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Formal concept analysis and linguistic hedges

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This paper presents an application of linguistic hedges to formal concept analysis of data with fuzzy attributes. Formal concept analysis aims at extraction of particular (bi-)clusters, called formal concepts, from data. The clusters link collections of objects (extents) and attributes (intents), and have a clear interpretation due to a simple verbal description of the concept-forming operators. We insert linguistic hedges such as "very" or "extremely" in the description of the operators. This way, linguistic hedges become parameters for formal concept analysis that control the number of clusters extracted from data. Namely, as we show theoretically as well as experimentally, stronger hedges result in a smaller number of clusters. The new concept-forming operators form Galois-like connections. We study their properties and axiomatize them. Then, we show that a concept lattice with hedges, i.e. the set of all formal concepts of the new operators, is indeed a complete lattice which is isomorphic to a particular ordinary concept lattice. We describe the isomorphism and its inverse. These mappings serve as translation procedures. As a consequence, we obtain a theorem characterizing the structure of concept lattices with hedges which generalizes the well-known main theorem of ordinary concept lattices. The isomorphism and its inverse enable us to compute a concept lattice with hedges using algorithms for ordinary concept lattices. We demonstrate by experiments that when selecting various hedges from the strongest to weaker ones, the reduction in size of the corresponding concept lattices is smooth. From a broader perspective, we argue that linguistic hedges represent mathematically and computationally a feasible way to parameterize methods for knowledge extraction from data that enables one to emphasize or suppress extracted patterns while keeping their interpretation.

Keywords: formal concept analysis, fuzzy logic, linguistic hedge

1. Introduction

Linguistic hedges, or simply hedges, are expressions such as "very", "roughly", or "extremely". Early in the development of fuzzy logic, Zadeh recognized that hedges play an important role in human reasoning and proposed a mathematical model of hedges (Zadeh 1972), see also (Zadeh 1975). Since then, hedges were studied in a number of papers from the point of view of applications of approximate reasoning (Klir and Yuan 1995) as well as from a linguistic and logical point of view (Hájek 1998, 2001, Klir and Yuan 1995, Lakoff 1973). The main role of hedges is that they modify the meaning of natural language expressions. Hedges such as "very" or "extremely" intensify meaning. For example, consider the expressions "country with a high rate of unemployment" and "country with an extremely high rate of unemployment". For a particular country, a degree to which the latter expression applies is certainly less than or equal to the degree to which the former expression applies. Therefore, the granule of countries represented by the latter expression is contained in the granule represented by the former expression. In this sense, intensifying hedges suppress the extent of granules to which they are applied. Analogously, de-intensifying hedges such as "roughly"'or "more or less" extend and emphasize the extent of granules.

In this paper, we develop an approach to control the number of formal concepts extracted in formal concept analysis from data tables describing objects (table rows) and their fuzzy attributes (table columns). FCA aims at extraction of particular (bi-)clusters, called formal concepts, from data. The clusters link collections

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of objects (concept extent) and attributes (concept intent), and are based on attribute sharing rather than distance. Moreover, the clusters have a clear interpretation due to a simple verbal description of the concept-forming operators. As a clustering method, FCA therefore has a linguistic flavor. If the attributes describing the input data are graded, or fuzzy, the collections of objects and attributes to which formal concepts apply are graded as well. The collection of formal concepts extracted from data is often large. In order to control the number of extracted formal concepts, we propose an approach in which we insert linguistic hedges such as "very" or "extremely" in the description of the concept-forming operators. This way, linguistic hedges become parameters for formal concept analysis. We show theoretically as well as experimentally that stronger hedges such as "extremely" result in a smaller number of more important formal concept extracted from data. Weaker hedges such as "very" result in a larger number of formal concepts among which are also less important ones. A user can therefore tune the process of data analysis by changing the hedge until he reaches the desired output. We prove several theorems regarding the parameterized formal concept analysis. We show that the new concept-forming operators form Galois-like connections, analyze their properties and provide their axiomatization. Then, we show that a concept lattice with hedges, i.e. the set of all formal concepts of the new operators, is a complete lattice, describe its structure, and show that it is isomorphic to a particular ordinary concept lattice. We describe the isomorphism and its inverse. These mappings serve as translation procedures between the framework of concept lattices with hedges and the framework of ordinary concept lattices. As a result, we obtain a theorem characterizing the structure of concept lattices with hedges. The theorem generalizes the well-known main theorem of ordinary concept lattices. In addition, the isomorphism and its inverse enable us to compute a concept lattice with hedges using algorithms for ordinary concept lattices. We include illustrative examples which demonstrate that when selecting various hedges from the strongest to weaker ones, the reduction in size of the corresponding concept lattices is smooth.

The paper is organized as follows. Section 2 surveys preliminaries from fuzzy logic and fuzzy sets, and from formal concept analysis. In Section 3, we present our method and perform its theoretical analysis, mainly an analysis of issues related to applications of formal concept analysis. Section 4 contains examples and experiments.

2. Preliminaries

2.1 *Fuzzy sets and fuzzy relations*

Fuzzy logic and fuzzy set theory are formal frameworks for a manipulation of a particular form of imperfection called fuzziness. Contrary to classical logic, fuzzy logic uses a scale L of truth degrees, a most common choice being $L = [0, 1]$ (real unit interval) or some subchain of $[0, 1]$. This enables us to consider intermediate truth degrees of propositions, e.g. "object x has attribute y" has a truth degree 0.8 indicating that the proposition is almost true. In addition to a set L of truth degrees, one has to pick an appropriate collection of logical connectives (implication, conjunction, ...). A general choice of a set of truth degrees plus logical connectives is represented by complete residuated lattices (equipped possibly with additional operations). The rest of this section presents an introduction to fuzzy logic notions we need in the sequel. Details can be found e.g. in (Belohlavek 2002, Goguen 1968–9, Hájek 1998, Klir and Yuan 1995).

A complete residuated lattice (Hájek 1998) is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$

such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

for each $a, b, c \in L$. Elements a of L are called truth degrees. Operations $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding residuum $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are:

$$\text{Łukasiewicz:} \quad \begin{array}{rcl} a \otimes b &=& \max(a + b - 1, 0), \\ a \rightarrow b &=& \min(1 - a + b, 1), \end{array} \tag{2}$$

$$\text{Gödel:} \quad \begin{array}{rcl} a \otimes b &=& \min(a, b), \\ a \rightarrow b &=& \begin{cases} 1 \text{ if } a \leq b, \\ b \text{ otherwise,} \end{cases} \end{array} \tag{3}$$

$$\text{Goguen (product):} \quad \begin{array}{rcl} a \otimes b &=& a \cdot b, \\ a \rightarrow b &=& \begin{cases} 1 \text{ if } a \leq b, \\ \frac{b}{a} \text{ otherwise.} \end{cases} \end{array} \tag{4}$$

Another common choice is a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of L and restrictions of Gödel operations on $[0, 1]$ to L.

Having $\mathbf{L}$, we define usual notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A\colon U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". If $U = \{u_1, \ldots, u_n\}$ then A can be denoted by $A = \{{}^{a_1}\!/u_1, \ldots, {}^{a_n}\!/u_n\}$ meaning that $A(u_i)$ equals a_i for each $i = 1, \ldots, n$. For brevity, we introduce the following convention: we write $\{\ldots, u, \ldots\}$ instead of $\{\ldots, {}^{1}\!/u, \ldots\}$, and we also omit elements of U whose membership degree is zero. For example, we write $\{u, {}^{0.5}\!/v\}$ instead of $\{{}^{1}\!/u, {}^{0.5}\!/v, {}^{0}\!/w\}$, etc.

Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. Binary $\mathbf{L}$-relations (binary fuzzy relations) between U and V can be thought of as $\mathbf{L}$-sets in the universe $U \times V$. That is, a binary $\mathbf{L}$-relation $R \in \mathbf{L}^{U \times V}$ between a set U and a set V is a mapping assigning to each $u \in U$ and each $v \in V$ a truth degree $R(u, v) \in L$ (a degree to which u and v are related by R). An $\mathbf{L}$-set $A \in \mathbf{L}^U$ is called crisp if $A(u) \in \{0, 1\}$ for each $u \in U$. Crisp $\mathbf{L}$-sets can be identified with (characteristic functions of) ordinary sets: crisp $\mathbf{L}$-set $A \in \mathbf{L}^U$ corresponds to the ordinary set $\{u \in U \mid A(u) = 1\}$. Therefore, for a crisp A, we also write $u \in A$ for $A(u) = 1$ and $u \notin A$ for $A(u) = 0$. An $\mathbf{L}$-set $A \in \mathbf{L}^U$ is called empty (denoted by $\emptyset$) if $A(u) = 0$ for each $u \in U$; $A \in \mathbf{L}^U$ is called full (denoted by U) if $A(u) = 1$ for each $u \in U$. An $\mathbf{L}$-set $A \in \mathbf{L}^U$ is called a singleton if there is $u \in U$ such that $A(v) = 0$ for each $v \neq u$. In such a case, we denote A by $\{{}^{a}\!/u\}$ where $a = A(u)$. Note that we allow $A(u) = 0$, i.e. a singleton may be an empty fuzzy set.

Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A,B) = \bigwedge_{u\in U}(A(u) \to B(u)) \tag{5}$$

of A in B generalizing the classical subsethood relation $\subseteq$. Described verbally, $S(A,B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. In the following we use well-known properties of residuated lattices and fuzzy structures which can be found in monographs (Belohlavek 2002, Hájek 1998). Throughout the rest of the paper, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

2.2 *Truth-stressing linguistic hedges*

For our purpose, we use intensifying linguistic hedges. Intensifying hedges such as "very" are conveniently modeled by unary functions of the set of truth degrees satisfying suitable properties which reflect our intuitive requirements. In our framework of residuated lattices, we use the concept of a truth-stressing hedge which is very close to the one used in (Hájek 1998, 2001). By a truth-stressing hedge (shortly, a hedge) on $\mathbf{L}$ we mean a unary mapping * on L satisfying

$$1^* = 1, \tag{6}$$

$$a^* \leq a, \tag{7}$$

$$(a \to b)^* \leq a^* \to b^*, \tag{8}$$

$$a^{**} = a^*, \tag{9}$$

for each $a, b \in L$. Hedge * can be seen as a (truth function of) unary logical connective "very", "extremely", etc., see (Hájek 1998, 2001). Properties (6)–(8) have natural interpretations. Namely, (6) can be seen as saying that a fully true formula is very true; (7) can be read: "if φ is very true, then φ is true"; and (8), which is equivalent to $(a \to b)^* \otimes a^* \leq b^*$, reads: "if $\varphi \Rightarrow \psi$ is very true and if φ is very true, then ψ is very true". Note that as a consequence of (6) and (8) we get monotony: if $a \leq b$ then $a^* \leq b^*$. (9) is a technical condition which we need and which can be thought of a little restricting but not very much. Namely, (9) says that propositions "very very φ" and "very φ" have equal truth degrees. A hedge may be regarded as an $\mathbf{L}^*$-interior operator (Belohlavek *et al.* 2005) mapping the largest element to itself, when one identifies $\mathbf{L}$ with $\mathbf{L}^{\{u\}}$.

Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^* = a$ ($a \in L$); (ii) globalization (Takeuti and Titani 1987):

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

A special case of a complete residuated lattice with hedge is a two-element Boolean algebra $\langle\{0,1\}, \wedge, \vee, \otimes, \to, {}^*, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of classical logic. That is, the operations $\wedge, \vee, \otimes, \to$ of $\mathbf{2}$ are the truth functions (interpretations) of the corresponding logical connectives of the classical logic and $0^* = 0$, $1^* = 1$.

2.3 *Formal concept analysis of data with fuzzy attributes*

In its basic setting, FCA can be applied to data with bivalent (crisp) attributes. For data with fuzzy attributes, several extensions have been proposed, see e.g. (Belohlavek and Vychodil 2005b) for an overview. We are interested in an approach presented e.g. in (Belohlavek 2002, Pollandt 1997) since it is the most developed one. In what follows, we present basic notions.

A data table with fuzzy attributes, which is the input to FCA, can be represented by a triplet $\langle X, Y, I\rangle$ where X is a finite set of objects, Y is a finite set of attributes, and $I \in \mathbf{L}^{X\times Y}$ is a binary fuzzy relation between X and Y assigning to each object $x \in X$ and each attribute $y \in Y$ the degree $I(x, y) \in L$ to which x has y. $\langle X, Y, I\rangle$ can be thought of as a table with rows and columns corresponding to objects $x \in X$ and attributes $y \in Y$, respectively, and table entries containing degrees $I(x, y)$, see e.g. Fig. 1 in Section 4.

For $A \in \mathbf{L}^X$, $B \in \mathbf{L}^Y$ (i.e. A is a fuzzy set of objects, B is a fuzzy set of attributes), we define fuzzy sets $A^{\Uparrow} \in \mathbf{L}^Y$ (fuzzy set of attributes) and $B^{\Downarrow} \in \mathbf{L}^X$ (fuzzy set of objects) by

$$A^{\Uparrow}(y) = \textstyle\bigwedge_{x\in X}(A(x) \to I(x, y)), \tag{11}$$

$$B^{\Downarrow}(x) = \textstyle\bigwedge_{y\in Y}(B(y) \to I(x, y)). \tag{12}$$

Described verbally, $A^{\Uparrow}$ is the fuzzy set of all attributes from Y shared by all objects from A (and similarly for $B^{\Downarrow}$), because according the basic principles of fuzzy logic, $A^{\Uparrow}(y)$ is the truth degree of the proposition "for each object $x \in X$: if x belongs to A then x has y". A formal concept of $\langle X, Y, I\rangle$ is any pair $\langle A, B\rangle$ of $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$ satisfying $A^{\Uparrow} = B$ and $B^{\Downarrow} = A$. That is, a formal concept consists of a fuzzy set A (so-called extent) of objects which fall under the concept and a fuzzy set B (so-called intent) of attributes which fall under the concept such that A is the fuzzy set of all objects from X sharing all attributes from B and, conversely, B is the fuzzy set of all attributes from Y shared by all objects from A. Formal concepts represent conceptual clusters hidden in the data table $\langle X, Y, I\rangle$. The notion of a formal concept is inspired by a traditional understanding of human concepts which is due to Port-Royal logic (Arnauld and Nicole 1662).

The collection $\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \mid A^{\Uparrow} = B, B^{\Downarrow} = A\}$, i.e. the collection of all formal concepts of $\langle X, Y, I\rangle$, can be equipped with a partial order $\leq$ modeling the subconcept-superconcept hierarchy (e.g., *dog* $\leq$ *mammal*) defined by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ (iff } B_2 \subseteq B_1). \tag{13}$$

Note that $^{\Uparrow}$ and $^{\Downarrow}$ form a so-called fuzzy Galois connection (Belohlavek 2002) and that $\mathcal{B}(X, Y, I)$ is in fact a set of all fixed points of $^{\Uparrow}$ and $^{\Downarrow}$. Under $\leq$, $\mathcal{B}(X, Y, I)$ happens to be a complete lattice, called a (fuzzy) concept lattice associated to $\langle X, Y, I\rangle$. The basic structure of fuzzy concept lattices is described by the so-called main theorem of concept lattices (Belohlavek 2002, 2004), the first part of which is given by the following theorem (Belohlavek 2004).

Theorem 2.1: *The set $\mathcal{B}(X, Y, I)$ equipped with $\leq$ a complete lattice where the infima and suprema are given by*

$$\textstyle\bigwedge_{j\in J}\langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\Downarrow\Uparrow}\rangle,$$

$$\textstyle\bigvee_{j\in J}\langle A_j, B_j\rangle = \langle (\bigcup_{j\in J} A_j)^{\Uparrow\Downarrow}, \bigcap_{j\in J} B_j\rangle.$$

For a detailed information on formal concept analysis of data tables with fuzzy attributes we refer to (Belohlavek 2002, Belohlavek and Vychodil 2005a). Formal concept analysis of data tables with binary attributes is thoroughly studied in (Carpineto and Romano 2004, Ganter and Wille 1999) where the reader can find theoretical foundations, methods and algorithms, and applications in various areas.

3. Concept lattices with hedges

3.1 *Definition of concept-forming operators and basic remarks*

Our main aim is to extend the above-presented approach to FCA of data with fuzzy attributes in order to be able to control, by means of parameters, the number of extracted formal concepts from the input data $\langle X, Y, I\rangle$. We use particular linguistic hedges, so-called (truth-stressing) hedges as the parameters, see Section 2.

Let $\mathbf{L}$ be a complete residuated lattice. Suppose that for each object $x \in X$ we are given a hedge $*_x$ on $\mathbf{L}$ and that for each attribute $y \in Y$ we are given a hedge $*_y$ on $\mathbf{L}$. For the sake of brevity, we will denote the indexed collection of all $*_x$'s by $*_X$ and the collection of all $*_y$'s by $*_Y$. For fuzzy sets $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$, consider fuzzy sets $A^{\uparrow} \in \mathbf{L}^Y$ and $B^{\downarrow} \in \mathbf{L}^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$) defined by

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x\in X}(A(x)^{*_x} \to I(x,y)) \tag{14}$$

and

$$B^{\downarrow}(x) = \textstyle\bigwedge_{y\in Y}(B(y)^{*_y} \to I(x,y)). \tag{15}$$

Since a hedge is a (truth function of) unary connective "very true", $A(x)^{*_x}$ is a truth degree of "it is very true that x is from A", the same applies to $B(y)^{*_y}$. Therefore, using basic rules of predicate fuzzy logic, $A^{\uparrow}(y)$ is the truth degree of "for each $x \in X$: if it is very true that x is from A then x has y". Similarly for $B^{\downarrow}$ with "very true" interpreted by $*_Y$. That is, $A^{\uparrow}$ is a fuzzy set of attributes common to all objects for which it is very true that they belong to A, and $B^{\downarrow}$ is a fuzzy set of objects sharing all attributes for which it is very true that they belong to B.

Remark 1 : In the ordinary setting, i.e. $L = \{0, 1\}$, the only hedge is the identity mapping on L. Therefore, our approach with hedges applied in the ordinary setting coincides with the ordinary approach. In other words, in the ordinary setting, hedges are hidden (degenerate) and bring thus nothing new. As we shall see, in a fuzzy setting, employing hedges is non-trivial and enhances FCA of data with fuzzy attributes with a useful feature.

The set

$$\mathcal{B}(X^{*_X}, Y^{*_Y}, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$$

of all fixpoints of $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ is called a (fuzzy) concept lattice with hedges, the pairs $\langle A, B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ are called formal concepts of $\langle X, Y, I\rangle$. $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ thus contains all pairs $\langle A, B\rangle$ such that A is the collection of all objects that have all the attributes of "very B", and B is the collection of all attributes that are shared by all the objects of "very A". Therefore, a verbal description of what is a formal concept in our new setting with hedges is essentially the same as in the ordinary case.

For the sake of brevity, we use also $\mathcal{B}(X^*, Y^*, I)$ instead of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. Also, we omit $*_X$ if each $*_x$ it is identity and write e.g. only $\mathcal{B}(X, Y^{*_Y}, I)$, etc. Given a formal concept $\langle A, B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, A and B are called the extent and intent of $\langle A, B\rangle$, respectively. In general, both A and B are fuzzy sets. This corresponds to the fact that in general, concepts apply to objects and attributes to intermediate degrees, not only 0 and 1.

For $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, put

$$\langle A_1, B_1\rangle \le \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ (iff } B_2 \subseteq B_1).$$

This defines a subconcept-superconcept hierarchy on $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. As we will see later on, $\le$ makes $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ indeed a lattice, justifying thus the term concept lattice with hedges.

The following example shows some relationships of concept lattices with hedges to selected approaches which can be found in the literature.

Example 3.1 (1) Let $*_x$ and $*_y$ be identities on L for each $x \in X$, $y \in Y$. Then $\mathcal{B}(X, Y, I)$, i.e. $\mathcal{B}(X^*, Y^*, I)$, is what is called a (fuzzy) concept lattice, see e.g. (Belohlavek 2002, Burusco and Fuentes-Gonzáles 1994, Pollandt 1997). An axiomatic characterization of mappings $^{\uparrow}$ and $^{\downarrow}$ can be found in (Belohlavek 1999, 2002).

(2) A crisply generated formal concept of $\langle X, Y, I\rangle$ is a formal concept $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ ($*_x$ and $*_y$ are identities) which is generated by a crisp set of attributes, i.e. there is $D \in \{0, 1\}^Y$ such that $A = D^{\Downarrow}$ and $B = A^{\Uparrow}$. Crisply generated formal concepts may be thought of as the important ones. The number of crisply generated concepts is considerably smaller than the number of all formal concepts, see (Belohlavek *et al.* 2005). Now, it can be shown that if each $*_x$ is identity and each $*_y$ is globalization on L, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is just the set of all crisply generated concepts.

(3) It can be shown (Belohlavek and Vychodil 2005b) that what is called a fuzzy concept lattice in (Ben Yahia and Jaoua 2001) is in fact a structure isomorphic to $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ with each $*_x$ being identity and each $*_y$ being globalization, respectively. If, on the other hand, each $*_x$ is globalization and each $*_y$ is identity, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic to what is called a one-sided fuzzy concept lattice in (Krajči 2003).

(4) Let $*_x$ be the same for all $x \in X$, $*_y$ be the same for all $y \in Y$. That is, we can arbitrarily choose one hedge $*_1$ for X and one hedge $*_2$ for Y, and set $*_x = *_1$ for each $x \in X$ and $*_y = *_2$ for each $y \in Y$. Then $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is what is called a concept lattice with hedges in (Belohlavek and Vychodil 2005a, 2007). This is how our present approach extends the one from (Belohlavek and Vychodil 2005a, 2007). Note that it has been shown in (Krajči 2005) that every concept lattice with hedges in the sense of (Belohlavek and Vychodil 2005a) is isomorphic to some generalized concept lattice (Krajči 2005). Still more general concept lattices have been studied in (Medina *et al.* 2009). It is an interesting topic for future research to see how the presented approach relates to (Krajči 2005) and (Medina *et al.* 2009).

(5) An attribute implication is an expression $A \Rightarrow B$ where $A, B \in \mathbf{L}^Y$ are fuzzy sets of attributes. The degree $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ to which $A \Rightarrow B$ is true in $\langle X, Y, I\rangle$ is defined by

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = \textstyle\bigwedge_{x\in X} S(A, I_x)^* \to S(B, I_x),$$

where $I_x \in \mathbf{L}^Y$ is a fuzzy set of attributes of object x, i.e. $I_x(y) = I(x, y)$, and $*$ is a hedge on L. Then, $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ is the truth degree of "each object from X

having all attributes from A has also all attributes from B". It can be shown that a set T of attribute implications is a basis of $\langle X, Y, I\rangle$, i.e. T semantically entails exactly the set of all attribute implications which are fully true (i.e., in degree 1) in $\langle X, Y, I\rangle$, if and only if the set of all models of T (a fuzzy set of attributes in which all implications of T are true) equals the set of all intents of formal concepts from $\mathcal{B}(X^{*_X}, Y, I)$ with each $*_x$ being equal to $*$, see (Belohlavek and Vychodil 2006) for an overview.

3.2 *Associated interior operators and their role as constraints*

Let us now take a closer look at formulas (14) and (15). Introduce operators $\mathrm{I}_X : \mathbf{L}^X \to \mathbf{L}^X$ and $\mathrm{I}_Y : \mathbf{L}^Y \to \mathbf{L}^Y$ by

$$(\mathrm{I}_X(A))(x) = (A(x))^{*_x} \tag{16}$$

and

$$(\mathrm{I}_Y(B))(y) = (B(y))^{*_y} \tag{17}$$

for any $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$. Then, (14) and (15) can be rewritten as

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x\in X}((\mathrm{I}_X(A))(x) \to I(x,y)) = (\mathrm{I}_X(A))^{\Uparrow}(y) \tag{18}$$

and

$$B^{\downarrow}(x) = \textstyle\bigwedge_{y\in Y}((\mathrm{I}_Y(B))(y) \to I(x,y)) = (\mathrm{I}_Y(B))^{\Downarrow}(x). \tag{19}$$

Therefore, $^{\uparrow}$ and $^{\downarrow}$ can be regarded as compositions of the original operators $^{\Uparrow}$ and $^{\Downarrow}$ and operators I_X and I_Y, respectively, i.e. as compositions of well-studied operators and new operators I_X and I_Y defined by hedges. The aim of this section is to explore the properties of operators I_X and I_Y.

Properties (6)–(9) of hedges suggest interior-like properties of I_X and I_Y. Because of symmetry, we proceed for I_X only.

Theorem 3.2: *For hedges ^{*_x} $(x \in X)$, the operator $\mathrm{I}_X : \mathbf{L}^X \to \mathbf{L}^X$ defined by* (16) *satisfies*

$$\mathrm{I}_X(\{^1/x\}) = \{^1/x\}, \tag{20}$$

$$\mathrm{I}_X(A) \subseteq A, \tag{21}$$

$$\mathrm{I}_X(A \to B) \subseteq \mathrm{I}_X(A) \to \mathrm{I}_X(B), \tag{22}$$

$$\mathrm{I}_X(A) = \mathrm{I}_X(\mathrm{I}_X(A)), \tag{23}$$

$$\mathrm{I}_X(\textstyle\bigcup_{x\in X}\{^{a_x}/x\}) = \textstyle\bigcup_{x\in X}\mathrm{I}_X(\{^{a_x}/x\}), \tag{24}$$

for each $A, B \in \mathbf{L}^X$, $x \in X$, $a_x \in X$.

Proof: (20): Follows from

$$(\mathrm{I}_X(\{^1/x\}))(x') = \begin{cases} (\{^1/x\}(x'))^{*_{x'}} = 1^{*_{x'}} = 1 & \text{for } x' = x \\ (\{^1/x\}(x'))^{*_{x'}} = 0^{*_{x'}} = 0 & \text{for } x' \neq x. \end{cases}$$

(21): Follows from $(\mathrm{I}_X(A))(x) = (A(x))^{*_x} \leq A(x)$.

(22): We have

$$(\mathrm{I}_X(A \to B))(x) = ((A \to B)(x))^{*_x} = (A(x) \to B(x))^{*_x} \leq$$
$$\leq A(x)^{*_x} \to B(x)^{*_x} = (\mathrm{I}_X(A) \to \mathrm{I}_X(B))(x).$$

(23): We have $(\mathrm{I}_X(A))(x) = (A(x))^{*_x} = (A(x))^{*_x *_x} = ((\mathrm{I}_X(A))(x))^{*_x} = (\mathrm{I}_X(\mathrm{I}_X(A)))(x)$.

(24): We have

$$(\mathrm{I}_X(\bigcup_{x\in X}\{{}^{a_x}/x\}))(x') = ((\bigcup_{x\in X}\{{}^{a_x}/x\})(x'))^{*_{x'}} = (\{{}^{a_{x'}}/x'\}(x'))^{*_{x'}} =$$
$$= {a_{x'}}^{*_{x'}} = (\bigcup_{x\in X}(\{{}^{{a_x}^{*_x}}/x\}))(x') = (\bigcup_{x\in X}\mathrm{I}_X(\{{}^{a_x}/x\}))(x').$$

The proof is finished. □

Corollary 3.3: I_X *satisfies*

$$\mathrm{I}_X(X) = X \tag{25}$$
$$A \subseteq B \textit{ implies } \mathrm{I}_X(A) \subseteq \mathrm{I}_X(B). \tag{26}$$

Proof: (25) follows easily from (20) and (24) by $\mathrm{I}_X(X) = \mathrm{I}_X(\bigcup_{x\in X}\{{}^1/x\}) = \bigcup_{x\in X}\mathrm{I}_X(\{{}^1/x\}) = \bigcup_{x\in X}\{{}^1/x\} = X$.

(26): If $A \subseteq B$ then $A \to B = X$. By (25) and (22) we get $X = \mathrm{I}_X(X) = \mathrm{I}_X(A \to B) \subseteq \mathrm{I}_X(A) \to \mathrm{I}_X(B)$. Since $X(x) = 1$, we conclude $(\mathrm{I}_X(A))(x) \leq (\mathrm{I}_X(B))(x)$ for each $x \in X$, i.e. $\mathrm{I}_X(A) \subseteq \mathrm{I}_X(B)$. □

Remark 2: (1) Note that an $\mathbf{L}_{\{1\}}$-interior operator (fuzzy interior operator) in X (Belohlavek and Funiokova 2004) is a mapping $\mathrm{I} : \mathbf{L}^X \to \mathbf{L}^X$ satisfying (21), (26), and (23). Therefore, I_X is a fuzzy interior operator.

(2) Note that I_X does not satisfy $S(A, B) \leq S(\mathrm{I}_X(A), \mathrm{I}_X(B))$ which a condition stronger than (26), see (Belohlavek and Funiokova 2004). Namely, if $X = \{x\}$, $A(x) = 1$, $B(x) = 0.5$, $\to$ is a Łukasiewicz implication and $*_x$ is globalization then $S(A, B) = 1 \to 0.5 = 0.5 \not\leq S(\mathrm{I}_X(A), \mathrm{I}_X(B)) = 1^{*_x} \to 0.5^{*_x} = 1 \to 0 = 0$.

Now, we proceed to obtain an axiomatic characterization of operator I_X which is induced by hedges $*_x$. According to Theorem 3.2 and Remark **2**, I_X is a fuzzy interior operator satisfying some further conditions. On the other hand, we have

Theorem 3.4: *Let* $\mathrm{I}_X : \mathbf{L}^X \to \mathbf{L}^X$ *satisfy* (20)–(24). *Then there exist hedges* $*_x$ $(x \in X)$ *satisfying* (16).

Proof: For each $x \in X$, let $*_x$ be defined by

$$a^{*_x} = (\mathrm{I}_X(\{{}^a/x\}))(x) \tag{27}$$

for each $x \in X$. First, we verify that each $*_x$ is a hedge, i.e., we verify (6)–(9).

(6): Follows directly from (20).

(7): Let $A = \{{}^a/x\}$. By (21), $a^{*_x} = (\mathrm{I}_X(\{{}^a/x\}))(x) = (\mathrm{I}_X(A))(x) \leq A(x) = a$.

(8): By (22),

$$(a \to b)^{*_x} = \mathrm{I}_X(\{{}^{a\to b}/x\})(x) = \mathrm{I}_X(\{{}^{a}/x\} \to \{{}^{b}/x\})(x) \leq$$
$$\leq (\mathrm{I}_X(\{{}^{a}/x\}) \to \mathrm{I}_X(\{{}^{b}/x\}))(x) = (\mathrm{I}_X(\{{}^{a}/x\}))(x) \to (\mathrm{I}_X(\{{}^{b}/x\}))(x) =$$
$$= a^{*_x} \to b^{*_x}.$$

(9): Let $X = \{x\}$. By (23) we have

$$a^{*_x} = (\mathrm{I}_X(\{{}^{a}/x\}))(x) = (\mathrm{I}_X(\mathrm{I}_X(\{{}^{a}/x\})))(x). \tag{28}$$

Now, since for $z \neq x$ we have $(\mathrm{I}_X(\{{}^{a}/x\}))(z) \leq \{{}^{a}/x\}(z) = 0$, $\mathrm{I}_X(\{{}^{a}/x\})$ is in fact a singleton and we have $\mathrm{I}_X(\{{}^{a}/x\}) = \{{}^{\mathrm{I}_X(\{{}^{a}/x\})(x)}/x\}$. Therefore,

$$(\mathrm{I}_X(\mathrm{I}_X(\{{}^{a}/x\})))(x) = (\mathrm{I}_X(\{{}^{(\mathrm{I}_X(\{{}^{a}/x\}))(x)}/x\}))(x) = (\mathrm{I}_X(\{{}^{a^{*_x}}/x\}))(x) = a^{*_x *_x}. \tag{29}$$

Putting (28) and (29) together, we get $a^{*_x} = a^{*_x *_x}$. Therefore, $*_x$ is a hedge.

Second, we verify (16). By (24), we have

$$(\mathrm{I}_X(A))(x) = (\mathrm{I}_X(\textstyle\bigcup_{z\in X}\{{}^{A(z)}/z\}))(x) = (\textstyle\bigcup_{z\in X}\mathrm{I}_X(\{{}^{A(z)}/z\}))(x) =$$
$$= \textstyle\bigvee_{z\in X}(\mathrm{I}_X(\{{}^{A(z)}/z\}))(x). \tag{30}$$

Since $\mathrm{I}_X(\{{}^{A(z)}/z\})(x) = 0$ for $z \neq x$, we have

$$\textstyle\bigvee_{z\in X}\mathrm{I}_X(\{{}^{A(z)}/z\})(x) = \mathrm{I}_X(\{{}^{A(x)}/x\})(x) = (A(x))^{*_x}. \tag{31}$$

Putting (30) and (31) together, we get (16). □

Theorem 3.2 and Theorem 3.4 yield immediately the following theorem.

Theorem 3.5 : *There is a one-to-one correspondence between indexed collections* $*_X$ *of hedges* $*_x$ $(x \in X)$ *and operators* I_X *satisfying* (20)–(24)*. The correspondence is defined by* (16) *and* (27)*.* □

Needless to say, an analogy of Theorem 3.5 holds true for hedges $*_y$ and for I_Y.

Remark 3 : Since $A^{\uparrow} = (\mathrm{I}_X(A))^{\Uparrow}$ and $B^{\downarrow} = (\mathrm{I}_Y(B))^{\Downarrow}$, operators $^{\uparrow}$ and $^{\downarrow}$ can be seen as resulting from the basic operators $^{\Uparrow}$ and $^{\Downarrow}$ by means of constraints imposed by fuzzy interior operators I_X and I_Y. While $\langle A, B\rangle$ is a formal concept of $\mathcal{B}(X, Y, I)$ iff A results by applying $^{\Downarrow}$ to B and B results by applying $^{\Uparrow}$ to A, one can see that $\langle A, B\rangle$ is a formal concept of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ iff A results by applying $^{\Downarrow}$ to the largest fixpoint of I_Y which is contained in B and B results by applying $^{\Uparrow}$ to the largest fixpoint of I_X which is contained in A. Therefore, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ differs from $\mathcal{B}(X, Y, I)$ in that $^{\Uparrow}$ and $^{\Downarrow}$ are constrained by I_X and I_Y. Fuzzy concept lattices with hedges thus represent a particular example of a general idea of reducing the size of concept lattices by imposing constraints.

3.3 *Properties of concept-forming operators and their axiomatization*

We are now going to explore properties of the concept-forming operators $^{\uparrow}$ and $^{\downarrow}$. The properties generalize well-known properties of Galois connections (Ore 1944) which are, in fact, the concept-forming operators in ordinary formal concept analysis. These properties provide us with theoretical insight into formal concept analysis and represent a mathematical core of formal concept analysis.

For the sake of brevity and in order to comply with our previous work (Belohlavek and Vychodil 2005a), we will also write

$$A^{*_X} \text{ instead of } \mathrm{I}_X(A) \qquad \text{and} \qquad B^{*_Y} \text{ instead of } \mathrm{I}_Y(B).$$

We need the following auxiliary result.

Lemma 3.6: *Any hedge* * *satisfies* $(\bigvee_{k\in K} a_k^*)^* = \bigvee_{k\in K} a_k^*$.

Proof: "$\leq$" follows from (7). "$\geq$" holds iff for any $k \in K$ we have $a_k^* \leq (\bigvee_{k\in K} a_k^*)^*$ which is true since $a_k^* = a_k^{**} \leq (\bigvee_{k\in K} a_k^*)^*$. □ □

Remark 4: A related equality, namely $(\bigvee_{k\in K} a_k)^* = \bigvee_{k\in K} a_k^*$ does not hold in general. Namely, consider $\mathbf{L}$ being a four element Boolean algebra with * being globalization. Let a and b be the two non-comparable elements of $\mathbf{L}$. Then $(a\vee b)^* = 1^* = 1 > 0 = a^* \vee b^*$.

Basic properties of $^\uparrow$ and $^\downarrow$ follow.

Theorem 3.7: *Operators* $^\uparrow$ *and* $^\downarrow$ *induced by* (14) *and* (15) *satisfy the following properties.*

(i) $S(A_1^{*_X}, A_2^{*_X}) \leq S(A_2^\uparrow, A_1^\uparrow)$,
$S(B_1^{*_Y}, B_2^{*_Y}) \leq S(B_2^\downarrow, B_1^\downarrow)$;
(ii) $A^{*_X} \subseteq A^{\uparrow\downarrow}$,
$B^{*_Y} \subseteq B^{\downarrow\uparrow}$;
(iii) $A_1 \subseteq A_2$ *implies* $A_2^\uparrow \subseteq A_1^\uparrow$,
$B_1 \subseteq B_2$ *implies* $B_2^\downarrow \subseteq B_1^\downarrow$;
(iv) $S(A^{*_X}, B^\downarrow) = S(B^{*_Y}, A^\uparrow)$;
(v) *if* $\bigcup_{j\in J} A_j^{*_X} = (\bigcup_{j\in J} A_j)^{*_X}$ *then* $(\bigcup_{j\in J} A_j)^\uparrow = \bigcap_{j\in J} A_j^\uparrow$,
if $\bigcup_{j\in J} B_j^{*_Y} = (\bigcup_{j\in J} B_j)^{*_Y}$ *then* $(\bigcup_{j\in J} B_j)^\downarrow = \bigcap_{j\in J} B_j^\downarrow$;
(vi) $A^\uparrow = A^{*_X\uparrow}$,
$B^\downarrow = B^{*_Y\downarrow}$;
(vii) $A^{\uparrow *_Y} \subseteq A^{\uparrow\downarrow\uparrow} \subseteq A^\uparrow$,
$B^{\downarrow *_X} \subseteq B^{\downarrow\uparrow\downarrow} \subseteq B^\downarrow$;
(viii) $(\bigcup_{j\in J} A_i^{*_X})^\uparrow = \bigcap_{j\in J} A_j^\uparrow$,
$(\bigcup_{j\in J} B_j^{*_Y})^\downarrow = \bigcap_{j\in J} B_j^\downarrow$;
(ix) $A^{\uparrow\downarrow} = A^{\uparrow\downarrow\uparrow\downarrow}$,
$B^{\downarrow\uparrow} = B^{\downarrow\uparrow\downarrow\uparrow}$.
(x) $A^{\uparrow *_Y} = A^{\uparrow\downarrow\uparrow *_Y}$,
$B^{\downarrow *_X} = B^{\downarrow\uparrow\downarrow *_X}$,

Proof: The proof is based on the results from (Belohlavek 1999), see also (Belohlavek 2002), and on the fact that $A^\uparrow = (A^{*_X})^\Uparrow$ and $B^\downarrow = (B^{*_Y})^\Downarrow$. Furthermore, Since $A^{*_X} \subseteq A$, we have $A^\Uparrow \subseteq A^\uparrow$ and $B^\Downarrow \subseteq B^\downarrow$. The last two inclusions will also be used in our proof.

(i): Since $S(A_1, A_2) \leq S(A_2^\Uparrow, A_1^\Uparrow)$ for any A_1, A_2, see (Belohlavek 1999), we have

$$S(A_1^{*_X}, A_2^{*_X}) \leq S((A_2^{*_X})^\Uparrow, (A_1^{*_X})^\Uparrow) = S(A_2^\uparrow, A_1^\uparrow).$$

The proof for $^\downarrow$ goes the same way.

(ii): Since $A \subseteq A^{\Uparrow\Downarrow}$ for any A, we get

$$A^{*_X} \subseteq (A^{*_X})^{\Uparrow\Downarrow} = A^{\uparrow\Downarrow} \subseteq A^{\uparrow\downarrow}.$$

Again, for $^\downarrow$, the proof is analogous.

(iii): Follows from (i) since if $A_1 \subseteq A_2$ then $A_1^{*_X} \subseteq A_2^{*_X}$ and so $1 = S(A_1^{*_X}, A_2^{*_X}) \leq S(A_2^\uparrow, A_1^\uparrow)$, hence $A_2^\uparrow \subseteq A_1^\uparrow$.

(iv): Since $S(A, B^\Downarrow) = S(B, A^\Uparrow)$ for any A, B, see (Belohlavek 1999), we have, in particular,

$$S(A^{*_X}, B^\downarrow) = S(A^{*_X}, (B^{*_Y})^\Downarrow) = S(B^{*_Y}, (A^{*_X})^\Uparrow) = S(B^{*_Y}, A^\uparrow).$$

(v): Since $(\bigcup_{j\in J} A_j)^\Uparrow = \bigcap_{j\in J} A_j^\Uparrow$, see (Belohlavek 1999), we have

$$\textstyle\bigcap_{j\in J} A_j^\uparrow = \bigcap_{j\in J}(A_j^{*_X})^\Uparrow = (\bigcup_{j\in J} A_j^{*_X})^\Uparrow = (\bigcup_{j\in J} A_j)^{*_X\Uparrow} = (\bigcup_{j\in J} A_j)^\uparrow.$$

The proof for B_j's is similar.

(vi): Since $A^{*_X} = (A^{*_X})^{*_X}$, we have

$$A^\uparrow = (A^{*_X})^\Uparrow = (A^{*_X*_X})^\Uparrow = (A^{*_X})^\uparrow;$$

and the same for $^\downarrow$.

(vii): Follows from (ii), (iii), and (vi).

(viii): By (vi) and Lemma 3.6, we have

$$\textstyle(\bigcup_{j\in J} A_j^{*_X})^\uparrow = (\bigcup_{j\in J} A_j^{*_X})^{*_X\Uparrow} = (\bigcup_{j\in J} A_j^{*_X})^\Uparrow = \bigcap_{j\in J}(A_j^{*_X})^\Uparrow = \bigcap_{j\in J} A_j^\uparrow;$$

the proof for $^\downarrow$ is similar.

(ix): Follows from (ii), (iii), (vi), and (vii): For A we have

$$A^{\uparrow\downarrow} = A^{\uparrow *_Y\downarrow} \supseteq A^{\uparrow\downarrow\uparrow\downarrow} \supseteq A^{*_X\uparrow\downarrow} = A^{\uparrow\downarrow}.$$

(x): A consequence of (vii), isotony of hedges, and the fact that $a^* \leq b$ iff $a^* \leq b^*$ for any hedge *. □ □

Remark 5: If one removes the hedge symbols from the expressions in (i), (ii), (iv), (v), (vii), (viii), and (x), one obtains expressions describing well-known properties of fuzzy concept-forming operators (fuzzy Galois connections). It is easy to see by counterexamples that such these properties are no longer true when general hedges are considered. As an example, consider $(\bigcup_{j\in J} B_j)^\downarrow = \bigcap_{j\in J} B_j^\downarrow$. Put $X = \{x\}$, $Y = \{y\}$ and take any residuate lattice on $L = [0,1]$ with $*_Y$ being globalization. Let $I(x,y) > 0$ and let $J = \{\frac{n}{n+1} \mid n \in \mathbf{N}\}$, $B_j(z) = j$. Then $(\bigcup_{j\in J} B_j)^\downarrow(x) = 1^{*_Y} \to I(x,y) = I(x,y) > 0 = \bigcap_{j\in J} B_j^\downarrow(x)$.

The following two theorems show some relationships between properties of $^\uparrow$ and $^\downarrow$.

Theorem 3.8: *Arbitrary operators* $^\uparrow : \mathbf{L}^X \to \mathbf{L}^Y$ *and* $^\downarrow : \mathbf{L}^Y \to \mathbf{L}^X$ *(i.e., not* *only those induced by* (14) *and* (15)) *satisfying*

$$a^{*_x} \to \{{}^b/y\}^{\downarrow}(x) = b^{*_y} \to \{{}^a/x\}^{\uparrow}(y), \tag{32}$$

$$(\bigcup_{j\in J} A_j^{*_X})^{\uparrow} = \bigcap_{j\in J} A_j^{\uparrow}, \tag{33}$$

$$(\bigcup_{j\in J} B_j^{*_Y})^{\downarrow} = \bigcap_{j\in J} B_j^{\downarrow}, \tag{34}$$

satisfy $S(A^{*_X}, B^{\downarrow}) = S(B^{*_Y}, A^{\uparrow})$ *as well.*

Proof: We have to show $S(A^{*_X}, B^{\downarrow}) \leq S(B^{*_Y}, A^{\uparrow})$, and $S(B^{*_Y}, A^{\uparrow}) \leq S(A^{*_X}, B^{\downarrow})$. Due to symmetry it suffices to check the first inequality only. Using adjointness, it is easy to see that we have to show that for any $y \in Y$ we have $S(A^{*_X}, B^{\downarrow}) \otimes B^{*_y}(y) \leq A^{\uparrow}(y)$.

First, notice that (32) implies $\{{}^1/y\}^{\downarrow}(x) = \{{}^1/x\}^{\uparrow}(y)$, and that (33) and (34) imply $A^{\uparrow} = A^{*_X\uparrow}$, $B^{\downarrow} = B^{*_Y\downarrow}$. That is, we have

$$\begin{aligned} A^{\uparrow}(y) = A^{*_X\uparrow}(y) = (\textstyle\bigcup_{x\in X}\{{}^{A(x)^{*_x}}/x\})^{\uparrow}(y) = \\ = \textstyle\bigcap_{x\in X}\{{}^{A(x)}/x\}^{\uparrow}(y) = \bigwedge_{x\in X}(1 \to \{{}^{A(x)}/x\}^{\uparrow}(y)) = \\ = \textstyle\bigwedge_{x\in X}(A(x)^{*_x} \to \{{}^1/y\}^{\downarrow}(x)). \end{aligned}$$

That is, in order to show $S(A^{*_X}, B^{\downarrow}) \otimes B^{*_Y}(y) \leq A^{\uparrow}(y)$, it suffices to show that for any $x \in X, y \in Y$ it we have

$$A(x)^{*_x} \otimes S(A^{*_X}, B^{\downarrow}) \otimes B(y)^{*_y} \leq \{{}^1/y\}^{\downarrow}(x),$$

which is true. Indeed,

$$\begin{aligned} & A(x)^{*_x} \otimes S(A^{*_X}, B^{\downarrow}) \otimes B(y)^{*_y} \leq \\ & \leq A(x)^{*_x} \otimes (A^{*_x}(x) \to B^{\downarrow}(x)) \otimes B(y)^{*_y} \leq \\ & \leq B^{\downarrow}(x) \otimes B(y)^{*_y} \leq B(y)^{*_y} \otimes \{{}^{B(y)}/y\}^{\downarrow}(x) = \\ & = B(y)^{*_y} \otimes (1 \to \{{}^{B(y)}/y\}^{\downarrow}(x)) = \\ & = B(y)^{*_y} \otimes (B(y)^{*_y} \to \{{}^1/x\}^{\uparrow}(y)) \leq \{{}^1/x\}^{\uparrow}(y) = \\ & = \{{}^1/y\}^{\downarrow}(x). \end{aligned}$$

□ □

Theorem 3.9: *For arbitrary operators* ${}^{\uparrow} : \mathbf{L}^X \to \mathbf{L}^Y$ *and* ${}^{\downarrow} : \mathbf{L}^Y \to \mathbf{L}^X$ (*i.e., not only those induced by* (14) *and* (15)) *which satisfy* $S(A^{*_X}, B^{\downarrow}) = S(B^{*_Y}, A^{\uparrow})$, *we have*

(i) $A^{*_X} \subseteq A^{\uparrow\downarrow}$,
$B^{*_Y} \subseteq B^{\downarrow\uparrow}$;
(ii) $S(A_1^{*_X}, A_2^{*_X}) \leq S(A_2^{\uparrow *_Y}, A_1^{\uparrow})$,
$S(B_1^{*_Y}, B_2^{*_Y}) \leq S(B_2^{\downarrow *_X}, B_1^{\downarrow})$;
(iii) $A_1 \subseteq A_2$ *implies* $A_2^{\uparrow *_Y} \subseteq A_1^{\uparrow}$,
$B_1 \subseteq B_2$ *implies* $B_2^{\downarrow *_X} \subseteq B_1^{\downarrow}$;
(iv) $a^{*_x} \to \{{}^b/y\}^{\downarrow}(x) = b^{*_y} \to \{{}^a/x\}^{\uparrow}(y)$;
(v) $\{{}^1/x\}^{\uparrow}(y) = \{{}^1/y\}^{\downarrow}(x)$;

(vi) $\{{}^{a}/x\}^{\uparrow}(y) = a^{*_x} \to \{{}^{1}/y\}^{\downarrow}(x) = a^{*_x} \to \{{}^{1}/x\}^{\uparrow}(y)$;
(vii) $\{{}^{b}/y\}^{\downarrow}(x) = b^{*_y} \to \{{}^{1}/x\}^{\uparrow}(y) = b^{*_y} \to \{{}^{1}/y\}^{\downarrow}(x)$;
(viii) *if* $a^{*_x} = a^{*_y}$ *then* $\{{}^{a}/x\}^{\uparrow}(y) = \{{}^{a}/y\}^{\downarrow}(x)$;
(ix) $\{{}^{a^{*_x}}/x\}^{\uparrow}(y) = \{{}^{a}/x\}^{\uparrow}(y)$,
$\{{}^{b^{*_y}}/y\}^{\downarrow}(x) = \{{}^{b}/y\}^{\downarrow}(x)$;
(x) *if* $(a \otimes b)^{*_x} = a^{*_x} \otimes b^{*_x}$ *then* $\{{}^{a\otimes b}/x\}^{\uparrow}(y) = a^{*_x} \to \{{}^{b}/x\}^{\uparrow}(y)$;
if $(a \otimes b)^{*_y} = a^{*_y} \otimes b^{*_y}$ *then* $\{{}^{a\otimes b}/y\}^{\downarrow}(x) = a^{*_y} \to \{{}^{b}/y\}^{\downarrow}(x)$.

Proof: (i) is equivalent to $S(A^{*_X}, A^{\uparrow\downarrow}) = 1$ which is true since

$$S(A^{*_X}, A^{\uparrow\downarrow}) = S(A^{\uparrow *_Y}, A^{\uparrow}) = 1,$$

as $A^{\uparrow *_Y} \subseteq A^{\uparrow}$; the proof is similar for $^{\downarrow}$.

(ii): Using (i) we have

$$S(A_1^{*_X}, A_2^{*_X}) \leq S(A_1^{*_X}, A_2^{\uparrow\downarrow}) = S(A_2^{\uparrow *_Y}, A_1^{\uparrow});$$

the proof is similar for $^{\downarrow}$.

(iii): Follows from (ii);

(iv): We have

$$\begin{aligned} a^{*_x} \to \{{}^{b}/y\}^{\downarrow}(x) &= S(\{{}^{a^{*_x}}/x\}, \{{}^{b}/y\}^{\downarrow}) = S(\{{}^{a}/x\}^{*_X}, \{{}^{b}/y\}^{\downarrow}) = \\ &= S(\{{}^{b}/y\}^{*_Y}, \{{}^{a}/x\}^{\uparrow}) = S(\{{}^{b^{*_y}}/y\}, \{{}^{a}/x\}^{\uparrow}) = \\ &= b^{*_y} \to \{{}^{a}/x\}^{\uparrow}(y). \end{aligned}$$

(v): Follows from (iv) by putting $a = 1$ and $b = 1$.

(vi): By (iv) and (v),

$$\{{}^{a}/x\}^{\uparrow}(y) = 1^{*_y} \to \{{}^{a}/x\}^{\uparrow}(y) = a^{*_x} \to \{{}^{1}/y\}^{\downarrow}(x) = a^{*_x} \to \{{}^{1}/x\}^{\uparrow}(y).$$

(vii): Similarly, by (iv) and (v),

$$\{{}^{b}/y\}^{\downarrow}(x) = 1^{*_x} \to \{{}^{b}/y\}^{\downarrow}(x) = b^{*_y} \to \{{}^{1}/x\}^{\uparrow}(y) = b^{*_y} \to \{{}^{1}/y\}^{\downarrow}(x).$$

(viii): By (vi) and (vii),

$$\{{}^{a}/x\}^{\uparrow}(y) = a^{*_x} \to \{{}^{1}/x\}^{\uparrow}(y) = a^{*_y} \to \{{}^{1}/y\}^{\downarrow}(x) = \{{}^{a}/y\}^{\downarrow}(x).$$

(ix): Follows from (vi) and (vii) using (9).

(x): Due to (vi) and the assumption, $\{{}^{a\otimes b}/x\}^{\uparrow}(y) = (a \otimes b)^{*_x} \to \{{}^{1}/x\}^{\uparrow}(y) = (a^{*_x} \otimes b^{*_x}) \to \{{}^{1}/x\}^{\uparrow}(y) = a^{*_x} \to (b^{*_x} \to \{{}^{1}/x\}^{\uparrow}(y)) = a^{*_x} \to \{{}^{b}/x\}^{\uparrow}(y)$. □ □

Since $S(A^{*_X}, B^{\downarrow}) = S(B^{*_Y}, A^{\uparrow})$ holds for the operators $^{\uparrow}$ and $^{\downarrow}$ induced by I (see Theorem 3.7 (iv)), all assertions of Theorem 3.9 are apply. Next, we are going to provide an axiomatic description of operators $^{\uparrow}$ and $^{\downarrow}$ induced by I.

Definition 3.10: A Galois connection with hedges $*_X = \{*_x \,|\, x \in X\}$ and $*_Y = \{*_y \,|\, y \in Y\}$ between sets X and Y is a pair $\langle{}^{\uparrow}, {}^{\downarrow}\rangle$ of mappings $^{\uparrow} : \mathbf{L}^X \to \mathbf{L}^Y$ and

${}^{\downarrow}: \mathbf{L}^Y \to \mathbf{L}^X$ satisfying

$$S(A^{*_X}, B^{\downarrow}) = S(B^{*_Y}, A^{\uparrow}),$$
$$(\bigcup_{j\in J} A_j^{*_X})^{\uparrow} = \bigcap_{j\in J} A_j^{\uparrow},$$
$$(\bigcup_{j\in J} B_j^{*_Y})^{\downarrow} = \bigcap_{j\in J} B_j^{\downarrow},$$

for $A, A_j \in \mathbf{L}^X$, $B, B_j \in \mathbf{L}^Y$.

It follows from Theorem 3.7 that mappings ${}^{\uparrow}$ and ${}^{\downarrow}$ induced by (14) and (15) form a Galois connection with hedges $*_X$ and $*_Y$. In the sequel, we denote the mappings induced by (14) and (15) by ${}^{\uparrow_I}$ and ${}^{\downarrow_I}$.

Lemma 3.11: *Let $\langle{}^{\uparrow},{}^{\downarrow}\rangle$ form a Galois connection with hedges $*_X = \{*_x \,|\, x \in X\}$ and $*_Y = \{*_y \,|\, y \in Y\}$. Then there exists an $\mathbf{L}$-relation $I \in \mathbf{L}^{X\times Y}$ such that $\langle{}^{\uparrow},{}^{\downarrow}\rangle = \langle{}^{\uparrow_I},{}^{\downarrow_I}\rangle$.*

Proof: Introduce I by

$$I(x,y) = \{{}^1/x\}^{\uparrow}(y) = \{{}^1/y\}^{\downarrow}(x)$$

which is correct due to Theorem 3.9. Take any $A \in \mathbf{L}^X$. We have

$$\begin{aligned} A^{\uparrow}(y) &= A^{*_X\uparrow}(y) = (\textstyle\bigcup_{x\in X}\{{}^{A^{*_x}(x)}/x\})^{\uparrow}(y) = \\ &= (\textstyle\bigcup_{x\in X}\{{}^{A(x)}/x\}^{*_X})^{\uparrow}(y) = (\bigcap_{x\in X}\{{}^{A(x)}/x\}^{\uparrow})(y) = \\ &= \textstyle\bigwedge_{x\in X}\{{}^{A(x)}/x\}^{\uparrow}(y) = \bigwedge_{x\in X}(A(x)^{*_x} \to \{{}^1/x\}^{\uparrow}(y)) = \\ &= \textstyle\bigwedge_{x\in X}(A(x)^{*_x} \to I(x,y)) = A^{\uparrow_I}(y), \end{aligned}$$

i.e. ${}^{\uparrow} = {}^{\uparrow_I}$. The proof of ${}^{\downarrow} = {}^{\downarrow_I}$ is similar. □ □

Lemma 3.11 says that each Galois connection with hedges is induced by some fuzzy relation. The following theorem shows that for given collections of hedges $*_X = \{*_x \,|\, x \in X\}$ and ${}^{*_Y} = \{*_y \,|\, y \in Y\}$, the above-described relationship between Galois connections with $*_X$ and $*_Y$, and fuzzy relations I, is one-to-one.

Theorem 3.12: *Let I be an $\mathbf{L}$-relation between X and Y, $\langle{}^{\uparrow},{}^{\downarrow}\rangle$ be a Galois connection with hedges $*_X = \{*_x \,|\, x \in X\}$ and $*_Y = \{*_y \,|\, y \in Y\}$. Then*

(i) *$\langle{}^{\uparrow_I},{}^{\downarrow_I}\rangle$ is a Galois connection with hedges $*_X$ and $*_Y$;*
(ii) *$I_{\langle\uparrow,\downarrow\rangle}$ defined by $I_{\langle\uparrow,\downarrow\rangle}(x,y) = \{{}^1/x\}^{\uparrow}(y) = \{{}^1/y\}^{\downarrow}(x)$ is an $\mathbf{L}$-relation between X and Y and we have*
(iii) *$\langle{}^{\uparrow},{}^{\downarrow}\rangle = \langle{}^{\uparrow_{I_{\langle\uparrow,\downarrow\rangle}}},{}^{\downarrow_{I_{\langle\uparrow,\downarrow\rangle}}}\rangle$ and $I = I_{\langle\uparrow_I,\downarrow_I\rangle}$, i.e. the mappings sending I to $\langle{}^{\uparrow_I},{}^{\downarrow_I}\rangle$ and $\langle{}^{\uparrow},{}^{\downarrow}\rangle$ to $I_{\langle\uparrow,\downarrow\rangle}$ are mutually inverse.*

Proof: By Theorem 3.7, Theorem 3.9, and Lemma 3.11 it suffices to prove $I = I_{\langle\uparrow_I,\downarrow_I\rangle}$. We have

$$I_{\langle\uparrow_I,\downarrow_I\rangle}(x,y) = \{{}^1/x\}^{\uparrow_I}(y) = \textstyle\bigwedge_{z\in X}(\{{}^{1^{*_x}}/x\}(z) \to I(z,y)) = I(x,y),$$

finishing the proof. □

3.4 *The structure of concept lattices with hedges*

A concept lattice (without hedges, i.e. with all $*_x$ and $*_y$ being identities) is a complete lattice with infima and suprema corresponding to conceptual specifications and generalizations. Moreover, theorems describing concept lattices up to an isomorphism are known, see (Ganter and Wille 1999) for crisp case and (Belohlavek 2002, 2004) for fuzzy setting, cf. Theorem 2.1. The question we are going to answer is: What is the structure of concept lattices with hedges, i.e. the structure of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$? The answer is not obvious. For instance, neither of the compound mappings ${}^{\uparrow\downarrow}$ and ${}^{\downarrow\uparrow}$ is a closure operator in general. Indeed, neither $A \subseteq A^{\uparrow\downarrow}$ nor $B \subseteq B^{\downarrow\uparrow}$ is true in general, cf. Theorem 3.7 (ii). In order to answer our question, we proceed as follows: First, we find an ordinary Galois connection $\langle {}^{\curlywedge}, {}^{\curlyvee} \rangle$ between sets such that $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic to the lattice of fixpoints of $\langle {}^{\curlywedge}, {}^{\curlyvee} \rangle$. In addition to that, we describe the isomorphism and its inverse. Second, since $\langle {}^{\curlywedge}, {}^{\curlyvee} \rangle$ is a Galois connection between sets, the lattice of its fixpoints obeys the main theorem of ordinary concept lattices. Applying the isomorphism and its inverse, and simplifying the formulas which we obtain this way, we get a theorem describing the structure of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. That is, we prove a main theorem of concept lattices with hedges by reducing our problem in fuzzy setting to the corresponding problem in the ordinary setting, see (Belohlavek 2001, Pollandt 1997) which serve as an inspiration for our approach.

For a hedge $* : L \to L$, denote by $\mathrm{fix}(*)$ the set of all fixpoints of $*$, i.e.

$$\mathrm{fix}(*) = \{a \in L \mid a^* = a\}.$$

It is easy to see that $\mathrm{fix}(*) = \{a^* \mid a \in L\}$. Recall that $*_X$ and $*_Y$ denote indexed collections of hedges $*_x$ ($x \in X$) and $*_y$ ($y \in Y$), and that for fuzzy sets $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$ we defined fuzzy sets $A^{*_X} \in \mathbf{L}^X$ and $B^{*_Y} \in \mathbf{L}^Y$ by

$$A^{*_X}(x) = (A(x))^{*_x} \quad \text{and} \quad B^{*_Y}(y) = (B(y))^{*_y}.$$

Put

$$\mathrm{fix}(*_X) = \textstyle\bigcup_{x \in X} \mathrm{fix}(*_x) \text{ and } \mathrm{fix}(*_Y) = \bigcup_{y \in Y} \mathrm{fix}(*_y).$$

Furthermore, for ordinary sets $C \subseteq X \times L$ and $D \subseteq Y \times L$, let us define ordinary sets $C^{*_X} \subseteq X \times \mathrm{fix}(*_X)$ and $D^{*_X} \subseteq Y \times \mathrm{fix}(*_Y)$ by

$$C^{*_X} = \{\langle x, a^{*_x} \rangle \mid \langle x, a \rangle \in C\} \quad \text{and} \quad D^{*_Y} = \{\langle y, b^{*_y} \rangle \mid \langle y, b \rangle \in D\}.$$

In the sequel we will take advantage of the following way to represent fuzzy sets by ordinary sets, see (Belohlavek 2002): For an $\mathbf{L}$-set A in universe U, denote by $\lfloor A \rfloor$ an ordinary subset of the Cartesian product $U \times L$ by

$$\lfloor A \rfloor = \{\langle u, a \rangle \in U \times L \mid a \leq A(u)\}. \tag{35}$$

Described verbally, $\lfloor A \rfloor$ can be seen as an area under the membership function $A\colon U \to L$. Conversely, for an ordinary set $A \subseteq U \times L$, denote by $\lceil A \rceil$ an $\mathbf{L}$-set in U defined by

$$\lceil A \rceil(u) = \bigvee\{a \in L \mid \langle u, a \rangle \in A\} \tag{36}$$

for each $u \in U$. We will make use of sets $X^{\times} \subseteq X \times \mathrm{fix}(*_X)$ and $Y^{\times} \subseteq Y \times \mathrm{fix}(*_Y)$ defined by

$$X^{\times} = \{\langle x, a\rangle \in X \times L \mid a^{*_x} = a\} \quad \text{and} \quad Y^{\times} = \{\langle y, b\rangle \in Y \times L \mid b^{*_y} = b\}.$$

That is, $X^{\times}$ is a subset of $X \times L$ which contains pairs $\langle x, a\rangle$ such that a is a fixpoint of $*_x$. Note also that we have $X^{\times} = (X \times L)^{*_X}$.

We need the following lemma.

Lemma 3.13: (i) *For $A \subseteq X^{\times}$, $B \subseteq Y^{\times}$ we have $A \subseteq \lfloor\lceil A\rceil^{*_X}\rfloor^{*_X}$ and $B \subseteq \lfloor\lceil B\rceil^{*_Y}\rfloor^{*_Y}$.*
(ii) *If $A = \lfloor A'\rfloor$ and $B = \lfloor B'\rfloor$ for some $A' \in \mathbf{L}^X$ and $B' \in \mathbf{L}^Y$ then $\lceil A^{*_X}\rceil = \lceil A\rceil^{*_X}$ and $\lceil B^{*_Y}\rceil = \lceil B\rceil^{*_Y}$.*

Proof: (i): Let $\langle x, a\rangle \in A$. Then $a \leq \lceil A\rceil(x)$, and hence $a = a^{*_x} \leq \lceil A\rceil^{*_X}(x)$. As a consequence, $\langle x, a\rangle \in \lfloor\lceil A\rceil^{*_X}\rfloor$ and since $a = a^{*_x}$, we get $\langle x, a\rangle \in \lfloor\lceil A\rceil^{*_X}\rfloor^{*_X}$. The proof for B is similar.

(ii): If $A = \lfloor A'\rfloor$ then clearly $A' = \lceil A\rceil$. Therefore, we need to show $\lceil A^{*_X}\rceil = A'^{*_X}$. We have

$$\lceil A^{*_X}\rceil(x) = \textstyle\bigvee_{\langle x,a\rangle \in A^{*_X}} a = \bigvee_{\langle x,a\rangle \in \lfloor A'\rfloor^{*_X}} a = (A'(x))^{*_x} = (A'^{*_X})(x);$$

the same for B. □ □

Recall that an ordinary Galois connection between sets U and V is a pair of mappings ${}^{\curlywedge} : 2^U \to 2^V$ and ${}^{\curlyvee} : 2^V \to 2^U$ such that

$$A_1 \subseteq A_2 \text{ implies } A_2^{\curlywedge} \subseteq A_1^{\curlywedge}, \tag{37}$$

$$B_1 \subseteq B_2 \text{ implies } B_2^{\curlyvee} \subseteq B_1^{\curlyvee}, \tag{38}$$

$$A \subseteq A^{\curlywedge\curlyvee}, \tag{39}$$

$$B \subseteq B^{\curlyvee\curlywedge}, \tag{40}$$

for each $A, A_1, A_2 \subseteq U$ and $B, B_1, B_2 \subseteq V$.

Now, with ${}^{\uparrow}$ and ${}^{\downarrow}$ defined by (14) and (15), define mappings ${}^{\curlywedge} : X^{\times} \to Y^{\times}$ and ${}^{\curlyvee} : Y^{\times} \to X^{\times}$ by

$$A^{\curlywedge} = \lfloor\lceil A\rceil^{\uparrow}\rfloor^{*_Y} \text{ and } B^{\curlyvee} = \lfloor\lceil B\rceil^{\downarrow}\rfloor^{*_X}, \tag{41}$$

for $A \in X^{\times}$ and $B \in Y^{\times}$.

Lemma 3.14: *The pair $\langle {}^{\curlywedge}, {}^{\curlyvee}\rangle$ defined by (41) forms a Galois connection between sets $X^{\times}$ and $Y^{\times}$.*

Proof: (37): $A_1 \subseteq A_2$ implies $\lceil A_1\rceil \subseteq \lceil A_2\rceil$ which implies $\lceil A_2\rceil^{\uparrow} \subseteq \lceil A_1\rceil^{\uparrow}$ which implies $\lfloor\lceil A_2\rceil^{\uparrow}\rfloor \subseteq \lfloor\lceil A_1\rceil^{\uparrow}\rfloor$ which implies $A_2^{\curlywedge} = \lfloor\lceil A_2\rceil^{\uparrow}\rfloor^{*_Y} \subseteq \lfloor\lceil A_1\rceil^{\uparrow}\rfloor^{*_Y} = A_1^{\curlywedge}$. Dually, we get (38).

(39): Using $B^{*_Y\downarrow} = B^{\downarrow}$, $\lceil A\rceil^{\uparrow\downarrow} \supseteq \lceil A\rceil^{*_X}$, and Lemma 3.13,

$$A^{\curlywedge\curlyvee} = \lfloor\lceil\lfloor\lceil A\rceil^{\uparrow}\rfloor^{*_Y}\rceil^{\downarrow}\rfloor^{*_X} = \lfloor\lceil\lfloor\lceil A\rceil^{\uparrow}\rfloor\rceil^{*_Y\downarrow}\rfloor^{*_X} =$$
$$= \lfloor\lceil A\rceil^{\uparrow *_Y\downarrow}\rfloor^{*_X} = \lfloor\lceil A\rceil^{\uparrow\downarrow}\rfloor^{*_X} \supseteq \lfloor\lceil A\rceil^{*_X}\rfloor^{*_X} \supseteq A.$$

Dually, we get (40). □ □

It is well known, see e.g. (Ganter and Wille 1999), that each Galois connection $\langle {}^{\curlywedge}, {}^{\curlyvee} \rangle$ between sets U and V is induced by some binary relation $I_{\langle \curlywedge, \curlyvee \rangle} \subseteq U \times V$. Namely, $I_{\langle \curlywedge, \curlyvee \rangle}$ is given by

$$\langle u, v \rangle \in I_{\langle \curlywedge, \curlyvee \rangle} \quad \text{iff} \quad v \in \{u\}^{\curlywedge}. \tag{42}$$

Then we have

$$\begin{aligned} A^{\curlywedge} &= \{v \in V \mid \text{ for each } u \in A : \langle u, v \rangle \in I_{\langle \curlywedge, \curlyvee \rangle}\}, \\ B^{\curlyvee} &= \{u \in U \mid \text{ for each } v \in B : \langle u, v \rangle \in I_{\langle \curlywedge, \curlyvee \rangle}\}, \end{aligned}$$

for any $A \subseteq U$ and $B \subseteq V$. Furthermore, in such a case, the set

$$\mathcal{B}(U, V, \langle {}^{\curlywedge}, {}^{\curlyvee} \rangle) = \{\langle A, B \rangle \in 2^U \times 2^V \mid A^{\curlywedge} = B, B^{\curlyvee} = A\}$$

of all fixpoints of $\langle {}^{\curlywedge}, {}^{\curlyvee} \rangle$ is, in fact, the ordinary concept lattice $\mathcal{B}(U, V, I_{\langle \curlywedge, \curlyvee \rangle})$, and obeys the so-called main theorem of concept lattices:

Theorem 3.15 (Ganter and Wille 1999) *(1) $\mathcal{B}(U, V, I_{\langle \curlywedge, \curlyvee \rangle})$ is under $\leq$, defined by $\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle$ iff $A_1 \subseteq A_2$, a complete lattice where the infima and suprema are given by*

$$\begin{aligned} \textstyle\bigwedge_{j \in J} \langle A_j, B_j \rangle &= \langle \textstyle\bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)^{\curlyvee \curlywedge} \rangle, \\ \textstyle\bigvee_{j \in J} \langle A_j, B_j \rangle &= \langle (\textstyle\bigcup_{j \in J} A_j)^{\curlywedge \curlyvee}, \bigcap_{j \in J} B_j \rangle. \end{aligned}$$

(2) Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K, \leq \rangle$ is isomorphic to $\mathcal{B}(U, V, I_{\langle \curlywedge, \curlyvee \rangle})$ iff there are mappings $\gamma : U \to K$, $\mu : V \to K$ such that

(i) *$\gamma(U)$ is $\bigvee$-dense in K, $\mu(V)$ is $\bigwedge$-dense in V;*
(ii) *$\gamma(u) \leq \mu(v)$ iff $\langle u, v \rangle \in I_{\langle \curlywedge, \curlyvee \rangle}$.*

Note that $M \subseteq K$ is called $\bigvee$-dense in K if for each $k \in K$ there is $M' \subseteq M$ such that $k = \bigvee M'$, i.e. each $k \in K$ is a supremum of some subset of M; dually, $M \subseteq K$ is $\bigwedge$-dense in K if for each $k \in K$ there is $M' \subseteq M$ such that $k = \bigwedge M'$.

Lemma 3.16: *The (ordinary) relation $I^{\times} = I_{\langle \curlywedge, \curlyvee \rangle}$ between $X^{\times}$ and $Y^{\times}$ corresponding to a Galois connection $\langle {}^{\curlywedge}, {}^{\curlyvee} \rangle$ defined by (41) is given by*

$$\langle \langle x, a \rangle, \langle y, b \rangle \rangle \in I^{\times} \textit{ iff } a \otimes b \leq I(x, y). \tag{43}$$

Proof: By (42), we have

$$\langle \langle x, a \rangle, \langle y, b \rangle \rangle \in I^{\times} \text{ iff } \langle y, b \rangle \in \{\langle x, a \rangle\}^{\curlywedge}.$$

By definition of ${}^{\curlywedge}$,

$$\langle y, b \rangle \in \{\langle x, a \rangle\}^{\curlywedge} \text{ iff } \langle y, b \rangle \in \lfloor \lceil \{\langle x, a \rangle\} \rceil^{\uparrow} \rfloor^{*_Y}.$$

Since $\lfloor\lceil\{\langle x,a\rangle\}\rceil^{\uparrow}\rfloor^{*_Y} = \lfloor\{^a/x\}^{\uparrow}\rfloor^{*_Y}$ and since the largest c such that $\langle y,c\rangle \in \lfloor\{^a/x\}^{\uparrow}\rfloor^{*_Y}$ is $c = (\{^a/x\}^{\uparrow}(y))^{*_y}$, one can see that

$$\langle y,b\rangle \in \lfloor\lceil\{\langle x,a\rangle\}\rceil^{\uparrow}\rfloor^{*_Y} \text{ iff } b \leq (\{^a/x\}^{\uparrow}(y))^{*_y}.$$

Since $b = b^{*_y}$, the latter inequality is equivalent to $b \leq \{^a/x\}^{\uparrow}(y)$. Now, using (14) and $a = a^{*_x}$, we get

$$\{^a/x\}^{\uparrow}(y) = a^{*_x} \rightarrow I(x,y) = a \rightarrow I(x,y),$$

whence

$$b \leq \{^a/x\}^{\uparrow}(y) \text{ iff } a \otimes b \leq I(x,y)$$

by adjointness. To sum up, the proof is finished. □ □

In the rest, $I^{\times}$ always denotes the relation defined by (43).

Theorem 3.17: *Each concept lattice with hedges $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ is isomorphic to the ordinary concept lattice $\mathcal{B}(X^{\times},Y^{\times},I^{\times})$. The isomorphism $h : \mathcal{B}(X^{*_X},Y^{*_Y},I) \rightarrow \mathcal{B}(X^{\times},Y^{\times},I^{\times})$ and its inverse $g : \mathcal{B}(X^{\times},Y^{\times},I^{\times}) \rightarrow \mathcal{B}(X^{*_X},Y^{*_Y},I)$ are given by*

$$h(\langle A,B\rangle) = \langle \lfloor A\rfloor^{*_X}, \lfloor B\rfloor^{*_Y}\rangle, \tag{44}$$

$$g(\langle A',B'\rangle) = \langle \lceil A'\rceil^{\uparrow\downarrow}, \lceil B'\rceil^{\downarrow\uparrow}\rangle. \tag{45}$$

Proof: The proof goes by showing that (a) h and g are defined correctly, (b) h and g are order-preserving, and (c) $g(h(\langle A,B\rangle)) = \langle A,B\rangle$ and $h(g(\langle A',B'\rangle)) = \langle A',B'\rangle$.

(a): We need to show that for $\langle A,B\rangle \in \mathcal{B}(X^{*_X},Y^{*_Y},I)$ and $\langle A',B'\rangle \in \mathcal{B}(X^{\times},Y^{\times},I^{\times})$ we have $h(\langle A,B\rangle) \in \mathcal{B}(X^{\times},Y^{\times},I^{\times})$ and $g(\langle A',B'\rangle) \in \mathcal{B}(X^{*_X},Y^{*_Y},I)$.

First, let $\langle A,B\rangle \in \mathcal{B}(X^{*_X},Y^{*_Y},I)$. In order to see $h(\langle A,B\rangle) \in \mathcal{B}(X^{\times},Y^{\times},I^{\times})$ we need to show $(\lfloor A\rfloor^{*_X})^{\curlywedge} = \lfloor A^{\uparrow}\rfloor^{*_Y}$ and $(\lfloor B\rfloor^{*_Y})^{\curlyvee} = \lfloor B^{\downarrow}\rfloor^{*_X}$. Using $\lceil\lfloor\cdots\rfloor\rceil = \cdots$, Lemma 3.13, and $\cdots^{*_X\uparrow} = \cdots^{\uparrow}$ we get

$$(\lfloor A\rfloor^{*_X})^{\curlywedge} = \lfloor\lceil\lfloor A\rfloor^{*_X}\rceil^{\uparrow}\rfloor^{*_Y} = \lfloor\lceil\lfloor A\rfloor\rceil^{*_X\uparrow}\rfloor^{*_Y} = \lfloor A^{*_X\uparrow}\rfloor^{*_Y} = \lfloor A^{\uparrow}\rfloor^{*_Y}.$$

Dually, $(\lfloor B\rfloor^{*_Y})^{\curlyvee} = \lfloor B^{\downarrow}\rfloor^{*_X}$.

Second, let $\langle A',B'\rangle \in \mathcal{B}(X^{\times},Y^{\times},I^{\times})$. To see $g(\langle A',B'\rangle) \in \mathcal{B}(X^{*_X},Y^{*_Y},I)$, we need to check $\lceil A'\rceil^{\uparrow\downarrow\uparrow} = \lceil B'\rceil^{\downarrow\uparrow}$ and $\lceil B'\rceil^{\downarrow\uparrow\downarrow} = \lceil A'\rceil^{\uparrow\downarrow}$. Using $\lceil\lfloor\cdots\rfloor\rceil = \cdots$, Lemma 3.13, and $\cdots^{*_X\uparrow} = \cdots^{\uparrow}$ we get

$$\lceil A'\rceil^{\uparrow\downarrow\uparrow} = \lceil A'\rceil^{\uparrow *_Y\downarrow\uparrow} = \lceil\lfloor\lceil A'\rceil^{\uparrow}\rfloor\rceil^{*_Y\downarrow\uparrow} = \lceil\lfloor\lceil A'\rceil^{\uparrow}\rfloor^{*_Y}\rceil^{\downarrow\uparrow} = \lceil A'^{\curlywedge}\rceil^{\downarrow\uparrow} = \lceil B'\rceil^{\downarrow\uparrow},$$

and dually for $\lceil B'\rceil^{\downarrow\uparrow\downarrow} = \lceil A'\rceil^{\uparrow\downarrow}$. This proves (a).

(b) For $A_1, A_2 \in \mathbf{L}^X$, $A_1 \subseteq A_2$ implies $\lfloor A_1\rfloor \subseteq \lfloor A_2\rfloor$ from which we get $\lfloor A_1\rfloor^{*_X} \subseteq \lfloor A_2\rfloor^{*_X}$, whence h is order-preserving. For $A'_1, A'_2 \subseteq X^{\times}$, $A'_1 \subseteq A'_2$ implies $\lceil A'_1\rceil \subseteq \lceil A'_2\rceil$ from which we get $\lceil A'_1\rceil^{\uparrow\downarrow} \subseteq \lceil A'_2\rceil^{\uparrow\downarrow}$, whence g is order-preserving.

(c): We have

$$g(h(\langle A,B\rangle)) = \langle \lceil \lfloor A\rfloor^{*_X}\rceil^{\uparrow\downarrow}, \lceil \lfloor B\rfloor^{*_Y}\rceil^{\downarrow\uparrow}\rangle = \langle \lceil \lfloor A\rfloor\rceil^{*_X\uparrow\downarrow}, \lceil \lfloor B\rfloor\rceil^{*_Y\downarrow\uparrow}\rangle =$$
$$= \langle A^{\uparrow\downarrow}, B^{\downarrow\uparrow}\rangle = \langle A,B\rangle.$$

Furthermore, since

$$h(g(\langle A',B'\rangle)) = \langle \lfloor \lceil A'\rceil^{\uparrow\downarrow}\rfloor^{*_X}, \lfloor \lceil B'\rceil^{\downarrow\uparrow}\rfloor^{*_Y}\rangle,$$

in order to prove $h(g(\langle A',B'\rangle)) = \langle A',B'\rangle$, we need to check $\lfloor \lceil A'\rceil^{\uparrow\downarrow}\rfloor^{*_X} = A'$ and $\lfloor \lceil B'\rceil^{\downarrow\uparrow}\rfloor^{*_Y} = B'$. We have

$$\lfloor \lceil A'\rceil^{\uparrow\downarrow}\rfloor^{*_X} = \lfloor \lceil \lfloor \lceil A'\rceil^{\uparrow}\rfloor\rceil^{*_Y\downarrow}\rfloor^{*_X} = \lfloor \lceil \lfloor \lceil A'\rceil^{\uparrow}\rfloor^{*_Y}\rceil^{\downarrow}\rfloor^{*_X} =$$
$$= \lfloor \lceil A'^{\curlywedge}\rceil^{\downarrow}\rfloor^{*_X} = A'^{\curlywedge\curlyvee} = A',$$

and dually for $\lfloor \lceil B'\rceil^{\downarrow\uparrow}\rfloor^{*_Y} = B'$. The proof is complete. $\square$

Remark 6: Theorem 3.17 is important also from the computational point of view. Namely, Theorem 3.17 says that in order to compute $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, one can use algorithms for computing ordinary concept lattices to obtain $\mathcal{B}(X^{\times}, Y^{\times}, I^{\times})$ and then mappings g and h to restore $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. For algorithms for computation of ordinary concept lattices, we refer the reader to (Carpineto and Romano 2004, Ganter and Wille 1999).

We need the following lemma.

Lemma 3.18: *If A_j are extents, i.e. $\langle A_j, B_j\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, for $j \in J$, then*

$$(\textstyle\bigcup_{j\in J} A_j^{*_X})^{\uparrow\downarrow} = (\bigcup_{j\in J} A_j)^{\uparrow\downarrow}.$$

Likewise, if B_j are intents, then

$$(\textstyle\bigcup_{j\in J} B_j^{*_Y})^{\downarrow\uparrow} = (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}.$$

Proof: Due to symmetry, we prove the first part only. We use the fact that A is an extent if and only if $A = A^{\uparrow\downarrow}$. "$\subseteq$": This part is obvious due to $\bigcup_{j\in J} A_j^{*_X} \subseteq \bigcup_{j\in J} A_j$ and the isotony of ${}^{\uparrow\downarrow}$ (the isotony follows by applying Theorem 3.7 (iii) twice). "$\supseteq$": As $A_j = A_j^{\uparrow\downarrow} = A_j^{*_X\uparrow\downarrow} \subseteq (\bigcup_{j\in J} A_j^{*_X})^{\uparrow\downarrow}$, one has $\bigcup_{j\in J} A_j \subseteq (\bigcup_{j\in J} A_j^{*_X})^{\uparrow\downarrow}$, hence $(\bigcup_{j\in J} A_j)^{\uparrow\downarrow} \subseteq (\bigcup_{j\in J} A_j^{*_X})^{\uparrow\downarrow\uparrow\downarrow} = (\bigcup_{j\in J} A_j^{*_X})^{\uparrow\downarrow}$ due to the isotony of ${}^{\uparrow\downarrow}$ Theorem 3.7 (ix). $\square$

Remark 7: Lemma 3.18 does not hold if one drops the assumptions that A_j and B_j are extents and intents, respectively. Consider the counterexample from Remark **5** with $I(x,y) < 1$. Then for any hedge $*_Y$, $(\bigcup_{j\in J} B_j^{*_Y})^{\downarrow\uparrow}(y) = I(x,y)$ while $(\bigcup_{j\in J} B_j)^{\downarrow\uparrow}(y) = 1$.

The following is our main theorem describing the structure of concept lattices with hedges.

Theorem 3.19 (main theorem for concept lattices with hedges)**:**
(1) $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ *equipped with* $\leq$ *is a complete lattice where the infima and*

suprema are given by

$$\bigwedge_{j\in J}\langle A_j, B_j\rangle = \langle(\bigcap_{j\in J} A_j)^{\uparrow\downarrow}, (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}\rangle, \tag{46}$$

$$\bigvee_{j\in J}\langle A_j, B_j\rangle = \langle(\bigcup_{j\in J} A_j)^{\uparrow\downarrow}, (\bigcap_{j\in J} B_j)^{\downarrow\uparrow}\rangle. \tag{47}$$

(2) *Moreover, an arbitrary complete lattice* $\mathbf{K} = \langle K, \leq\rangle$ *is isomorphic to* $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ *iff there are mappings* $\gamma : X^\times \to K$, $\mu : Y^\times \to K$ *such that*

(i) $\gamma(X^\times)$ *is* $\bigvee$*-dense in* K, $\mu(Y^\times)$ *is* $\bigwedge$*-dense in* V*;*
(ii) $\gamma(x, a) \leq \mu(y, b)$ *iff* $a \otimes b \leq I(x, y)$*.*

Proof: The idea of our proof is the following. By Theorem 3.17, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic to $\mathcal{B}(X^\times, Y^\times, I^\times)$. Then, we apply Theorem 3.15 to $\mathcal{B}(X^\times, Y^\times, I^\times)$. After that, we "translate" the formulas and conditions which we obtain this way using h and g from Theorem 3.17 and simplify them by using our previous results from this paper. This gives us the formulas and conditions we need to prove. The details follow.

(1): Theorem 3.17 and Theorem 3.15 yield that $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is a complete lattice with infima and suprema corresponding to infima and suprema in $\mathcal{B}(X^\times, Y^\times, I^\times)$. Due to duality, we verify (46) only. For $\langle A_j, B_j\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, the corresponding formal concept in $\mathcal{B}(X^\times, Y^\times, I^\times)$ is $h(\langle A_j, B_j\rangle) = \langle\lfloor A_j\rfloor^{*_X}, \lfloor B_j\rfloor^{*_Y}\rangle$. Therefore, using Theorem 3.15, the formal concept corresponding to $\bigwedge_{j\in J}\langle A_j, B_j\rangle$ is

$$h(\bigwedge_{j\in J}\langle A_j, B_j\rangle) = \bigwedge_{j\in J}\langle\lfloor A_j\rfloor^{*_X}, \lfloor B_j\rfloor^{*_Y}\rangle = \langle\bigcap_{j\in J}\lfloor A_j\rfloor^{*_X}, (\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y})^{\curlyvee\curlywedge}\rangle.$$

To check (46), it suffices to show

$$g(\langle\bigcap_{j\in J}\lfloor A_j\rfloor^{*_X}, (\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y})^{\curlyvee\curlywedge}\rangle) = \langle(\bigcap_{j\in J} A_j)^{\uparrow\downarrow}, (\bigcup_{j\in J} B_j^{*_Y})^{\downarrow\uparrow}.\rangle \tag{48}$$

Namely, (46) then follows due to Lemma 3.18. To check (48), denote

$$g(\langle\bigcap_{j\in J}\lfloor A_j\rfloor^{*_X}, (\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y})^{\curlyvee\curlywedge}\rangle) = \langle g(\bigcap_{j\in J}\lfloor A_j\rfloor^{*_X}), g((\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y})^{\curlyvee\curlywedge})\rangle.$$

For the first coordinate of (48): First, observe that $\bigcap_{j\in J}\lfloor A_j\rfloor^{*_X} = \lfloor\bigcap_{j\in J} A_j\rfloor^{*_X}$. Indeed, $\langle x, a\rangle \in \bigcap_{j\in J}\lfloor A_j\rfloor^{*_X}$ iff for each $j \in J$, $\langle x, a\rangle \in \lfloor A_j\rfloor^{*_X}$ iff for each $j \in J$, $a \leq A_j^{*_X}(x)$. Since $a = a^{*_X}$, the latter is equivalent to $a \leq A_j(x)$ for each $j \in J$ which is equivalent to $a \leq (\bigcap_{j\in J} A_j)(x)$ which is equivalent to $\langle x, a\rangle \in \lfloor\bigcap_{j\in J} A_j\rfloor$ which is equivalent to $\langle x, a\rangle \in \lfloor\bigcap_{j\in J} A_j\rfloor^{*_X}$ by virtue of $a = a^{*_X}$. Second, using this observation and using Lemma 3.13, $\cdots^{*_X\uparrow} = \cdots^{\uparrow}$, and $\lceil\lfloor\cdots\rfloor\rceil = \cdots$, we have

$$\begin{aligned} g(\bigcap_{j\in J}\lfloor A_j\rfloor^{*_X}) &= \lceil\bigcap_{j\in J}\lfloor A_j\rfloor^{*_X}\rceil^{\uparrow\downarrow} = \lceil\lfloor\bigcap_{j\in J} A_j\rfloor^{*_X}\rceil^{\uparrow\downarrow} = \lceil\lfloor\bigcap_{j\in J} A_j\rfloor\rceil^{*_X\uparrow\downarrow} = \\ &= \lceil\lfloor\bigcap_{j\in J} A_j\rfloor\rceil^{\uparrow\downarrow} = (\bigcap_{j\in J} A_j)^{\uparrow\downarrow}. \end{aligned}$$

For the second coordinate of (48): First, observe that $\lceil\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y}\rceil = \bigcup_{j\in J} B_j^{*_Y}$. Indeed,

$$\begin{aligned} &\lceil\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y}\rceil(y) = \bigvee\{a \mid \langle y, a\rangle \in \bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y}\} = \\ &= \bigvee\{a \mid \text{ exists } j \in J : \langle y, a\rangle \in \lfloor B_j\rfloor^{*_Y}\} = \\ &= \bigvee\{a \mid \text{ exists } j \in J : a \leq B_j^{*_Y}(y)\} = \bigvee_{j\in J} B_j^{*_Y}(y) = (\bigcup_{j\in J} B_j^{*_Y})(y). \end{aligned}$$

By virtue of this observation and using Lemma 3.13, we get

$$
\begin{aligned}
&g((\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y})^{\curlyvee\curlywedge}) = \lceil(\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y})^{\curlyvee\curlywedge}\rceil^{\downarrow\uparrow} = \\
&= \lceil\lfloor\lceil\lfloor\lceil(\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y})\rceil^{\downarrow}\rfloor^{*_X}\rceil^{\uparrow}\rfloor^{*_Y}\rceil^{\downarrow\uparrow} = \\
&= \lceil\lfloor\lceil\lfloor\lceil(\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y})\rceil^{\downarrow}\rfloor\rceil^{*_X\uparrow}\rfloor\rceil^{*_Y\downarrow\uparrow} = \lceil\lfloor\lceil\lfloor\lceil(\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y})\rceil^{\downarrow}\rfloor\rceil^{\uparrow}\rfloor\rceil^{\downarrow\uparrow} = \\
&= \lceil(\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y})\rceil^{\downarrow\uparrow\downarrow\uparrow} = \lceil(\bigcup_{j\in J}\lfloor B_j\rfloor^{*_Y})\rceil^{\downarrow\uparrow} = (\bigcup_{j\in J} B_j^{*_Y})^{\downarrow\uparrow}.
\end{aligned}
$$

This proves (48). This finishes the proof of (1).

(2) results by a direct application of Theorem 3.17, Theorem 3.15, and Lemma 3.16. □ □

Remark 8: Note that the intersection of extents from $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ need not be an extent. Hence, one cannot replace $(\bigcap_{j\in J} A_j)^{\uparrow\downarrow}$ by $\bigcap_{j\in J} A_j$ in (46), i.e. one does not thave the same formulas as in the case without hedges. Namely, consider the following example. Take the Łukasiewicz structure on $[0,1]$, let each $*_x$ be the globalization, let each $*_y$ be the identity, and consider the formal context described by the following table:

I	y_1	y_2
x_1	0.5	0
x_2	0	0.5

Then $A_1 = \{{}^1/x_1\}^{\uparrow\downarrow} = \{{}^1/x_1, {}^{0.5}/x_2\}$ and $A_2 = \{{}^1/x_2\}^{\uparrow\downarrow} = \{{}^{0.5}/x_1, {}^1/x_2\}$ are both extents but $A_1 \cap A_2 = \{{}^{0.5}/x_1, {}^{0.5}/x_2\}$ is not an extent since $(A_1 \cap A_2)^{\uparrow\downarrow} = \{{}^0/x_1, {}^0/x_2\} \neq A_1 \cap A_2$. Clearly, the same applies to intents.

We are now going to investigate a particular case of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ in which either all $*_x$'s are identities or all $*_y$'s are identities. Because of symmetry, we proceed just for the case when each $*_x$ is an identity and denote the corresponding concept lattice with hedges by $\mathcal{B}(X, Y^{*_Y}, I)$. This is an important special case. If each $*_y$ is globalization (first boundary possibility), $\mathcal{B}(X, Y^{*_Y}, I)$ is just the lattice of crisply generated concepts (Belohlavek *et al.* 2005). If each $*_y$ is identity (second boundary possibility), $\mathcal{B}(X, Y^{*_Y}, I)$ is the original fuzzy concept lattice (Belohlavek 2004, Pollandt 1997). In general, $*_Y$ (possibly between globalization and identity) controls the meaning of "having all attributes from (the intent) B". Loosely speaking, paying attention to $\langle A, B\rangle \in \mathcal{B}(X, Y^{*_Y}, I)$ means that we do not put any restriction on extents A (any closed fuzzy set of objects of objects is good), but use $*_y$'s to impose restrictions on intents B.

Our first theorem shows that formal concepts from $\mathcal{B}(X, Y^{*_Y}, I)$ are particular concepts from the fuzzy concept lattice $\mathcal{B}(X, Y, I)$.

Theorem 3.20: *$\mathcal{B}(X, Y^{*_Y}, I) \subseteq \mathcal{B}(X, Y, I)$. Moreover,*

$$
\begin{aligned}
\mathcal{B}(X, Y^{*_Y}, I) = \{\langle A, B\rangle \in \mathcal{B}(X, Y, I) \;\mid\; & A = D^{\downarrow} \text{ for some } D \in \mathbf{L}^Y \text{ with} \\
& D(y) \in \mathrm{fix}(*_y) \text{ for } y \in Y\}.
\end{aligned}
$$

Proof: "$\subseteq$": If $\langle A, B\rangle \in \mathcal{B}(X, Y^{*_Y}, I)$ then clearly, for $D = B^{*_Y}$ we have $D(y) \in \mathrm{fix}(*_y)$ for all $y \in Y$ and $A = D^{\downarrow}$. Furthermore, $A^{\Uparrow} = A^{\uparrow} = B$ and $A = B^{\downarrow} = B^{*_Y\Downarrow} \supseteq B^{\Downarrow} = A^{\Uparrow\Downarrow} \supseteq A$, whence $B^{\Downarrow} = A$. That is, $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$.

"$\supseteq$": Let $\langle A,B\rangle \in \mathcal{B}(X,Y,I)$ such that $A = D^{\downarrow}$ for some $D \in \mathbf{L}^Y$ such that $D(y) \in \mathrm{fix}(*_y)$ for all $y \in Y$. We need to show $\langle A,B\rangle \in \mathcal{B}(X,Y^{*_Y},I)$ for which it is sufficient to see $A = B^{\downarrow}$. From $D(y) \in \mathrm{fix}(*_y)$ we get $D = D^{*_Y}$. Since, moreover, ^{*_X} is identity, Theorem 3.7 (vii) yields, $D^{*_Y\Downarrow} = D^{\downarrow} = D^{\downarrow\uparrow\downarrow} = D^{\Downarrow\Uparrow*_Y\Downarrow}$. Hence,

$$A = D^{\downarrow} = D^{*_Y\Downarrow} = D^{\Downarrow\Uparrow*_Y\Downarrow} = B^{*_Y\Downarrow} = B^{\downarrow},$$

finishing the proof. □

Example 3.21 In the general case, $\mathcal{B}(X^{*_X},Y^{*_Y},I)$ need not be a subset of $\mathcal{B}(X,Y,I)$. Namely, take the Łukasiewicz structure on $[0,1]$, let all $*_x$ and $*_y$ be globalizations, and consider the following data table

I	y_1	y_2
x_1	1	0.5
x_2	0.7	0.1

One can check that for $A = \{{}^{1}/x_1, {}^{0.7}/x_2\}$ and $B = \{{}^{1}/y_1, {}^{0.5}/y_2\}$, $\langle A,B\rangle \in \mathcal{B}(X^{*_X},Y^{*_Y},I)$ but $\langle A,B\rangle \notin \mathcal{B}(X,Y,I)$.

The next theorem shows that the smaller the sets of fixpoints of $*_y$'s, the larger the reduction.

Theorem 3.22: *Let* $*_{Y1} = \{*_{y_1} \,|\, y \in Y\}$ *and* $*_{Y2} = \{*_{y_2} \,|\, y \in Y\}$ *satisfy* $\mathrm{fix}(*_{y_1}) \subseteq \mathrm{fix}(*_{y_2})$ *for each* $y \in Y$*. Then* $\mathcal{B}(X,Y^{*_1},I) \subseteq \mathcal{B}(X,Y^{*_2},I)$*.*

Proof: Follows immediately from Theorem 3.20. □

Theorem 3.23: *If each* $*_x$ *is identity, formula* (46) *simplifies to any of the following forms:*

$$\textstyle\bigwedge_{j\in J}\langle A_j,B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}\rangle, \tag{49}$$

$$\textstyle\bigwedge_{j\in J}\langle A_j,B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\Downarrow\Uparrow}\rangle. \tag{50}$$

Proof: First, we show that in this case, $(\bigcap_j A_j)^{\uparrow\downarrow} = \bigcap_j A_j$: On the one hand, $\bigcap_j A_j \subseteq (\bigcap_j A_j)^{\Uparrow\Downarrow} = (\bigcap_j A_j)^{\uparrow\Downarrow} \subseteq (\bigcap_j A_j)^{\uparrow\downarrow}$. On the other hand, $(\bigcap_j A_j)^{\uparrow\downarrow} \subseteq \bigcap_j A_j$ iff for each $j \in J$ we have $(\bigcap_j A_j)^{\uparrow\downarrow} \subseteq A_j$ which is true. Indeed, $(\bigcap_j A_j) \subseteq A_j$ implies $(\bigcap_j A_j)^{\uparrow\downarrow} \subseteq A_j^{\uparrow\downarrow} = A_j$ since A_j is an extent. We verified (49).

Now, we have $\langle \bigcap_j A_j, (\bigcup_j B_j)^{\downarrow\uparrow}\rangle \in \mathcal{B}(X,Y^{*_Y},I)$, hence also $\langle \bigcap_j A_j, (\bigcup_j B_j)^{\downarrow\uparrow}\rangle \in \mathcal{B}(X,Y,I)$ due to Theorem 3.20. Since the intent corresponding to $\bigcap_j A_j$ in $\mathcal{B}(X,Y,I)$ is $(\bigcup_j B_j)^{\Downarrow\Uparrow}$ (see e.g. (Belohlavek 2002, 2004)), we verified (50). □

As a corollary, we get the following assertion.

Theorem 3.24: *If each* $*_x$ *is identity,* $\mathcal{B}(X,Y^{*_Y},I)$ *is a* $\bigwedge$*-sublattice of* $\mathcal{B}(X,Y,I)$*.*

Proof: Follows from Theorem 3.23 and the fact that the infimum of $\langle A_j,B_j\rangle$'s in $\mathcal{B}(X,Y,I)$ is given by (50), see (Belohlavek 2002). □

The following example shows that $\mathcal{B}(X, Y^{*_Y}, I)$ need not be a $\bigvee$-sublattice of $\mathcal{B}(X, Y, I)$.

Example 3.25 Take a Łukasiewicz structure on $[0, 1]$ (but this works for Gödel and product as well), let $*_Y$ be globalization, and consider the following data table

I	y_1	y_2	y_3
x_1	0.3	0.5	0.4
x_2	0.2	0.6	0.1

Then both $B_1 = \{{}^1/y_1, {}^1/y_2\}^{\downarrow\uparrow} = \{{}^1/y_1, {}^1/y_2, {}^{0.9}/y_3\}$ and $B_2 = \{{}^1/y_2, {}^1/y_3\}^{\downarrow\uparrow} = \{{}^{0.9}/y_1, {}^1/y_2, {}^1/y_3\}$ are intents in $\mathcal{B}(X, Y^{*_Y}, I)$. However, since $B_1 \cap B_2 \neq (B_1 \cap B_2)^{\downarrow\uparrow}$, suprema in $\mathcal{B}(X, Y^{*_Y}, I)$ and $\mathcal{B}(X, Y, I)$ are different.

The following theorem shows that if all $*_x$'s and $*_y$'s are globalizations (boundary case, largest restriction), $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is in fact isomorphic to an ordinary concept lattice given by 1-cut ${}^1I = \{\langle x, y\rangle \mid I(x, y) = 1\}$ of I. Note that the data table $\langle X, Y, {}^1I\rangle$ results from $\langle X, Y, I\rangle$ by keeping entries with 1's and deleting (replacing by 0) all other entries.

Theorem 3.26 : *If all $*_x$'s and $*_y$'s are globalizations, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic to (ordinary) concept lattice $\mathcal{B}(X, Y, {}^1I)$.*

Proof : Obvious from definitions. □

4. Experiments and examples

In this section we present examples which illustrate reduction of the size of fuzzy concept lattices by means of constraints imposed by hedges.

For the sake of illustration, we took a data table[1] describing imaginary holidays and their attributes. The original table contains entries denoted "--", "-", "blank", "+", and "++". The degrees represented ratings for each attribute, "--" and "++" representing "the worst" and "the best", respectively. Symbolic ratings such as these are a typical example of the Likert scale of degrees of satisfaction and they can be easily represented by degrees from linearly ordered finite residuated lattices. In our case, we take a five-element Łukasiewicz chain as our structure of truth degrees: We let $L = \{0, 0.25, 0.5, 0.75, 1\}$ be the set of truth degrees (0 corresponds to "--", 0.25 corresponds to "-", and so on), $\wedge$ and $\vee$ being minimum and maximum, respectively. L, equipped with $\wedge$ and $\vee$, forms a five-element linearly ordered lattice with 0 and 1 being the least and the greatest element, respectively, with ordering of truth degrees given by $0 < 0.25 < 0.5 < 0.75 < 1$. The adjoint operations $\otimes$ and $\to$ are defined as follows: $a \otimes b = \max(a+b-1, 0)$ and $a \to b = \min(1-a+b, 1)$, cf. (2). Our structure of truth degrees is a finite subalgebra of the standard Łukasiewicz algebra defined on the real unit interval, see (Belohlavek 2002, Gerla 2001, Hájek 1998).

This way, we get a data table with fuzzy attributes $\langle X, Y, I\rangle$ describing holidays and their graded properties which is depicted in Fig. 1. The set X of objects consists of "holiday types", the set Y of attributes consists of four attributes "happy kids",

[1] The table is taken from `http://www.mycoted.com/Comparison_tables`.

	happy kids	low cost	happy adults	easy travel
walking holiday	0.25	0.75	1	0.75
cruise holiday	0.5	0.25	0.5	0.5
beach holiday	0.75	0.25	0.75	0.25
stay at home	0.25	1	0.5	1
holiday camp	1	0.25	0.25	0.25

Figure 1. Illustrative data table with fuzzy attributes

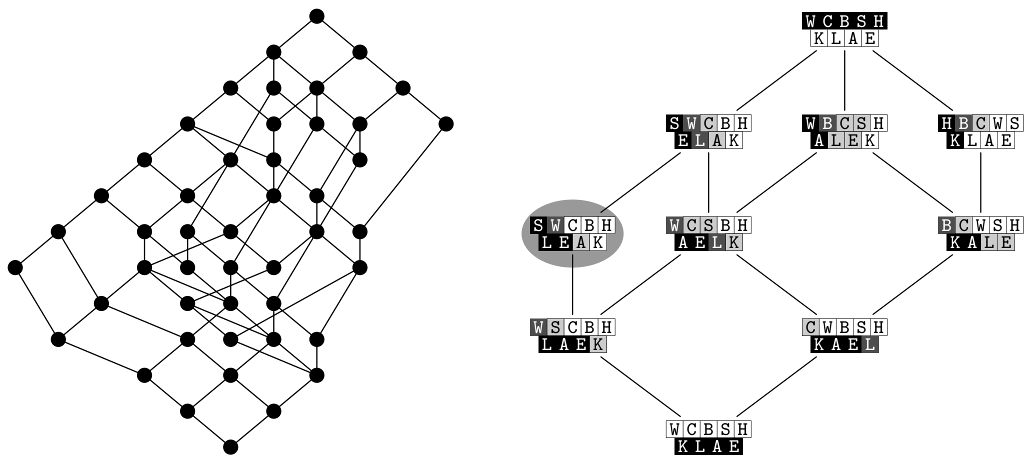

Figure 2. Fuzzy concept lattices (Hasse diagrams of partially ordered formal concepts extracted from table in Fig.1).

"low cost", "happy adults", and "easy travel". The table entries indicate to which degree a given type of holiday has the corresponding attribute. For instance, the row corresponding to "walking holiday" means: "The cost is quite low, the adults are (fully) happy, travel is quite easy, and kids are not too happy.".

The fuzzy concept lattice $\mathcal{B}(X, Y, I)$ generated from this data table contains 47 formal concepts (clusters). The hierarchy of fuzzy concepts is depicted in Fig. 2 (left). As one can see, $\mathcal{B}(X, Y, I)$ is large and hardly graspable by users. Therefore, we try to reduce the number of formal concepts by employing constraints by hedges. That is, instead of $\mathcal{B}(X, Y, I)$, we now consider parameterized concept lattices $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ with hedges $*_x$ and $*_y$ playing the role of parameters.

Our $\mathbf{L}$ admits five hedges. The hedges are depicted in Fig. 3. Arrows in Fig. 3 indicate values a^{*_i} for $a \in L$ and $i = 1, \ldots, 5$. For instance, for $*_2 \colon L \to L$ we have $0^{*_2} = 0.25^{*_2} = 0$, $0.5^{*_2} = 0.75^{*_2} = 0.5$, and $1^{*_2} = 1$. Since $|X| = 5$ and $|Y| = 4$, there are $5^{5+4} = 1953125$ possibilities to select $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. For illustration, we take each $*_x$ to be identity (i.e., $a^{*_x} = a$ for each $a \in L$) and inspect the reduction of $\mathcal{B}(X, Y, I)$ depending on various choices of $*_y$. Since $|Y| = 4$, there are $5^4 = 625$ possible constrained concept lattices $\mathcal{B}(X, Y^{*_Y}, I)$. Due to Theorem 3.20, each $\mathcal{B}(X, Y^{*_Y}, I)$ is a subset of the whole $\mathcal{B}(X, Y, I)$. Moreover, due to Theorem 3.22, stronger hedges $*_y$ lead to stronger restrictions and thus to smaller concept lattices $\mathcal{B}(X, Y^{*_Y}, I)$. If we compute concept lattices corresponding to all possible choices of $*_y$'s, we arrive at 81 distinct sets of fuzzy concepts with sizes varying from 10 (least) up to 47 (greatest) concepts. Concept hierarchies corresponding to the distinct sets of fuzzy concepts are depicted in Fig. 5. Line diagrams in Fig. 5 are sorted by the number of their edges. As we can see, there is a smooth transition from the least (most concise) hierarchy depicted in the top-left corner to the greatest (most detailed) hierarchy depicted in the bottom-right corner of the figure.

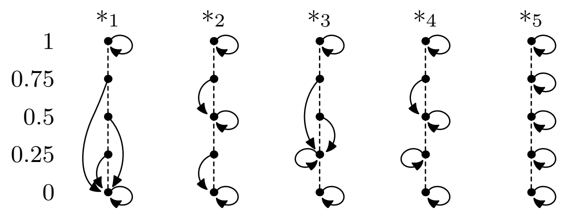

Figure 3. Hedges on a five-element Łukasiewicz chain

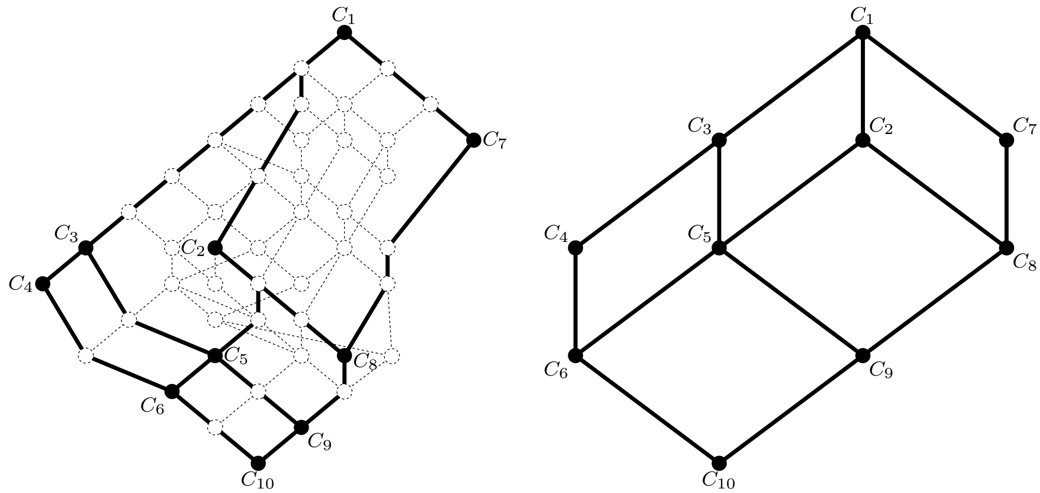

Figure 4. $\mathcal{B}(X, Y^{*_Y}, I)$ drawn as a $\bigwedge$-sublattice of $\mathcal{B}(X, Y, I)$.

The least fuzzy concept lattice $\mathcal{B}(X, Y^{*_Y}, I)$ is depicted in more detail in Fig. 2 (right). In this figure we have used the following method of describing formal concepts, i.e., conceptual clusters corresponding to the nodes of the diagram: each cluster is labeled by its extent (objects that fall under the concept) and intent (attributes that fall under the concept). Objects in the extents and attributes in the intents are depicted by color bars indicating degrees to which objects and attributes (represented by their abbreviations, see underlined letters in Fig. 1) fall under the concept (the darker the background color, the higher the degree). For example, the highlighted concept in Fig. 2 (right) can be described as follows: The black "L" and "E" say that this is a concept of "low cost and easy travel holidays". From the extent of the node we can see that "S" is black and "W" is dark, i.e. "staying at home" falls fully under that concept and "walking holiday" falls almost fully under that concept. On the other hand, light "C", "B", and "H" can be interpreted so that "cruise holiday, beach holiday, and holiday camp do not fall under the concept of low cost and easy travel holidays very much". Using concept lattice of hierarchically ordered formal concepts, users can compare various groups of holidays (holiday concepts) by traversing through the hierarchy. The subconcept-superconcept hierarchy allows us to find, for any two or more concepts, a concept that is more general or more specific.

According to Theorem 3.24, each $\mathcal{B}(X, Y^{*_Y}, I)$ is a $\bigwedge$-sublattice of $\mathcal{B}(X, Y, I)$. It is useful to draw Hasse diagram of the reduced concept lattice $\mathcal{B}(X, Y^{*_Y}, I)$ over the Hasse diagram of the original concept lattice $\mathcal{B}(X, Y, I)$ to see positions of important concepts in $\mathcal{B}(X, Y, I)$. This is illustrated in Fig. 4 where we have drawn $\mathcal{B}(X, Y, I)$ and the least $\mathcal{B}(X, Y^{*_Y}, I)$ together. Dark nodes labeled $C_1, \ldots, C_{10}$ represent concepts which are present in both $\mathcal{B}(X, Y, I)$ and $\mathcal{B}(X, Y^{*_Y}, I)$, the white nodes represent concepts which are in $\mathcal{B}(X, Y, I)$, only. Likewise, dotted lines represent edges of in $\mathcal{B}(X, Y, I)$ whereas dark bold lines indicate edges in $\mathcal{B}(X, Y^{*_Y}, I)$. Note that $\mathcal{B}(X, Y^{*_Y}, I)$ is not a $\bigvee$-sublattice of $\mathcal{B}(X, Y, I)$ which is easily seen from

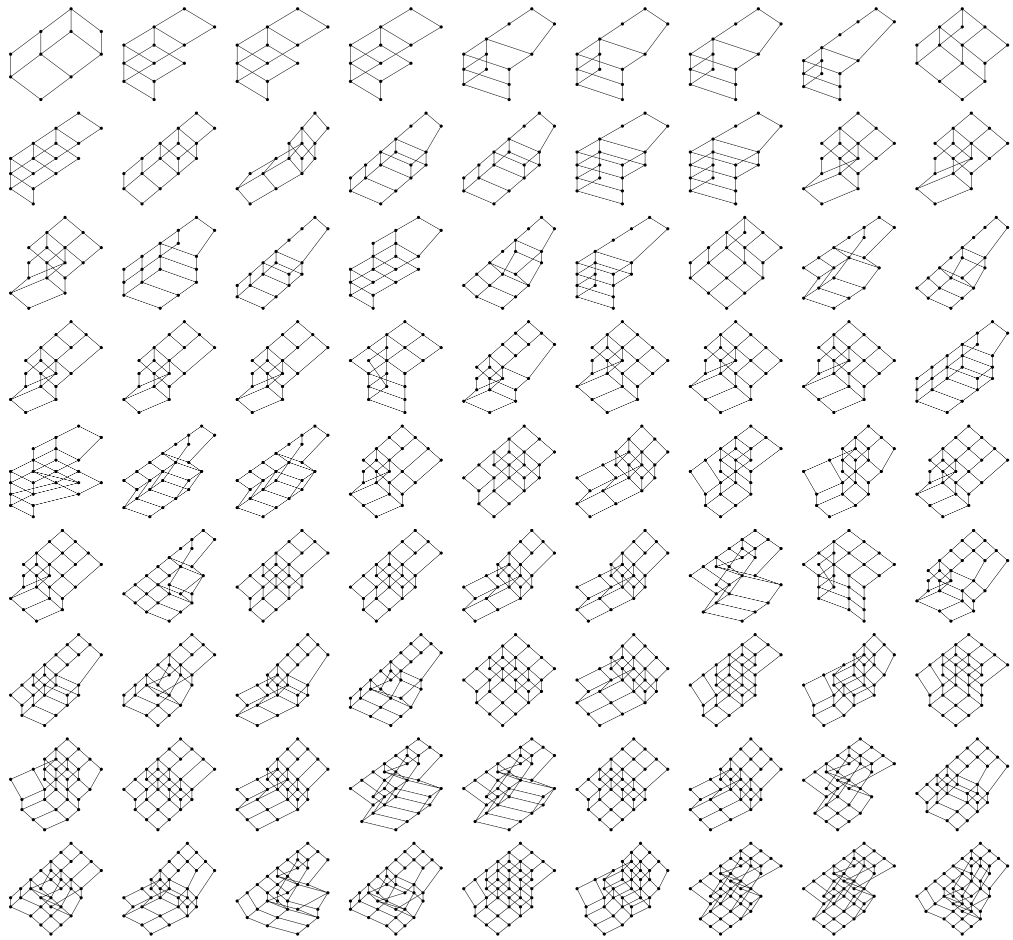

Figure 5. Parameterized reduction of a fuzzy concept lattice

the picture (take, e.g., concepts C_2 and C_3).

The smoothness of transition from one hierarchy to another, which can be seen in Fig. 5, is a consequence of the fact that similar hedges yield similar structures of formal concepts. This important property can also be proved (we have formulas for estimation of degree of similarity of concept lattices $\mathcal{B}(X, Y^{*_{Y_1}}, I)$ and $\mathcal{B}(X, Y^{*_{Y_2}}, I)$ based on a particularly defined degree of similarity of $*_{y_1}$ and $*_{y_2}$). Note that the choice of hedges $*_X$ and $*_Y$ is up to the user. The user thus needs not to define the hedges. But hedges are simple unary functions on the scale L of truth degrees and can easily be pre-computed automatically. Therefore, the user's role is to say "use stronger hedges" if the resulting concept lattice is too large for the user's purpose, or to say "use weaker hedges" is the concept lattice is too small and the user wants to see more details.

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