 1), and from $({}^1B)^{\downarrow}(x) = \bigwedge_{y\in {}^1B} I(x, y)$.

2.2 Independence of the Choice of Fuzzy Logical Connectives

The next step is to observe that restricting ourselves to crisply generated concepts, one is no more dependent (almost) on the logical connectives defined on the scale L of truth degrees. To formulate this precisely, let us denote the concept lattice over the structure $\mathbf{L}$ of truth degrees by $\mathcal{B}^{\mathbf{L}}(X, Y, I)$ and denote $\mathcal{B}_c^{\mathbf{L}}(X, Y, I)$ the set of all crisply generated concepts of $\mathcal{B}^{\mathbf{L}}(X, Y, I)$. Suppose we have two structures $\mathbf{L}_1$ and $\mathbf{L}_2$ with a common set L of truth degrees, i.e. $\mathbf{L}_1 = \langle L, \otimes_1, \rightarrow_1, \ldots\rangle$ and $\mathbf{L}_2 = \langle L, \otimes_2, \rightarrow_2, \ldots\rangle$, and a data table (formal fuzzy context) $\langle X, Y, I\rangle$ which is filled with truth degrees from L.

Lemma 3. *Let $\mathbf{L}_1$ and $\mathbf{L}_2$ have a common set L of truth degrees, let $\langle X, Y, I\rangle$ be a formal fuzzy context with truth degrees from L. Then there is an isomorphism between $\mathcal{B}_c^{\mathbf{L}_1}(X, Y, I)$ and $\mathcal{B}_c^{\mathbf{L}_2}(X, Y, I)$ such that for the corresponding formal concepts $\langle A_1, B_1\rangle \in \mathcal{B}_c^{\mathbf{L}_1}(X, Y, I)$ and $\langle A_2, B_2\rangle \in \mathcal{B}_c^{\mathbf{L}_2}(X, Y, I)$ we have $A_1 = A_2$ and ${}^1B_1 = {}^1B_2$.*

Proof. Denote by ${}^{\downarrow_i}$ and ${}^{\uparrow_i}$ the operators generated by $\rightarrow_i$ ($i = 1, 2$). Recall that for each residuated implication connective $\rightarrow$ we have $1 \rightarrow a = a$. Therefore, for each crisp $B \subseteq Y$ we have $B^{\downarrow_i}(x) = \bigwedge_{y\in Y}(B(y) \rightarrow_i I(x, y)) = \bigwedge_{y\in B}(1 \rightarrow_i I(x, y)) = \bigwedge_{y\in B} I(x, y)$. That is, $B^{\downarrow_i}$ does not depend on $\rightarrow_i$. Therefore, if $\langle A, B\rangle \in \mathcal{B}_c^{\mathbf{L}_1}(X, Y, I)$ is crisply generated then from Lemma 1 we have $({}^1B)^{\downarrow_1} = A$ and since ${}^1B$ is crisp, also $({}^1B)^{\downarrow_2} = A$. This shows that $\langle A, D\rangle$, for $D = A^{\uparrow_2}$, is a crisply generated formal concept from $\mathcal{B}_c^{\mathbf{L}_2}(X, Y, I)$. Clearly, ${}^1B \subseteq {}^1D$. If ${}^1B \subset {}^1D$, i.e. D is larger than B, then Lemma 1 gives $({}^1D)^{\downarrow_2} = A$ and so $({}^1D)^{\downarrow_1} = A$

which is impossible since by Lemma 1, ${}^1B$ is the largest one with $({}^1B)^{\downarrow_1} = A$. In a similar way one shows that if $\langle A, B\rangle \in \mathcal{B}_c^{\mathbf{L}_2}(X, Y, I)$ is crisply generated then $\langle A, A^{\uparrow_1}\rangle \in \mathcal{B}^{\mathbf{L}_1}(X, Y, I)$ is crisply generated as well and ${}^1B = {}^1A^{\uparrow_1}$. The assertion then immediately follows.

Note that in general, $\mathcal{B}^{\mathbf{L}_1}(X, Y, I)$ and $\mathcal{B}^{\mathbf{L}_2}(X, Y, I)$ may have different number of formal concepts, i.e. the choice of fuzzy logical connectives matters. Lemma 3 shows that their crisply generated parts $\mathcal{B}_c^{\mathbf{L}_1}(X, Y, I)$ and $\mathcal{B}_c^{\mathbf{L}_2}(X, Y, I)$ are isomorphic. That is, if we consider only crisply generated concepts, the choice of fuzzy logical connectives, in a sense, does not matter.

2.3 Computing All Crisply Generated Formal Concepts

We now present an algorithm for generating $\mathcal{B}_c(X, Y, I)$. Going directly by definition, i.e. creating $\langle B^{\downarrow}, B^{\downarrow\uparrow}\rangle$ for each crisp $B \in 2^X$, has exponential time complexity and thus, cannot be used. Our algorithm is inspired by Ganter's Next Closure algorithm [12–p. 67] for generating an ordinary concept lattice, i.e. generating all formal concepts in lexicographic order. This idea can be adopted to fuzzy setting to generate all crisply generated formal fuzzy concepts.

The idea of our algorithm is to introduce a linear ordering $<$ on $\mathcal{B}_c(X, Y, I)$ such that for a given $\langle A, B\rangle \in \mathcal{B}_c(X, Y, I)$, we can compute its immediate successor w.r.t. to $<$. Since a formal concept $\langle A, B\rangle$ is uniquely given by its intent B, it is sufficient to generate all intents B. By $\mathrm{Int}_c(I)$ we denote all intents of crisply generated fuzzy concepts, i.e. $\mathrm{Int}_c(I) = \{B \mid \langle A, B\rangle \in \mathcal{B}_c(X, Y, I)$ for some $A \in L^X\}$. We suppose that $Y = \{1, \ldots, n\}$; $L = \{0 = a_1, a_2, \ldots, a_k = 1\}$ such that if $a_i \leq a_j$ in $\mathbf{L}$ then $i \leq j$ (that is, the ordering of elements of L by indices extends their ordering in $\leq$ in $\mathbf{L}$; such an indexing is always possible and is automatically satisfied if $\mathbf{L}$ is linearly ordered and we index the elements in L using this order from the least to the greatest element, i.e. $a_1 \leq a_2 \leq \cdots \leq a_k$). For $i = 1, \ldots, n$, introduce a relation $<_i$ on L^Y by

$$B_1 <_i B_2 \text{ iff } ({}^1B_1)(i) = 0, ({}^1B_2)(i) = 1, \text{ and } ({}^1B_1)(j) = ({}^1B_2)(j) \text{ for } j < i.$$

Furthermore, we put

$$B_1 < B_2 \text{ iff } B_1 <_i B_2 \text{ for some } i.$$

That is, $B_1 < B_2$ iff the first element of Y on which ${}^1B_1$ and ${}^1B_2$ differ, belongs to B_2; i.e. $B_1 < B_2$ means that ${}^1B_1$ is lexicographically smaller than ${}^1B_2$.

Lemma 4. *$<$ is a strict total order on $\mathrm{Int}_c(I)$ which extends $\subset$.*

Proof. Easy to see since every $B_1, B_2 \in \mathrm{Int}_c(I)$ with common 1-cuts (i.e. with ${}^1B_1 = {}^1B_2$) are equal. Indeed, By Lemma 1, if ${}^1B_1 = {}^1B_2$ then $B_1 = ({}^1B_1)^{\downarrow\uparrow} = ({}^1B_2)^{\downarrow\uparrow} = B_2$.

Furthermore, for $B \in L^Y$ and $i \in \{1, \ldots, n\}$, we put

$$B \oplus_c i := (({}^1B \cap \{1, \ldots, i-1\}) \cup \{i\})^{\downarrow\uparrow}.$$

That is, we obtain $B \oplus_c i$ by taking the 1-cut of B, cutting off the elements $i, \ldots, n$, joining with element i and applying the closure $^{\downarrow\uparrow}$.

Lemma 5. *The following assertions are true.*

(1) If $B <_i D_1$, $B <_j D_2$, and $i < j$ then $D_2 <_i D_1$;
(2) if $B <_i D$ and $D = D^{\downarrow\uparrow}$ then $({}^1B) \oplus_c i \subseteq D$;
(3) if $B <_i D$ and $D = D^{\downarrow\uparrow}$ then $B <_i ({}^1B) \oplus_c i$.

Proof. (1) follows directly from definition. (2) From $B <_i D$ we have $D(i) = 1$ and ${}^1B \cap \{1, \ldots, i-1\} \subseteq D$. Putting $C_1 = {}^1B \cap \{1, \ldots, i-1\}$, $C_2 = \{\,{}^1\!/i\}$, we thus have $C_1 \cup C_2 \subseteq D$, whence $({}^1B) \oplus_c i = (C_1 \cup C_2)^{\downarrow\uparrow} \subseteq D^{\downarrow\uparrow} = D$. (3) From $B <_i D$ we have ${}^1B \cap \{1, \ldots, i-1\} = {}^1D \cap \{1, \ldots, i-1\}$. Using (2) we get $({}^1B) \oplus_c i \subseteq D$, and so ${}^1({}^1B \oplus_c i) \cap \{1, \ldots, i-1\} \subseteq {}^1D \cap \{1, \ldots, i-1\} = {}^1B \cap \{1, \ldots, i-1\}$. On the other hand, ${}^1({}^1B \oplus_c i) \cap \{1, \ldots, i-1\} \supseteq {}^1({}^1B \cap \{1, \ldots, i-1\})^{\downarrow\uparrow} \cap \{1, \ldots, i-1\} \supseteq {}^1({}^1B \cap \{1, \ldots, i-1\}) \cap \{1, \ldots, i-1\} \supseteq {}^1B \cap \{1, \ldots, i-1\}$. Therefore, ${}^1B \cap \{1, \ldots, i-1\} = {}^1({}^1B \oplus_c i) \cap \{1, \ldots, i-1\}$. Finally, by (2), $1 = ({}^1B \oplus_c i)(i) \leq D(i)$, i.e. $D(i) = 1 = a_k$ proving $B <_i ({}^1B) \oplus_c i$.

Theorem 2. *For $B \in L^Y$, the least crisply generated intent $B^{+_c} \in \mathrm{Int}_c(I)$ which is greater than B is given by*

$$B^{+_c} = B \oplus_c i$$

where i is the greatest element with $B <_i B \oplus_c i$.

Proof. Let B^{+_c} be the required successor of B w.r.t. $<$. We have $B < B^{+_c}$, i.e $B <_i B^{+_c}$ for some i. By Lemma 5 (3), $B <_i {}^1B \oplus_c i$. By Lemma 5 (2), ${}^1B \oplus_c i \subseteq B^{+_c}$ and thus ${}^1B \oplus_c i \leq B^{+_c}$ (i.e. ${}^1B \oplus_c i < B^{+_c}$ or ${}^1B \oplus_c i = B^{+_c}$), and so $B <_i {}^1B \oplus_c i \leq B^{+_c}$. Since B^{+_c} is the successor of B, we have $B^{+_c} = {}^1B \oplus_c i$. It remains to show that i is the greatest element with $B <_i {}^1B \oplus_c i$, i.e. ${}^1B <_i {}^1({}^1B \oplus_c i)$. If ${}^1B <_j {}^1({}^1B \oplus_c j)$ for $i < j$ then Lemma 5 (1) yields ${}^1({}^1B \oplus_c j) <_i {}^1({}^1B \oplus_c i)$, i.e. ${}^1B \oplus_c j < {}^1B \oplus_c i$ which is a contradiction to ${}^1B \oplus_c i = B^{+_c} < {}^1B \oplus_c j$ (since B^{+_c} is the immediate of B).

Theorem 2 leads to the following algorithm.

INPUT: $\langle X, Y, I\rangle$, OUTPUT: $\mathrm{Int}_c(I)$

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store(B)
while B ≠ Y do
  B := B^{+c}
  store(B)
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The time complexity of computing from B the next crisply generated intent B^{+_c} is $O(|X| \cdot |Y|^2)$. Therefore, our algorithm has polynomial time delay complexity [15] (generating crisply generated intents, one generates the successor B^{+_c} of B in polynomial time $O(|X| \cdot |Y|^2)$). The time complexity of the algorithm is thus $O(|\mathrm{Int}_c(I)| \cdot |X| \cdot |Y|^2)$.

Remark 1. Note that in [8] we presented an algorithm for generating all formal fuzzy concepts of $\mathcal{B}(X, Y, I)$. This algorithm is inspired by Ganter's Next Closure which is its particular case. Using this algorithm, we can generate $\mathcal{B}_c(X, Y, I)$ in the following way: Generate all $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ and for each such $\langle A, B\rangle$, test by Lemma 1 whether $\langle A, B\rangle$ is crisply generated. Comprated to this, the algorithm presented here generates $\mathcal{B}_c(X, Y, I)$ directly, going from one crisply generated concept to the next one. We demonstrate the speed-up in Section 3.

2.4 Crisply Generated Fuzzy Concepts as Fixed Points of Fuzzy Galois-Like Mappings

It is well-known that ordinary formal concepts $\langle X, Y, I\rangle$ are exactly the fixed points of a Galois connection formed by (the concept derivation operators) ${}^{\uparrow} : 2^X \to 2^Y$ and ${}^{\downarrow} : 2^Y \to 2^X$ induced by I [19, 12]. Moreover, each Galois connection between X and Y is induced by some relation $I \in 2^{X\times Y}$. In [2], this fact was generalized to the setting of fuzzy logic: Call a fuzzy Galois connection between X and Y any pair $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ of mappings ${}^{\uparrow} : L^X \to L^Y$ and ${}^{\uparrow} : L^Y \to L^X$ satisfying

$$S(A_1, A_2) \leq S(A_2^{\uparrow}, A_1^{\uparrow}) \tag{6}$$

$$S(B_1, B_2) \leq S(B_2^{\downarrow}, B_1^{\downarrow}) \tag{7}$$

$$A \subseteq A^{\uparrow\downarrow} \tag{8}$$

$$B \subseteq B^{\downarrow\uparrow}, \tag{9}$$

for each $A, A_1, A_2 \in L^X$ and $B, B_1, B_2 \in L^Y$. It was proved in [2] that given $\langle X, Y, I\rangle$, the pair $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ defined by (1) and (2) is a fuzzy Galois connection and, conversely, each fuzzy Galois connection is induced by some $\langle X, Y, I\rangle$ by (1) and (2). The relationship between fuzzy Galois connections and fuzzy relations between X and Y is one-to-one.

A natural question arises as to whether crisply generated fuzzy concepts can be thought of as fixed points of suitable mappings, possibly axiomatically definable. In the following, we present a positive answer. In fact, what we are going to present is a special case of a more general case of so-called (fuzzy) Galois connections with hedges [10]. However, to keep our discussion in the framework of crisply generated concepts, we do not go to the more general notions of [10] and present the results with proofs for our special case.

Consider mappings ${}^{\triangle} : L^X \to L^Y$ and ${}^{\triangledown} : L^Y \to L^X$ resulting from $\langle X, Y, I\rangle$ by

$$A^{\triangle}(y) = \bigwedge_{x\in X} (A(x) \to I(x, y)) \tag{10}$$

and

$$B^{\triangledown}(x) = \bigwedge_{y\in Y} ({}^{1}B(y) \to I(x, y)). \tag{11}$$

Note that we have $A^{\triangle} = A^{\uparrow}$ and $B^{\triangledown} = ({}^{1}B)^{\downarrow}$ where ${}^{\uparrow}$ and ${}^{\downarrow}$ are defined by (1) and (2). Now, denote by $\mathcal{B}\left(X, {}^{1}Y, I\right)$ the set of all fixed points of $\langle {}^{\triangle}, {}^{\triangledown}\rangle$, i.e.

$$\mathcal{B}\left(X, {}^{1}Y, I\right) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\triangle} = B,\ B^{\triangledown} = A\}.$$

Theorem 3. $\mathcal{B}\left(X, {}^1Y, I\right) = \mathcal{B}_c\left(X, Y, I\right)$*, i.e. crisply generated fuzzy concepts are exactly the fixed points of* $^\triangle$ *and* $^\triangledown$*.*

Proof. "$\subseteq$": If $\langle A, B\rangle \in \mathcal{B}\left(X, {}^1Y, I\right)$ then $A^\triangle = B$ and $B^\triangledown = A$, i.e. $A^\uparrow = B$ and $({}^1B)^\downarrow = A$. Therefore, $\langle A, B\rangle \in \mathcal{B}_c\left(X, Y, I\right)$, by definition.

"$\supseteq$": Let $\langle A, B\rangle \in \mathcal{B}_c\left(X, Y, I\right)$, i.e. $A^\uparrow = B$, $B^\downarrow = A$, and $A = D^\downarrow$ for some crisp $D \subseteq Y$. We need to verify $A^\triangle = B$ and $B^\triangledown = A$. By Lemma 1, it clearly suffices to check $({}^1B)^\downarrow = A$, i.e. $B^\triangledown = A$. As $A = D^\downarrow$ and $B = B^{\downarrow\uparrow}$, we need to verify $D^\downarrow = ({}^1D^{\downarrow\uparrow})^\downarrow$. But $D^\downarrow = ({}^1D)^\downarrow = ({}^{11}D^{\downarrow\uparrow})^\downarrow$. Indeed, the first equality follows from the fact that D is crisp and thus ${}^1D = D$. For the second equality, $({}^1D)^\downarrow \subseteq ({}^1({}^1D)^{\downarrow\uparrow})^\downarrow$ follows from $F \subseteq ({}^1F^\uparrow)^\downarrow$ for $F = ({}^1D)^\downarrow$ (easy), and $({}^1D)^\downarrow \supseteq ({}^1({}^1D)^{\downarrow\uparrow})^\downarrow$ follows from $D = {}^1D$, from ${}^1D \subseteq ({}^1D)^{\downarrow\uparrow}$, and from the fact that if $E \subseteq F$ then ${}^1E^\downarrow \supseteq {}^1F^\downarrow$ (just put $E = D$ and $F = ({}^1D)^{\downarrow\uparrow}$). Hence, $\langle A, B\rangle \in \mathcal{B}\left(X, {}^1Y, I\right)$.

Now, we turn to the investigation of the properties of $^\triangle$ and $^\triangledown$ and the problem of axiomatization of these properties.

Lemma 6. $^\triangle$ *and* $^\triangledown$ *defined by (10) and (11) satisfy*

$$S(A, B^\triangledown) = S({}^1B, A^\triangle) \qquad (12)$$

$$(\bigcup_{j\in J} A_j)^\triangle = \bigcap_{j\in J} A_j^\triangle \qquad (13)$$

for every $A, A_j \in L^X$ *and* $B \in L^Y$*.*

Proof. We have

$$S(A, B^\triangledown) = \bigwedge_{x\in X}(A(x) \to (\bigwedge_{y\in Y} {}^1B(y) \to I(x, y))) =$$

$$= \bigwedge_{y\in Y}\bigwedge_{x\in X}({}^1B(y) \to (A(x) \to I(x, y))) =$$

$$= \bigwedge_{y\in Y}({}^1B(y) \to (\bigwedge_{x\in X} A(x) \to I(x, y))) = S({}^1B, A^\triangle),$$

proving (12). (13) is a consequence of properties of fuzzy Galois connections [2].

Definition 2. *A pair* $\langle ^\triangle, ^\triangledown\rangle$ *of mappings satisfying (12) and (13) is called a c-Galois connection between* X *and* Y*.*

The following are some consequences of (12).

Lemma 7. *If* $^\triangle : L^X \to L^Y$ *and* $^\triangledown : L^Y \to L^X$ *satisfy (12) then*

$$(\bigcup_{j\in J} {}^1B_j)^\triangledown = \bigcap_{j\in J} B_j^\triangledown \qquad (14)$$

$$B^\triangledown = ({}^1B)^\triangledown \qquad (15)$$

$$\{^a/x\}^\triangle(y) = a \to \{^1/x\}^\triangle(y) \qquad (16)$$

$$\{^a/y\}^\triangledown(x) = a \to \{^1/y\}^\triangledown(x) \qquad (17)$$

for any $B, B_j \in L^Y$*,* $x \in X$*,* $y \in Y$*,* $a \in L$*.*

Proof. (14): We show that $S(A, (\bigcup_i {}^1B_i)^\bigtriangledown) = 1$ iff $S(A, \bigcap_i ({}^1B)_i^\bigtriangledown) = 1$ for each $A \in L^X$. Using (12) we have $S(A, (\bigcup_i {}^1B_i)^\bigtriangledown) = S({}^1(\bigcup_i {}^1B_i), A^\bigtriangleup) = S((\bigcup_i {}^1B_i), A^\bigtriangleup)$. As a result, we have $S(A, (\bigcup_i {}^1B_i)^\bigtriangledown) = 1$ iff $S((\bigcup_i {}^1B_i), A^\bigtriangleup) = 1$ iff for each i we have ${}^1B_i \subseteq A^\bigtriangleup$ iff for each i we have $S({}^1B_i, A^\bigtriangleup)$ iff for each i we have $S(A, B_i^\bigtriangledown)$ iff $S(A, \bigcap_i B_i^\bigtriangledown)$.

(15) is just (14) for $|J| = 1$.

(16) and (17) follow from ${}^1b \to \{ {}^a/x \}^\bigtriangleup(y) = a \to \{ {}^b/y \}^\bigtriangledown(x)$ and $\{ {}^1/x \}^\bigtriangleup(y) = \{ {}^1/y \}^\bigtriangledown(x)$ which we now verify. First, ${}^1b \to \{ {}^a/x \}^\bigtriangleup(y) = S(\{ {}^{1b}/y \}, \{ {}^a/x \}^\bigtriangleup) = S(\{ {}^{1\,b}/y \}, \{ {}^a/x \}^\bigtriangleup) = S(\{ {}^a/x \}, \{ {}^b/y \}^\bigtriangledown) = a \to \{ {}^b/y \}^\bigtriangledown(x)$ (here ${}^1b = 1$ for $b = 1$ and ${}^1b = 0$ otherwise). For $\{ {}^1/x \}^\bigtriangleup(y) = \{ {}^1/y \}^\bigtriangledown(x)$ just put $a = b = 1$ in the foregoing equality.

Lemma 8. *Let $\langle {}^\bigtriangleup, {}^\bigtriangledown \rangle$ be a c-Galois connection. Then there is a fuzzy relation $I \in L^{X \times Y}$ such that $\langle {}^\bigtriangleup, {}^\bigtriangledown \rangle = \langle {}^{\bigtriangleup_I}, {}^{\bigtriangledown_I} \rangle$ where ${}^{\bigtriangleup_I}$ and ${}^{\bigtriangledown_I}$ are induced by I by (10) and (11).*

Proof. Let I be defined by $I(x, y) = \{ {}^1/x \}^\bigtriangleup(y) = \{ {}^1/y \}^\bigtriangledown(x)$. Then using (16), it is straightforward to show $A^\bigtriangleup = A^{\bigtriangleup_I}$. Furthermore, using (14) and (15), and (17) we get

$$B^\bigtriangledown(x) = {}^1B^\bigtriangledown(x) = (\bigcup_{y \in Y} \{ {}^{1}B(y)/y \})^\bigtriangledown(x) =$$
$$= (\bigcup_{y \in Y} {}^1\{ {}^{B(y)}/y \})^\bigtriangledown(x) = (\bigcap_{y \in Y} \{ {}^{B(y)}/y \}^\bigtriangledown)(x) = \bigwedge_{y \in Y} \{ {}^{B(y)}/y \}^\bigtriangledown(x) =$$
$$= \bigwedge_{y \in Y} {}^1B(y) \to \{ {}^1/y \}^\bigtriangledown(x) = \bigwedge_{y \in Y} {}^1B(y) \to I(x, y) = B^{\bigtriangledown_I}(x).$$

Next, we have the desired one-to-one correspondence between fuzzy relations and c-Galois connections.

Theorem 4. *Let $I \in L^{X \times Y}$ be a fuzzy relation, let ${}^{\bigtriangleup_I}$ and ${}^{\bigtriangledown_I}$ be defined by (10) and (11). Let $\langle {}^\bigtriangleup, {}^\bigtriangledown \rangle$ be a c-Galois connection. Then*

(1) $\langle {}^{\bigtriangleup_I}, {}^{\bigtriangledown_I} \rangle$ satisfy (12) and (13).
(2) $I_{\langle \bigtriangleup, \bigtriangledown \rangle}$ defined as in the proof of Lemma 8 is a fuzzy relation and we have
(3) $\langle {}^\bigtriangleup, {}^\bigtriangledown \rangle = \langle {}^{\bigtriangleup_{I_{\langle \bigtriangleup, \bigtriangledown \rangle}}}, {}^{\bigtriangledown_{I_{\langle \bigtriangleup, \bigtriangledown \rangle}}} \rangle$ and $I = I_{\langle \bigtriangleup_I, \bigtriangledown_I \rangle}$.

Proof. Due to the previous results, it remains to check $I = I_{\langle \bigtriangleup_I, \bigtriangledown_I \rangle}$. We have $I_{\langle \bigtriangleup_I, \bigtriangledown_I \rangle}(x, y) = \{ {}^1/x \}^{\bigtriangleup_I}(y) = \bigwedge_{z \in X} \{ {}^1/x \}(z) \to I(z, y) = I(x, y)$, completing the proof.

Coming back to conditions (6)–(9), one can easily see that they are in general not satisfied by a c-Galois connection ${}^\bigtriangleup, {}^\bigtriangledown$. The next lemma shows properties of c-Galois connections which are analogous to (6)–(9).

Lemma 9. *For a c-Galois connection* $\langle^{\triangle},^{\triangledown}\rangle$, *we have*

$$S(A_1, A_2) \leq S(A_2^{\triangle}, A_1^{\triangle}) \tag{18}$$

$$S({}^1B_1, {}^1B_2) \leq S(B_2^{\triangledown}, B_1^{\triangledown}) \tag{19}$$

$$A \subseteq A^{\triangle\triangledown} \tag{20}$$

$${}^1B \subseteq B^{\triangledown\triangle} \tag{21}$$

Proof. By direct verification.

2.5 Main Theorem for $\mathcal{B}_c\,(X,Y,I)$

Now we present a version of main theorem of concept lattices for $\mathcal{B}_c\,(X,Y,I)$. Due to the limited scope of the paper, we present only sketch of proof.

Note (see [6]) that for a fuzzy set $E \in L^U$, $\lfloor E \rfloor$ is a subset of $U \times L$ defined by $\lfloor E \rfloor = \{\langle u, a\rangle \in U \times L \mid a \leq E(u)\}$. Conversely, for $F \subseteq U \times L$, $\lceil F \rceil$ is a fuzzy set in U defined by $\lceil F \rceil(u) = \bigvee\{a \mid \langle u, a\rangle \in F\}$. Now, for a fuzzy relation $I \in L^{X\times Y}$, define an ordinary relation I^+ between $X \times L$ and Y by $\langle\langle x, a\rangle, y\rangle \in I^+ \quad \text{iff} \quad a \leq I(x, y)$.

Theorem 5. *The set* $\mathcal{B}_c\,(X,Y,I)$ *equipped with* $\leq$ *is a complete lattice where infima and suprema are given by* (4) *and*

$$\bigvee_{j\in J} \langle A_j, B_j\rangle = \langle \lceil (\bigcup_{j\in J} \lfloor A_j \rfloor)^{\uparrow\downarrow} \rceil, \lceil \bigcap_{j\in J} \lfloor {}^1B_j \rfloor \rceil\rangle\ . \tag{22}$$

Moreover, an arbitrary complete lattice $\mathbf{V} = \langle V, \wedge, \vee\rangle$ *is isomorphic to some* $\mathcal{B}_c\,(X,Y,I)$ *iff there are mappings* $\gamma : X \times L \to V$, $\mu : Y \to V$ *such that* $\gamma(X, L)$ *is* $\bigvee$*-dense in* V, $\mu(Y)$ *is* $\bigwedge$*-dense in* V, *and* $a \leq I(x, y)$ *iff* $\gamma(x, a) \leq \mu(y)$.

Proof. Sketch: Analogously as in [4], we can find a bijection between c-Galois connections between X and Y, and ordinary Galois connections between $X \times L$ and Y. Under this bijection, I (fuzzy relation corresponding an c-Galois connection) corresponds to I^+ (ordinary relation corresponding to a Galois connection) and the corresponding c-Galois connection and Galois connection have isomorphic lattices of fixed points. One of them is our $\mathcal{B}_c\,(X,Y,I)$, the other one is $\mathcal{B}\,(X \times L, Y, I^+)$. Now, $\mathcal{B}\,(X \times L, Y, I^+)$ is an ordinary concept lattice, and thus obeys Wille's Main Theorem [20]. Translating the Main Theorem to $\mathcal{B}_c\,(X,Y,I)$ then gives our theorem.

Corollary 1. $\mathcal{B}_c\,(X,Y,I)$ *is a* $\bigwedge$*-subsemilattice of* $\mathcal{B}\,(X,Y,I)$.

Remark 2. $\mathcal{B}_c\,(X,Y,I)$ need not be a $\bigvee$-subsemilattice of $\mathcal{B}\,(X,Y,I)$. One can verify by taking $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$, and $I(x_1, y_1) = 0.3$, $I(x_1, y_2) = 0.5$, $I(x_1, y_3) = 0.4$, $I(x_2, y_1) = 0.2$, $I(x_2, y_2) = 0.6$, $I(x_2, y_3) = 0.1$.

2.6 Crisply Generated Concepts and One-Sided Fuzzy Concepts

In [22], the authors deal with the following. Let $\langle X, Y, I\rangle$ be a fuzzy context (with $L = [0, 1]$). Define mappings $f : 2^X \to L^X$ (assigning a *fuzzy set* $f(A) \in L^Y$ of attributes to a *set* $A \subseteq X$ of objects) and $h : L^X \to 2^X$ (assigning a *set* $h(B) \subseteq$ of objects to a *fuzzy set* $B \in L^Y$ of attributes) by

$$f(A)(y) = \bigwedge_{x \in A} I(x, y) \tag{23}$$

and

$$h(B) = \{x \in X \mid \text{for each } y \in Y : B(y) \le I(x, y)\}. \tag{24}$$

The same definition was later "rediscovered" by Krajči [17]. Pairs $\langle A, B\rangle \in 2^X \times L^Y$ are called one-sided fuzzy concepts (A is a set, B is a fuzzy set) in [17]. By direct computation one can verify that $\langle A, B\rangle$ is a one sided fuzzy concept iff it is of the form $\langle A, B\rangle = \langle {}^1A', B\rangle$ for some fuzzy concept $\langle A', B\rangle \in \mathcal{B}(X, Y, I)$ which is "crisply generated by extents", i.e. such that for some $\mathbf{L}$ we have $B = C^{\uparrow}$ (and $A' = C^{\uparrow\downarrow}$) for some set $C \subseteq X$. Therefore, up to exchanging roles of extents and intents, [22, 17] in fact deal with particular formal fuzzy concepts (crisply generated by extents) from $\mathcal{B}(X, Y, I)$, only that instead of $\langle A, B\rangle$ they consider $\langle {}^1A, B\rangle$. As a consequence, our result presented in this paper apply in an appropriate modification to one-sided fuzzy concepts of [22, 17].

3 Examples and Experiments

Tab. 1 describes economic indexes of selected countries, transformed to $[0, 1]$ to get a formal fuzzy context. Using minimum-based fuzzy logical operations, the corresponding concept lattice $\mathcal{B}(X, Y, I)$ contains 304 formal concepts and is depicted in Fig. 1. The corresponding set $\mathcal{B}_c(X, Y, I)$ of all crisply generated fuzzy concepts contains 27 formal concepts and is depicted in Fig. 2. As we are interested only in the reduction of the size of the concept lattice, we omit the descriptions of formal concepts.

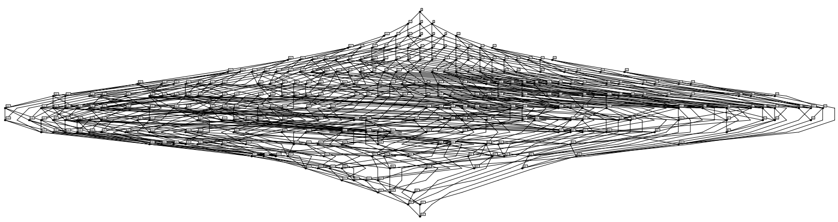

Fig. 1. Concept lattice corresponding to data from Tab. 1

Next we show results of experiments demonstrating the factor of reduction. That is, we are interested in the ratio $r = |\mathcal{B}_c(X, Y, I)|/|\mathcal{B}(X, Y, I)|$ (the smaller, the larger the reduction). Tab. 2 shows the values of r for 10 experiments

Table 1. Economic indexes: data table with fuzzy attributes

	1	2	3	4	5	6	7
1 Czech	0.4	0.4	0.6	0.2	0.2	0.4	0.2
2 Hungary	0.4	1.0	0.4	0.0	0.0	0.4	0.2
3 Poland	0.2	1.0	1.0	0.0	0.0	0.0	0.0
4 Slovakia	0.2	0.6	1.0	0.0	0.2	0.2	0.2
5 Austria	1.0	0.0	0.2	0.2	0.2	1.0	1.0
6 France	1.0	0.0	0.6	0.4	0.4	0.6	0.6
7 Italy	1.0	0.2	0.6	0.0	0.2	0.6	0.4
8 Germany	1.0	0.0	0.6	0.2	0.2	1.0	0.6
9 UK	1.0	0.2	0.4	0.0	0.2	0.6	0.6
10 Japan	1.0	0.0	0.4	0.2	0.2	0.4	0.2
11 Canada	1.0	0.2	0.4	1.0	1.0	1.0	1.0
12 USA	1.0	0.2	0.4	1.0	1.0	0.2	0.4

attributes: 1 - high gross domestic product per capita (USD), 2 - high consumer price index (1995=100) , 3 - high unemployment rate (percent - ILO), 4 - high electricity production per capita (kWh), 5 - high energy consumption per capita (GJ), 6 - high export per capita (USD), 7 - high import per capita (USD)

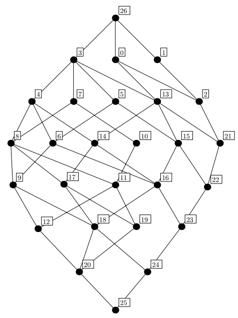

Fig. 2. Crisply generated formal concepts corresponding to data from Tab. 1

(columns) run over randomly generated formal contexts (rows) with the number of objects equal to the number of attributes (from 5 to 25 objects/attributes) and with $|L| = 11$ (11 truth degrees). Moreover, we show average and dispersion of r. We can see that the dispersion is low and that r decreases with growing size of data. Further experiments need to be run to show in more detail the behavior of r. In the second experiment, we randomly generated tables with 20 objects and 20 attributes, $|L| = 11$ with minimum-based fuzzy conjunction, each object with 10 attributes with a degree > 0 (and 10 attributes with a degree $=0$); of the ten attribute with nonzero degrees, we varied the number of attributes, from 1 to 10 (rows), with degree $= 1$; columns represent experiments; we consider average and dispersion of r, see Tab. 3. In the third experiment, we randomly generated tables with 20 objects and 20 attributes, $|L| = 11$ with minimum-based fuzzy conjunction, each object with varying number of attributes with a

Table 2. Behavior of r (average Av, dispersion Var) in dependence on the size of input data table (rows); columns correspond to experiments

	1	2	3	4	5	6	7	8	9	10	Av	Var
5	0.58	0.4	0.38	0.53	0.48	0.38	0.43	0.41	0.48	0.33	0.441	0.0733
6	0.31	0.31	0.38	0.43	0.38	0.32	0.43	0.42	0.36	0.38	0.372	0.0443
7	0.46	0.37	0.31	0.48	0.45	0.27	0.41	0.43	0.4	0.37	0.395	0.0635
...	...	...	...	...	...	...	...	...	...	...	...	...
23	0.09	0.11	0.1	0.1	0.1	0.1	0.1	0.09	0.1	0.11	0.099	0.0066
24	0.1	0.09	0.08	0.1	0.09	0.1	0.09	0.09	0.09	0.08	0.090	0.0079
25	0.08	0.07	0.09	0.08	0.09	0.09	0.08	0.07	0.09	0.08	0.081	0.0074

Table 3. Dependence of r (average Av, dispersion Var) on the number of 1's in object attributes (rows); columns correspond to experiments

	1	2	3	4	5	6	7	8	9	10	Av	Var
1	0.1	0.08	0.07	0.08	0.08	0.09	0.1	0.08	0.08	0.08	0.084	0.0100
2	0.09	0.1	0.11	0.11	0.08	0.09	0.1	0.08	0.09	0.09	0.094	0.0086
3	0.12	0.11	0.11	0.1	0.11	0.11	0.12	0.1	0.1	0.11	0.107	0.0084
4	0.12	0.13	0.11	0.14	0.14	0.13	0.13	0.12	0.14	0.14	0.130	0.0093
5	0.18	0.15	0.15	0.16	0.16	0.16	0.16	0.17	0.16	0.17	0.162	0.0095
6	0.18	0.21	0.17	0.17	0.2	0.2	0.2	0.18	0.2	0.22	0.193	0.0151
7	0.24	0.24	0.26	0.26	0.28	0.26	0.22	0.27	0.25	0.28	0.256	0.0193
8	0.36	0.33	0.34	0.36	0.35	0.35	0.33	0.34	0.37	0.35	0.347	0.0121
9	0.54	0.59	0.55	0.56	0.53	0.48	0.51	0.56	0.52	0.55	0.539	0.0298
10	1	1	1	1	1	1	1	1	1	1	1,000	0.0000

Table 4. Dependence of r (average Av, dispersion Var) on the number of nonzero values in attributes (rows); columns correspond to experiments

	1	2	3	4	5	6	7	8	9	10	Av	Var
1	0.59	0.59	0.64	0.56	0.62	0.56	0.58	0.63	0.61	0.63	0.600	0.0273
2	0.55	0.55	0.47	0.49	0.59	0.43	0.51	0.47	0.55	0.44	0.504	0.0503
3	0.62	0.43	0.57	0.5	0.48	0.54	0.47	0.48	0.61	0.45	0.514	0.0642
...	...	...	...	...	...	...	...	...	...	...	...	...
13	0.06	0.07	0.08	0.07	0.08	0.07	0.07	0.08	0.06	0.05	0.067	0.0089
14	0.03	0.05	0.04	0.06	0.04	0.07	0.04	0.04	0.04	0.03	0.044	0.0113
15	0.04	0.05	0.06	0.07	0.05	0.04	0.06	0.08	0.05	0.07	0.058	0.0117

degree > 0 (the number varies from 1 to 15, rows); columns represent experiments; we consider average and dispersion of r, see Tab. 4. Next, we randomly generated input data tables with 20 objects and 20 attributes with varying $|L|$ for $|L| = 3, 6, 11, 16, 21, 31$ (rows), see Tab. 5; columns represent experiments; we consider average and dispersion of r.

In the last experiment, we observed the speed-up of the algorithm described in Section 2.3 compared to just using [8] and testing which concepts are crisply generated, see Remark 1. We randomly generated several data tables with di-

Table 5. Dependence of r (average Av, dispersion Var) on the number $|L|$ of truth degrees (rows); columns correspond to experiments

	1	2	3	4	5	6	7	8	9	10	Av	Var
3	0.36	0.34	0.34	0.34	0.33	0.3	0.34	0.42	0.33	0.36	0.346	0.0286
6	0.18	0.21	0.19	0.26	0.23	0.21	0.24	0.17	0.18	0.18	0.205	0.0312
11	0.17	0.23	0.16	0.21	0.17	0.18	0.2	0.17	0.19	0.22	0.187	0.0244
16	0.13	0.15	0.22	0.14	0.18	0.17	0.17	0.15	0.16	0.16	0.163	0.0220
21	0.14	0.18	0.15	0.15	0.15	0.15	0.16	0.13	0.13	0.16	0.150	0.0151
26	0.14	0.2	0.14	0.14	0.13	0.16	0.16	0.11	0.11	0.18	0.147	0.0280
31	0.17	0.14	0.13	0.1	0.15	0.14	0.13	0.15	0.16	0.21	0.147	0.0276

mensions 50 objects × 50 attributes to 70 objects × 70 attributes with $|L| = 6$ under a constraint that each object has 10 attributes with a nonzero degree and 4 of these equal 1. The graph in Fig. 3 demonstrates the speed-up in dependence on the size of input data (50 to 70), i.e. the ratio T/T_c where T is the time needed for computing the whole $\mathcal{B}(X, Y, I)$ and using Lemma 1 to test whether each concept is crisply generated, and T_c is the time needed by the algorithm from Section 2.3.

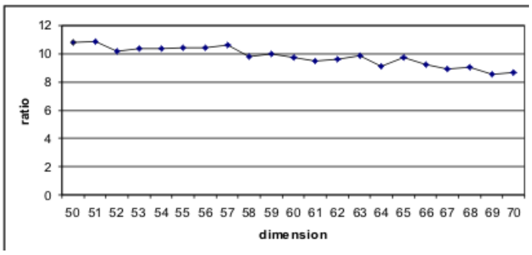

Fig. 3. Speed-up of algorithm from Section 2.3, see Remark 1

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