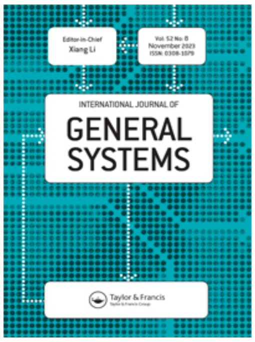

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On the concept of fuzzy order II: Antisymmetry

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ABSTRACT
In the second part of our paper, we explore antisymmetry of fuzzy orders. We provide a unifying definition of antisymmetry, which generalizes three existing variants of antisymmetry examined in the literature, along with the corresponding generalized definition of fuzzy order. We prove that all the particular instances of the generalized definition, which include the three basic ones, are mutually equivalent. We also examine distinctive properties of the three basic notions of fuzzy order.

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1. Preliminaries

We assume that the reader is familiar with the first part of our paper (Belohlavek and Urbanec 2023), to which we refer simply by "part I." Part I contains preliminaries in fuzzy logic in its Appendix and the notions and results we use in the present paper. We only recall the two definitions of fuzzy order analyzed in part I:

Definition 1 (Höhle, Blanchard, Bodenhofer): A fuzzy order on a set U equipped with a fuzzy equality relation $\approx$ is a binary fuzzy relation $\lesssim$ on U satisfying

$$u \approx v \leq u \lesssim v \qquad (\approx\text{-reflexivity}),$$
$$(u \lesssim v) \otimes (v \lesssim w) \leq u \lesssim w \qquad (\text{transitivity}),$$
$$(u \lesssim v) \otimes (v \lesssim u) \leq u \approx v \qquad (\otimes\text{-antisymmetry}),$$

for each $u, v, w \in U$. (Note: Höhle and Blanchard's as well as Bodenhofer's original definitions actually assume, more generally, that $\approx$ is a fuzzy equivalence rather than fuzzy equality; this is discussed in part I.)

Definition 2 (Höhle, Belohlavek): A fuzzy order on a set U equipped with a fuzzy equality relation $\approx$ is a binary fuzzy relation $\lesssim$ on U compatible with $\approx$, i.e. fulfilling

$$(u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \leq u_2 \lesssim v_2,$$

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for every $u_1, u_2, v_1, v_2 \in U$, which satisfies

$$u \lesssim u = 1 \qquad \text{(reflexivity)},$$
$$(u \lesssim v) \otimes (v \lesssim w) \leq u \lesssim w \qquad \text{(transitivity)},$$
$$(u \lesssim v) \wedge (v \lesssim u) \leq u \approx v \qquad (\wedge\text{-antisymmetry}),$$

for each $u, v, w \in U$.

2. Antisymmetry reconsidered

In view of part I (cf. Remark 11), antisymmetry represents the only essential difference between the two notions of fuzzy order expressed by Definitions 1 and 2. In this section, we explore antisymmetry in detail.

We first consider what we call crisp antisymmetry, a version of antisymmetry used in the literature in definitions of fuzzy order which do not employ fuzzy equality. Given the three variants of antisymmetry, namely the $\otimes$-antisymmetry, $\wedge$-antisymmetry, and crisp antisymmetry, we then provide a generalization of these variants. It turns out that in addition to the three variants, the generalized notion of antisymmetry renders a variety of other particular forms of antisymmetry. Importantly, we prove that all these forms are, in a sense, equivalent, and hence it is basically a matter of one's preference which concept of antisymmetry to use in the definition of fuzzy order. We then provide considerations of distinguishing properties of the various versions of antisymmetry, and thus various notions of fuzzy order. We conclude by a discussion regarding future research in fuzzy order.

2.1. Crisp antisymmetry and avoiding fuzzy equality

We now examine in detail a possible approach to fuzzy orders that avoids explicit reliance on the notion of fuzzy equality. This approach turns out to be almost equivalent to the approach utilizing the notion of fuzzy equality as codified by Definitions 1 and 2. Its possible shortcoming, in our view, consists in that it is not as clean compared to the approach utilizing the notion of fuzzy equality from a logical and an epistemic viewpoint, both of which have been explained in part I. Its advantage, however, is that the corresponding definition is simpler compared to Definitions 1 and 2.

The approach seems to have appeared for the first time in a study by Blanchard (1983), who examined Szpilrajn's embedding theorem in a fuzzy setting and introduced for this purpose several notions of fuzzy order. In particular, the notion Blanchard calls 4-fuzzy ordering is that of a fuzzy relation $\lesssim$ on a universe U satisfying reflexivity, i.e. $u \lesssim u = 1$, transitivity w.r.t. $\wedge$, i.e. $(u \lesssim v) \wedge (v \lesssim w) \leq u \lesssim w$, and the following form of antisymmetry, we shall call *crisp antisymmetry*:

$$(u \lesssim v) = 1 \text{ and } (v \lesssim u) = 1 \text{ imply } u = v, \tag{1}$$

for any $u, v \in U$.[1] In (1), $u = v$ means that u equals v, hence crisp antisymmetry provides a straightforward generalization of ordinary antisymmetry. Note that Blanchard only used the real unit interval $[0, 1]$ as the set of truth degrees and the minimum $\wedge$ on $[0, 1]$, i.e. infimum, as a truth function of conjunction, hence the employment of $\wedge$ in Blanchard's definition of transitivity. Blanchard seems not to have continued this approach to fuzzy order in her further work. Instead, she later employed the notion of fuzzy order proposed in her paper (Höhle and Blanchard 1985), which we discussed in part I.

Independently, the same notion of fuzzy order, i.e. not referring to fuzzy equality and using crisp antisymmetry has been proposed by Fan (2001), who used it in his further studies (Xie, Zhang, and Fan 2009; Zhang and Fan 2005; Zhang, Xie, and Fan 2009).[2] Fan uses the so-called frames as the structures of truth degrees, and hence uses the infimum $\wedge$ as the truth function of conjunction, as Blanchard does, rather than a more general $\otimes$ employed in our framework of residuated lattices.[3]

The following is the obvious generalization of the definition by Blanchard and Fan to the framework of general complete residuated lattices; it appeared in the works of Yao (Yao 2010; Yao and Lu 2009):

Definition 3 (Blanchard, Fan): A fuzzy order on a set U is a binary fuzzy relation $\lesssim$ on U satisfying

$$u \lesssim u = 1 \qquad \text{(reflexivity)},$$

$$(u \lesssim v) \otimes (v \lesssim w) \leq u \lesssim w \qquad \text{(transitivity)},$$

$$(u \lesssim v) = 1 \text{ and } (v \lesssim u) = 1 \text{ imply } u = v \qquad \text{(crisp antisymmetry)},$$

for each $u, v, w \in U$. The pair $\langle U, \lesssim \rangle$ shall be called a fuzzy ordered set (according to Definition 3).

Let us now consider the relationship of Definition 3 to Definitions 1 and 2. The possibility to avoid fuzzy equality in Definitions 1 and 2 has been observed in the respective early papers by Belohlavek and Bodenhofer. Thus Belohlavek (2001, 2002, 2004) observed and utilized the observation that a fuzzy order according to Definition 2 satisfies

$$u \approx v = (u \lesssim v) \wedge (v \lesssim u), \tag{2}$$

i.e. $\approx$ is uniquely determined by $\lesssim$. Bodenhofer made various observations on the relationship between $\lesssim$ and $\approx$ as regards Definition 1 too (see Section 3.3 in part I) and made comments regarding a possible omission of fuzzy equality (Bodenhofer 2003, end of Section 5). Later on, Xie, Zhang, and Fan (2009) for $\otimes = \wedge$ and Yao (2010) for general complete residuated lattices made the following observation on the relationship between Definitions 2 and 3:

Lemma 1: (a) *If $\langle U, \approx, \lesssim \rangle$ is a fuzzy ordered set according to Definition 2, then $\langle U, \lesssim \rangle$ is a fuzzy ordered set according to Definition* 3.
(b) *If $\langle U, \lesssim \rangle$ is a fuzzy ordered set according to Definition 3, then $\approx$ defined by* (2) *is a fuzzy equality and $\langle U, \approx, \lesssim \rangle$ is a fuzzy ordered set according to Definition* 2.

We now provide an observation analogous to Lemma 1 regarding the relationship between Definitions 1 and 3:

Lemma 2: (a) *If $\langle U, \approx, \lesssim \rangle$ is a fuzzy ordered set according to Definition* 1, *then $\langle U, \lesssim \rangle$ is a fuzzy ordered set according to Definition* 3.

(b) *If* $\langle U, \lesssim \rangle$ *is a fuzzy ordered set according to Definition* 3*, then* $\approx$ *defined by*

$$u \approx v = (u \lesssim v) \otimes (v \lesssim u) \tag{3}$$

is a fuzzy equality and $\langle U, \approx, \lesssim \rangle$ *is a fuzzy ordered set according to Definition* 1*.*

Proof: (a): Since reflexivity of $\lesssim$ follows from $\approx$-reflexivity of $\lesssim$ and reflexivity of $\approx$, it remains to verify crisp antisymmetry. If $u \lesssim v = 1$ and $v \lesssim u = 1$ then $\otimes$-antisymmetry yields

$$1 = 1 \otimes 1 = (u \lesssim v) \otimes (v \lesssim u) \leq u \approx v,$$

i.e. $u \approx v = 1$. Since $\approx$ is a fuzzy equality, it is separable, whence $u = v$.

(b): It is straightforward to check that $\approx$ defined by (3) is a fuzzy equivalence (this also follows from Lemma 6). If $u \approx v = 1$, then $(u \lesssim v) \otimes (v \lesssim u) = 1$, hence $u \lesssim v = 1$ and $v \lesssim u = 1$, from which $u = v$ follows due to crisp antisymmetry, verifying that $\approx$ is separable, and thus a fuzzy equality. The claim now follows from Theorem 4(a) in part I. ■

Remark 1: It is clear that the two constructions in (a) and (b) of Lemma 1, bringing $\langle U, \approx, \lesssim \rangle$ to $\langle U, \lesssim \rangle$ and vice versa, are mutually inverse.

On the other hand, the constructions in (a) and (b) of Lemma 2 are not mutually inverse because $\approx$ defined by (3) is but one of the possible fuzzy equalities described by Theorem 4 (a3) in part I. In this regard, one may generalize (b) in Lemma 2 as follows:

(b') If $\langle U, \lesssim \rangle$ is a fuzzy ordered set according to Definition 3, then if $\approx$ is a fuzzy equality satisfying (a3) of Theorem 4 in part I, then $\langle U, \approx, \lesssim \rangle$ is a fuzzy ordered set according to Definition 1.

2.2. A unifying concept of antisymmetry

2.2.1. Unification of $\otimes$-, $\wedge$-, and crisp antisymmetry

We shall consider binary operations on a given complete lattice $\langle L, \leq, 0, 1 \rangle$. Following a recent common practice, we call a t-norm on $\langle L, \leq, 0, 1 \rangle$ a binary operation $\otimes : L \times L \to L$ which is commutative, associative, order-preserving, and has 1 as its neutral element, i.e. $1 \otimes a = a$ for each $a \in L$. In this generalized meaning, classical t-norms are just t-norms on $\langle [0, 1], \leq, 0, 1 \rangle$; moreover, the operation $\otimes$ of any complete residuated lattice $\langle L, \wedge, \vee, \otimes, \to, 0, 1 \rangle$ is a t-norm on $\langle L, \leq, 0, 1 \rangle$.

In addition, we employ more general conjunction-like operations $\odot$ which satisfy

$$a \odot b = b \odot a, \tag{4}$$

$$a_1 \odot a_2 \leq b_1 \odot b_2, \text{ whenever } a_1 \leq b_1 \text{ and } a_2 \leq b_2, \tag{5}$$

$$a \odot 1 \leq a, \text{ and} \tag{6}$$

$$1 \odot 1 = 1. \tag{7}$$

Obviously, every t-norm satisfies these conditions. We need the following properties.

Lemma 3: *Assume* (4)–(7). *Then*

$$a \odot b \leq a \wedge b, \tag{8}$$

$$a \odot b = 1 \text{ implies } a = 1 \text{ and } b = 1. \tag{9}$$

Proof: (8): (5) and (6) imply $a \odot b \leq a \odot 1 \leq a$. Using (4), one similarly obtains $a \odot b \leq 1 \odot b = b \odot 1 \leq b$. Putting these together, we get $a \odot b \leq a \wedge b$.

(9): In view of (8), if $a \odot b = 1$ then $a \wedge b = 1$, from which $a = 1 = b$ readily follows. ■

Consider now the following notion. Let $\odot$ satisfy (4)–(7). A binary fuzzy relation $\lesssim$ on a set U equipped with a fuzzy equality $\approx$ satisfies $\odot$-*antisymmetry* if

$$(u \lesssim v) \odot (v \lesssim u) \leq u \approx v \tag{10}$$

for each $u, v \in U$.

While both $\otimes$-antisymmetry and $\wedge$-antisymmetry are obviously particular cases of $\odot$-antisymmetry, the same holds true for the seemingly different notion of crisp antisymmetry:

Lemma 4: *Consider the binary operation* $\bullet$ *on L and the fuzzy relation* $\approx$ *on U defined by*

$$a \bullet b = \begin{cases} 1 & \text{for } a = 1 \text{ and } b = 1, \\ 0 & \text{otherwise;} \end{cases} \qquad u \approx v = \begin{cases} 1 & \text{for } u = v, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Then $\bullet$ *satisfies* (4)–(7) *and* $\approx$ *is a fuzzy equality (the crisp fuzzy equality). Moreover, a binary fuzzy relation* $\lesssim$ *on U satisfies crisp antisymmetry if and only if it satisfies* $\bullet$-*antisymmetry.*

Proof: Straightforward by a direct verification of the conditions involved. ■

Remark 2: The operation $\bullet$ defined by (11) is the smallest operation satisfying (4)–(7) in that any $\odot$ verifying (4)–(7) satisfies $a \bullet b \leq a \odot b$ for any $a, b \in L$.

2.3. Constructing a fuzzy equality from $\lesssim \odot \lesssim^{-1}$

Notice that the crisp fuzzy equality $\approx$ in Lemma 4, which is involved in the condition of $\bullet$-antisymmetry, may in fact be obtained from $\lesssim$ by

$$u \approx v = (u \lesssim v) \bullet (v \lesssim u). \tag{12}$$

In view of Lemma 1, Lemma 2, and Lemma 4, and in particular the relationships (2), (3) and (12), respectively, we now explore – for the subsequent considerations on antisymmetry in general – the role of the fuzzy relation $\lesssim \odot \lesssim^{-1}$, which we denote $\equiv_\odot$, i.e.

$$u \equiv_\odot v = (u \lesssim v) \odot (v \lesssim u).$$

The following observation is immediate.

Lemma 5: *For $\odot$ satisfying* (4)–(7), $\equiv_\odot$ *is separable if and only if* $\lesssim$ *satisfies crisp antisymmetry.*

Proof: The $\Rightarrow$-part follows from (7). The $\Leftarrow$-part follows from (9). ■

Note that Lemma 5, which holds true for any fuzzy relation $\lesssim$, in fact provides a reformulation of crisp antisymmetry in terms of the fuzzy relation $\equiv_\odot$ derived from $\lesssim$.

We now recall a result by Bodenhofer (2000), which is related to our problem. For this purpose, recall the concept of dominance and an important result by De Baets and Mesiar (1998), on which Bodenhofer's result is based. A t-norm $\odot$ dominates a t-norm $\otimes$ (Klement, Mesiar, and Pap 2000) if

$$(a \odot b) \otimes (c \odot d) \leq (a \otimes c) \odot (b \otimes d) \tag{13}$$

for every $a, b, c, d \in L$; this is denoted by $\otimes \ll \odot$. De Baets and Mesiar proved that $\odot$ dominates $\otimes$ if and only if the $\odot$-intersection of any two $\otimes$-transitive fuzzy relations is $\otimes$-transitive. Here, the $\odot$-intersection $R \odot S$ of R and S is defined by $(R \odot S)(x, y) = R(x, y) \odot S(x, y)$, and $\otimes$-transitivity of R means $R(x, y) \otimes R(y, z) \leq R(x, z)$. The following lemma presents the above-mentioned result by Bodenhofer (2000, Theorem 17):

Lemma 6: *Let* $\lesssim$ *be a reflexive and transitive fuzzy relation on* U *and let* $\odot$ *be a t-norm dominating* $\otimes$*. Then* $\lesssim$ *is a fuzzy order on* U *equipped with a fuzzy equivalence* $\equiv_\odot$ *in the sense of Definition* 1 *(i.e. the original variant with fuzzy equivalence instead of fuzzy equality).*

Remark 3: (a) While Bodenhofer (2000) proves his Theorem 17 (i.e. Lemma 6) directly, the theorem follows from the equivalence of conditions (a1) and (a3) in Theorem 4 in part I, which is, as mentioned in part I, essentially the content of Bodenhofer's Theorem 18. Namely, since $\odot$ dominates $\otimes$, we have $\otimes \leq \odot$, hence $(u \lesssim v) \otimes (v \lesssim u) \leq (u \lesssim v) \odot (v \lesssim u) = u \approx_\odot v$, verifying (a3).

(b) In view of Lemma 5, the claim of Lemma 6 may be altered to fit Definition 1: Let $\lesssim$ be a reflexive and transitive fuzzy relation on U and let $\odot$ be a t-norm dominating $\otimes$. If $\lesssim$ satisfies crisp antisymmetry then $\equiv_\odot$ is a fuzzy equality and $\lesssim$ is a fuzzy order on U equipped with $\equiv_\odot$ according to Definition 1.

The obstacle we now face in proceeding with $\odot$-antisymmetry for a general $\odot$ satisfying (4)–(7) is that the fuzzy relation $\equiv_\odot$ need not be transitive if $\odot$ does not dominate $\otimes$.

Example 1: Let $U = \{u, v, w\}$ and let $L = [0, 1]$ with $\otimes$ being any of the Gödel, Goguen, and Łukasiewicz t-norm, and let $\odot$ be the drastic product $\otimes_D$, i.e.

$$a \otimes_D b = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{else.} \end{cases}$$

Notice that $\otimes_D$ satisfies (4)–(7). Let now $\lesssim$ be defined as follows:

$\lesssim$	u	v	w
u	1	1	0.7
v	0.7	1	0.7
w	0.7	1	1

As one easily checks, $\lesssim$ is $\otimes$-transitive. Nevertheless, the fuzzy relation $\equiv_{\otimes_D}$ is not $\otimes$-transitive. Namely,

$$(u \equiv_{\otimes_D} v) \otimes (v \equiv_{\otimes_D} w)$$
$$= 0.7 \otimes 0.7 \not\leq 0 = 0.7 \otimes_D 0.7 = (u \lesssim w) \otimes_D (w \lesssim u) = (u \equiv_{\otimes_D} w).$$

A natural way out is to consider the transitive closure of $\equiv_\odot$ rather than $\equiv_\odot$. Recall that the transitive closure $\mathrm{Tra}(R)$ of a binary fuzzy relation $R : U \times U \to L$, i.e. the least transitive fuzzy relation containing R, satisfies

$$\mathrm{Tra}(R) = \bigvee_{n=1}^{\infty} R^n = R \vee R \circ R \vee R \circ R \circ R \vee \ldots$$

where $(R \circ S)(u, v) = \bigvee_{x \in U} R(u, x) \otimes S(x, v)$.

Clearly, $\equiv_\odot$ is symmetric and since $1 \odot 1 = 1$, reflexivity of $\lesssim$ implies reflexivity of $\equiv_\odot$. Now, since the transitive closure preserves reflexivity and symmetry, we obtain:

Lemma 7: $\mathrm{Tra}(\equiv_\odot)$ *is reflexive and symmetric, whenever* $\lesssim$ *is reflexive.*

Since we are interested in fuzzy equalities, i.e. require separability, the following example demonstrating that the transitive closure does not preserve separability seems to present a problem:

Example 2: Consider $L = [0, 1]$ and the fuzzy relation R on the set

$$U = \{u, v\} \cup \{x_{11}\} \cup \{x_{21}, x_{22}\} \cup \{x_{31}, x_{32}, x_{33}\} \cup \cdots \cup \{x_{i1}, \ldots, x_{ii}\} \cup \cdots$$

defined by $R(y, z) = 0$ for every $y, z \in U$ except for

$$R(u, x_{i1}) = R(x_{i1}, x_{i2}) = \cdots = R(x_{ii}, v) = 1 - 1/i{+}1 \quad \text{for each } i = 1, 2, \ldots$$

For $R^n = R \circ \cdots \circ R$ (n times), one easily checks that for $\otimes$ being the Gödel t-norm,

$$R(u, v) = 0,\ R^2(u, v) = 1 - 1/2,\ R^3(u, v) = 1 - 1/3, \ldots,\ R^n(u, v) = 1 - 1/n, \ldots,$$

and thus

$$[\mathrm{Tra}(R)](u, v) = \left(\bigvee_{n=1}^{\infty} R^n\right)(u, v) = \bigvee_{n=1}^{\infty} (1 - 1/n) = 1.$$

Hence, while R is separable, $\mathrm{Tra}(R)$ is not. A similar example may be obtained for the Goguen and the Łukasiewicz t-norm.

Now, the particular structure of $\equiv_\odot$ enables us to prove that the possible problem of losing separability by the transitive closure does not materialize in our setting:

Lemma 8: *Let* $\lesssim$ *be transitive. If* $\equiv_\odot$ *is separable then* $\mathrm{Tra}(\equiv_\odot)$ *is separable.*

Proof: Let us first check that for each $i = 1, 2, \ldots,$ one has

$$u \equiv_\odot^i v \;\le\; (u \lesssim v) \wedge (v \lesssim u). \tag{14}$$

Indeed, due to (5) and (6), $x \equiv_\odot y \le x \lesssim y$, which along with the transitivity of $\lesssim$ yields

$$\begin{aligned} u \equiv_\odot^i v &= \bigvee_{x_1,\ldots,x_{i-1}\in U} ((u \equiv_\odot x_1) \otimes (x_1 \equiv_\odot x_2) \otimes \cdots \otimes (x_{i-1} \equiv_\odot v)) \\ &\le \bigvee_{x_1,\ldots,x_{i-1}\in U} ((u \lesssim x_1) \otimes (x_1 \lesssim x_2) \otimes \cdots \otimes (x_{i-1} \lesssim v)) \\ &\le \bigvee_{x_1,\ldots,x_{i-1}\in U} u \lesssim v \\ &= u \lesssim v. \end{aligned}$$

In a similar manner, one obtains $u \equiv_\odot^i v \le v \lesssim u$, from which (14) readily follows. Now,

$$[\mathrm{Tra}(\equiv_\odot)](u,v) = \bigvee_{i=1}^{\infty} u \equiv_\odot^i v \le \bigvee_{i=1}^{\infty} ((u \lesssim v) \wedge (v \lesssim u)) \le (u \lesssim v) \wedge (v \lesssim u).$$

It follows that if $[\mathrm{Tra}(\equiv_\odot)](u,v) = 1$ then $(u \lesssim v) \wedge (v \lesssim u) = 1$, whence $u \lesssim v = 1$ and $v \lesssim u = 1$. Condition (7) then yields $u \equiv_\odot v = (u \lesssim v) \odot (v \lesssim u) = 1 \odot 1 = 1$, from which $u = v$ follows by the separability of $\equiv_\odot$. ■

2.4. Main result: equivalence of definitions of fuzzy order

In view of the notions and observations in the preceding paragraphs, we now proceed toward a general concept of fuzzy order and our main result in this section. For this purpose, we consider the following fuzzy relations on a given universe U:

$\lesssim$ … a reflexive and transitive fuzzy relation on U,
$\equiv_\odot$ … a fuzzy relation defined by

$$u \equiv_\odot v = (u \lesssim v) \odot (v \lesssim u), \tag{15}$$

$\approx_\odot$ … the transitive closure of $\equiv_\odot$, i.e.

$$u \approx_\odot v = [\mathrm{Tra}(\equiv_\odot)](u,v). \tag{16}$$

We first summarize and extend the previous observations regarding $\equiv_\odot$ and $\approx_\odot$:

Lemma 9: *Let* $\odot$ *satisfy* (4)–(7) *and* $\lesssim$ *be a reflexive and transitive fuzzy relation on* U.

(a) $\approx_\odot$ *is a fuzzy equivalence on* U.

(b) *The following conditions are equivalent:*
   (b1) $\approx_\odot$ *is a fuzzy equality;*
   (b2) $\equiv_\odot$ *is separable;*
   (b3) $\lesssim$ *satisfies crisp antisymmetry.*
(c) *If* $\odot$ *is a t-norm which dominates* $\otimes$*, then* $\equiv_\odot = \approx_\odot$.

Proof: (a): The claim follows from Lemma 7.

(b1) $\Rightarrow$ (b3): Let $u \lesssim v = 1$ and $v \lesssim u = 1$. Due to (7), $u \equiv_\odot v = (u \lesssim v) \odot (v \lesssim u) = 1 \odot 1 = 1$. Since $u \equiv_\odot v \leq [\mathrm{Tra}(\equiv_\odot)](u, v) = u \approx_\odot v$, we obtain $u \approx_\odot v = 1$, whence $u = v$ due to separability of $\approx_\odot$.

(b3) $\Rightarrow$ (b2): If $u \equiv_\odot v = 1$ then (9) yields $u \lesssim v = 1$ and $v \lesssim u = 1$, hence $u = v$ using crisp antisymmetry of $\lesssim$.

(b2) $\Rightarrow$ (b1): The claim follows from (a) and Lemma 8.

(c): This is Bodenhofer's observation based on De Baets and Mesiar (1998); cf. Lemma 6. ■

Remark 4: For $\bullet$ defined by (11), $\equiv_\bullet$ is transitive, which is obvious because $\equiv_\bullet$ is the crisp equality, cf. Lemma 4. Hence, $\equiv_\bullet = \approx_\bullet$, even though $\bullet$ does not meet the assumption of Lemma 9 (c) because $\bullet$ is not a t-norm. Nevertheless, $\bullet$ still satisfies the dominance condition (13), with $\odot = \bullet$ and any t-norm $\otimes$, which is easily seen to imply transitivity of $\bullet$-intersection of arbitrary $\otimes$-transitive fuzzy relations. Note that in this sense, not only $\bullet$ dominates $\otimes$, but also $\otimes$ dominates $\bullet$. Yet $\bullet \neq \otimes$, which cannot happen with t-norms because if a t-norm $\odot$ both dominates and is dominated by a t-norm $\otimes$, then $\odot = \otimes$.

In the present perspective, the following concept provides a natural generalization of the three notions of fuzzy order presented in Definitions 1, 2, and 3:

Definition 4: Let $\odot$ satisfy (4)–(7). A fuzzy order on a set U equipped with a fuzzy equality relation $\approx$ is a binary fuzzy relation $\lesssim$ on U compatible with $\approx$, i.e. satisfying

$$(u_1 \lesssim v_1) \otimes (u_1 \approx u_2) \otimes (v_1 \approx v_2) \ \leq \ u_2 \lesssim v_2,$$

for every $u_1, u_2, v_1, v_2 \in U$, which, moreover, fulfills

$$\begin{aligned} u \lesssim u &= 1 && \text{(reflexivity)},\\ (u \lesssim v) \otimes (v \lesssim w) &\leq u \lesssim w && \text{(transitivity)},\\ (u \lesssim v) \odot (v \lesssim u) &\leq u \approx v && (\odot\text{-antisymmetry}), \end{aligned}$$

for each $u, v, w \in U$.

Remark 5: Definition 4 generalizes the notion of fuzzy order according to Definition 2 and, in view of Theorem 4 (a) in part I, the notion of fuzzy order according to Definition 1 as well. Since for a crisp fuzzy equality, compatibility is trivially satisfied, it also essentially generalizes Definition 3 (cf. Lemma 4). Namely,

- for $\odot = \otimes$, Definition 4 yields Definition 1;
- for $\odot = \wedge$, Definition 4 yields Definition 2;
- for $\odot = \bullet$, Definition 4 yields Definition 3.

To describe relationships among the discussed definitions of fuzzy orders, as well as among the respective variants of antisymmetry, we first present a theorem providing a number of mutually equivalent possibilities to define the general notion of fuzzy order according to Definition 4. Next, we present a theorem claiming that the notions of fuzzy order according to Definitions 1, 2, 3, and 4 are essentially mutually equivalent.[4]

Theorem 1: *Let $\lesssim$ be a reflexive and transitive fuzzy relation on U. The following conditions are equivalent:*

(a) *There exists $\odot$ satisfying* (4)–(7) *and a fuzzy equality $\approx$ such that $\lesssim$ is a fuzzy order on U equipped with $\approx$ according to Definition* 4.
(b) *For each $\odot$ satisfying* (4)–(7) *there exists a fuzzy equality $\approx$ such that $\lesssim$ is a fuzzy order on U equipped with $\approx$ according to Definition* 4.
(c) *There exists $\odot$ satisfying* (4)–(7) *such that $\lesssim$ is a fuzzy order on U equipped with $\approx_\odot$ according to Definition* 4.
(d) *For each $\odot$ satisfying* (4)–(7), *$\lesssim$ is a fuzzy order on U equipped with $\approx_\odot$ according to Definition* 4.
(e) *There exists $\odot$ satisfying* (4)–(7) *and a fuzzy equality $\approx$ on U such that $\equiv_\odot \le \approx \le \equiv_\wedge$.*
(f) *For each $\odot$ satisfying* (4)–(7) *there exists a fuzzy equality $\approx$ on U such that $\equiv_\odot \le \approx \le \equiv_\wedge$.*

Proof: We prove the claim by verifying the following implications:

(b) $\Rightarrow$ (a) and (d) $\Rightarrow$ (c) are obvious.

(a) $\Rightarrow$ (e): For $\odot$ and $\approx$ from (a), $\equiv_\odot \le \approx$ is just the $\odot$-antisymmetry of $\lesssim$ while $\approx \le \equiv_\wedge$ is a consequence of $\approx$-reflexivity of $\lesssim$ due to Lemma 2 (c) in part I. Note that $\approx$-reflexivity of $\lesssim$ follows from the reflexivity and compatibility of $\lesssim$ due to Corollary 2 in part I.

(f) $\Rightarrow$ (b): For an arbitrary $\odot$ satisfying (4)–(7), consider a fuzzy equality $\approx$ implied by (f). Like in the proof of "(a) $\Rightarrow$ (e)," the $\odot$-antisymmetry of $\lesssim$ is expressed by $\equiv_\odot \le \approx$, while the compatibility of $\lesssim$ with $\approx$ follows – due to Corollary 2 in part I – from $\approx$-reflexivity of $\approx$, which itself is expressed by $\approx \le \equiv_\wedge$.

(c) $\Rightarrow$ (e): Consider an $\odot$ implied by (c). We verify the two inequalities in (e) for $\approx$ being $\approx_\odot$. Clearly, $\equiv_\odot \le \mathrm{Tra}(\equiv_\odot) = \approx_\odot$, checking the first inequality in (e). Now, (8) clearly implies $\equiv_\odot \le \equiv_\wedge$, hence $\mathrm{Tra}(\equiv_\odot) \le \mathrm{Tra}(\equiv_\wedge)$. Since Lemma 9 (c) implies $\mathrm{Tra}(\equiv_\wedge) = \equiv_\wedge$, we obtain

$$\approx_\odot = \mathrm{Tra}(\equiv_\odot) \le \mathrm{Tra}(\equiv_\wedge) = \equiv_\wedge,$$

verifying the second inequality in (e).

(f) $\Rightarrow$ (d): Consider an arbitrary $\odot$ satisfying (4)–(7) and a fuzzy equality $\approx$ implied by (f). First, $\equiv_\odot \le \approx$, which holds due to (f), expresses the $\odot$-antisymmetry of $\lesssim$. Second, as $\approx_\odot = \mathrm{Tra}(\equiv_\odot) \le \mathrm{Tra}(\approx) = \approx$, the second inequality of (f), and Lemma 2 (c) in part I, imply

$$\approx_\odot \le \approx \le \equiv_\wedge,$$

hence $\lesssim$ is $\approx_\odot$-reflexive. The compatibility of $\lesssim$ and $\approx_\odot$ now follows from Corollary 2 in part I.

(e) ⇒ (f): Consider an arbitrary $\odot$ satisfying (4)–(7). We check that the fuzzy relation $\approx = \approx_{\odot}$ satisfies the conditions in (f).

Due to (8), $(u \lesssim v) \odot (v \lesssim u) \leq (u \lesssim v) \wedge (v \lesssim u)$, i.e. $\equiv_{\odot} \leq \approx_{\wedge}$, whence $\mathrm{Tra}(\equiv_{\odot}) \leq \mathrm{Tra}(\equiv_{\wedge})$. Since $\approx_{\odot} = \mathrm{Tra}(\equiv_{\odot})$ and $\approx_{\wedge} = \mathrm{Tra}(\equiv_{\wedge})$, using $\equiv_{\odot} \leq \mathrm{Tra}(\equiv_{\odot})$ and Lemma 9 (c) we obtain

$$\equiv_{\odot} \leq \approx_{\odot} \leq \approx_{\wedge} = \equiv_{\wedge},$$

verifying the required inequality in (f). It remains to check that $\approx_{\odot}$ is indeed a fuzzy equality. Since $\approx_{\odot}$ is a fuzzy equivalence due to Lemma 7 and $\approx_{\odot} = \mathrm{Tra}(\equiv_{\odot})$, it remains to verify the separability of $\approx_{\odot}$.

Consider an operation $*$ and a fuzzy equality $\sim$ implied by (e). Due to Lemma 8, it suffices to check the separability of $\equiv_{\odot}$. Let thus $u \equiv_{\odot} v = 1$, i.e. $(u \lesssim v) \odot (v \lesssim u) = 1$. Due to (9), $(u \lesssim v) = 1$ and $(v \lesssim u) = 1$, whence (7) yields

$$1 = (u \lesssim v) * (v \lesssim u) = u \equiv_{*} v.$$

The first inequality in (e) now implies $u \equiv_{*} v \leq u \equiv v$, whence $u \approx_{\odot} v = 1$, from which $u = v$ follows by the separability of $\approx_{\odot}$. ∎

The second theorem reveals the equivalence of the four definitions of a fuzzy order:

Theorem 2: *Let $\lesssim$ be a reflexive and transitive fuzzy relation on U. Each of the following conditions is equivalent to any of conditions (a)–(f) in Theorem* 1. *(Thus, in particular, the following conditions are mutually equivalent.)*

(a) $\lesssim$ *is a fuzzy order according to Definition* 1 *for some fuzzy equality* $\approx$.
(b) $\lesssim$ *is a fuzzy order according to Definition* 2 *for some fuzzy equality* $\approx$.
(c) $\lesssim$ *is a fuzzy order according to Definition* 3.

Proof: Obviously, any of (a), (b), and (c) implies condition (a) of Theorem 1. Conversely, condition (b) of Theorem 1 obviously implies (a) and (b) of the present theorem. Using Lemma 4, a moment's reflection shows that it also implies (c) of the present theorem. The claim now follows from the mutual equivalence of conditions (a) and (b) of Theorem 1. ∎

Remark 6: (a) It is apparent that in addition to the above mutually equivalent conditions for $\lesssim$ to form a fuzzy order, other conditions may be obtained.

(b) Other definitions of the general notion of fuzzy order may be formulated. For instance, in view of the above results, one may verify that the following conditions are equivalent for a fuzzy relation $\lesssim$ on U for any $\odot$ satisfying (4)–(7):
  (b1) $\lesssim$ is a fuzzy order according to Definition 4 for some fuzzy equality $\approx$;
  (b2) $\lesssim$ is transitive and the induced fuzzy relation $\equiv_{\odot}$ is reflexive and separable;
  (b3) $\lesssim$ is transitive and the induced fuzzy relation $\approx_{\odot}$ is a fuzzy equality.

3. Distinctive properties of various notions of antisymmetry and fuzzy order

In view of the results of the preceding section, the choice of the operation $\odot$, which is employed in the general concept of $\odot$-antisymmetry and the notion of fuzzy order according to Definition 4, does not essentially matter and is rather a matter of one's preference. In this section, though, we look at the question of which significant properties distinguish the three basic notions of fuzzy order codified by Definitions 1, 2, and 3, which correspond to $\otimes$-antisymmetry, $\wedge$-antisymmetry and, $\bullet$-antisymmetry (or, equivalently, crisp antisymmetry), respectively.

First view: Ordering of aggregation operations $\odot$

Clearly, a partial order $\leq$ can be defined on the class of all operations $\odot$ on L satisfying (4)–(7) by putting

$$\odot_1 \ \leq\ \odot_2 \quad \text{if and only if} \quad a \odot_1 b \ \leq\ a \odot_2 b \ \text{ for every } a, b \in L.$$

The following claim implies that from this viewpoint, $\wedge$-antisymmetry and $\bullet$-antisymmetry, and hence fuzzy orders according to Definitions 2 and 3, have distinct roles:

Theorem 3: *Let $\odot$ satisfy (4)–(7) and let $\lesssim$ be a fuzzy order according to Definition 4 for some fuzzy equality $\approx$.*

(a) *The operation $\bullet$ defined by (11) is the smallest operation satisfying (4)–(7); hence $\bullet$ is the smallest operation $\odot$ for which $\lesssim$ is a fuzzy order according to Definition 4 for some fuzzy equality $\approx$.*
(b) *The operation $\wedge$ is the largest operation $\odot$ satisfying (4)–(7); hence $\wedge$ is the largest operation $\odot$ for which $\lesssim$ is a fuzzy order according to Definition 4 for some fuzzy equality $\approx$.*

Proof: (a): The first part follows from the definition of $\bullet$ and property (7) of the considered operations $\odot$. The second part is a direct consequence of Theorem 2 and the first part.

(b): The first part follows from property (8) in Lemma 3. The second part follows again from Theorem 2 and the first part. ■

Second view: $\odot$ *as logical connective*

Since the degree $(u \lesssim v) \odot (v \lesssim u)$ is interpreted as a degree to which u is less than or equal to v *and* v is less than or equal to u, the operation $\odot$ satisfying (4)–(7) is naturally interpreted as conjunctive aggregation. Since it is well established that adjointness of conjunction w.r.t. implication is an essential property from a logical view (Belohlavek 2002; Goguen 1969; Gottwald 2001; Hájek 1998), the following immediate observation points out a distinguished position of $\otimes$-antisymmetry and of the notion of fuzzy order according to Definition 1:

Theorem 4: *Of all the operations $\odot$ satisfying (4)–(7) for which a given fuzzy relation $\lesssim$ is a fuzzy order according to Definition 4 for some fuzzy equality $\approx$, $\otimes$ is the only one that* *satisfies adjointness w.r.t.* $\to$, *i.e.*

$$a \odot b \leq c \quad \textit{iff} \quad a \leq b \to c \quad \textit{for every } a, b, c \in L.$$

Proof: The proof follows from Theorem 1 and the following well-known argument showing that in each residuated lattice, $\otimes$ is the only binary operation satisfying adjointness w.r.t. $\to$: First, $\otimes$ satisfies adjointness due to the definition of a complete residuated lattice; second, if $\odot$ satisfies adjointness, then for each $a, b, c \in L$, $a \otimes b \leq c$ iff $a \leq b \to c$ iff $a \odot b \leq c$, from which it follows that $a \otimes b = a \odot b$. ■

Note also that for fuzzy orders according to Definitions 1 and 2, which employ $\otimes$- and $\wedge$-antisymmetry, respectively, one need not extend the language of residuated lattices because both $\otimes$ and $\wedge$ are residuated lattice operations. For fuzzy orders according to Definition 4, which employs $\odot$-antisymmetry for a general $\odot$, the presence of $\odot$ means that the language of residuated lattices needs to be extended unless $\odot$ is definable by the residuated lattice operations. That is to say, while fuzzy orders according to Definitions 1 and 2 may be developed within the framework of complete residuated lattices, fuzzy orders according to Definition 4 require a richer framework of complete residuated lattices equipped with an additional operation.

Third view: Uniqueness of fuzzy equality

Let $\odot$ satisfy (4)–(7) and let $\lesssim$ be a fuzzy order on U equipped with $\approx$ according to Definition 4. According to Theorem 1, the set of all fuzzy equalities $\sim$ for which $\lesssim$ is a fuzzy order on U equipped with $\sim$ forms the interval

$$\mathcal{I}_{\odot} = \{\sim \mid \sim \text{ is a fuzzy equality and } a \approx_{\odot} b \leq a \sim b \leq a \approx_{\wedge} b \text{ for every } a, b \in L\}$$

in the set of all fuzzy equalities on U partially ordered by inclusion of fuzzy relations. The following theorem reveals another distinct feature of $\wedge$ and fuzzy orders with $\wedge$-antisymmetry according to Definition 2:

Theorem 5: *Let* **L** *be an arbitrary complete residuated lattice and let U have at least two elements. Then $\wedge$ is the only operation $\odot$ satisfying* (4)–(7) *such that for each fuzzy order $\lesssim$ according to Definition* 4*, the interval $\mathcal{I}_{\odot}$ is a singleton. Hence, $\wedge$ is the only operation satisfying* (4)–(7) *for which $\approx$ is uniquely determined by $\lesssim$.*

Proof: Due to Theorem 1, $\mathcal{I}_{\wedge}$ is a singleton. On the other hand, let $\odot$ be different from $\wedge$. We prove the claim by constructing a fuzzy order for which $\mathcal{I}_{\odot}$ is not a singleton.

Since $\odot \neq \wedge$, there exist $a, b \in L$ such that

$$a \odot b < a \wedge b.$$

Pick two distinct elements $u, v \in U$ and consider the fuzzy relation $\lesssim$ on U defined by

$$x \lesssim x = 1 \text{ for each } x \in U, \quad u \lesssim v = a, \quad v \lesssim u = b, \quad \text{and} \quad x \lesssim y = 0 \text{ otherwise.}$$

Define fuzzy relations $\sim_1$ and $\sim_2$ on U by

$$x \sim_1 y = (x \lesssim y) \odot (y \lesssim x) \quad \text{and} \quad x \sim_2 y = (x \lesssim y) \wedge (y \lesssim x),$$

for any $x, y \in U$. One may observe that $\sim_1$ and $\sim_2$ are two distinct fuzzy equalities on U. Note that the separability of $\sim_1$ and $\sim_2$ follows from $1 \odot 1 = 1$ and $a \odot b < a \wedge b$, since these assumptions imply that $a \neq 1$ or $b \neq 1$, hence $u \sim_1 v \neq 1$ and $u \sim_2 v \neq 1$, verifying that $x \sim_1 y = 1$ implies $x = y$ and $x \sim_2 y = 1$ implies $x = y$ for any $x, y \in U$.

As $\lesssim$ is clearly $\sim_1$-reflexive and $\sim_2$-reflexive, $\lesssim$ is reflexive and compatible with $\sim_1$ as well as with $\sim_2$ due to Corollary 2 of part I. Now, $\lesssim$ is obviously transitive, satisfies $\odot$-antisymmetry w.r.t. $\sim_1$, and due to (8), also w.r.t. $\sim_2$. We obtained that $\lesssim$ is a fuzzy order on U equipped with $\sim_1$ as well as a fuzzy order on U equipped with $\sim_2$ according to Definition 4. Therefore, $\mathcal{I}_\odot$ contains both $\sim_1$ and $\sim_2$, and is hence not a singleton. ■

It is well known and trivial fact that for any ordinary order $\leq$, the equality relation $=$ is determined by $\leq$ as follows:

$$u = v \text{ if and only if } u \leq v \text{ and } v \leq u.$$

In view of Theorem 5, a generalization of this property is satisfied only for the notion of fuzzy order according to Definition 2, revealing thus a distinct role of $\wedge$-antisymmetry.

Fourth view: Indistinguishability with respect to hierarchy

In addition to the distinct features of $\wedge$ and $\bullet$ established above, one may derive further distinct properties of these two aggregation operations by the following rationale.

A fuzzy order $\lesssim$ represents a graded hierarchy of the objects on the underlying universe U. It is hence natural to ask which objects are indistinguishable with respect to the hierarchy. Such an indistinguishability is naturally conceived as a fuzzy relation $\sim$ on U satisfying at least the following properties: $\sim$ is reflexive, symmetric, and is included in $\lesssim$. Reflexivity and symmetry are implied by the obvious requirements that any $u \in U$ is indistinguishable from itself and that if u is indistinguishable from v, then v is indistinguishable from u. Inclusion of $\sim$ in $\lesssim$ is crucial for our argument below and we derive this requirement intuitively as follows: Since $\lesssim$ is reflexive, u is less than or equal to u for each u. One hence expects that if u is indistinguishable from v, then u is less than or equal to v as well, since the other possibility, i.e. u not being less than or equal to v, would distinguish u from v.

Now, of all the possible indistinguishabilities w.r.t. the hierarchy represented by $\lesssim$, one is naturally interested in the largest one, which is most informative (the least one is intuitively expected to be the crisp identity).

In a fuzzy setting, reflexivity, symmetry, and inclusion of $\sim$ in $\lesssim$ mean $u \sim u = 1$, $u \sim v = v \sim u$, and $u \sim v \leq u \lesssim v$ for any $u, v \in U$. The following observation reveals distinct roles of $\wedge$ and $\bullet$ from the present viewpoint:

Theorem 6: *Let $\lesssim$ be reflexive and transitive fuzzy relation on U.*

(a) *The largest reflexive and symmetric fuzzy relation contained in $\lesssim$ (i.e. the most informative indistinguishability w.r.t. $\lesssim$ in the sense above) is $\equiv_\wedge$, which is also the largest reflexive, symmetric, and transitive fuzzy relation contained in $\lesssim$.*
(b) *The least reflexive, symmetric, and transitive fuzzy relation contained in $\lesssim$ is $\equiv_\bullet$.*

Proof: Since by definition, $u \equiv_\wedge v = (u \lesssim v) \wedge (v \lesssim u)$, the first part in (a) follows from the following claim, which is easy to verify: For any binary fuzzy relation R, the relation S_R defined by

$$S_R(u, v) = R(u, v) \wedge R(v, u)$$

is the largest symmetric fuzzy relation contained in R. The second part is due to the fact established above that $\equiv_\wedge$ is reflexive and transitive.

(b) is trivial because $\approx_\bullet$ is the crisp equality. ■

4. Conclusions and future topics

4.1. Conclusions

In our two-part paper, we thoroughly consider the existing definitions of fuzzy order in which antisymmetry is formulated with respect to a generalized equality on the underlying universe. We review the current approaches, which exist in the literature for quite some time (Belohlavek 2001, 2002, 2004; Blanchard 1983; Bodenhofer 1999, 2000, 2003; Höhle 1987; Höhle and Blanchard 1985) but have not been examined from the perspective we provide in our treatment.

We first present a detailed account of the development of the variants of the considered notion of fuzzy order along with a number of historical remarks starting with the initial paper by Zadeh (1971). Second, we provide various kinds of observations to enhance the current understanding of the examined notion of fuzzy order, and analyze relationships between the existing variants of this notion. Third, we study in detail the notion of antisymmetry, which is arguably the least understood of the conditions required by the existing definitions of fuzzy order.

The most important results regarding antisymmetry is a unifying concept of antisymmetry along with the resulting generalization of the concept of fuzzy order and our theorems according to which – contrary to the present understanding – the existing variants of the notion of fuzzy order are mutually equivalent and are equivalent to our generalized concept of fuzzy order. The latter is due to a new perspective that we present, which is different from the current view according to which fuzzy orders with $\otimes$-antisymmetry are more general than fuzzy orders with $\wedge$-antisymmetry. The new perspective consists in asking:

Which fuzzy relations may be regarded as fuzzy orders?

We regard such a perspective more suitable compared to the one considered implicitly in some previous works, namely one based on the question: Given a fixed fuzzy equality, which fuzzy relations may be regarded as fuzzy orders? We also identify several properties that distinguish the existing variants of the notion of fuzzy order.

4.2. Future topics

The present results open a general problem of whether and to what extent it matters which of the variants of the notion of fuzzy order examined in this paper one employs in the development of further areas involving the notion of fuzzy order. For instance, whether and to what extent this matters in the development of complete lattices, closure structures, fixed point theory, and other topics in the setting of fuzzy logic. We obtained several results along these lines already and shall present them in future publications.

For illustration, let us consider the concept of a complete lattice in the setting of fuzzy logic as developed by Belohlavek (2001, 2002, 2004); see also Höhle (1987) for a closely related earlier approach. Let $\lesssim$ be a fuzzy order on a set U equipped with a fuzzy equality $\approx$ in the sense of Definition 2, which notion represents the framework for the considerations on complete lattices we are about to recall (Belohlavek 2001, 2002, 2004).

For any fuzzy set $A \in L^U$ define the fuzzy sets $\mathcal{L}(A) \in L^U$ and $\mathcal{U}(A) \in L^U$ of lower and upper cones of A, respectively, by

$$[\mathcal{L}(A)](u) = \bigwedge_{v \in U} (A(v) \to u \lesssim v) \text{ and } [\mathcal{U}(A)](u) = \bigwedge_{v \in U} (A(v) \to v \lesssim u).$$

Furthermore, define for any $A \in L^U$ the fuzzy sets $\inf(A)$ and $\sup(A)$ of infima and suprema by

$$\inf(A) = \mathcal{L}(A) \wedge \mathcal{UL}(A) \quad \text{and} \quad \sup(A) = \mathcal{U}(A) \wedge \mathcal{LU}(A), \tag{17}$$

where $\mathcal{UL}(A)$ and $\mathcal{LU}(A)$ stand for $\mathcal{U}(\mathcal{L}(A))$ and $\mathcal{L}(\mathcal{U}(A))$, respectively. That is, $[\inf(A)](u) = [\mathcal{L}(A)](u) \wedge [\mathcal{UL}(A)](u)$ for each $u \in U$ and analogously for $\sup(A)$.

Now, a fuzzy ordered set $\langle U, \approx, \lesssim \rangle$ in the sense of Definition 2 is called a complete lattice if for every $A \in L^U$, both $\inf(A)$ and $\sup(A)$ are $\approx$-singletons (Belohlavek 2001, 2002, 2004). Note that a $\approx$-singleton is a fuzzy set $A \in L^U$ for which there exists $u \in U$ such that for each $v \in U$ one has $A(v) = u \approx v$; there exist other, equivalent definitions of $\approx$-singletons. If A is a $\approx$-singleton, u is the only element for which $A(u) = 1$, hence one may also speak of the $\approx$-singleton determined by u. If $\inf(A)$ is a $\approx$-singleton, the unique element $u \in U$ for which $[\inf(A)](u) = 1$ is called the infimum of A; the same applies to suprema. Note that in the original works (Belohlavek 2001, 2002, 2004), complete lattices as defined above are called completely lattice **L**-ordered sets and that in some subsequent works, they are called simply fuzzy lattices by other authors.

While the theory of complete lattices and related structures in a fuzzy setting has been advanced considerably, the purpose of the present illustration is to briefly point out a natural possibility to reconsider the above notions from the viewpoint of the general definition of a fuzzy order provided by Definition 4. For the sake of our illustration we refrain to the case in which $\lesssim$ is a fuzzy order on a set U equipped with the fuzzy equality $\approx_\otimes$ according to Definition 1, and hence also a fuzzy order according Definition 4 (for $\odot = \otimes$). Such a setting is very close to the one of fuzzy orders according to Definition 2, i.e. to the setting in which the theory of complete lattices has been developed as mentioned above. For instance, $u \approx_\otimes v = (u \lesssim v) \otimes (v \lesssim u)$ is analogous to the equality $u \approx v = (u \lesssim v) \wedge (v \lesssim u)$ implied by Definition 2. Yet, this setting does not impose any restriction on $\lesssim$ itself; cf. Theorem 1, its part (d), and Theorem 2.

To obtain a sound variant of the above notion of a complete lattice for our setting with a fuzzy ordered set $\langle U, \approx_\otimes, \lesssim \rangle$ according to Definition 1, one may proceed in several ways, of which we present the following one. For a fuzzy set $A \in L^U$, put

$$\inf_\otimes(A) = \mathcal{L}(A) \otimes \mathcal{UL}(A) \quad \text{and} \quad \sup_\otimes(A) = \mathcal{U}(A) \otimes \mathcal{LU}(A),$$

with $\mathcal{L}$ and $\mathcal{U}$ defined as above. Let us call $\langle U, \approx_\otimes, \lesssim \rangle$ a complete lattice if both $\inf_\otimes(A)$ and $\sup_\otimes(A)$ are $\approx_\otimes$-singletons for each $A \in L^U$. This definition is directly analogous to (17); in a sense, $\otimes$ replaces $\wedge$ in appropriate places.

It has been established (Belohlavek 2004, Lemma 11) that for a fuzzy ordered set $\langle U, \approx, \lesssim \rangle$ according to Definition 2, the following conditions are equivalent for any fuzzy set $A \in L^U$:

(a) $\inf(A)$ is a $\approx$-singleton;
(b) there exists $u \in U$ such that $[\inf(A)](u) = 1$.

Adopting the proof of this Lemma 11, we obtain an analogous property for the present setting: For a fuzzy ordered set $\langle U, \approx_\otimes, \lesssim \rangle$ according to Definition 1, the following conditions are equivalent for any fuzzy set $A \in L^U$:

(a') $\inf(A)$ is a $\approx_\otimes$-singleton;
(b') there exists $u \in U$ such that $[\inf_\otimes(A)](u) = 1$.

In view of Theorem 1 and Theorem 2, and the obvious equivalence of the above conditions (b) and (b'), we obtain the following result: For a reflexive and transitive relation $\lesssim$ on U, the following conditions are equivalent:

(i) $\langle U, \approx, \lesssim \rangle$ is a fuzzy ordered set according to Definition 2 that forms a complete lattice in the sense of (Belohlavek 2001, 2002, 2004).
(ii) $\langle U, \approx_\otimes, \lesssim \rangle$ is a fuzzy ordered set according to Definition 1 that forms a complete lattice in the sense of the above definition with $\inf_\otimes$ and $\sup_\otimes$.

This result is one of several possible ways expressing the fact that being a complete lattice is invariant with respect to the two possible notions of a fuzzy order involved. A proper study of such an invariance in general and of its ramifications for the theory of complete lattices thus presents a topic for further research. Note that relevant results, which need to be reconsidered in the present perspective, have been obtained by Martinek (2008, 2011).

Notes

1. In fact, Blanchard in general defines the notion of a fuzzy order on a fuzzy set defined on the universe U. The notion we describe corresponds to the case of a fuzzy order on a set, i.e. when the fuzzy set on U is identified with U.
2. With respect to this notion of fuzzy order, Zhang and Fan (2005) cite an earlier paper by L. Fan, Q.-Y. Zhang, W.-Y. Xiang, and C.-Y. Zheng, "An L-fuzzy approach to quantitative domain (I) (generalized ordered set valued in frame and adjunction theory)", *Fuzzy Systems Math.* 14 (2000), 6–7, written in Chinese, which we were not able to obtain.
3. A frame, or a complete Heyting algebra, is a complete lattice satisfying $a \wedge (\bigvee_j b_j) = \bigvee_j (a \wedge b_j)$. That is, a frame may be regarded as a complete residuated lattice in which $\otimes$ coincides with the infimum $\wedge$.
4. As is easily seen, further variations of the claims of the two theorems may be formulated. For instance, a variation of Theorem 1 may be proved for fuzzy orders according to Definition 1 as well as for fuzzy orders according to Definition 2.

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