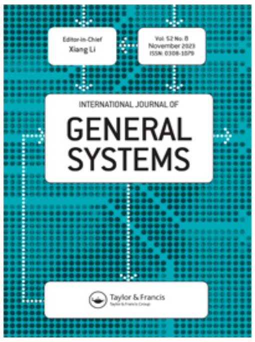

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On the concept of fuzzy order I: Remarks and observations

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ABSTRACT
We consider the concept of fuzzy order in which antisymmetry is intrinsically connected to a many-valued equality on the underlying universe. We examine the origins of this concept, provide remarks and observations on the existing studies, and prove new results. In part I, we scrutinize the existing approaches to the examined concept of fuzzy order and present remarks and results to elucidate the available notions and findings, as well as to provide a deeper insight into several issues. In part II, we explore antisymmetry.

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1. Aim of this paper

The concept of order is one of the basic concepts accompanying human reasoning. Correspondingly, orders – known also as partial orders or orderings – became a widely studied kind of relations, which are utilized across a variety of fields. Recall that a classical order on a set U is a binary relation $\leq$ on U that is reflexive, antisymmetric, and transitive, i.e. satisfies $u \leq u$; $u \leq v$ and $v \leq u$ implies $u = v$; and $u \leq v$ and $v \leq w$ implies $u \leq w$ for all $u, v, w \in U$. In addition to a classical, bivalent setting, the concept of order makes a good sense in a more general setting, in which bivalence is replaced by graduality. For instance, instead of conceiving inclusion, which is a particular example of order, as bivalent, one may consider an entity as being included in another entity to a certain degree. It hence comes as no surprise that generalized orders – known as fuzzy orders – in which ordering is a matter of degree represent a thoroughly studied subject.

Since the pioneering paper by Zadeh (1971),[1] a number of definitions of the concept of fuzzy order have been proposed.[2] In our paper, we are concerned with the arguably most developed approach to fuzzy orders, pursued originally by Ulrich Höhle, Nicole Blanchard, Ulrich Bodenhofer, and Radim Belohlavek. The distinctive feature of this approach is the treatment of antisymmetry: The approach assumes that the set on which a fuzzy order is defined is equipped with a fuzzy relation that generalizes ordinary equality, which is involved in classical antisymmetry. This approach actually subsumes two particular definitions of antisymmetry, which shall be examined in detail below.

Even though a number of papers on fuzzy orders have been published after the pioneering works by the above-mentioned authors, some basic questions still remain open

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or answered partially only. Most important among them is the basic question of what is actually an appropriate definition of fuzzy order.

Note at this point that such question should not be regarded as a quest for "the right" definition of fuzzy order, which might rightfully be considered as ill posed. Namely, in the more general setting of fuzzy logic, different situations may require different definitions of fuzzy order, each of which may serve the intended purpose in the particular situation. Yet, all of these definitions may indeed be proper generalizations of the classical concept of order.[3]

Correspondingly, rather than looking for "the right" definition of fuzzy order, the question mentioned in the previous paragraph is to be understood as a question of ramifications of and relationships between possible definitions of fuzzy orders. Exploration of this question as regards the above approach to fuzzy order, i.e. involving antisymmetry with respect to generalized equality, is the primary purpose of our paper.

2. Definitions of fuzzy order

Throughout the rest of this paper, we assume a framework for dealing with fuzzy sets and fuzzy relations that is based on complete residuated lattices used as the structures of truth degrees. For details, we refer to Appendix. In particular, we denote an arbitrary complete residuated lattice by $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$.

2.1. Preliminary considerations

A transfer of an ordinary concept to a fuzzy setting, i.e. a generalization of a concept defined in the framework of classical logic to a fuzzy logic framework, involves two aspects: The first one is obvious and requires that the generalized concept indeed be a generalization of the ordinary concept. The second one, which is somewhat vague and much less trivial, asks that the generalized concept be useful and behave naturally. In this broader perspective, which forms the starting point of our considerations, the first aspect concerns mathematical correctness, while the second pertains to mathematical practice.

However, trivial the imperative of the first aspect appears, it is worth noting that it may be understood several ways. Our understanding, which has become common in the past two decades or so, is as follows. A definition of the generalized concept assumes a general structure $\mathbf{L}$ of truth degrees and is expressed by appropriate conditions. For instance, a binary fuzzy relation $R : U \times U \rightarrow L$ is called transitive if the condition

$$R(u, v) \otimes R(v, w) \leq R(u, w) \tag{1}$$

holds true for each $u, v, w \in U$. Now, one may consider the structure $\mathbf{L}$ of truth degrees as a parameter and consider the definition for arbitrary $\mathbf{L}$. Those $\mathbf{L}$'s include, e.g. the real unit interval $L = [0, 1]$ and Łukasiewicz operations, but also – as a very particular case – the two-valued Boolean algebra $\mathbf{L} = \mathbf{2}$ of classical logic in which $L = \{0, 1\}$, i.e. the classical truth values 0 and 1 are the only recognized degrees of truth. Taking $\mathbf{L} = \mathbf{2}$ means that the fuzzy relation R is in fact a two-valued relation and may thus be identified with the corresponding ordinary relation $o(R) = \{\langle u, v\rangle \mid R(u, v) = 1\}$. As one then easily checks, R is transitive in the sense of definition (1) if and only if the ordinary relation $o(R)$ is classically transitive, i.e. $\langle u, v\rangle \in o(R)$ and $\langle v, w\rangle \in o(R)$ imply $\langle u, w\rangle \in o(R)$ for each $u, v, w \in U$. It is in this sense that the definition (1) of transitivity of a fuzzy relation *generalizes* the classical definition.[4]

Usefulness and natural behavior, i.e. the second aspect of generalizing a classical concept to a fuzzy setting, basically implies that the generalized concept be useful in modeling of reality, have nice properties, and be connected to other concepts in the generalized framework in an analogous way the classical concept is in the classical framework. For instance, when generalizing the concept of an equivalence relation to a fuzzy setting, it is desired that the generalized concept of fuzzy equivalence provides a reasonable model of indistinguishability in a setting that allows for gradual indistinguishability, and that it is naturally connected to an appropriately defined concept of a fuzzy partition.

It is immediate that meeting the requirement for the generalized concept to be indeed a generalization of the corresponding ordinary concept does not imply that the second requirement is satisfied, i.e. usefulness and natural behavior of the generalized concept. To meet both of the above-outlined criteria, one needs to "experiment" and "play" with the generalized concept, i.e. explore its properties in the generalized framework and possibly modify its definition, until a generalized concept comes up that is useful and behaves naturally from the viewpoint of the concerned needs.[5]

According to the above rationale, to define a reasonable concept of fuzzy order not only requires to provide a generalization of classical orders but also to examine thoroughly the properties of such generalization with regard to notions which are relevant when considering gradual ordering. Since classical orders are reflexive, antisymmetric, and transitive relations, it appears reasonable to define generalized conditions of reflexivity, antisymmetry, and transitivity, and define fuzzy orders as fuzzy relations that satisfy these generalized conditions. Generalizing reflexivity and transitivity appears immediate: reflexivity of a fuzzy relation $R : U \times U \to L$ means $R(u,u) = 1$ for each $u \in U$, while transitivity of R is defined by (1). These two definitions have been proven useful and naturally behaving by a great number of studies. Generalizing antisymmetry, however, is much less immediate.

To illustrate our point, let us recall the pioneering paper by Zadeh (1971), in which a fuzzy relation R is considered antisymmetric if

$$R(u,v) > 0 \text{ and } R(v,u) > 0 \text{ imply } u = v \tag{2}$$

for each $u, v \in U$. As one easily checks, this definition generalizes classical antisymmetry, and hence is mathematically correct. However, it has serious drawbacks, of which we present the following one.

In the classical setting, one of the most important examples of orders is represented by inclusion $\subseteq$ of sets: $\subseteq$ is a reflexive, antisymmetric, and transitive relation, i.e. the pair $\langle 2^U, \subseteq \rangle$ is an ordered set, for any set U. However, the graded inclusion $\tilde{\subseteq}$ of fuzzy sets on U, defined by (A3) in Appendix, does not satisfy Zadeh's antisymmetry (2).[6] From the above viewpoint, the reason is that Zadeh did not put his definition to a proper test, i.e. did not derive his definition from natural examples and did not consider it in a proper context of relevant mathematical considerations. In a sense, (2) provides a formalistic approach to antisymmetry, which is mathematically correct but has a rather limited use.

This example is not to criticize Zadeh, who typically had been deriving his notions from natural examples, nor to criticize several other papers and even textbooks, such as the widely circulated (Klir and Yuan 1995), which adopted Zadeh's definition, or proposed different definitions, with similar drawbacks.[7] Rather, we intend to emphasize the importance of putting notions generalized to the setting of fuzzy logic to proper tests involving concepts and theories with which the generalized notions shall interact properly.

2.2. Definitions of fuzzy order on a set with generalized equality

From today's perspective, the approach we examine in our paper may be regarded as alleviating the drawbacks of formalistic attempts as the one outlined above. The key idea of this approach is to consider as fuzzy (graded, many-valued) not only the order relation on the universe set U but also the equality relation on U. That is to say, one considers a universe U, a fuzzy relation $\lesssim$ generalizing classical order, and a fuzzy relation $\approx$ generalizing classical equality. Even without further exploration, such approach appears well thought out because the classical theory of ordered sets refers to equality on many occasions including the definition of antisymmetry.

Remark 1: From an epistemic viewpoint, it is even tempting – when considering a fuzzy order $\lesssim$ on U – to assume that $\lesssim$ is defined on a set U on which a generalized equality $\approx$ is given already. This view agrees with the classical situation in which equality $=$ is implicitly understood as being given on the considered universe set. In drawing conclusions of this sort, though, one has to be careful because an alternative view is also possible in which even in a fuzzy setting, U may be regarded as equipped with classical equality $=$ only, and both $\lesssim$ and $\approx$ may be understood as further entities with the provision that in the classical case, $=$ coincides with $\approx$.[8]

The approach we explore has been initiated in the pioneering works of Höhle, Blanchard, Bodenhofer, and Belohlavek; see e.g. (Belohlavek 2001, 2002, 2004; Blanchard 1989; Bodenhofer 2000, 2003; Bodenhofer and Klawonn 2004; Höhle 1987; Höhle and Blanchard 1985). It appeared for the first time in the paper by Höhle and Blanchard (1985) and was apparently rediscovered later by Bodenhofer and Belohlavek, whose notions of fuzzy order differ from each other in the antisymmetry condition. While Bodenhofer's antisymmetry coincides with that of Höhle and Blanchard (1985), Belohlavek's antisymmetry is different and essentially coincides with antisymmetry proposed in yet another paper by Höhle (1987) whose purpose is a study of Dedekind's construction of real numbers in a fuzzy setting. Both Bodenhofer and Belohlavek studied their notions of fuzzy order in several subsequent papers. They both were motivated by rather different goals and have not examined the relationships of their two notions of fuzzy order to any deeper extent.

Let us note at this point that a related notion of fuzzy order has independently been introduced by Blanchard (1983) and Fan (2001). This notion does not involve fuzzy equality on the underlying universe set U and is formulated in a slightly restricted framework. Nevertheless, its generalization to the framework of complete residuated lattices turns out to be equivalent in a sense with the approach utilizing fuzzy equalities. We present details on this topic in part II (Belohlavek and Urbanec 2023).

We now provide the two definitions of fuzzy order on a set with a generalized equality. We provide them basically in the forms present in the works of Bodenhofer and Belohlavek, respectively, since these forms are most common in the literature; the definitions which appeared in the works by Höhle are just mild variations of the definitions we present. Detailed comments on the definitions are presented below.

Definition 1 (Höhle, Blanchard, Bodenhofer): A fuzzy order on a set U equipped with a fuzzy equality relation $\approx$ is a binary fuzzy relation $\precsim$ on U satisfying

$$u \approx v \ \leq \ u \precsim v \qquad (\approx\text{-reflexivity}),$$
$$(u \precsim v) \otimes (v \precsim w) \ \leq \ u \precsim w \qquad (\text{transitivity}),$$
$$(u \precsim v) \otimes (v \precsim u) \ \leq \ u \approx v \qquad (\otimes\text{-antisymmetry}),$$

for each $u, v, w \in U$. (Note: Höhle and Blanchard's as well as Bodenhofer's original definitions actually assume, more generally, that $\approx$ is a fuzzy equivalence rather than fuzzy equality; this is discussed below.)

Definition 2 (Höhle, Belohlavek): A fuzzy order on a set U equipped with a fuzzy equality relation $\approx$ is a binary fuzzy relation $\precsim$ on U compatible with $\approx$, i.e. fulfilling

$$(u_1 \precsim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \ \leq \ u_2 \precsim v_2,$$

for every $u_1, u_2, v_1, v_2 \in U$, which satisfies

$$u \precsim u \ = \ 1 \qquad (\text{reflexivity}),$$
$$(u \precsim v) \otimes (v \precsim w) \ \leq \ u \precsim w \qquad (\text{transitivity}),$$
$$(u \precsim v) \wedge (v \precsim u) \ \leq \ u \approx v \qquad (\wedge\text{-antisymmetry}),$$

for each $u, v, w \in U$.

Remark 2 (nomenclature): (a) If distinction is needed, we shall call fuzzy orders according to Definitions 1 and 2 fuzzy orders with $\otimes$-antisymmetry and fuzzy orders with $\wedge$-antisymmetry, respectively.

(b) Various terms are used in the literature, e.g. partial ordering in L-underdeterminate sets (Höhle and Blanchard 1985), T-E-ordering (Bodenhofer),[9] partial ordering on an I-valued set (Höhle 1987), and **L**-order on a set with **L**-equality (Belohlavek). Also note that instead of complete residuated lattices, Höhle and Blanchard (1985) use somewhat more particular structures.[10] Höhle (1987) and Bodenhofer use $L = [0, 1]$ with a left-continuous t-norm.[11] Let us also mention that Höhle (1987) uses a more general concept of fuzzy equality inspired by category-theoretical considerations, in which the degree $u \approx u$ may be strictly smaller than 1 and is interpreted as an extent to which u exists.

(c) The structure consisting of U, $\approx$, and $\precsim$ as described in the definitions is called a fuzzy ordered set, and is denoted $\langle U, \approx, \precsim \rangle$, or $\langle \langle U, \approx \rangle, \precsim \rangle$ if the distinct role of $\approx$, e.g. resulting from epistemic preference as discussed in Remark 1, is to be emphasized. In the latter case, one naturally speaks of a fuzzy order $\precsim$ on a set with fuzzy equality $\langle U, \approx \rangle$.

Remark 3 (basic relationships): (a) One may observe two distinctions when comparing Definition 1 with Definition 2. First, the definitions use different forms of antisymmetry, with the stronger $\wedge$-antisymmetry implying the weaker $\otimes$-antisymmetry. Basic relationships of these two forms of antisymmetry are addressed in Section 3.3; a thorough consideration of antisymmetry is presented in the second part of this paper. Second, Definition 1 requires $\approx$-reflexivity of the fuzzy order $\lesssim$ while Definition 2 requires that $\lesssim$ be reflexive and compatible with $\approx$. Note at this point that in presence of the other conditions, these two requirements are equivalent and that we discuss this relationship in Section 3.2. It hence follows from the facts just mentioned that Definition 1 delineates a more general notion of fuzzy order than Definition 2 in the following sense: If $\langle U, \approx, \lesssim \rangle$ is a fuzzy ordered set according to Definition 2, it also is a fuzzy ordered set according to Definition 1, but not vice versa.[12] This view, however, is thoroughly reconsidered in part II of our paper, in which an alternative view is provided.

(b) As noted in Definition 1, the original definitions of Höhle, Blanchard, and Bodenhofer assume that $\approx$ is a fuzzy equivalence rather than a fuzzy equality. A fuzzy equality is a more particular concept than a fuzzy equivalence since it additionally satisfies separation; see Appendix. We nevertheless assume that $\approx$ is a fuzzy equality in Definition 1 since this assumption yields, in a sense, a cleaner generalization of the notion of order to the setting of fuzzy logic, which may moreover be better compared to the notion of fuzzy order from Definition 2; see Section 3.1 for details. Still, we shall speak of Definition 1 as the definition by Höhle, Blanchard, and Bodenhofer, as no confusion arises in view of the present remark.

Note also that in their definition of fuzzy order, Höhle and Blanchard (1985) in fact use – somewhat misleadingly – the term "L-equality" to denote a fuzzy equivalence.

(c) With respect to the problem with Zadeh's antisymmetry and graded inclusion, it is immediate and nowadays well known that for any set X, graded inclusion $\subsetsim$ becomes a fuzzy order with $\otimes$-antisymmetry on $U = L^X$ when one considers $A \approx B = (A \subsetsim B) \otimes (B \subsetsim A)$ for fuzzy sets $A, B \in L^X$, and becomes a fuzzy order with $\wedge$-antisymmetry on L^X when one considers $A \approx B = (A \subsetsim B) \wedge (B \subsetsim A)$. Note that in both cases, $\approx$ is a fuzzy equality on L^X. These observations appear in their respective forms in Belohlavek (2002); Bodenhofer (1999); Höhle (1987).

(d) It is well known and important for algebraic investigations of logic that the truth function $\rightarrow$ of classical implication defines an order $\leq$ on the set $L = \{0, 1\}$ of classical truth values by letting $a \leq b$ iff $a \rightarrow b = 1$. In other words, $\rightarrow$ is the characteristic function of $\leq$. A natural generalization of this property holds true in the present framework (recall: the two-valued Boolean algebra is a particular case of a complete residuated lattice): For any complete residuated lattice $\mathbf{L}$, the function $\rightarrow$ (i.e. residuum, or truth function of implication) is a fuzzy order with $\otimes$-antisymmetry on the set L of truth degrees equipped with the fuzzy equality defined by $a \leftrightarrow_{\otimes} b = (a \rightarrow b) \otimes (b \rightarrow a)$. Furthermore, $\rightarrow$ is a fuzzy order with $\wedge$-antisymmetry when the fuzzy equality is defined by $a \leftrightarrow_{\wedge} b = (a \rightarrow b) \wedge (b \rightarrow a)$. These observations appear in works by Belohlavek (2002); Bodenhofer (1999); Höhle and Blanchard (1985).[13] Note that the examples in the present condition (d) may be regarded as special cases of those of condition (c) of the present remark, because L may be identified with L^U for a singleton $U = \{u\}$, in which case $\subsetsim$ becomes $\rightarrow$ and $\approx$ becomes $\leftrightarrow_{\otimes}$ or $\leftrightarrow_{\wedge}$, respectively.

(e) For $L = [0, 1]$ (or, more generally, a linearly ordered L), $\otimes = \wedge$, and $\approx$ coinciding with ordinary equality (i.e. $u \approx v = 1$ for $u = v$ and $u \approx v = 0$ for $u \neq v$), both

$\otimes$-antisymmetry and $\wedge$-antisymmetry are equivalent to Zadeh's antisymmetry (2), which – for $\otimes$-antisymmetry – is mentioned by Bodenhofer (1999) and Höhle and Blanchard (1985).

Remark 4 (historical comments): (a) Definition 1 was – with the conditions listed above but, as noted, with fuzzy equivalence rather than fuzzy equality – proposed for the first time by Höhle and Blanchard (1985), who aimed to improve and further study the concept of order in the setting of fuzzy logic originally introduced by Zadeh (1971). This definition was later reinvented by Bodenhofer, who was apparently not aware of Höhle and Blanchard's work. Bodenhofer does not cite this work in his first papers (1999; 2000), but cites it in his next paper (2003), in which he acknowledges Höhle and Blanchard's historical priority.

(b) Definition 2 was proposed for the first time by Höhle (1987) with a more particular choice of structures **L** of truth degrees (namely, complete residuated lattices on $[0,1]$) but with a more general concept of fuzzy equality; cf. Remark 2 (b). It was later reinvented by Belohlavek who was not aware of Höhle's paper.

(c) It is worth noting that the motivation in the works investigating the notion of fuzzy order according to Definition 1, i.e. by Höhle and Blanchard (1985) and by Bodenhofer, was basically a general study of fuzzy order. The motivation in the first works exploring fuzzy orders according to Definition 2, i.e. by Höhle (1987) and by Belohlavek, was more particular, namely to study certain ordered structures determined by binary fuzzy relations. In particular, Höhle studied the so-called Dedekind cuts, while Belohlavek studied so-called concept lattices; both of these structures are strongly related (put briefly, Dedekind cuts are a particular case of concept lattices).

(d) Interestingly, Höhle (1987) does not comment on and does not cite his previous definition of fuzzy order (Höhle and Blanchard 1985), i.e. does not mention why he changed his definition for the purpose of his 1987 paper.

(e) Even though neither Bodenhofer nor Belohlavek were initially familiar with Höhle's work on fuzzy orders, both were strongly influenced by Höhle's work on fuzzy logic.

(f) Bodenhofer and Belohlavek discussed their works on fuzzy order at the FSTA 1998 conference in Liptovský Ján, at which point most of the results of their first papers were worked out, but they never got to comparing their approaches. Note also that Belohlavek's first paper (Belohlavek 2004) got stuck in the production process: As is apparent from the acknowledgment in this paper and from Belohlavek (2001), the 2004 paper was submitted in 2000.

In the remainder of this paper, we shall examine the properties, relationships, and ramifications of the two notions of fuzzy order in detail.

3. Observations and results

We now present our observations on the two notions of fuzzy order. We also aim at providing and putting in context the existing results and, in particular, attempt to clarify relationships between the various conditions involved by providing clean statements.

To provide a deeper insight, we not only consider the conditions involved in the definitions of fuzzy order but also consider truth degrees to which these conditions are satisfied, such as the degree to which a fuzzy relation $\precsim$ is reflexive or transitive. This is because considerations of these degrees and the respective relationships provide a deeper understanding of the concerned notions. We therefore start by recalling the definition of the degrees of relevant properties of fuzzy relations.

For binary fuzzy relations R and $\approx$ on U, we define

$$\mathrm{ref}(R) = \bigwedge_{u\in U} R(u,u), \tag{3}$$

$$\mathrm{ref}_{\approx}(R) = \bigwedge_{u,v\in U} ((u \approx v) \to R(u,v)), \tag{4}$$

$$\mathrm{sym}(R) = \bigwedge_{u,v\in U} (R(u,v) \to R(v,u)), \tag{5}$$

$$\mathrm{tra}(R) = \bigwedge_{u,v,w\in U} ((R(u,v) \otimes R(v,w)) \to R(u,w)), \tag{6}$$

$$\wedge\text{-ant}(R) = \bigwedge_{u,v\in U} ((R(u,v) \wedge R(v,u)) \to (u \approx v)), \tag{7}$$

$$\otimes\text{-ant}(R) = \bigwedge_{u,v\in U} ((R(u,v) \otimes R(v,u)) \to (u \approx v)), \tag{8}$$

$$\mathrm{comp}(R) = \bigwedge_{u_1,u_2,v_1,v_2\in U} ((R(u_1,v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2)) \to R(u_2,v_2)). \tag{9}$$

The degrees $\mathrm{ref}(R)$, $\mathrm{ref}_{\approx}(R)$, $\mathrm{sym}(R)$, $\mathrm{tra}(R)$, $\wedge$-ant(R), $\otimes$-ant(R), and $\mathrm{comp}(R)$ are called the degree of reflexivity, $\approx$-reflexivity, symmetry, transitivity, $\wedge$-antisymmetry, and $\otimes$-antisymmetry of R, and the compatibility of R with $\approx$, respectively.

Remark 5: (a) The above degrees have a clear meaning. For instance, $\mathrm{ref}(R)$ and $\mathrm{sym}(R)$ are just the truth degrees of the first-order formulas[14]

$$(\forall x)r(x,x) \quad \text{and} \quad (\forall x)(\forall y)(r(x,y) \Rightarrow r(y,x)),$$

respectively, i.e. formulas verbally described as “for each x, x is related to x” and “for each x and y, if x is related to y then y is related to x”; similarly for the other degrees.

(b) Observe that the degree $\mathrm{ref}_{\approx}(R)$ is just the degree $\approx \subsetapprox R$ to which $\approx$ is included in R; cf. (A3) in Appendix.

(c) Since $\bigwedge_{j\in J} a_j = 1$ iff $a_j = 1$ for each $j \in J$, we obtain that $\mathrm{ref}(R) = 1$ if and only if R is reflexive. The same holds for the other properties, hence the degrees of the properties naturally generalize the respective bivalent properties.

(d) Grades of properties (or graded properties) of fuzzy relations were studied by Gottwald (1993, 2001) and Belohlavek (2002), and were later systematically examined within the effort by Běhounek and Cintula (2006).

(e) Alternatively, one can consider a fuzzy relation a-reflexive for a given truth degree $a \in L$ if $a \leq R(u,u)$ for each $u \in U$. One may then check that

$$\mathrm{ref}(R) = \bigvee\{a \in L \mid \mathrm{R\ is}\ a\text{-reflexive}\};$$

the same holds true for the other properties.

(f) The notations $\wedge$-ant(R) and $\otimes$-ant(R) assume that $\approx$ is obvious from the context; alternatively, one could use "$\wedge$-$\approx$-ant(R)" and "$\otimes$-$\approx$-ant(R)." The same applies to comp($\lesssim$).

3.1. Fuzzy order on a set with fuzzy equality versus fuzzy equivalence

As briefly discussed in Remark 3 (b), Definition 1 differs from the original definition of fuzzy order by Bodenhofer (1999) and Höhle and Blanchard (1985) in that it assumes that $\approx$ is a fuzzy equality rather than fuzzy equivalence. We now briefly examine this distinction since it is conceptually significant and has not been properly addressed in the literature.

In our view, assuming a fuzzy equivalence instead of fuzzy equality in Definition 1 does not represent a direct generalization of the notion of order to a fuzzy setting. Rather, it represents a generalization that proceeds along two lines simultaneously. First, the two-valued Boolean algebra is replaced by the more general complete residuated lattice. Second, equality is replaced by equivalence.

This is also apparent when one examines what results from these two notions – i.e. fuzzy order on a set with a fuzzy equality per Definition 1 and fuzzy order on a set with a fuzzy equivalence per the original definition by Höhle, Blanchard, and Bodenhofer – if the definitions are considered within the classical setting. Consider thus both definitions with the structure **L** of truth degrees being the two-element Boolean algebra **2** of classical logic.

On the one hand, the notion resulting from Definition 1 coincides with the classical notion of order because a fuzzy equality becomes classical equality, and the defining conditions become classical reflexivity, transitivity, and antisymmetry.

On the other hand, the notion which results from the definition of a fuzzy order on a set with a fuzzy equivalence is not the notion of a classical order. Rather, it is a notion of classical relation $\leq$ on a set U, on which a classical equivalence $\equiv$ is defined, such that $\leq$ contains $\equiv$, is transitive, and satisfies a generalized form of antisymmetry in that $u \leq v$ and $v \leq u$ implies $u \equiv v$. As $\leq$ contains $\equiv$ and as $\equiv$ is reflexive, $\leq$ is reflexive as well. Moreover, since $\equiv$ is contained in $\leq$, we obtain that

$$u \equiv v \quad \text{if and only if} \quad u \leq v \text{ and } v \leq u.$$

In terms of standard notions of ordered sets (Birkhoff 1967; Blyth 2005; Davey and Priestley 2002; Grätzer 2007), this means that $\leq$ is a quasiorder (preorder in an alternative terminology; i.e. is reflexive and transitive) and $\equiv$ is just the equivalence that is used to make the quasiorder to an order by a well-known factorization.

Let us point out that it is clear from Bodenhofer's papers that he was aware of this property of the definition of fuzzy order assuming fuzzy equivalence (Bodenhofer 2000, 2003). In particular, Bodenhofer (2003, 123) says:

> Although this is most often not mentioned explicitly, many orderings in classical mathematics are in fact only preorderings that may be understood as orderings by considering some factorization. . . . In contrast to most classical cases, however, we do not use the projection of a given preordering to the factor set with respect to the underlying equivalence relation defined by the symmetric kernel, but include the equivalence relation in the axioms of the ordering explicitly. This might look like a significant deviation from the classical formulation, however, the two ways are logically equivalent.

Although we basically agree with Bodenhofer's remarks, we find it necessary to obey the maxim, according to which a generalization of a classical concept to the setting of fuzzy logic needs to behave as explained above. That is, the generalized concept needs to become the original classical concept when considered in the classical setting, i.e. when the considered structure of truth degrees is the two-element Boolean algebra. This is why we prefer Definition 1 assuming a fuzzy equality instead of fuzzy equivalence. In addition to the above reason, ramifications of Definition 1 and Definition 2 may more directly be compared when both definitions assume a fuzzy equality.

The notion of a (fuzzy) order on a set with a (fuzzy) equivalence also implies some inconvenient properties compared to the ordinary notion of order. An example is the fact that important distinguished elements, such as largest and smallest elements or suprema and infima are then not unique. Rather, they are unique just up to the equivalence. We illustrate this property by the following example in the classical setting.

Example 1: Let $U = \{u, v, w\}$, let a classical equivalence $\equiv$ be given by the equivalence classes $\{u\}$ and $\{v, w\}$. Then the relation $\leq$ given by $u \leq u$, $v \leq v$, $w \leq w$, $u \leq v$, $u \leq w$, $v \leq w$, and $w \leq v$ is an order on a set with an equivalence in the sense of Höhle, Blanchard, and Bodenhofer. Defining naturally a smallest element x as an element such that $x \leq y$ for every y, and dually for a largest element, it is immediate that u is the only smallest element. On the other hand, both v and w are largest, even though these are two distinct elements.

With respect to the last paragraph, let us note that the non-uniqueness of distinguished elements can be handled by developing the theory in an appropriate manner, but the resulting theory is not likely to be straightforward. This is apparent e.g. from studies of the notion of a lattice in quasiordered sets initiated by Chajda (1992).

Remark 6: It is easy to check the following claim: If $\lesssim$ is a fuzzy order on the set U with a fuzzy equivalence $\approx$ in the sense of Höhle, Blanchard, and Bodenhofer, one may – generalizing in a straightforward manner the well-known classical construction of order from a quasiorder – consider the factor set $U' = U/E$ of U by the ordinary equivalence E defined by

$$\langle u, v \rangle \in E \quad \text{if and only if} \quad u \approx v = 1,$$

and define fuzzy relations $\lesssim'$ and $\approx'$ on U' by

$$[u]_E \lesssim' [v]_E \;=\; u \lesssim v \quad \text{and} \quad [u]_E \approx' [v]_E \;=\; u \approx v,$$

for any equivalence classes $[u]_E$ and $[v]_E$ in U'. Then $\lesssim'$ is a fuzzy order on the set U' equipped with a fuzzy equality $\approx'$ according to Definition 1.

3.2. Reflexivity and compatibility

An immediate difference between Definitions 1 and 2 consists in their condition of reflexivity. Both generalize classical reflexivity in that in the classical setting, i.e. **L** being the two-element Boolean algebra, both coincide with classical reflexivity. However, while $\approx$-reflexivity required by Definition 1 is stronger than reflexivity of Definition 2, the latter requires compatibility of the fuzzy order $\lesssim$ with $\approx$.[15]

Remark 7: (a) From the epistemic viewpoint mentioned in Remark 1, it seems natural, if not necessary, to assume compatibility of $\lesssim$ with $\approx$. Compatibility generalizes the axiom of equality of classical logic, which in the context of order relations reads: If u_1 is less than or equal to v_1, u_1 equals u_2, and v_1 equals v_2, then u_2 is less than or equal to v_2. In the setting involving degrees, compatibility is compelling particularly when degrees of equality are interpreted as degrees of indistinguishability. Compatibility then says that the following formula is true (i.e. its truth degree equals 1): If u_1 is less than or equal to v_1, u_1 is indistinguishable from u_2, and v_1 is indistinguishable from v_2, then u_2 is less than or equal to v_2. Validity of such formula seems an unavoidable condition.

(b) Interestingly, compatibility has not been mentioned by Höhle and Blanchard (1985), nor in the first works by Bodenhofer; Bodenhofer actually considers compatibility considerably later (Bodenhofer and Demirci 2008). On the other hand, compatibility has been a common condition utilized in modern studies of fuzzy relational systems in the early 2000s; see, e.g. the books by Belohlavek (2002), Gottwald (2001), and Hájek (1998).

In spite of the seemingly different conditions, i.e. $\approx$-reflexivity vs. reflexivity and compatibility, $\approx$-reflexivity turns out to be equivalent to reflexivity and compatibility given the context of both definitions.[16] The argument was observed for the first time by Belohlavek and Vychodil (2005, Lemma 1.82) in the context of fuzzy equivalences on sets with fuzzy equalities and later, independently, by Bodenhofer and Demirci (2008) in the context of fuzzy orders. Since, as we shall see below, this relationship is of considerable importance, we consider it thoroughly, namely by taking into account the degrees of the properties of fuzzy relations. We start by the following lemma, in which $\text{tra}(\lesssim)^2$ stands for $\text{tra}(\lesssim) \otimes \text{tra}(\lesssim)$ and analogously for $\text{ref}_{\approx}(\lesssim)^2$.

Lemma 1: *Let $\lesssim$ and $\approx$ be arbitrary fuzzy relations on a given set U. Then*

$$\text{ref}(\approx) \otimes \text{ref}_{\approx}(\lesssim) \leq \text{ref}(\lesssim), \tag{10}$$

$$\text{sym}(\approx) \otimes \text{tra}(\lesssim)^2 \otimes \text{ref}_{\approx}(\lesssim)^2 \leq \text{comp}(\lesssim), \tag{11}$$

$$\text{ref}(\approx) \otimes \text{ref}(\lesssim) \otimes \text{comp}(\lesssim) \leq \text{ref}_{\approx}(\lesssim). \tag{12}$$

In the following as well as in the subsequent proofs, we shall use – with no further notice – common properties of infima and suprema, as well as properties of complete residuated lattices (Belohlavek 2002; Gottwald 2001; Novák, Perfilieva, and Močkoř 1999).

Proof: Inequality (10) holds true iff for each $u \in U$, $\text{ref}(\approx) \otimes \text{ref}_{\approx}(\lesssim) \leq u \lesssim u$, which is indeed the case, as

$$\text{ref}(\approx) \otimes \text{ref}_{\approx}(\lesssim) \leq (u \approx u) \otimes ((u \approx u) \to (u \lesssim u)) \leq u \lesssim u.$$

To check (11), we need to verify that for each $u_1, u_2, v_1, v_2 \in U$ one has

$$\text{sym}(\approx) \otimes \text{tra}(\lesssim)^2 \otimes \text{ref}_{\approx}(\lesssim)^2 \leq ((u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2)) \to u_2 \lesssim v_2,$$

which is equivalent to

$$\mathrm{sym}(\approx) \otimes \mathrm{tra}(\lesssim)^2 \otimes \mathrm{ref}_{\approx}(\lesssim)^2 \otimes (u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq u_2 \lesssim v_2,$$

which holds true. Indeed, observe first (easy, by standard arguments) that

$$(u \approx v) \otimes \mathrm{sym}(\approx) \leq v \approx u,$$
$$(u \approx v) \otimes \mathrm{ref}_{\approx}(\lesssim) \leq u \lesssim v, \text{ and}$$
$$(u \approx v) \otimes (v \approx w) \otimes \mathrm{tra}(\lesssim) \leq u \approx w.$$

Now,

$$\begin{aligned}
&\mathrm{sym}(\approx) \otimes \mathrm{tra}(\lesssim)^2 \otimes \mathrm{ref}_{\approx}(\lesssim)^2 \otimes (u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2)\\
&= (u_1 \approx u_2) \otimes \mathrm{sym}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \otimes (u_1 \lesssim v_1) \otimes (v_1 \approx v_2) \otimes \mathrm{ref}_{\approx}(\lesssim) \otimes \mathrm{tra}(\lesssim)^2\\
&\leq (u_2 \approx u_1) \otimes \mathrm{ref}_{\approx}(\lesssim) \otimes (u_1 \lesssim v_1) \otimes (v_1 \lesssim v_2) \otimes \mathrm{tra}(\lesssim)^2\\
&\leq (u_2 \lesssim u_1) \otimes (u_1 \lesssim v_2) \otimes \mathrm{tra}(\lesssim)\\
&\leq u_2 \lesssim v_2,
\end{aligned}$$

completing the proof of (11).

Verifying (12) amounts to checking

$$\mathrm{ref}(\approx) \otimes \mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim) \leq (u \approx v) \to (u \lesssim v),$$

i.e.

$$\mathrm{ref}(\lesssim) \otimes \mathrm{ref}(\approx) \otimes (u \approx v) \otimes \mathrm{comp}(\lesssim) \leq u \lesssim v,$$

for each $u, v \in U$, which is easy as

$$\begin{aligned}
&\mathrm{ref}(\lesssim) \otimes \mathrm{ref}(\approx) \otimes (u \approx v) \otimes \mathrm{comp}(\lesssim)\\
&\leq (u \lesssim u) \otimes (u \approx u) \otimes (u \approx v) \otimes \mathrm{comp}(\lesssim) \leq u \lesssim v,
\end{aligned}$$

with the last equality holding due to the definition of comp($\lesssim$). ■

Remark 8: (a) The meaning of inequalities (10), (11), and (12) may easily be described verbally as follows; justification may either be intuitive or formal as explained in (b) below. In particular, (10) means that if $\approx$ is reflexive and $\lesssim$ is $\approx$-reflexive, then $\lesssim$ is reflexive, even when this implication is interpreted in a way in which degrees of being reflexive and $\approx$-reflexive are taken into account. In the same vein, (11) is interpreted as claiming that if $\approx$ is symmetric, $\lesssim$ is transitive and $\approx$-reflexive, then $\lesssim$ is compatible with $\approx$.[17] Finally, (12) means that if $\approx$ is reflexive and $\lesssim$ is reflexive and compatible with $\approx$, then $\lesssim$ is $\approx$-reflexive.

(b) To explain a formal justification of the meaning of the above inequalities, consider (10); for the other inequalities, one proceeds analogously. There is a first-order formula expressing inequality (10) syntactically, which is:

$$[(\forall x)(x = x) \& (\forall x)(x = x \Rightarrow x \leq x)] \Rightarrow (\forall x)(x \leq x). \tag{13}$$

Namely, consider a first-order structure with universe U such that the relation symbols $=$ and $\leq$ are interpreted by the fuzzy relations $\approx$ and $\lesssim$, and connectives & and

$\Rightarrow$ are interpreted by $\otimes$ and $\rightarrow$. Due to basic semantic rules of first-order fuzzy logic (Belohlavek 2002; Gottwald 2001; Hájek 1998), the truth degrees of the subformulas $(\forall x)(x = x)$, $(\forall x)(x = x \Rightarrow x \le x)$, and $(\forall x)(x \le x)$ are then just equal to $\text{ref}(\approx)$, $\text{ref}_{\approx}(\lesssim)$, and $\text{ref}(\lesssim)$, respectively. Hence, the truth degree of formula (13) is equal to

$$[\text{ref}(\approx) \otimes \text{ref}_{\approx}(\lesssim)] \rightarrow \text{ref}(\lesssim). \quad (14)$$

From the properties of $\rightarrow$, it now follows that formula (13) is true, i.e. the truth degree (14) equals 1, if and only if inequality (10) is satisfied.

(c) One may easily observe that the exponents in the inequality (11) tell us how many times the degree of the respective property is used in the proof of the inequality. For instance, the exponent 2 in $\text{tra}(\lesssim)^2$ indicates that the degree of transitivity is used twice in the proof. This demonstrates an interesting added value of analyzing relationships among the concerned properties of fuzzy relations by looking at the degrees to which the properties are satisfied.

(d) In (11), neither of the exponents of 2 in $\text{tra}(\lesssim)^2$ and in $\text{ref}_{\approx}(\lesssim)^2$ may be reduced to 1. Indeed, consider the residuated lattice **L** to be the real unit interval $L = [0, 1]$ equipped with the Łukasiewicz connectives (cf. Appendix). Let $U = \{u_1, u_2, u_3, u_4\}$ and let $\approx$ and $\lesssim$ be given by the following tables:

$\approx$	u_1	u_2	u_3	u_4
u_1	1	1	0	0
u_2	1	1	0	0
u_3	0	0	1	1
u_4	0	0	1	1

$\lesssim$	u_1	u_2	u_3	u_4
u_1	1	0.8	0.6	0.2
u_2	0.8	1	1	0.6
u_3	0	0	1	0.8
u_4	0	0	0.8	1

As one may verify, $\text{sym}(\approx) = 1$, $\text{tra}(\lesssim) = 0.8$, $\text{ref}_{\approx}(\lesssim) = 0.8$, and $\text{comp}(\lesssim) = 0.2$. Now we have

$$\text{sym}(\approx) \otimes \text{tra}(\lesssim) \otimes \text{ref}_{\approx}(\lesssim)^2 = 1 \otimes 0.8 \otimes 0.8^2 = 0.4 \not\le 0.2 = \text{comp}(\lesssim)$$

and

$$\text{sym}(\approx) \otimes \text{tra}(\lesssim)^2 \otimes \text{ref}_{\approx}(\lesssim) = 1 \otimes 0.8^2 \otimes 0.8 = 0.4 \not\le 0.2 = \text{comp}(\lesssim).$$

Let us now consider some corollaries of Lemma 1, which result by strengthening the assumptions.

Corollary 1: *Let $\approx$ be reflexive and symmetric (in particular, a fuzzy equality) and $\lesssim$ be transitive. Then:*

$$\text{ref}_{\approx}(\lesssim) \le \text{ref}(\lesssim),$$
$$\text{ref}_{\approx}(\lesssim)^2 \le \text{comp}(\lesssim),$$
$$\text{ref}(\lesssim) \otimes \text{comp}(\lesssim) \le \text{ref}_{\approx}(\lesssim).$$

Proof: Trivial given Lemma 1, because reflexivity and symmetry of $\approx$ is equivalent to $\text{ref}(\approx) = 1$ and $\text{sym}(\approx) = 1$, respectively, and transitivity of $\lesssim$ is equivalent to $\text{tra}(\lesssim) = 1$. ■

Considering as a particular case the full satisfaction of the properties involved in Corollary 1, we obtain the above-mentioned claim:

Corollary 2 (Belohlavek and Vychodil 2005, Bodenhofer and Demirci 2008): *Let $\approx$ be a fuzzy equality and $\lesssim$ be transitive. Then $\lesssim$ is $\approx$-reflexive if and only if $\lesssim$ is reflexive and compatible with $\approx$.*

Proof: By a moment's reflection from Corollary 1 taking into account that $\lesssim$ is $\approx$-reflexive, reflexive, and compatible with $\approx$ iff $\mathrm{ref}_{\approx}(\lesssim) = 1$, $\mathrm{ref}(\lesssim) = 1$, and $\mathrm{comp}(\lesssim) = 1$, respectively. ■

Remark 9: (a) Observe that Corollary 2 is in the form of a logical equivalence, which may be rephrased as follows: Given the assumptions (i.e. $\approx$ a fuzzy equality and $\lesssim$ transitive), we have $\mathrm{ref}_{\approx}(\lesssim) = 1$ if and only if $\mathrm{ref}(\lesssim) = 1$ and $\mathrm{comp}(\lesssim) = 1$.

(b) On the other hand, Corollary 1, i.e. a direct "graded generalization" of Corollary 2, is in the form of three inequalities of truth degrees expressing three implications regarding graded properties of fuzzy relations. Namely, corresponding to the three inequalities of Corollary 1 are three implications regarding graded properties of $\lesssim$ and $\approx$, respectively, which have their truth degree equal to 1. In particular, using the basic rules of semantics of first-order fuzzy logic, it may be shown that:
- $\mathrm{ref}_{\approx}(\lesssim) \leq \mathrm{ref}(\lesssim)$ holds true iff the truth degree of the formula "if $\lesssim$ is $\approx$-reflexive then $\lesssim$ is reflexive" equals 1, i.e. the formula is fully true;
- $\mathrm{ref}_{\approx}(\lesssim)^2 \leq \mathrm{comp}(\lesssim)$ holds true iff the truth degree of the formula "if $\lesssim$ is $\approx$-reflexive and $\lesssim$ is $\approx$-reflexive then $\lesssim$ is compatible" equals 1; and
- $\mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim) \leq \mathrm{ref}_{\approx}(\lesssim)$ holds true iff the truth degree of the formula "if $\lesssim$ is reflexive and $\lesssim$ is compatible then $\lesssim$ is $\approx$-reflexive" equals 1.

(c) Observe that if $\otimes$ is idempotent, which is e.g. the case of the two-element Boolean algebra, then the three inequalities may readily be replaced by a single equality, namely

$$\mathrm{ref}_{\approx}(\lesssim) = \mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim). \tag{15}$$

This equality expresses the fact that the formula "$\lesssim$ is $\approx$-reflexive if and only if $\lesssim$ is reflexive and $\lesssim$ is compatible" regarding graded properties of $\lesssim$ and $\approx$ has its truth degree equal to 1.

(d) In general, however, the three inequalities of Corollary 1 may not be expressed by the single equality (15); for instance the fuzzy relations in Remark 8 (d) do not satisfy (15).

Remark 10: One may formulate other corollaries of Lemma 1. As an example, the following corollary concerns crisp properties of $\lesssim$ and $\approx$, as does Corollary 2, but is more informative than Corollary 2:

Let $\lesssim$ and $\approx$ be arbitrary fuzzy relations. Then

(a) if $\approx$ is reflexive and $\lesssim$ is $\approx$-reflexive, then $\lesssim$ is reflexive; if $\approx$ is symmetric and $\lesssim$ is transitive and $\approx$-reflexive, then $\lesssim$ is compatible with $\approx$;
(b) if $\approx$ is reflexive and $\lesssim$ is reflexive and compatible with $\approx$ then $\lesssim$ is $\approx$-reflexive.

Remark 11: Clearly, Corollary 2 implies that both Definition 1 and Definition 2 of fuzzy order may be rephrased so that they differ in the condition of antisymmetry only. That is, the condition of $\approx$-reflexivity in Definition 1 may equivalently be replaced by reflexivity and compatibility, and, conversely, the latter two conditions in Definition 2 may be replaced by $\approx$-reflexivity (cf. Theorem 4).

3.3. Constraints regarding fuzzy equality

Both the $\otimes$-antisymmetry and $\wedge$-antisymmetry may be regarded as lower bounds for the fuzzy equality $\approx$. This view opens the question of exploring constraints pertaining to $\approx$. A basic answer was provided by Bodenhofer (2000, Theorem 18) who proved the following claim, which he phrased for fuzzy equivalences instead of equalities:

If $\precsim$ is a reflexive and transitive fuzzy relation on U and $\approx$ is a fuzzy equality on U, then $\precsim$ is a fuzzy order according to Definition 1 if and only if

$$(u \precsim v) \otimes (v \precsim u) \;\leq\; u \approx v \;\leq\; (u \precsim v) \wedge (v \precsim u) \tag{16}$$

for every $u, v \in U$.

An easy inspection of the proof reveals that a corresponding theorem for the notion of fuzzy order with $\wedge$-antisymmetry is obtained when replacing (16) by the equality

$$u \approx v \;=\; (u \precsim v) \wedge (v \precsim u),$$

the validity of which for fuzzy orders according to Definition 2 was observed by Belohlavek (2002).

The results we just mentioned, though, involve a redundancy in that the assumption of reflexivity of $\precsim$ may be dropped. The redundancy may be regarded as resulting from a lack of awareness of the relationship of reflexivity and compatibility versus $\approx$-reflexivity (cf. Section 3.2). In fact, some easy observations on the notions involved render a non-redundant generalization of the above-mentioned results. Below we provide such observations in a more general manner which, as in Section 3.2, take into account the degrees of the properties of fuzzy relations. Then we obtain a proper formulation of the above-mentioned results as simple consequences. We start with the following observation.

Lemma 2: *Let $\approx$ and $\precsim$ be arbitrary binary fuzzy relations on U.*

(a) *One has*

$$\text{sym}(\approx) \;\leq\; \text{ref}_{\approx}(\precsim) \leftrightarrow \bigwedge_{u,v\in U} [u \approx v \rightarrow ((u \precsim v) \wedge (v \precsim u))]. \tag{17}$$

(b) *If $\approx$ is symmetric (in particular, a fuzzy equality), then*

$$\text{ref}_{\approx}(\precsim) = \bigwedge_{u,v\in U} [u \approx v \rightarrow ((u \precsim v) \wedge (v \precsim u))]. \tag{18}$$

(c) *If $\approx$ is symmetric (in particular, a fuzzy equality), then*

$$\precsim \text{ is } \approx\text{-reflexive} \quad \text{iff} \quad u \approx v \leq (u \precsim v) \wedge (v \precsim u). \tag{19}$$

Proof: (a) Since for any $a, b, c \in L$, $a \leq b \leftrightarrow c$ is equivalent to $a \leq b \rightarrow c$ and $a \leq c \rightarrow b$, i.e. – due to adjointness – to $a \otimes b \leq c$ and $a \otimes c \leq b$, we need to verify

$$\mathrm{sym}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \leq \bigwedge_{u,v \in U} [u \approx v \rightarrow ((u \lesssim v) \wedge (v \lesssim u))] \tag{20}$$

and

$$\mathrm{sym}(\approx) \otimes \bigwedge_{u,v \in U} [u \approx v \rightarrow ((u \lesssim v) \wedge (v \lesssim u))] \leq \mathrm{ref}_{\approx}(\lesssim). \tag{21}$$

Check (20) first. Since

$$[u \approx v \rightarrow ((u \lesssim v) \wedge (v \lesssim u))] = (u \approx v \rightarrow u \lesssim v) \wedge (u \approx v \rightarrow v \lesssim u),$$

(20) holds true iff for each $u, v \in U$,

$$\mathrm{sym}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \leq (u \approx v \rightarrow u \lesssim v) \tag{22}$$

and

$$\mathrm{sym}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \leq (u \approx v \rightarrow v \lesssim u). \tag{23}$$

While (22) is trivial due to the definition of $\mathrm{ref}_{\approx}(\lesssim)$, (23) is equivalent to

$$(u \approx v) \otimes \mathrm{sym}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \leq v \lesssim u,$$

which is true because

$$\begin{aligned} &(u \approx v) \otimes \mathrm{sym}(\approx) \otimes \mathrm{ref}_{\approx}(\lesssim) \\ &\leq (u \approx v) \otimes (u \approx v \rightarrow v \approx u) \otimes (v \approx u \rightarrow v \lesssim u) \leq v \lesssim u. \end{aligned}$$

Checking (21) is straightforward: As $a \rightarrow (b \wedge c) \leq a \rightarrow b$, we have

$$\mathrm{sym}(\approx) \otimes \bigwedge_{u,v \in U} [u \approx v \rightarrow ((u \lesssim v) \wedge (v \lesssim u))] \leq \bigwedge_{u,v \in U} [u \approx v \rightarrow u \lesssim v] = \mathrm{ref}_{\approx}(\lesssim).$$

(b) follows from (a) because if $\approx$ is symmetric, we have $\mathrm{sym}(\approx) = 1$, hence

$$\mathrm{ref}_{\approx}(\lesssim) \leftrightarrow \bigwedge_{u,v \in U} [u \approx v \rightarrow ((u \lesssim v) \wedge (v \lesssim u))] = 1.$$

Now, since $a \leftrightarrow b = 1$ iff $a = b$, equality (18) readily follows.

(c) follows from (b) because $\approx$-reflexivity of $\lesssim$ means $\mathrm{ref}_{\approx}(\lesssim) = 1$ and because $a \rightarrow b = 1$ is equivalent to $a \leq b$. ■

As an immediate consequence of Lemma 2 (c) and the definition of $\otimes$- and $\wedge$-antisymmetry, we have:

Corollary 3: *Let $\lesssim$ be a fuzzy relation and $\approx$ be a fuzzy equality on U.*

(a) $\lesssim$ *is* $\approx$*-reflexive and* $\otimes$*-antisymmetric iff*

$$(u \lesssim v) \otimes (v \lesssim u) \ \leq \ u \approx v \ \leq \ (u \lesssim v) \wedge (v \lesssim u).$$

(b) $\lesssim$ *is* $\approx$*-reflexive and* $\wedge$*-antisymmetric iff*

$$u \approx v \ = \ (u \lesssim v) \wedge (v \lesssim u).$$

The preceding corollary along with the equivalence of $\approx$-reflexivity to reflexivity and compatibility for transitive fuzzy relations (cf. Corollary 2) immediately yield the announced non-redundant and hence more informative rephrasement of the result by Bodenhofer (2000, Theorem 18) and its counterpart for fuzzy orders according to Definition 2, which we mentioned above.

Theorem 4: *Let* $\lesssim$ *be a transitive fuzzy relation and* $\approx$ *be a fuzzy equality on U.*

(a) *The following conditions are equivalent:*
  (a1) $\lesssim$ *is a fuzzy order according to Definition 1.*
  (a2) $\lesssim$ *is reflexive,* $\otimes$*-antisymmetric, and compatible with* $\approx$*.*
  (a3) $(u \lesssim v) \otimes (v \lesssim u) \ \leq \ u \approx v \ \leq \ (u \lesssim v) \wedge (v \lesssim u)$.
(b) *The following conditions are equivalent:*
  (b1) $\lesssim$ *is a fuzzy order according to Definition 2.*
  (b2) $\lesssim$ *is* $\approx$*-reflexive and* $\wedge$*-antisymmetric.*
  (b3) $u \approx v \ = \ (u \lesssim v) \wedge (v \lesssim u)$.

Remark 12: (a) Theorem 4 basically presents equivalent conditions for a transitive fuzzy relation $\lesssim$ to become a fuzzy order. In particular, it shows that one such condition is expressed by a simple constraint regarding $\lesssim$ and the fuzzy equality $\approx$. Note also that Theorem 4 lets us regard the claims as trivial consequences of the definitions of and previous observations on the individual properties of fuzzy orders.
(b) Compared to Theorem 18 by Bodenhofer (2000), part (a) of Theorem 4 is stronger. Namely, as mentioned in the beginning of this section, Bodenhofer claims in his Theorem 18 that being a fuzzy order (according to Definition 1) is equivalent to the inequality (a3) of Theorem 4 for any *reflexive and transitive* $\lesssim$. Theorem 4, on the other hand, makes it explicit that the assumption of reflexivity for $\lesssim$ may be dropped. This is worth mentioning because reflexivity is actually implied by one of the properties of fuzzy orders, namely $\approx$-reflexivity. In this respect, Theorem 4 is nonredundant and properly separates the role of the individual properties.
(c) On the other hand, let us note that as is clear from the proof of his Theorem 18, Bodenhofer (2000) was aware of the fact that the inequality $u \approx v \leq (u \lesssim v) \wedge (v \lesssim u)$ is equivalent to $\approx$-reflexivity for any transitive $\lesssim$, rather than any reflexive and transitive $\lesssim$.

We conclude this section by presenting a possible generalization of the previous theorem which takes the degrees of validity into account. We start with the following graded generalization of Corollary 3.

Lemma 3: *Let $\approx$ and $\lesssim$ be fuzzy relations, and let $\approx$ be symmetric (in particular, a fuzzy equality).*

(a)

$$\mathrm{ref}_{\approx}(\lesssim) \;\wedge\; \otimes\text{-ant}(\lesssim) = \bigwedge_{u,v\in U} \big([((u \lesssim v) \otimes (v \lesssim u)) \to u \approx v] \wedge [u \approx v \to ((u \lesssim v) \wedge (v \lesssim u))]\big).$$

(b)

$$\mathrm{ref}_{\approx}(\lesssim) \;\wedge\; \wedge\text{-ant}(\lesssim) = \bigwedge_{u,v\in U} [u \approx v \leftrightarrow ((u \lesssim v) \wedge (v \lesssim u))].$$

Proof: (a) Due to Lemma 2 (b) and the definition of $\otimes$-antisymmetry,

$$\mathrm{ref}_{\approx}(\lesssim) \wedge \otimes\text{-ant}(\lesssim) = \otimes\text{-ant}(\lesssim) \wedge \mathrm{ref}_{\approx}(\lesssim) =$$
$$= \bigwedge_{u,v\in U} ((u \lesssim v) \otimes (v \lesssim u)) \to u \approx v] \wedge \bigwedge_{u,v\in U} [u \approx v \to ((u \lesssim v) \wedge (v \lesssim u))].$$

The required equality now follows because

$$\bigwedge_{i\in I}(a_i \to b_i) \wedge \bigwedge_{i\in I}(b_i \to c_i) = \bigwedge_{i\in I}[(a_i \to b_i) \wedge (b_i \to c_i)]$$

holds in any complete residuated lattice.

(b) Lemma 2 (b) again and the definition of $\wedge$-antisymmetry yield

$$\mathrm{ref}_{\approx}(\lesssim) \;\wedge\; \wedge\text{-ant}(\lesssim)$$
$$= \bigwedge_{u,v\in U} [u \approx v \to ((u \lesssim v) \wedge (v \lesssim u))] \wedge \bigwedge_{u,v\in U} [((u \lesssim v) \wedge (v \lesssim u)) \to u \approx v],$$

from which the required equality follows because

$$\bigwedge_{i\in I}(a_i \to b_i) \wedge \bigwedge_{i\in I}(b_i \to a_i) = \bigwedge_{i\in I}(a_i \leftrightarrow b_i).$$

■

For brevity, we now only consider a graded generalization of part (b) in Theorem 4. Note that in this respect, Lemma 3 may be interpreted as claiming that the degree to which $u \approx v$ equals $(u \lesssim v) \wedge (v \lesssim u)$ coincides with the infimum of the degree of $\approx$-reflexivity of $\lesssim$ and the degree of $\wedge$-antisymmetry of $\lesssim$. We now obtain the following possible generalization of Theorem 4 (b):

Lemma 4 (generalization of Theorem 4 (b)): *Let $\lesssim$ be a transitive fuzzy relation and $\approx$ be symmetric (in particular, a fuzzy equality).*

(b_{12})

$$(\mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim)) \ \wedge \ \wedge\text{-ant}(\lesssim) \leq \mathrm{ref}_{\approx}(\lesssim) \ \wedge \ \wedge\text{-ant}(\lesssim)$$

$$\mathrm{ref}_{\approx}(\lesssim)^2 \ \wedge \ \wedge\text{-ant}(\lesssim) \leq (\mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim)) \ \wedge \ \wedge\text{-ant}(\lesssim)$$

(b_{23})

$$\mathrm{ref}_{\approx}(\lesssim) \ \wedge \ \wedge\text{-ant}(\lesssim) = \bigwedge_{u,v\in U} [u \approx v \leftrightarrow ((u \lesssim v) \wedge (v \lesssim u))]$$

(b_{13})

$$(\mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim)) \ \wedge \ \wedge\text{-ant}(\lesssim) \leq \bigwedge_{u,v\in U} [u \approx v \leftrightarrow ((u \lesssim v) \wedge (v \lesssim u))]$$

$$\Big(\bigwedge_{u,v\in U} [u \approx v \leftrightarrow ((u \lesssim v) \wedge (v \lesssim u))]\Big)^2 \leq (\mathrm{ref}(\lesssim) \otimes \mathrm{comp}(\lesssim)) \ \wedge \ \wedge\text{-ant}(\lesssim)$$

Proof: The claims are direct consequences of Lemma 1, Lemma 3 (b), and the properties of complete residuated lattices. ■

Remark 13: (a) Note that Lemma 4 implies (b) of Theorem 4. In particular, part (b_{12}) of Lemma 4 implies the equivalence of (b1) with (b2) in Theorem 4. In a similar manner, part (b_{23}) and (b_{13}) of Lemma 4 imply the equivalence of (b2) with (b3) and of (b1) with (b3) in Theorem 4, respectively.

Indeed, if $\lesssim$ satisfies (b1), i.e. is reflexive, transitive, $\wedge$-antisymmetric, and compatible with $\approx$ then $\mathrm{ref}(\lesssim) = 1$, $\mathrm{comp}(\lesssim) = 1$, and $\wedge\text{-ant}(\lesssim) = 1$, from which it follows by the first inequality in (b_{12}) that $\mathrm{ref}_{\approx}(\lesssim) = 1$ and $\wedge\text{-ant}(\lesssim) = 1$, i.e. $\lesssim$ is $\approx$-reflexive and $\wedge$-antisymmetric, establishing (b2). Similarly, the second inequality in (b_{12}) implies that (b2) implies (b1). For (b_{23}) and (b_{13}), one proceeds analogously.

(b) Clearly, other graded generalizations of Theorem 4 may be obtained, e.g. generalizations employing a general degree $\mathrm{tra}(\lesssim)$ of transitivity of $\lesssim$ instead of assuming transitivity of $\lesssim$ as in Lemma 4. Like Lemma 4, such generalizations would have a form of a set of inequalities, since it is unclear how the two pairs of inequalities in (b_{12}) and (b_{13}) might be expressed by two equalities, one implying the equivalence of (b1) with (b2) and the second the equivalence of (b1) with (b3).

Notes

1. It is worth noting that before Zadeh, many-valued orders were considered by Menger (1951) as part of his probabilistic approach to relations.
2. We identified over a thousand papers on fuzzy order in Scopus (papers containing "fuzzy order" or "fuzzy lattice" in the title, abstract, or keywords).
3. This is a common situation encountered in many areas of mathematics: A given concept defined in a given framework might have several different meaningful generalizations in a more general framework.
4. See Belohlavek, Dauben, and Klir (2017) for a detailed exposition of generalization to the framework of fuzzy logic.
5. Clearly, for different purposes, the concerned needs may be different. It may hence well be that there co-exist several different generalized concepts, each of which is a generalization of the given classical concept, is useful and behaves naturally for the particular purpose.

6. This is now well known: Take e.g. the Łukasiewicz structure on $L = [0, 1]$, $U = \{u, v\}$, $A = \{^{0.1}/u, ^{0.9}/v\}$, and $B = \{^{0.9}/u, ^{0.1}/v\}$. Then $A \subsetneq B = 0.2 > 0$ and $B \subsetneq A = 0.2 > 0$ but $A \neq B$. Several other notions of degree of inclusion of fuzzy sets proposed in the literature violate Zadeh's antisymmetry as well.
7. A slightly more general condition for antisymmetry is used e.g. by Gottwald (1993) and by Fodor and Roubens (1994), where antisymmetry asks that $R(u, v) \otimes R(v, u) > 0$ imply $u = v$ for each $u, v \in U$, where $\otimes$ is a chosen t-norm. As with Zadeh's antisymmetry, graded inclusion does not satisfy this condition.
8. By coincidence of $=$ with $\approx$ we mean that $u \approx v = 1$ for $u = v$ and $u \approx v = 0$ for $u \neq v$.
9. Bodenhofer uses T and E for $\otimes$ and $\approx$, respectively.
10. They use completely lattice-ordered commutative semigroups whose identity element is the largest element (such structures are equivalent to complete residuated lattices) satisfying additionally for non-empty $A \subseteq L$ that $\bigvee A = 1$ implies $\bigvee\{a \otimes a \mid a \in A\} = 1$.
11. Note that $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ with L being the real unit interval $[0, 1]$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous t-norm and $a \rightarrow b = \max\{c \mid a \otimes c \leq b\}$. Hence, the only restriction compared to the framework of complete residuated lattices is $L = [0, 1]$.
12. Put $U = \{u, v\}$ and let $\approx$ and $\lesssim$ be defined by $u \approx u = v \approx v = 1$, $u \approx v = v \approx u = 0.5$, $u \lesssim u = v \lesssim v = 1$, $u \lesssim v = 0.8$, and $v \lesssim u = 0.6$. Then for the Łukasiewicz structure on $L = [0, 1]$, $\lesssim$ is a fuzzy order according to Definition 1 but not according to Definition 2, as $(u \lesssim v) \wedge (v \lesssim u) = 0.8 \wedge 0.6 = 0.6 \not\leq 0.5 = u \approx v$.
13. It is worth noting that Höhle and Blanchard (1985) consider $\otimes$-antisymmetry but take $\leftrightarrow_\wedge$ as the fuzzy equality.
14. More precisely, truth degrees in a first order structure in which the relation symbol r is interpreted by the fuzzy relation R.
15. For a fuzzy equality $\approx$ on $U = \{u, v\}$ defined by $u \approx u = v \approx v = 1$ and $u \approx v = v \approx u = 0.5$, the fuzzy relation $\lesssim$ defined by $u \lesssim u = v \lesssim v = 1$ and $u \lesssim v = v \lesssim u = 0$ is reflexive but not $\approx$-reflexive, demonstrating that $\approx$-reflexivity of $\lesssim$ is stronger than reflexivity.
16. We observed in n. 15 that reflexivity of $\lesssim$ does not imply $\approx$-reflexivity. Observe, moreover, that compatibility of a (possibly transitive, and $\otimes$-antisymmetric or $\wedge$-antisymmetric) fuzzy relation $\lesssim$ with a fuzzy equality $\approx$ does not imply $\approx$-reflexivity of $\lesssim$ either (just take U and $\approx$ as in n. 15 and the empty fuzzy relation $\emptyset$ for $\lesssim$).
17. To take the exponents ^{2} properly into account, the precise meaning of (11) is: if $\approx$ is symmetric and $\lesssim$ is transitive and $\lesssim$ is transitive and $\lesssim$ is $\approx$-reflexive and $\lesssim$ is $\approx$-reflexive, then $\lesssim$ is compatible with $\approx$. Namely, conjunction is not idempotent in general in fuzzy logic, hence the two appearances of "$\lesssim$ is transitive" as well as "$\lesssim$ is $\approx$-reflexive" (two appearances because the exponents in $\mathrm{tra}(\lesssim)^2$ and $\mathrm{ref}_\approx(\lesssim)^2$ are equal to 2).

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Appendix: Residuated lattices, fuzzy sets, and fuzzy relations

Structures of truth degrees

Unlike classical logic, which uses a two-element set $L = \{0, 1\}$ of truth values and classical truth functions of logical connectives, i.e. uses a fixed structure of truth values, neither the set of truth degrees nor the truth functions of logical connectives are fixed in fuzzy logic. A modern approach in fuzzy logic assumes a general set L of truth degrees and general truth functions of logical connectives satisfying some natural basic conditions, i.e. assumes a general structure $\mathbf{L}$ of truth degrees. This assumption thus delineates a class of structures, which includes various particular structures such as the real unit interval $L = [0, 1]$ equipped with the Łukasiewicz connectives. A given theory or method, such as a theory of fuzzy equivalence relations, is then developed for the general assumptions, i.e. for a general $\mathbf{L}$, and is hence valid also for any of the particular structures.

Since the seminal work by Goguen (1967, 1969), it proved useful to assume that the structure $\mathbf{L}$ of truth degrees forms a complete residuated lattice (Belohlavek 2002; Belohlavek, Dauben, and Klir 2017; Gottwald 2001; Hájek 1998; Novák, Perfilieva, and Močkoř 1999), i.e. an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$$

such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy the so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{A1}$$

for each $a, b, c \in L$. The elements a of L are called truth degrees and $\otimes$ and $\rightarrow$ are considered as the truth functions of (many-valued) conjunction and implication, respectively. The biresiduum in $\mathbf{L}$ is the binary operation defined by

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a), \tag{A2}$$

and is interpreted as the truth function of (many-valued) equivalence.

Examples of complete residuated lattices, particularly those with L being $[0, 1]$ or a finite subchain of $[0, 1]$ which are based on t-norms and their residua, are well known. A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a continuous (or at least left-continuous) t-norm (i.e. a commutative, associative, and isotone operation on $[0, 1]$ with 1 acting as a neutral element) with the corresponding $\rightarrow$ given by

$$a \rightarrow b = \max\{c \mid a \otimes c \leq b\}.$$

The three most important pairs of adjoint operations on the unit interval are: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$); Gödel ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $a \rightarrow b = b$ if $a > b$); and Goguen ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $a \rightarrow b = b/a$ if $a > b$).

Another common choice is a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \dots, a_n = 1\} \subseteq [0,1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of $L = \{a_0 = 0, a_1, \dots, a_n = 1\} \subseteq [0,1]$ and the restrictions of the Gödel operations from $[0,1]$ to L.

Importantly, a special case of a complete residuated lattice is the two-element Boolean algebra $\langle \{0,1\}, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of classical logic. This is important because for the particular case $\mathbf{L} = \mathbf{2}$, the developed notions and results essentially become the ordinary notions. In particular, the notions regarding fuzzy sets and fuzzy relations (cf. the next section) may be identified with the corresponding notions regarding classical sets and classical relations.

Fuzzy sets and relations

Given a complete residuated lattice $\mathbf{L}$, we define the usual notions: An $\mathbf{L}$-set (fuzzy set, or L-set if one need not emphasize the operations on L) A in a universe U is a mapping $A\colon U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A." Let L^U denote the collection of all $\mathbf{L}$-sets in U. Binary $\mathbf{L}$-relations (binary fuzzy relations) between U and V are naturally just $\mathbf{L}$-sets in the universe $U \times V$, i.e. mappings $R : U \times V \to L$.

The basic operations with $\mathbf{L}$-sets are based on the residuated lattice operations and are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in L^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$; to emphasize that $\cap$ arises from $\wedge$, one also writes $A \wedge B$ instead of $A \cap B$.

A fuzzy set $A \in L^U$ is called crisp if $A(u) = 0$ or $A(u) = 1$ for each $u \in U$. Each crisp fuzzy set $A \in L^U$ may be obviously identified with the ordinary subset $\{u \in U \mid A(u) = 1\}$ of U; a crisp fuzzy set is in fact the characteristic function of the corresponding ordinary subset of U. Note also that all $\mathbf{2}$-sets are crisp and hence $\mathbf{2}$-sets and operations with $\mathbf{2}$-sets can be identified with ordinary sets and operations with ordinary sets, respectively. It is a common practice not to distinguish crisp fuzzy sets in U from the corresponding ordinary subsets of U if there is no danger of confusion.

For $a \in L$ and $u \in U$, we denote by $\{a/u\}$ the $\mathbf{L}$-set A in U, called a singleton, for which $A(x) = a$ if $x = u$ and $A(x) = 0$ if $x \neq u$. A crisp singleton $\{1/u\}$ may be identified with a one-element ordinary subset $\{u\}$ of U.

Given $A, B \in L^U$, we define the degree $A \subsetapprox B$ of inclusion of A in B by

$$A \subsetapprox B = \bigwedge\nolimits_{u \in U} \big(A(u) \rightarrow B(u)\big), \tag{A3}$$

which is also denoted $S(A, B)$ in the literature, and the degree of equality of A and B by

$$A \approx B = \bigwedge\nolimits_{u \in U} \big(A(u) \leftrightarrow B(u)\big). \tag{A4}$$

Note that (A3) generalizes the ordinary subsethood relation $\subseteq$. Described verbally, $A \subsetapprox B$ represents the degree to which every element of A is an element of B. In particular, we write $A \subseteq B$ iff $A \subsetapprox B = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. Likewise, (A4) generalizes the ordinary equality $=$ of sets, and $A \approx B$ represents the degree to which every element belongs to A iff it belongs to B. Clearly, $A = B$ iff $A \approx B = 1$.

An $\mathbf{L}$-equivalence (fuzzy equivalence) on U is a binary fuzzy relation $\approx$ on U, i.e. $\approx\colon U \times U \to L$, satisfying for each $u, v, w \in U$ the conditions

$$u \approx u = 1, \tag{A5}$$

$$u \approx v = v \approx u, \tag{A6}$$

$$(u \approx v) \otimes (v \approx w) \leq u \approx w, \tag{A7}$$

called reflexivity, symmetry, and transitivity, respectively. An **L**-equality is an **L**-equivalence satisfying the condition of separation, i.e.

$$u \approx v = 1 \quad \text{implies} \quad u = v, \tag{A8}$$

for each $u, v \in U$.

A binary fuzzy relation $R : U \times U \to L$ is called compatible with a fuzzy equivalence $\approx$ on U if

$$R(u_1, v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq R(u_2, v_2). \tag{A9}$$

Put verbally, compatibility reads that if u_1 and v_1 are related by R, u_1 is equivalent to u_2, and v_1 is equivalent to v_2, then u_2 and v_2 are related by R as well.

In some contexts, it is convenient to speak of $\otimes$-transitivity, rather than transitivity, of a fuzzy relation to emphasize that the connective "and" is interpreted by the truth function $\otimes$, and to distinguish this condition from, e.g. $\wedge$-transitivity, which would read $(u \approx v) \wedge (v \approx w) \leq u \approx w$. This manner of emphasizing the truth functions is common in the literature and we adopt it when needed. For further details on fuzzy sets, we refer to the books by Belohlavek (2002); Belohlavek, Dauben, and Klir (2017); Gottwald (2001); Hájek (1998); Novák, Perfilieva, and Močkoř (1999).