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Algebras with fuzzy equalities

Radim Bělohlávek*, Vilém Vychodil

Department of Computer Science, Palacký University, Tomkova 40, CZ-779 00 Olomouc, Czech Republic

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Abstract

An algebra with fuzzy equality is a set with operations on it that is equipped with similarity $\approx$, i.e. a fuzzy equivalence relation, such that each operation f is compatible with $\approx$. Described verbally, compatibility says that each f yields similar results if applied to pairwise similar arguments. On the one hand, algebras with fuzzy equalities are structures for the equational fragment of fuzzy logic and have been studied from this point of view before. On the other hand, they are the formal counterpart to the intuitive idea of having functions that are not allowed to map similar objects to dissimilar ones. The present paper aims at developing fundamental points of algebras with fuzzy equalities: we introduce the notion of an algebra with fuzzy equality, present natural examples, compare the notion with other approaches, and introduce and develop basic structural notions (subalgebras, morphisms, products, direct unions). In a follow-up paper [Vychodil, Direct limits and reduced products of algebras with fuzzy equalities, submitted for publication], we deal with advanced topics in algebras with fuzzy equalities (direct limits, reduced products).
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1. The concept of algebra with fuzzy equality

Functions operating on a set so that close (similar) elements are mapped to close elements have traditionally been the subject of study of calculus and functional analysis. The concept of closeness has been almost exclusively formalized using the notion of a metric. On the other hand, the very essence of

* Corresponding author. Tel.: +420 585 634 700; fax: +420 585 411 643.
E-mail addresses: radim.belohlavek@upol.cz (R. Bělohlávek), vilem.vychodil@upol.cz (V. Vychodil).

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the problem calls for a logical treatment. Namely, formulated verbally, the condition of mapping similar elements to similar ones reads “if arguments of a function are pairwise similar then the results are similar as well”. From a logical point of view, the situation is described by a logical formula that is traditionally being called the compatibility axiom (or congruence axiom). Therefore, congruence relations which are the relations satisfying the compatibility axioms provide us with a logico-algebraic means for handling the above problem. The appropriateness of such an approach is, however, seriously questionable. Namely, congruences are bivalent relations in that every two elements either are congruent or not. Contrary to that, closeness (or similarity) is a graded notion—any two elements are close to some degree. With the emergence of fuzzy logic, the ability of logic to treat the problem of functions preserving in a natural way a given similarity on the universe set changed. Namely, instead of 0 and 1 only, fuzzy logic allows one to have a whole scale of truth degrees and, consequently, one can model graded similarity in fuzzy logic. The above mentioned compatibility axiom, compared to its crisp (two-valued) interpretation, becomes much less trivial in fuzzy logic since its meaning depends on the choice of a conjunction operation (usually a t-norm) and has a numerical (if truth degrees are numbers) significance.

In our previous papers, we used algebras with fuzzy equalities as structures for the equational fragment of first-order fuzzy logic. In [4], we developed a syntactico-semantically complete logic for reasoning about fuzzy equalities and in [6], we provided an algebraic characterization of the class of all models of a given fuzzy set of identities. The main aim of this paper is to look closer at the very notion of an algebra with fuzzy equality.

Our motivation is twofold. First, as shown in [4,6], the notion of an algebra with fuzzy equality is a nontrivial and thus interesting one from the purely logico-mathematical point of view. Second, as mentioned above and as we will see in the beginning of Section 3.1, the concept of an algebra with fuzzy equality is the immediate semantic structure enabling us to study the problem of functions mapping similar elements to similar elements from the very natural point of view. In addition to that, our paper represents another attempt to application of fuzzy approach to universal algebra. Fuzzy approach to various universal algebraic concepts started with Rosenfeld’s fuzzy groups [27]. Rosenfeld’s approach has been applied to several algebraic structures and concepts, see e.g. [12], and also generalized to arbitrary universal algebras, see Di Nola [14]. We comment on fuzzy algebras in Remark 3.2(2). Another attempt started by [24] and was further developed in [28] where one can find further references. Our approach is close to this attempt. However, there are important differences which we comment on later in the end of Section 3.2.

Furthermore, our approach may be considered an alternative to so-called metric algebras [31,32], the aim of which is also to provide a formal framework for treating functions mapping similar elements to similar ones. However, contrary to our approach, metric algebras use the notion of a metric which leads to conditions which are intuitively not very clear (we comment on this in Remark 3.2(1)).

Our paper is organized as follows. Section 2 introduces preliminaries, Section 3 contains the results. Section 3.1 introduces the notion of an algebra with fuzzy equality and presents motivating examples. Section 3.2 deals with subalgebras, congruences, morphisms, and related concepts. Direct and subdirect products, and subdirect representation of algebras with fuzzy equalities are dealt with in Section 3.3. Section 3.4 delas with direct unions. Section 3.5 studies operators on classes of algebras with fuzzy equalities, free algebras, and varieties.

The paper is an extension of our short conference paper [7]. Compared to [7], the present paper contains full proofs, more examples, more detailed remarks, and several further results. Since algebras with fuzzy equalities are special first-order fuzzy structures, we can sometimes use the general notions and results developed for general fuzzy structures. If this is the case, we omit proofs and refer to [5]. However, we introduce the notions and formulate results in their setting for algebras with fuzzy equalities since they are simpler than in their form for general fuzzy structures (this pertains particularly to morphisms, congruences, and direct products).

2. Preliminaries

We pick complete residuated lattices as the structures of truth values. Complete residuated lattices, being introduced in the 1930s in ring theory, were introduced into the context of fuzzy logic by Goguen [15,16]. Fundamental contribution to formal fuzzy logic using residuated lattices as the structures of truth values is due to Pavelka [26]. Later on, various logical calculi were investigated using residuated lattices or particular types of residuated lattices. A thorough information about the role of residuated lattices in fuzzy logic can be obtained from monographs [17,18,25].

In the following, **L** denotes an arbitrary complete residuated lattice. Recall that a (complete) residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes, \rightarrow$ form an adjoint pair, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$ is valid for each $a, b, c \in L$. In what follows, **L** always refers to a complete residuated lattice. All properties of complete residuated lattices used in the sequel are well known and can be found in any of the above mentioned monographs. Note that particular types of residuated lattices (distinguishable by identities) include Boolean algebras, Heyting algebras, algebras of Girard's linear logic, MV-algebras, Gödel algebras, product algebras, and more generally, BL-algebras (see [18,21]).

An **L**-set (or fuzzy set with truth degrees in **L**) in a universe set U is any mapping $A: U \rightarrow L$, $A(u) \in L$ being interpreted as the truth value of "u belongs to A". An **L**-set A in U is said to be finite, if the subset $\{u \mid A(u) > 0\} \subseteq U$ is finite. Every finite **L**-set A in U can be denoted by

$$A = \{A(u_1)/u_1, \ldots, A(u_n)/u_n\},$$

where $\{u_1, \ldots, u_n\} = \{u \mid A(u) > 0\}$. Analogously, an n-ary **L**-relation (or fuzzy relation with truth degrees in **L**) on a universe set U is an **L**-set in the universe set U^n, e.g. a binary relation R on U is a mapping $R: U \times U \rightarrow L$. For a binary **L**-relation R, an **L**-relation R^{-1} is defined by $R^{-1}(u, v) = R(v, u)$. For binary **L**-relations R_1, R_2 on U, the $\circ$-composition of R_1 and R_2 is a binary **L**-relation on U defined by

$$(R_1 \circ R_2)(u, v) = \bigvee_{w \in U} R_1(u, w) \otimes R_2(w, v).$$

Recall that an **L**-equivalence (fuzzy equivalence) relation E on a set U is a mapping $E: U \times U \rightarrow L$ satisfying

(i) $E(u, u) = 1$ (reflexivity),
(ii) $E(u, v) = E(v, u)$ (symmetry),
(iii) $E(u, v) \otimes E(v, w) \leqslant E(u, w)$ (transitivity)

for every $u, v, w \in U$. An **L**-equivalence on U where $E(u, v) = 1$ implies $u = v$ will be called an **L**-equality. A mapping $f: U^n \rightarrow U$, $n \in \mathbb{N}$, is said to be compatible with a binary **L**-relation R on U

if for arbitrary $u_1, v_1, \dots, u_n, v_n \in U$ we have

$$R(u_1, v_1) \otimes \cdots \otimes R(u_n, v_n) \leqslant R(f(u_1, \dots, u_n), f(v_1, \dots, v_n)).$$

Compatibility has a natural verbal description. It says "if u_1 and v_1 are R-related and $\cdots$ and u_n and v_n are R-related then $f(u_1, \dots, u_n)$ and $f(v_1, \dots, v_n)$ are R-related". For further information on **L**-equivalences, compatibility of **L**-relations, and their role in fuzzy reasoning see e.g. [22,21,18,5].

For **L**-sets A and B in U we define

$$(A \approx B) = \bigwedge_{u \in U} (A(u) \leftrightarrow B(u)) \quad \text{(degree of equality of } A \text{ and } B)$$

and

$$S(A, B) = \bigwedge_{u \in U} (A(u) \to B(u)) \quad \text{(degree of subsethood of } A \text{ in } B).$$

Note, that $a \leftrightarrow b$ is an abbreviation for $(a \to b) \wedge (b \to a)$. Clearly, $(A \approx B) = S(A, B) \wedge S(B, A)$. Furthermore, we write $A \subseteq B$ (A is a subset of B) if $S(A, B) = 1$, i.e. for each $u \in U$, $A(u) \leqslant B(u)$.

An **L**-set A in U is called crisp if $A(u) \in \{0, 1\}$ for each $u \in U$. Crisp **L**-sets in U correspond in an obvious way to ordinary subsets of U (crisp **L**-sets are the characteristic functions of ordinary subsets). Therefore, we will sometimes identify crisp **L**-sets with their corresponding ordinary subsets.

3. Algebras with fuzzy equalities

3.1. Definition and examples

A *type* is a triplet $\langle \approx, F, \sigma \rangle$ where $\approx \notin F$ and σ is a mapping $\sigma : F \cup \{\approx\} \to \mathbb{N}_0$ with $\sigma(\approx) = 2$. Each $f \in F$ is called a *function symbol,* $\approx$ is a relation symbol called a *symbol for fuzzy equality.* The mapping σ assigns the arity $\sigma(f)$ to every function symbol $f \in F$. The symbol $\approx$ stands always for a binary relation symbol. If there is no danger of confusion, a type will be denoted simply by F.

Definition 3.1. An *algebra with* **L***-equality of type* $\langle \approx, F, \sigma \rangle$ is a triplet $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ such that $\langle M, F^{\mathbf{M}} \rangle$ is an algebra of type $\langle F, \sigma \rangle$ and $\approx^{\mathbf{M}}$ is an **L**-equality on M such that each $f^{\mathbf{M}} \in F^{\mathbf{M}}$ is compatible with $\approx^{\mathbf{M}}$. Algebra $\langle M, F^{\mathbf{M}} \rangle$ denoted by ske($\mathbf{M}$) is called a *skeleton* [1] *of* $\mathbf{M}$.

That is, $F^{\mathbf{M}} = \{f^{\mathbf{M}} : M^{\sigma(f)} \to M \mid f \in F\}$, i.e., $F^{\mathbf{M}}$ contains a $\sigma(f)$-ary function $f^{\mathbf{M}}$ on M for each $f \in F$. Recall that the compatibility condition says that

$$(a_1 \approx^{\mathbf{M}} b_1) \otimes \cdots \otimes (a_n \approx^{\mathbf{M}} b_n) \leqslant f^{\mathbf{M}}(a_1, \dots, a_n) \approx^{\mathbf{M}} f^{\mathbf{M}}(b_1, \dots, b_n)$$

for each n-ary $f \in F$ and every $a_1, b_1, \dots, a_n, b_n \in M$. An algebra with **L**-equality will also be simply called an **L**-algebra. If **L** is obvious, we say also an algebra with fuzzy equality. Following common usage, we sometimes identify an **L**-algebra $\mathbf{M}$ with its support set M. Moreover, we write $\langle M, \approx^{\mathbf{M}}, f_1^{\mathbf{M}}, \dots, f_n^{\mathbf{M}} \rangle$ instead of $\langle M, \approx^{\mathbf{M}}, \{f_1^{\mathbf{M}}, \dots, f_n^{\mathbf{M}}\} \rangle$ and sometimes also omit the superscript $\mathbf{M}$ from $\approx^{\mathbf{M}}$, $F^{\mathbf{M}}$, and $f^{\mathbf{M}}$.

[1] The term skeleton comes from [19].

In the following, we often call the degree $(a \approx^{\mathbf{M}} b)$ the degree of similarity between a and b which due to the intended interpretation of $\approx^{\mathbf{M}}$.

The concept of an **L**-algebra is by no means artificial. First, it is obvious that the compatibility axiom expresses a natural constraint on the operations, i.e. mapping similar to similar. If one takes, e.g., $L = [0, 1]$, this constraint has a numerical character. Moreover, this constraint is expressed in a simple fragment of first-order fuzzy logic ("if a_1 and b_1 are similar and $\cdots$ and a_n and b_n are similar then $f^{\mathbf{M}}(a_1, \ldots, a_n)$ and $f^{\mathbf{M}}(b_1, \ldots, b_n)$ are similar" translates to a first-order formula). Therefore, unlike ordinary algebras, **L**-algebras **M** have two nontrivial parts. First, the "functional part", which is an ordinary algebra $\langle M, F^{\mathbf{M}}\rangle$. Second, the "relational part" which is a set equipped with a fuzzy equality, $\langle M, \approx^{\mathbf{M}}\rangle$.

Remark 3.1. (1) For $\mathbf{L} = \mathbf{2}$ (two-element Boolean algebra), the only **L**-equality $\approx$ on U is the ordinary identity relation, i.e. $(u \approx v) = 1$ for $u = v$, $(u \approx v) = 0$ for $u \neq v$. It is therefore easy to see that **2**-algebras coincide with ordinary algebras [10,11]. This is the first way the notion of an **L**-algebra generalizes the notion of an (ordinary) algebra.

(2) Taking this point of view, one may consider the structure **L** of truth values as a parameter and think of the theory of **L**-algebras as a parametrized theory. Setting $\mathbf{L} = \mathbf{2}$, we get the theory of ordinary universal algebras. As we will see in the following, several results are valid for each **L** (each complete residuated lattice). However, there are results which are true only for **L**'s satisfying special properties and so, additional properties of **L** are important.

(3) Obviously, taking arbitrary **L** and a crisp **L**-equality $\approx^{\mathbf{M}}$, **L**-algebras $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ can be identified with ordinary algebras. This is the second way the notion of an **L**-algebra generalizes the notion of an algebra.

The following are natural examples of **L**-algebras.

Example 3.1. Let U be a set equipped with an **L**-equality $\approx^U$. Let $M = S(U)$ be the set of all permutations of U compatible with $\approx^U$, i.e. the set of all bijective mappings π on U for which we have $(u \approx^U v) \leqslant (\pi(u) \approx^U \pi(v))$. The triplet $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, \circ^{\mathbf{M}}\rangle$, where

$$\pi \approx^{\mathbf{M}} \sigma = \bigwedge_{u,v} \Big((u \approx^U v) \rightarrow (\pi(u) \approx^U \sigma(v))\Big) \tag{1}$$

and $\circ^{\mathbf{M}}$ denotes the composition of permutations, is an **L**-algebra. Note that $\pi \approx^{\mathbf{M}} \sigma$ is the degree to which it is true that similar elements are mapped to similar ones. Moreover, the restriction of **M** to bijective mappings satisfying the property $(u \approx^U v) = (\pi(u) \approx^U \pi(v))$ is a group with fuzzy equality. To prove this assertion, we first show that

$$\pi \approx^{\mathbf{M}} \sigma = \bigwedge_{u} \Big(\pi(u) \approx^U \sigma(u)\Big). \tag{2}$$

On the one hand, we clearly have

$$\begin{aligned}\pi \approx^{\mathbf{M}} \sigma &= \bigwedge_{u,v} \Big((u \approx^U v) \rightarrow (\pi(u) \approx^U \sigma(v))\Big) \leqslant \bigwedge_{u} \Big((u \approx^U u) \rightarrow (\pi(u) \approx^U \sigma(u))\Big) \\ &= \bigwedge_{u} \Big(1 \rightarrow (\pi(u) \approx^U \sigma(u))\Big) = \bigwedge_{u} \Big(\pi(u) \approx^U \sigma(u)\Big).\end{aligned}$$

On the other hand,

$$\bigwedge_u \Big(\pi(u) \approx^U \sigma(u)\Big) \leqslant \pi \approx^{\mathbf{M}} \sigma$$

holds true iff for each $u, v \in U$ we have

$$\bigwedge_u \Big(\pi(u) \approx^U \sigma(u)\Big) \leqslant \Big((u \approx^U v) \to (\pi(u) \approx^U \sigma(v))\Big),$$

i.e. iff

$$\Big(u \approx^U v\Big) \otimes \bigwedge_u \Big(\pi(u) \approx^U \sigma(u)\Big) \leqslant \pi(u) \approx^U \sigma(v)$$

which is true. Indeed, compatibility of σ yields

$$\Big(u \approx^U v\Big) \otimes \bigwedge_u \Big(\pi(u) \approx^U \sigma(u)\Big) \leqslant \Big(\sigma(u) \approx^U \sigma(v)\Big) \otimes \Big(\pi(u) \approx^U \sigma(u)\Big) \leqslant \pi(u) \approx^U \sigma(v).$$

It is now routine to check that $\approx^{\mathbf{M}}$ is an $\mathbf{L}$-equality on M: reflexivity and symmetry follow directly from definition, and for transitivity we have on account of (2) for each $u \in U$ that

$$\Big(\pi \approx^{\mathbf{M}} \sigma\Big) \otimes \Big(\sigma \approx^{\mathbf{M}} \theta\Big) \leqslant \Big(\pi(u) \approx^U \sigma(u)\Big) \otimes \Big(\sigma(u) \approx^U \theta(u)\Big) \leqslant \pi(u) \approx^U \theta(u),$$

that is

$$\Big(\pi \approx^{\mathbf{M}} \sigma\Big) \otimes \Big(\sigma \approx^{\mathbf{M}} \theta\Big) \leqslant \bigwedge_u \Big(\pi(u) \approx^U \theta(u)\Big) = \pi \approx^{\mathbf{M}} \theta,$$

proving transitivity of $\approx^{\mathbf{M}}$. If $(\pi \approx^{\mathbf{M}} \sigma) = 1$ then, using (2), $(\pi(u) \approx^U \sigma(u)) = 1$ for each $u \in U$. Since $\approx^U$ is an $\mathbf{L}$-equality, we have that $\pi(u) = \sigma(u)$ for each $u \in U$, whence $\pi = \sigma$. Therefore, $\approx^{\mathbf{M}}$ is an $\mathbf{L}$-equality on $S(U)$. To verify that $\circ^{\mathbf{M}}$ is compatible with $\approx^{\mathbf{M}}$ take any $\pi, \pi', \varrho, \varrho' \in M$. We have

$$\Big(\pi \approx^{\mathbf{M}} \pi'\Big) \otimes \Big(\varrho \approx^{\mathbf{M}} \varrho'\Big) \leqslant \Big(\pi \circ \varrho \approx^{\mathbf{M}} \pi' \circ \varrho'\Big)$$

iff for each $u \in U$ we have

$$\Big(\pi \approx^{\mathbf{M}} \pi'\Big) \otimes \Big(\varrho \approx^{\mathbf{M}} \varrho'\Big) \leqslant \Big(\varrho(\pi(u)) \approx^U \varrho'(\pi'(u))\Big)$$

which is true:

$$\begin{aligned}\Big(\pi \approx^{\mathbf{M}} \pi'\Big) \otimes \Big(\varrho \approx^{\mathbf{M}} \varrho'\Big) &\leqslant \Big(\pi(u) \approx^U \pi'(u)\Big) \otimes \Big(\varrho(\pi'(u)) \approx^U \varrho'(\pi'(u))\Big) \\ &\leqslant \Big(\varrho(\pi(u)) \approx^U \varrho(\pi'(u))\Big) \otimes \Big(\varrho(\pi'(u)) \approx^U \varrho'(\pi'(u))\Big) \\ &\leqslant \varrho(\pi(u)) \approx^U \varrho'(\pi'(u)).\end{aligned}$$

To sum up, $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, \circ^{\mathbf{M}} \rangle$ is an algebra with fuzzy equality.

Example 3.2. Let $C : L^U \to L^U$ be an $\mathbf{L}$-closure operator on X. Furthermore, let $\mathcal{S}_{\mathrm{C}} = \{A \in L^X \mid C(A) = A\}$ be the system of all closed fuzzy sets of C, see [2]. Then $\mathcal{S}_{\mathrm{C}}$ is a complete lattice with respect to $\subseteq$ where infima $\bigwedge$ coincide with intersections and for suprema $\bigvee$ we have $\bigvee_{i\in I} A_i = C(\bigcup_{i\in I} A_i)$. It is easy to verify that

$$\bigwedge_{i\in I}(A_i \approx B_i) \leqslant \left(\bigcap_{i\in I} A_i \approx \bigcap_{i\in I} B_i\right) = \left(\bigwedge_{i\in I} A_i \approx \bigwedge_{i\in I} B_i\right), \tag{3}$$

$$\bigwedge_{i\in I}(A_i \approx B_i) \leqslant \left(\bigcup_{i\in I} A_i \approx \bigcup_{i\in I} B_i\right). \tag{4}$$

Hence, using (4) it follows that

$$\begin{aligned}\bigwedge_{i\in I}(A_i \approx B_i) &\leqslant \left(\bigcup_{i\in I} A_i \approx \bigcup_{i\in I} B_i\right) = S\left(\bigcup_{i\in I} A_i, \bigcup_{i\in I} B_i\right) \wedge S\left(\bigcup_{i\in I} B_i, \bigcup_{i\in I} A_i\right)\\ &\leqslant S\left(C\left(\bigcup_{i\in I} A_i\right), C\left(\bigcup_{i\in I} B_i\right)\right) \wedge S\left(C\left(\bigcup_{i\in I} B_i\right), C\left(\bigcup_{i\in I} A_i\right)\right)\\ &= \left(C\left(\bigcup_{i\in I} A_i\right) \approx C\left(\bigcup_{i\in I} B_i\right)\right) = \left(\bigvee_{i\in I} A_i \approx \bigvee_{i\in I} B_i\right).\end{aligned}$$

For $I = \{1, 2\}$ we can use (3) and the previous inequality to get

$$(A_1 \approx B_1) \otimes (A_2 \approx B_2) \leqslant (A_1 \wedge A_2) \approx (B_1 \wedge B_2),$$
$$(A_1 \approx B_1) \otimes (A_2 \approx B_2) \leqslant (A_1 \vee A_2) \approx (B_1 \vee B_2).$$

Therefore, $\langle \mathcal{S}_C, \approx, \wedge, \vee\rangle$ is an **L**-algebra. Particularly, since the identity mapping is an **L**-closure operator C (sending A to A) for which $\mathcal{S}_C = L^X$, we get that $\langle L^X, \approx, \cap, \cup\rangle$ is an **L**-algebra, i.e. a lattice with fuzzy equality.

Example 3.3. Let X and Y be a set of objects and a set of attributes, respectively, and I be an **L**-relation between X and Y such that $I(x, y)$ is the degree to which the object x has the attribute y. Furthermore, let us define for all **L**-sets $A \in L^X$, $B \in L^Y$ the **L**-sets $A^{\uparrow} \in L^Y$, $B^{\downarrow} \in L^X$ by

$$A^{\uparrow}(y) = \bigwedge_{x\in X} (A(x) \to I(x, y)),$$
$$B^{\downarrow}(x) = \bigwedge_{y\in Y} (B(y) \to I(x, y)).$$

Now, put $\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$ and define for $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X, Y, I)$ a binary relation $\leqslant$ by $\langle A_1, B_1\rangle \leqslant \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (or, iff $B_2 \subseteq B_1$; both ways are equivalent). The structure $\langle \mathcal{B}(X, Y, I), \leqslant\rangle$ is called a *fuzzy concept lattice* induced by X, Y, and I. The elements of $\langle A, B\rangle$ of $\mathcal{B}(X, Y, I)$ are naturally interpreted as concepts hidden in (the input data represented by) I. Namely, $A^{\uparrow} = B$ and $B^{\downarrow} = A$ says that B is the collection of all attributes shared by all objects from A, and A is the collection of all objects sharing all attributes from B. Note that these conditions represent exactly the definition of a concept as developed in the so-called Port-Royal logic; A and B are called the *extent* and the *intent* of the concept $\langle A, B\rangle$, respectively, and represent the collection of all objects and all attributes covered by the particular concept. Furthermore, $\leqslant$ models the natural subconcept–superconcept hierarchy—concept $\langle A_1, B_1\rangle$ is a subconcept of $\langle A_2, B_2\rangle$ iff each object from A_1 belongs to A_2 (dually for attributes).

Put $\mathrm{Ext}(X, Y, I) = \{A \in L^X \mid \langle A, B\rangle \in \mathcal{B}(X, Y, I)$ for some $B \in L^Y\}$, i.e. $\mathrm{Ext}(X, Y, I)$ consists of all extents of the concepts from $\mathcal{B}(X, Y, I)$. Now, it can be shown (see [2,3]) that for a fuzzy concept lattice $\mathcal{B}(X, Y, I)$, $\mathrm{Ext}(X, Y, I)$ is an **L**-closure system in X and that for each **L**-closure system $\mathcal{S}$ in X there are some X, Y, and I, such that $\mathcal{S} = \mathrm{Ext}(X, Y, I)$. This means that $\langle \mathcal{B}(X, Y, I), \approx, \wedge, \vee\rangle$, where $(\langle A_1, B_1\rangle \approx \langle A_2, B_2\rangle) = (A_1 \approx A_2)$ and $\wedge$ and $\vee$ are the infimum and supremum induced by the corresponding operations in $\mathrm{Ext}(X, Y, I)$, is an **L**-algebra (see Example 3.2).

Remark 3.2. (1) The idea of extending universal algebra by adding further constrains is not new. Particularly, in [32] (see also [31]), the author investigates the so-called *metric algebras* which are basically algebras with a metric on the support set. It is interesting to note that the author introduces the notion of an atomic inequality which is an expression of the form "$\varrho(p, q) \preccurlyeq a$" where ϱ denotes a metric, p, q are terms, and $a \in [0, \infty]$. An atomic inequality is said to be ε-true in a given metric algebra if $\varrho(p, q) \leqslant a + \varepsilon$ in the metric algebra. The analogy to the fuzzy logic approach is obvious. Furthermore, an important notion of [32] is that of an *equicontinuity of operations* of metric algebras: An operation f is equicontinuous if the implication

$$\varrho(x_1, y_1) \preccurlyeq 0 \wedge \cdots \wedge \varrho(x_n, y_n) \preccurlyeq 0 \;\Rightarrow\; \varrho(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \preccurlyeq 0$$

is satisfied equicontinuously. This means that for each $\varepsilon > 0$ there is a $\delta > 0$ such that for any interpretation we have that if each $\varrho(a_i, b_i) \preccurlyeq 0$ is δ-true then $\varrho(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \preccurlyeq 0$ is ε-true. Let us note that even if the idea behind equicontinuity of operations in metric algebras is similar to that of compatibility (with a given **L**-equality), the theory of metric algebras deals with a completely different kind of restrictions on operations. In our opinion, the explicit formulation of the requirement in the framework of fuzzy logic (**L**-algebras) is more natural.

(2) Let us comment more on so-called fuzzy (sub)algebras, see e.g. [9,12,14,23,29,33]. Recall that a fuzzy (sub)algebra in an ordinary algebra $\mathbf{M} = \left\langle M, F^{\mathbf{M}}\right\rangle$ is an **L**-set A in M such that for each n-ary operation $f^{\mathbf{M}} \in F^{\mathbf{M}}$ we have

$$A(a_1) \otimes \cdots \otimes A(a_n) \leqslant A\left(f^{\mathbf{M}}(a_1, \ldots, a_n)\right)$$

for every $a_1, \ldots, a_n \in M$. That is, the notion of a fuzzy subalgebra results by fuzzification of the concept of a subalgebra and there is no fuzzy equality on the universe M considered. On the other hand, as we will see, a subalgebra of an algebra with fuzzy equality is again an algebra with fuzzy equality (i.e. with a crisp set as a universe). In a sense, the difference between the theory of fuzzy (sub)algebras and the presented theory of algebras with fuzzy equalities lies in what becomes fuzzy: in fuzzy (sub)algebras, it is the universe set, while in algebras with fuzzy equalities, it is the equality on the universe. As we will see, we are able to develop most parts of universal algebra in the setting of algebras with fuzzy equalities. On the other hand, we are not aware of a development of these parts in the setting of fuzzy (sub)algebras. Nevertheless, fuzzy (sub)algebras and algebras with fuzzy equalities are complementary and it might thus be interesting to look at the combination of both of these approaches, i.e. to consider fuzzy (sub)algebras A in an algebra $\mathbf{M}$ with fuzzy equality $\approx^{\mathbf{M}}$, possibly such that A is compatible with $\approx^{\mathbf{M}}$, i.e. $A(a) \otimes (a \approx^{\mathbf{M}} b) \leqslant A(b)$.

(3) In a series of papers, see e.g. [13], Demirci considers sets equipped with fuzzy equivalence/equality relations and compatible fuzzy functions. Fuzzy functions are particular fuzzy relations that need to satisfy certain natural properties which make them behave like functions. Ordinary functions can be seen

as particular fuzzy functions. Our approach is thus a particular approach of Demirci's. However, Demirci considers only binary fuzzy functions. Moreover, he does not consider general structural algebraic notions which are developed in our paper. A development of universal algebraic results in the setting of Demirci might be an interesting problem. Note, however, that with fuzzy functions, things get technically more complicated. For instance, it is not immediately obvious how terms and terms functions should look like in the setting of fuzzy functions. More generally, structures with fuzzy functions should possibly be studied in a framework of a suitable predicate fuzzy logic. This would provide a unifying framework and make it possible to use general results of predicate fuzzy logic. Also, this would perhaps suggest a way to develop logical calculi for reasoning about fuzzy identities in Demirci's setting analogous the calculi developed in our setting (see [4,8]).

3.2. Subalgebras, congruences and morphisms

The goal of this section is to investigate subalgebras, morphisms, congruences, and factor algebras, i.e. the very fundamental algebraic constructions. All principal properties of such constructions, being developed in universal algebra, will generalize for every complete residuated lattice $\mathbf{L}$ as the structure of truth values. Note that subalgebras, congruences, and morphisms of $\mathbf{L}$-algebras are particular cases of substructures, congruences, and morphisms of general fuzzy structures as developed in [5]. We will therefore use results of [5] when possible. Note however, that due to a special nature of $\mathbf{L}$-algebras, several results for $\mathbf{L}$-algebras are new and have no counterparts for general fuzzy structures.

Remark 3.3. Before delving into the basic structural notions, an important remark is in order. As mentioned above, each algebra $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ can be thought of as having two parts, a skeleton ske($\mathbf{M}$) (functional part) and a set with $\mathbf{L}$-equality $\langle M, \approx^{\mathbf{M}} \rangle$ (relational part) which are connected via the compatibility condition. Traditional universal algebra deals with the functional part only. However, it is obvious that when developing structural properties of $\mathbf{L}$-algebras, we will face situations where (1) only the functional part is important, (2) only the relational part is important, (3) both the functional and the relational parts are important. For (1), we can obviously use well-known results established in universal algebra. For example, we can omit the parts of proofs dealing only with the functional part. We apply this argument in detail in the proof of Theorem 3.4 and, subsequently, in short by saying that the particular result in question "follows from the ordinary case".

Definition 3.2. Let $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ be an $\mathbf{L}$-algebra of type F. An $\mathbf{L}$-algebra $\mathbf{N} = \langle N, \approx^{\mathbf{N}}, F^{\mathbf{N}} \rangle$ is called a *subalgebra of* $\mathbf{M}$, if $\emptyset \neq N \subseteq M$, every function $f^{\mathbf{N}} \in F^{\mathbf{N}}$ is a restriction of $f^{\mathbf{M}}$ to N, and $\approx^{\mathbf{N}}$ is a restriction of $\approx^{\mathbf{M}}$ to N. A *subuniverse of* $\mathbf{M}$ is any subset $N \subseteq M$ which is closed under all operations of $\mathbf{M}$.

That is, $N \subseteq M$ is a subuniverse of $\mathbf{M}$ iff for each n-ary $f^{\mathbf{M}} \in F^{\mathbf{M}}$ and every $a_1, \ldots, a_n \in N$ we have $f^{\mathbf{M}}(a_1, \ldots, a_n) \in N$. The collection of all subuniverses of $\mathbf{M}$ will be denoted by Sub($\mathbf{M}$). For convenience, we write $\mathbf{N} \in$ Sub($\mathbf{M}$) if $\mathbf{N}$ is a subalgebra of $\mathbf{M}$. It is almost immediate that there is a bijective correspondence between subalgebras and nonempty subuniverses of $\mathbf{M}$. Namely, a support set of a subalgebra of $\mathbf{M}$ is a nonempty subuniverse and, conversely, a nonempty subuniverse of $\mathbf{M}$ equipped with the restrictions of $f^{\mathbf{M}}$'s and the restriction of $\approx^{\mathbf{M}}$ is a subalgebra of $\mathbf{M}$.

It follows from the ordinary case that Sub(**M**) is closed under arbitrary intersections and, therefore, it is a closure system under the subsethood relation $\subseteq$. We denote the corresponding closure operator by $[\ ]_{\mathbf{M}}$, i.e. for $N \subseteq M$, $[N]_{\mathbf{M}}$ is the least subuniverse of **M** containing N and we have

$$[N]_{\mathbf{M}} = \bigcap \left\{ N' \mid N' \in \mathrm{Sub}(\mathbf{M}) \text{ and } N \subseteq N' \right\}. \tag{5}$$

If $[N]_{\mathbf{M}}$ is nonempty then the corresponding subalgebra of **M** is called the *subalgebra of* **M** *generated by N*. If *N* is finite then the corresponding subalgebra of **M** is said to be *finitely generated.*

In ordinary case, congruences correspond to abstract views on algebraic systems: one can put the congruent (i.e. equivalent from a certain abstract point of view) elements together and form a factor algebra which provides a view on the original algebra under which congruent elements are indistinguishable. In fuzzy setting, it is natural to consider graded indistinguishability of elements. This leads to the notion of a congruence on an **L**-algebra.

Definition 3.3. Let **M** be an **L**-algebra of type *F*. An **L**-relation θ on *M* is said to be a *congruence on* **M** if θ satisfies the following conditions,

(i) θ is an **L**-equivalence relation on *M*,
(ii) $(a \approx^{\mathbf{M}} b) \leqslant \theta(a, b)$ for arbitrary $a, b \in M$ (shortly $\approx^{\mathbf{M}} \subseteq \theta$),
(iii) all functions $f^{\mathbf{M}} \in F^{\mathbf{M}}$ are compatible with θ.

Remark 3.4. (1) In what follows, the (ordinary) sets of all **L**-equivalences and congruences on an **L**-algebra **M** are denoted by $\mathrm{Eq}_{\mathbf{L}}(M)$ and $\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. Evidently, $\mathrm{Con}_{\mathbf{L}}(\mathbf{M}) \subseteq \mathrm{Eq}_{\mathbf{L}}(M)$.

(2) It is immediate that congruences on **2**-algebras correspond to ordinary congruences on algebras since condition (ii) of Definition 3.3 is trivially satisfied by ordinary congruence relations, generalizing thus the well-known concept of a congruence.

(3) Note that condition (ii) may be equivalently replaced by

$$\theta(a, b) \otimes (a \approx^{\mathbf{M}} a') \otimes (b \approx^{\mathbf{M}} b') \leqslant \theta(a', b'), \tag{6}$$

which is supposed to hold for all $a, a', b, b' \in M$. Indeed, from (ii) we get $\theta(a, b) \otimes (a \approx^{\mathbf{M}} a') \otimes (b \approx^{\mathbf{M}} b') \leqslant \theta(a, b) \otimes \theta(a, a') \otimes \theta(b, b') \leqslant \theta(a', b')$, by transitivity and symmetry of θ. Conversely, (6) and reflexivity of θ and $\approx^{\mathbf{M}}$ yield $(a \approx^{\mathbf{M}} b) = \theta(a, a) \otimes (a \approx^{\mathbf{M}} a) \otimes (a \approx^{\mathbf{M}} b) \leqslant \theta(a, b)$.

Remark 3.5. Let **L** be a complete residuated lattice with $\otimes$ being $\wedge$ (i.e. **L** is a complete Heyting algebra). If $L = [0, 1]$, this means that $\otimes$ is the minimum which is probably the most common choice of "fuzzy conjunction". It is straightforward to verify that a binary **L**-relation on *M* is an **L**-equality if and only if each *a*-cut ${}^{a}\theta$, $a \in L$ is an ordinary equivalence relation and ${}^{1}\theta$ is the identity relation.

Furthermore, for an **L**-algebra **M**, θ satisfies the compatibility condition w.r.t. the operations of **M** if and only if each ${}^{a}\theta$ satisfies the compatibility condition w.r.t. operations of the corresponding ordinary algebra $\langle M, F^{\mathbf{M}} \rangle$. Indeed, for $\langle a_1, b_1 \rangle \in {}^{a}\theta, \ldots, \langle a_n, b_n \rangle \in {}^{a}\theta$ we have $a \leqslant \theta(a_i, b_i)$ for every $i = 1, \ldots, n$, thus

$$a \leqslant \theta(a_1, b_1) \otimes \cdots \otimes \theta(a_n, b_n) \leqslant \theta \left(f^{\mathbf{M}}(a_1, \ldots, a_n), f^{\mathbf{M}}(b_1, \ldots, b_n) \right)$$

by the assumption $\otimes = \wedge$, i.e. $\left\langle f^{\mathbf{M}}(a_1, \ldots, a_n), f^{\mathbf{M}}(b_1, \ldots, b_n)\right\rangle \in {}^a\theta$. Conversely, put $a = \theta(a_1, b_1) \otimes \cdots \otimes \theta(a_n, b_n)$. Then we have $\langle a_i, b_i\rangle \in {}^a\theta$ for every $i = 1, \ldots, n$, and so $\langle f^{\mathbf{M}}(a_1, \ldots, a_n), f^{\mathbf{M}}(b_1, \ldots, b_n)\rangle \in {}^a\theta$. Hence, it follows that $a \leqslant \theta\left(f^{\mathbf{M}}(a_1, \ldots, a_n), f^{\mathbf{M}}(b_1, \ldots, b_n)\right)$.

Therefore, for a congruence θ on an $\mathbf{L}$-algebra $\mathbf{M}$, $\mathcal{S}_\theta = \{{}^a\theta \mid a \in L\}$ is a system of binary relations on M indexed by L which satisfies: (i) $a \leqslant b$ implies ${}^b\theta \subseteq {}^a\theta$; (ii) for each $a, b \in M$ there exists a greatest $a \in L$ with $\langle a, b\rangle \in {}^a\theta$; (iii) for each $a \in L$, ${}^a\theta$ is a congruence on $\left\langle M, F^{\mathbf{M}}\right\rangle$ containing ${}^a{\approx^{\mathbf{M}}}$. One may verify that conversely, if $\mathcal{S} = \{\theta_a \mid a \in L\}$ is a system of binary relations on M satisfying (i)–(iii) then putting $\theta_{\mathcal{S}}(a, b) = \bigvee\{a \mid \langle a, b\rangle \in \theta_a\}$, $\theta_{\mathcal{S}}$ is a congruence on $\mathbf{M}$. Furthermore, $\theta = \theta_{\mathcal{S}_\theta}$ and $\mathcal{S} = \mathcal{S}_{\theta_{\mathcal{S}}}$.

This way of looking at congruences of $\mathbf{L}$-algebras (but only when $\otimes$ is $\wedge$!) gives the following interpretation: An ordinary congruence on an ordinary algebra may be thought of as representing an abstract view of the algebra—a view under which one does not distinguish congruent elements. A congruence θ on an $\mathbf{L}$-algebra may be thought of as a hierarchic system (the hierarchy supplied by $\mathbf{L}$) of abstract views ${}^a\theta$. Since each ${}^a\theta$ is a congruence on $\left\langle M, F^{\mathbf{M}}\right\rangle$, one may form the ordinary factor algebra $\left\langle M, F^{\mathbf{M}}\right\rangle / {}^a\theta$. Clearly, the smaller a, the bigger ${}^a\theta$, and so the coarser the factorization.

This fact has nontrivial applications. In [1], a method of factorization of concept lattices is presented. As shown in Example 3.3, a fuzzy concept lattice $\mathcal{B}(X, Y, I)$ can be naturally turned into an $\mathbf{L}$-algebra $\langle \mathcal{B}(X, Y, I), \approx, \wedge, \vee\rangle$. A fuzzy concept lattice $\mathcal{B}(X, Y, I)$ represents a set of natural clusters (concepts) contained in the input data $\langle X, Y, I\rangle$. If there are too many clusters in $\mathcal{B}(X, Y, I)$, the fuzzy concept lattice is hardly graspable by human mind. If $\otimes$ is $\wedge$, one can form the factor lattice $\mathcal{B}(X, Y, I)/{}^a{\approx}$ for any a. The factor lattice $\mathcal{B}(X, Y, I)/{}^a{\approx}$ may be thought of as a simplified version of $\mathcal{B}(X, Y, I)$. The choice of a controls the coarseness of the factorization. The point of [1] is that, in fact, one may form the factor lattice even for general $\otimes$ (not necessarily $\wedge$) in which case ${}^a{\approx}$ is a tolerance relation (i.e., reflexive and symmetric) which is compatible with the lattice structure of $\mathcal{B}(X, Y, I)$. Since (complete) lattices can be factorized even by compatible tolerance relations, a method for general $\otimes$ is available, see [1].

By definition, $\mathrm{Con}_{\mathbf{L}}(\mathbf{M}) \subseteq \mathrm{Eq}_{\mathbf{L}}(M)$. The following theorem shows that $\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ inherits even the complete lattice structure of $\mathrm{Eq}_{\mathbf{L}}(M)$.

Theorem 3.1. *$\langle \mathrm{Con}_{\mathbf{L}}(\mathbf{M}), \subseteq\rangle$ is a complete sublattice of $\langle \mathrm{Eq}_{\mathbf{L}}(M), \subseteq\rangle$.*

Proof. Take an index set I and a family $\{\theta_i \mid \theta_i \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M}),\ i \in I\}$ of congruences. Obviously, $\bigwedge_{i\in I}\theta_i \in \mathrm{Eq}_{\mathbf{L}}(M)$ and $(a \approx^{\mathbf{M}} b) \leqslant \left(\bigwedge_{i\in I}\theta_i\right)(a, b)$. For every $f^{\mathbf{M}} \in F^{\mathbf{M}}$ and arbitrary $a_1, b_1, \ldots, a_n, b_n \in M$, we have,

$$\bigotimes_{j=1}^{n}\left(\bigwedge_{i\in I}\theta_i\left(a_j, b_j\right)\right) \leqslant \bigwedge_{i\in I}\left(\bigotimes_{j=1}^{n}\theta_i\left(a_j, b_j\right)\right) \leqslant \bigwedge_{i\in I}\theta_i\left(f^{\mathbf{M}}(a_1, \ldots, a_n), f^{\mathbf{M}}(b_1, \ldots, b_n)\right),$$

thus every $f^{\mathbf{M}} \in F^{\mathbf{M}}$ is compatible with $\bigwedge_{i\in I}\theta_i$. Hence, $\langle \mathrm{Con}_{\mathbf{L}}(\mathbf{M}), \subseteq\rangle$ is closed under arbitrary infima showing that infima in $\langle \mathrm{Con}_{\mathbf{L}}(\mathbf{M}), \subseteq\rangle$ coincide with infima in $\langle \mathrm{Eq}_{\mathbf{L}}(M), \subseteq\rangle$.

To complete the proof we have to show that $\langle \mathrm{Con}_{\mathbf{L}}(\mathbf{M}), \subseteq\rangle$ and $\langle \mathrm{Eq}_{\mathbf{L}}(M), \subseteq\rangle$ agree on their suprema. First note that the supremum of θ_i $(i \in I)$ in $\mathrm{Eq}_{\mathbf{L}}(M)$ equals $\bigvee_{i_1,\ldots,i_k\in I}\theta_{i_1}\circ\cdots\circ\theta_{i_k}$. Indeed, denote $\bigvee_{i_1,\ldots,i_k\in I}\theta_{i_1}\circ\cdots\circ\theta_{i_k}$ by θ. One can see that each θ_i, $i \in I$ is contained in θ (i.e. $\theta_i \subseteq \theta$), θ is contained in any $\mathbf{L}$-equivalence containing all θ_i's, and that θ is reflexive (due to reflexivity of θ_i) and symmetric (due to symmetry of θ_i and commutativity of $\otimes$). Moreover,

$$\begin{aligned}\theta(a,b)\otimes\theta(b,c) &= \bigvee_{\substack{i_1,\ldots,i_k\in I\\ j_1,\ldots,j_l\in I}} \left(\theta_{i_1}\circ\cdots\circ\theta_{i_k}\right)(a,b)\otimes\left(\theta_{j_1}\circ\cdots\circ\theta_{j_l}\right)(b,c)\\ &\leqslant \bigvee_{\substack{i_1,\ldots,i_k\in I\\ j_1,\ldots,j_l\in I}} \left(\left(\theta_{i_1}\circ\cdots\circ\theta_{i_k}\right)\circ\left(\theta_{j_1}\circ\cdots\circ\theta_{j_l}\right)\right)(a,c)\\ &\leqslant \bigvee_{i_1,\ldots,i_m\in I} \left(\theta_{i_1}\circ\cdots\circ\theta_{i_m}\right)(a,c)=\theta(a,c),\end{aligned}$$

thus θ is also transitive showing that θ is the supremum of θ_i's in $\mathrm{Eq}_{\mathbf{L}}(M)$. Now it suffices to show that $\theta\in\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. Condition (ii) of Definition 3.3 is satisfied trivially. For (iii) take any n-ary function $f^{\mathbf{M}}\in F^{\mathbf{M}}$ and arbitrary $a_1,b_1,\ldots,a_n,b_n\in M$. We have

$$\begin{aligned}&\theta(a_1,b_1)\otimes\cdots\otimes\theta(a_n,b_n)\\ &\quad=\bigotimes_{j=1}^{n}\left(\bigvee_{i_{j1},\ldots,i_{jk_j}\in I}\left(\theta_{i_{j1}}\circ\cdots\circ\theta_{i_{jk_j}}\right)(a_j,b_j)\right)\\ &\quad=\bigvee_{\substack{i_{11},\ldots,i_{1k_1}\in I\\ \vdots\quad\vdots\\ i_{n1},\ldots,i_{nk_n}\in I}}\ \bigvee_{\substack{c_{11},\ldots,c_{1(k_1-1)}\in M\\ \vdots\quad\vdots\\ c_{n1},\ldots,c_{n(k_n-1)}\in M}}\ \bigotimes_{j=1}^{n}\left(\theta_{i_{j1}}\left(a_j,c_{j1}\right)\otimes\theta_{i_{j2}}\left(c_{j1},c_{j2}\right)\otimes\cdots\otimes\theta_{i_{jk_j}}\left(c_{j(k_j-1)},b_j\right)\right).\end{aligned}$$

Now for any of $\theta_{i_{j1}}(a_j,c_{j1}),\theta_{i_{j2}}(c_{j1},c_{j2}),\ldots,\theta_{i_{jk_j}}(c_{j(k_j-1)},b_j)$, $j\in J$ we have

$$\begin{aligned}\theta_{i_{11}}(a_1,c_{11})&\leqslant\theta_{i_{11}}\left(f^{\mathbf{M}}(a_1,\ldots,a_n),f^{\mathbf{M}}(c_{11},a_2,\ldots,a_n)\right),\\ &\ \ \vdots\\ \theta_{i_{n1}}(a_n,c_{n1})&\leqslant\theta_{i_{n1}}\left(f^{\mathbf{M}}(c_{11},c_{21},\ldots,c_{(n-1)1},a_n),f^{\mathbf{M}}(c_{11},\ldots,c_{n1})\right),\\ \theta_{i_{12}}(c_{11},c_{12})&\leqslant\theta_{i_{12}}\left(f^{\mathbf{M}}(c_{11},\ldots,c_{n1}),f^{\mathbf{M}}(c_{12},c_{21},\ldots,c_{n1})\right),\\ &\ \ \vdots\\ \theta_{i_{nk_n}}(c_{n(k_n-1)},b_n)&\leqslant\theta_{i_{nk_n}}\left(f^{\mathbf{M}}(b_1,\ldots,b_{(n-1)},c_{n(k_n-1)}),f^{\mathbf{M}}(b_1,\ldots,b_n)\right).\end{aligned}$$

Hence, it is possible to finish the proof with

$$\begin{aligned}&\bigvee_{\substack{i_{11},\ldots,i_{1k_1}\in I\\ \vdots\quad\vdots\\ i_{n1},\ldots,i_{nk_n}\in I}}\ \bigvee_{\substack{c_{11},\ldots,c_{1(k_1-1)}\in M\\ \vdots\quad\vdots\\ c_{n1},\ldots,c_{n(k_n-1)}\in M}}\ \bigotimes_{j=1}^{n}\left(\theta_{i_{j1}}(a_j,c_{j1})\otimes\theta_{i_{j2}}(c_{j1},c_{j2})\otimes\cdots\otimes\theta_{i_{jk_j}}(c_{j(k_j-1)},b_j)\right)\\ &\quad\leqslant\bigvee_{l_1,\ldots,l_m\in I}\left(\theta_{l_1}\circ\cdots\circ\theta_{l_m}\right)\left(f^{\mathbf{M}}(a_1,\ldots,a_n),f^{\mathbf{M}}(b_1,\ldots,b_n)\right)\\ &\quad=\theta\left(f^{\mathbf{M}}(a_1,\ldots,a_n),f^{\mathbf{M}}(b_1,\ldots,b_n)\right).\end{aligned}$$

Altogether, $\theta\in\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. □

Lemma 3.1. *For congruences* $\theta_1, \theta_2 \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ *the following conditions are equivalent*:

(i) $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$,
(ii) $\theta_1 \vee \theta_2 = \theta_1 \circ \theta_2$,
(iii) $\theta_2 \circ \theta_1 \subseteq \theta_1 \circ \theta_2$.

Proof. Analogously as in the ordinary case. □

An important notion is that of a congruence generated by a finite collection of pairs. The natural meaning of this notion is the following: we are given a collection of pairs $\langle a_i, b_i\rangle$ of elements of M and are looking for the least congruence containing all $\langle a_i, b_i\rangle$'s. In other words, we are looking for the least abstract view on $\mathbf{M}$ under which all a_i and b_i are indistinguishable. In fuzzy setting, it is natural to provide additionally the degrees of required indistinguishability of a_i and b_i. In what follows, we present tractable descriptions of congruences generated by prescribed pairs of elements and their required indistinguishability degrees.

Definition 3.4. Let $\mathbf{M}$ be an $\mathbf{L}$-algebra. For a binary $\mathbf{L}$-relation R in M we denote by $\theta(R)$ the least congruence on $\mathbf{M}$ containing R. Particularly, for any $b_1, c_1, \ldots, b_k, c_k \in M$ and arbitrary truth values $a_1, \ldots, a_k \in L$ we denote $\theta(\{a_1/\langle b_1, c_1\rangle, \ldots, a_k/\langle b_k, c_k\rangle\})$ by $\theta(a_1/\langle b_1, c_1\rangle, \ldots, a_k/\langle b_k, c_k\rangle)$. It is the least congruence $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ such that $\theta(b_1, c_1) \geqslant a_1, \ldots, \theta(b_k, c_k) \geqslant a_k$, i.e. $\theta(a_1/\langle b_1, c_1\rangle, \ldots, a_k/\langle b_k, c_k\rangle)$ is generated by the finite $\mathbf{L}$-relation $R = \{a_1/\langle b_1, c_1\rangle, \ldots, a_k/\langle b_k, c_k\rangle\}$. A congruence $\theta(a/\langle b, c\rangle) \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ is called a *principal congruence on* $\mathbf{M}$.

Remark 3.6. Note that due to Theorem 3.1, $\theta(R)$ exists for every $\mathbf{L}$-relation $R \in L^{M\times M}$. Namely, $\theta(R) = \bigcap\{\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M}) \mid R \subseteq \theta\}$.

Lemma 3.2. *The following are properties of principal congruences*:

(i) $\theta(a/\langle b, c\rangle) = \theta(a/\langle c, b\rangle)$,
(ii) $\theta(a_1/\langle b_1, c_1\rangle, \ldots, a_k/\langle b_k, c_k\rangle) = \theta(a_1/\langle b_1, c_1\rangle) \vee \cdots \vee \theta(a_k/\langle b_k, c_k\rangle)$,
(iii) $\theta = \bigvee_{b,c\in M} \theta(\theta(b, c)/\langle b, c\rangle) = \bigcup_{b,c\in M} \theta(\theta(b, c)/\langle b, c\rangle)$.

Proof. (i) follows directly from the definition.

(ii) It is easily seen that $\theta(a_1/\langle b_1, c_1\rangle, \ldots, a_k/\langle b_k, c_k\rangle)$ is always contained in $\theta(a_1/\langle b_1, c_1\rangle) \vee \cdots \vee \theta(a_k/\langle b_k, c_k\rangle)$. For the converse inequality observe that $\theta(a_i/\langle b_i, c_i\rangle) \subseteq \theta(a_1/\langle b_1, c_1\rangle, \ldots, a_k/\langle b_k, c_k\rangle)$ for every $i = 1, \ldots, k$. Thus, property (ii) holds.

(iii) $\bigcup_{b,c\in M} \theta(\theta(b, c)/\langle b, c\rangle) \subseteq \bigvee_{b,c\in M} \theta(\theta(b, c)/\langle b, c\rangle) \subseteq \theta \subseteq \bigcup_{b,c\in M} \theta(\theta(b, c)/\langle b, c\rangle)$, proving the claim. □

Terms (of a given type F in variables X) and *term functions* (on an $\mathbf{L}$-algebra $\mathbf{M}$) are defined as in the ordinary case [11] and are denoted by $t, s, \ldots,$ and $t^{\mathbf{M}}, s^{\mathbf{M}}, \ldots,$ respectively. The set of all terms of type F in variables X is denoted by $T(X)$. For a variable $x \in X$ and $t \in T(X)$ let $|t|_x$ denote the *number of occurrences* of x in t. For $t \in T(X)$ we often write $t(x_1, \ldots, x_k)$ instead of t to indicate that the variables occurring in t are among $x_1, \ldots, x_k \in X$. The following assertion is a consequence of [5, Theorem 3.87].

Theorem 3.2. *Let $t(x_1, \ldots, x_n)$ be a term of type F over X and let $\mathbf{M}$ be an $\mathbf{L}$-algebra of type F. Then*

$$\theta(a_1, b_1)^{|t|_{x_1}} \otimes \cdots \otimes \theta(a_n, b_n)^{|t|_{x_n}} \leqslant \theta\left(t^{\mathbf{M}}(a_1, \ldots, a_n), t^{\mathbf{M}}(b_1, \ldots, b_n)\right)$$

for each $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ and $a_1, b_1, \ldots, a_n, b_n \in M$.

Remark 3.7. (1) Estimation (i) of Theorem 3.2 is degenerate and has no interesting meaning in the ordinary case. In fuzzy case, however, (i) provides a nontrivial (possibly numerical, when truth values are numbers) estimation of similarity (degree of equivalence) of results of term functions which can be interpreted as describing sensitivity of a term function.

(2) Since $\approx^{\mathbf{M}}$ is a congruence relation, applying (i) of Theorem 3.2 to $\approx^{\mathbf{M}}$ gives

$$\left(a_1 \approx^{\mathbf{M}} b_1\right)^{|t|_{x_1}} \otimes \cdots \otimes \left(a_n \approx^{\mathbf{M}} b_n\right)^{|t|_{x_n}} \leqslant t^{\mathbf{M}}(a_1, \ldots, a_n) \approx^{\mathbf{M}} t^{\mathbf{M}}(b_1, \ldots, b_n). \tag{7}$$

(3) If $\otimes$ is $\wedge$, the powers $|t|_{x_i}$ can be removed from $\theta(a_i, b_i)^{|t|_{x_i}}$ and $(a_i \approx^{\mathbf{M}} b_i)^{|t|_{x_i}}$. This has the following consequence. In general, the estimation describing sensitivity of a term function depends on the structure of the corresponding term, hence the number of occurrences of variables is important. If $\otimes = \wedge$, the structure of the term does not play any role.

The following theorem generalizes the well-known Mal'cev lemma describing principal congruences. Note that although on verbal level (and for the truth degree $a = 1$) the theorem reads the same way as in the ordinary case, the proof is technically much more involved in fuzzy setting.

Theorem 3.3. *Suppose $\mathbf{M}$ is an $\mathbf{L}$-algebra of type F and let $c, d \in M$. For every set of variables $X = \{x, y_1, \ldots, y_k\}$, terms $p_1, \ldots, p_n \in T(X)$ and arbitrary elements $s_1, t_1, \ldots, s_n, t_n \in M$ such that $\{s_j, t_j\} = \{c, d\}$, $1 \leqslant j \leqslant n$ and $b, b', e_1, \ldots, e_k \in M$ let $\Pi(b, b', p_1, \ldots, p_n, \overline{s}, \overline{t}, \overline{e})$ denote*

$$b \approx^{\mathbf{M}} p_1^{\mathbf{M}}(s_1, \overline{e}) \otimes \bigotimes_{i=1}^{n-1} \left(p_i^{\mathbf{M}}(t_i, \overline{e}) \approx^{\mathbf{M}} p_{i+1}^{\mathbf{M}}(s_{i+1}, \overline{e})\right) \otimes p_n^{\mathbf{M}}(t_n, \overline{e}) \approx^{\mathbf{M}} b',$$

where $\overline{e}$ is an abbreviation for $e_1, \ldots, e_k$, $\overline{s}$ is an abbreviation for $s_1, \ldots, s_n$, and $\overline{t}$ is an abbreviation for $t_1, \ldots, t_n$. Then for $\theta(a/\langle c, d\rangle) \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ and every $b, b' \in M$ we have

$$\theta(a/\langle c, d\rangle)(b, b') = \bigvee_{\substack{X=\{x,y_1,\ldots,y_k\};\ e_1,\ldots,e_k \in M \\ p_1,\ldots,p_n \in T(X);\ \{s_j,t_j\}=\{c,d\},\ 1 \leqslant j \leqslant n}} \left(a^{\sum_{i=1}^{n} |p_i|_x} \otimes \Pi(b, b', p_1, \ldots, p_n, \overline{s}, \overline{t}, \overline{e})\right). \tag{8}$$

Proof. First, let θ^* be an $\mathbf{L}$-relation on M defined by the right side of (8). We have to check that $\theta(a/\langle c, d\rangle) = \theta^*$. In what follows we use the fact that if for every a_i there is b_j with $a_i \leqslant b_j$ and for every b_i there is a_j such that $b_i \leqslant a_j$ then $\bigvee_{i \in I} a_i = \bigvee_{j \in J} b_j$.

"$\theta(a/\langle c, d\rangle) \subseteq \theta^*$": It is sufficient to prove that $\theta^*(c, d) \geqslant a$ and $\theta^* \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. The first condition is obvious. Indeed, take a term $p(x) = x$, elements $s_1 = c$, $t_1 = d$, and observe that

$$\theta^*(c, d) \geqslant a^{|p|_x} \otimes \left(c \approx^{\mathbf{M}} p^{\mathbf{M}}(c)\right) \otimes \left(p^{\mathbf{M}}(d) \approx^{\mathbf{M}} d\right) = a^1 \otimes 1 \otimes 1 = a.$$

Thus, $\theta^*(c,d) \geqslant a$. Moreover, take $b, b' \in M$ and a binary term $p(x,y) = y$. It follows that

$$\theta^*(b,b') \geqslant a^{|p|_x} \otimes \left(b \approx^{\mathbf{M}} p^{\mathbf{M}}(c,b')\right) \otimes \left(p^{\mathbf{M}}(d,b') \approx^{\mathbf{M}} b'\right) = a^0 \otimes \left(b \approx^{\mathbf{M}} b'\right) \otimes 1 = b \approx^{\mathbf{M}} b',$$

i.e. $b \approx^{\mathbf{M}} b' \leqslant \theta^*(b,b')$. Clearly, we can put $b = b'$, thus the foregoing inequality yields $1 = b \approx^{\mathbf{M}} b \leqslant \theta^*(b,b)$. Hence, θ^* is reflexive.

Furthermore, for every set of variables $X = \{x, y_1, \ldots, y_k\}$, arbitrary terms $p_1, \ldots, p_n \in T(X)$ and elements $s_1, t_1, \ldots, s_n, t_n \in M$ such that $\left\{s_j, t_j\right\} = \{c,d\}$, $1 \leqslant j \leqslant n$ and $b, b', e_1, \ldots, e_k \in M$ it is easy to see that

$$\begin{aligned} b &\approx^{\mathbf{M}} p_1^{\mathbf{M}}(s_1, \overline{e}) \otimes \bigotimes_{i=1}^{n-1} \left(p_i^{\mathbf{M}}(t_i, \overline{e}) \approx^{\mathbf{M}} p_{i+1}^{\mathbf{M}}(s_{i+1}, \overline{e})\right) \otimes p_n^{\mathbf{M}}(t_n, \overline{e}) \approx^{\mathbf{M}} b' \\ &= b' \approx^{\mathbf{M}} p_n^{\mathbf{M}}(t_n, \overline{e}) \otimes \bigotimes_{i=0}^{n-2} \left(p_{n-i}^{\mathbf{M}}(s_{n-i}, \overline{e}) \approx^{\mathbf{M}} p_{n-i-1}^{\mathbf{M}}(t_{n-i-1}, \overline{e})\right) \otimes p_1^{\mathbf{M}}(s_1, \overline{e}) \approx^{\mathbf{M}} b. \end{aligned}$$

Hence, the following equality holds,

$$a^{\sum_{i=1}^n |p_i|_x} \otimes \Pi(b, b', p_1, \ldots, p_n, \overline{s}, \overline{t}, \overline{e}) = a^{\sum_{i=1}^n |p_i|_x} \otimes \Pi(b', b, p_n, \ldots, p_1, \underline{t}, \underline{s}, \overline{e}),$$

where $\underline{s}$ is an abbreviation for $s_n \ldots, s_1$ and $\underline{t}$ is an abbreviation for $t_n, \ldots, t_1$. This equality ensures that $\theta^*(b,b') = \theta^*(b',b)$, i.e. θ^* is symmetric.

Consider $b, b', b'' \in M$, two sets of variables $X_1 = \left\{x_1, y_{11}, \ldots, y_{1k_1}\right\}$, $X_2 = \left\{x_2, y_{21}, \ldots, y_{2k_2}\right\}$, terms $p_{11}, \ldots, p_{1n_1} \in T(X_1)$, $p_{21}, \ldots, p_{2n_2} \in T(X_2)$ and elements $s_{11}, t_{11}, \ldots, s_{1n_1}, t_{1n_1} \in M$, $s_{21}, t_{21}, \ldots, s_{2n_2}, t_{2n_2} \in M$ abbreviated by $\overline{s_1}, \overline{t_1}, \overline{s_2}, \overline{t_2}$ such that $\left\{s_{1j}, t_{1j}\right\} = \{c,d\}$, $\{s_{2l}, t_{2l}\} = \{c,d\}$, $1 \leqslant j \leqslant n_1$, $1 \leqslant l \leqslant n_2$. Moreover, let us have $e_{11}, \ldots, e_{1k_1} \in M$, $e_{21}, \ldots, e_{2k_2} \in M$ abbreviated by $\overline{e_1}$ and $\overline{e_2}$. Without loss of generality, we can assume that $x_1 = x_2$ and $(X_1 - \{x_1\}) \cap (X_2 - \{x_2\}) = \emptyset$ (if this is not so, one can rename variables in X_1 and X_2 accordingly). In other words, all terms $p_{11}, \ldots, p_{1n_1}, p_{21}, \ldots, p_{2n_2}$ can be thought of as terms in variables $X_1 \cup X_2$. Using transitivity of $\approx^{\mathbf{M}}$ we have

$$\begin{aligned} &\left(b \approx^{\mathbf{M}} p_{11}^{\mathbf{M}}(s_{11}, \overline{e_1}) \otimes \bigotimes_{i=1}^{n_1-1} \left(p_{1i}^{\mathbf{M}}(t_{1i}, \overline{e_1}) \approx^{\mathbf{M}} p_{1(i+1)}^{\mathbf{M}}(s_{1(i+1)}, \overline{e_1})\right) \otimes p_{1n_1}^{\mathbf{M}}(t_{1n_1}, \overline{e_1}) \approx^{\mathbf{M}} b'\right) \\ &\otimes \left(b' \approx^{\mathbf{M}} p_{21}^{\mathbf{M}}(s_{21}, \overline{e_2}) \otimes \bigotimes_{i=1}^{n_2-1} \left(p_{2i}^{\mathbf{M}}(t_{2i}, \overline{e_2}) \approx^{\mathbf{M}} p_{2(i+1)}^{\mathbf{M}}(s_{2(i+1)}, \overline{e_2})\right) \otimes p_{2n_2}^{\mathbf{M}}(t_{2n_2}, \overline{e_2}) \approx^{\mathbf{M}} b''\right) \\ &\quad \leqslant b \approx^{\mathbf{M}} p_{11}^{\mathbf{M}}(s_{11}, \overline{e_1}) \otimes \bigotimes_{i=1}^{n_1-1} \left(p_{1i}^{\mathbf{M}}(t_{1i}, \overline{e_1}) \approx^{\mathbf{M}} p_{1(i+1)}^{\mathbf{M}}(s_{1(i+1)}, \overline{e_1})\right) \\ &\quad \otimes p_{1n_1}^{\mathbf{M}}(t_{1n_1}, \overline{e_1}) \approx^{\mathbf{M}} p_{21}^{\mathbf{M}}(s_{21}, \overline{e_2}) \otimes \bigotimes_{i=1}^{n_2-1} \left(p_{2i}^{\mathbf{M}}(t_{2i}, \overline{e_2}) \approx^{\mathbf{M}} p_{2(i+1)}^{\mathbf{M}}(s_{2(i+1)}, \overline{e_2})\right) \\ &\quad \otimes p_{2n_2}^{\mathbf{M}}(t_{2n_2}, \overline{e_2}) \approx^{\mathbf{M}} b''. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} &a^{\sum_{i=1}^{n_1} |p_{1i}|_{x_1}} \otimes \Pi(b, b', p_{11}, \ldots, p_{1n_1}, \overline{s_1}, \overline{t_1}, \overline{e_1}) \otimes a^{\sum_{i=1}^{n_2} |p_{2i}|_{x_2}} \\ &\quad \otimes \Pi(b', b'', p_{21}, \ldots, p_{2n_2}, \overline{s_2}, \overline{t_2}, \overline{e_2}) \end{aligned}$$

$$\leqslant a^{\sum_{i=1}^{n_1}|p_{1i}|_{x_1}} \otimes a^{\sum_{i=1}^{n_2}|p_{2i}|_{x_2}} \otimes \Pi(b, b'', p_{11}, \dots, p_{1n_1}, p_{21}, \dots, p_{2n_2}, \overline{s}, \overline{t}, \overline{e})$$
$$= a^{\left(\sum_{i=1}^{n_1}|p_{1i}|_{x_1} + \sum_{i=1}^{n_2}|p_{2i}|_{x_2}\right)} \otimes \Pi(b, b'', p_{11}, \dots, p_{1n_1}, p_{21}, \dots, p_{2n_2}, \overline{s}, \overline{t}, \overline{e}),$$

where $\overline{e}$ is an abbreviation for elements $e_{11}, \dots, e_{1k_1}, e_{21}, \dots, e_{2k_2}$, $\overline{s}$ denotes $s_{11}, \dots, s_{1n_1}, s_{21}, \dots, s_{2n_2}$, analogously for $\overline{t}$. Taking into account the definition of θ^*, we can conclude that $\theta^*(b, b') \otimes \theta^*(b', b'') \leqslant \theta^*(b, b'')$, i.e. θ^* is transitive.

Now we will check compatibility with operations of $\mathbf{M}$. Take any m-ary $f^{\mathbf{M}} \in F^{\mathbf{M}}$ and elements $a_1, b_1, \dots, a_m, b_m \in M$. For every $1 \leqslant i \leqslant m$ let us have a set $X_i = \{x_i, y_{i1}, \dots, y_{ik_i}\}$ of variables, terms $p_{i1}, \dots, p_{in_i} \in T(X_i)$, arbitrary elements $s_{i1}, t_{i1}, \dots, s_{in_i}, t_{in_i} \in M$ such that $\{s_{ij}, t_{ij}\} = \{c, d\}$ $(1 \leqslant j \leqslant n_i)$ denoted by $\overline{s_i}, \overline{t_i}$ and elements $e_{i1}, \dots, e_{ik_i} \in M$ denoted by $\overline{e_i}$. Again, we can assume that X_i, X_j are pairwise disjoint for every $1 \leqslant i, j \leqslant m$, $i \neq j$, i.e. for every $1 \leqslant i \leqslant m$, $1 \leqslant l \leqslant n_i$ we have $p_{il} \in T(X')$, where $X' = X_1 \cup \dots \cup X_m$.

Put $n = \max(n_1, \dots, n_m)$. We are going to show that we can enlarge every sequence $p_{i1}, \dots, p_{in_i}$ of terms by defining new terms and suitable elements to obtain a sequence $p_{i1}, \dots, p_{in_i}, p_{i(n_i+1)}, \dots, p_{in}$ which satisfies certain conditions. After that, all sequences have the same length, so it will be possible to apply the compatibility of $f^{\mathbf{M}}$ with $\approx^{\mathbf{M}}$ simultaneously. First of all, put $X = X' \cup \{z_1, \dots, z_m\}$, where $z_1, \dots, z_m \notin X'$. All terms p_{ij} can be thought as terms of the form $p_{ij}(x_1, y_{11}, \dots, y_{1k_1}, x_2, y_{21}, \dots, y_{mk_m}, z_1, \dots, z_m)$. For every sequence $p_{i1}, \dots, p_{in_i}$ such that $n_i < n$, let $p_{ij} = z_i$ for all $n_i < j \leqslant n$. We can conclude that

$$a^{\sum_{j=1}^{n_i}|p_{ij}|_{x_i}} \otimes \Pi(a_i, b_i, p_{i1}, \dots, p_{in_i}, \overline{s_i}, \overline{t_i}, \overline{e_i})$$
$$= a^{\sum_{j=1}^{n}|p_{ij}|_{x_i}} \otimes \Pi(a_i, b_i, p_{i1}, \dots, p_{in}, \overline{s_i'}, \overline{t_i'}, \overline{e}),$$

where $\overline{e}$ denotes $e_{11}, \dots, e_{mk_m}, b_1, \dots, b_m$, further, $\overline{s_i'}$ denotes $s_{i1}, \dots, s_{in}$, and $\overline{t_i'}$ denotes elements $t_{i1}, \dots, t_{in}$ such that $\{s_{ij}, t_{ij}\} = \{c, d\}$, $n_i < j \leqslant n$. Indeed, since the new terms $p_{ij} = z_i$ do not have any occurrence of x_i, we have $a^{\sum_{j=1}^{n_i}|p_{ij}|_{x_i}} = a^{\sum_{j=1}^{n}|p_{ij}|_{x_i}}$ and the values of $p_{ij}^{\mathbf{M}}$'s do not depend on newly defined elements s_{ij}, t_{ij} for $j > n_i$. Thus,

$$p_{in_i}^{\mathbf{M}}(t_{in_i}, \overline{e_i}) \approx^{\mathbf{M}} b_i = \left(p_{in_i}^{\mathbf{M}}(t_{in_i}, \overline{e_i}) \approx^{\mathbf{M}} b_i\right) \otimes \underbrace{1 \otimes 1 \otimes \cdots \otimes 1}_{(n-n_i)\text{-times}}$$
$$= \left(p_{in_i}^{\mathbf{M}}(t_{in_i}, \overline{e}) \approx^{\mathbf{M}} p_{i(n_i+1)}^{\mathbf{M}}(s_{i(n_i+1)}, \overline{e})\right)$$
$$\otimes \left(p_{i(n_i+1)}^{\mathbf{M}}(t_{i(n_i+1)}, \overline{e}) \approx^{\mathbf{M}} p_{i(n_i+2)}^{\mathbf{M}}(s_{i(n_i+2)}, \overline{e})\right) \otimes \cdots \otimes \left(p_{in}^{\mathbf{M}}(t_{in}, \overline{e}) \approx^{\mathbf{M}} b_i\right).$$

Hence, it is possible to state

$$\bigotimes_{j=1}^{m}\left(a_j \approx^{\mathbf{M}} p_{j1}^{\mathbf{M}}(s_{j1}, \overline{e}) \otimes \bigotimes_{i=1}^{n-1}\left(p_{ji}^{\mathbf{M}}(t_{ji}, \overline{e}) \approx^{\mathbf{M}} p_{j(i+1)}^{\mathbf{M}}(s_{j(i+1)}, \overline{e})\right) \otimes p_{jn}^{\mathbf{M}}(t_{jn}, \overline{e}) \approx^{\mathbf{M}} b_j\right)$$
$$\leqslant f^{\mathbf{M}}(a_1, \dots, a_m) \approx^{\mathbf{M}} f^{\mathbf{M}}\left(p_{11}^{\mathbf{M}}(s_{11}, \overline{e}), \dots, p_{m1}^{\mathbf{M}}(s_{m1}, \overline{e})\right)$$
$$\otimes \bigotimes_{i=1}^{n-1}\left(f^{\mathbf{M}}\left(p_{1i}^{\mathbf{M}}(t_{1i}, \overline{e}), \dots, p_{mi}^{\mathbf{M}}(t_{mi}, \overline{e})\right)\right.$$
$$\left.\approx^{\mathbf{M}} f^{\mathbf{M}}\left(p_{1(i+1)}^{\mathbf{M}}(s_{1(i+1)}, \overline{e}), \dots, p_{m(i+1)}^{\mathbf{M}}(s_{m(i+1)}, \overline{e})\right)\right)$$
$$\otimes f^{\mathbf{M}}\left(p_{1n}^{\mathbf{M}}(t_{1n}, \overline{e}), \dots, p_{mn}^{\mathbf{M}}(t_{mn}, \overline{e})\right) \approx^{\mathbf{M}} f^{\mathbf{M}}(b_1, \dots, b_m).$$

Moreover, the value of every $f^{\mathbf{M}}\left(p_{1i}^{\mathbf{M}}(s_{1i},\overline{e}),\ldots,p_{mi}^{\mathbf{M}}(s_{mi},\overline{e})\right)$ can be expressed as the resulting value of the term $f(p_{1i}(x_1,\overline{y}), p_{2i}(x_{2i},\overline{y}),\ldots,p_{mi}(x_{mi},\overline{y}))$, in variables $(X\cup\left\{x_{ji}\mid j\geqslant 2\right\})-\{x_2,\ldots,x_m\}$ for elements $\overline{e}$ together with elements $\left\{t_{ji},s_{ji}\mid j\geqslant 2\right\}$. In the case of $f^{\mathbf{M}}\left(p_{1i}^{\mathbf{M}}(t_{1i},\overline{e}),\ldots,p_{mi}^{\mathbf{M}}(t_{mi},\overline{e})\right)$, we can proceed analogously. Thus,

$$\begin{aligned}
&\bigotimes_{i=1}^{m} a^{\sum_{j=1}^{n_i}|p_{ij}|_{x_i}}\otimes \Pi(a_i,b_i,p_{i1},\ldots,p_{in_i},\overline{s_i},\overline{t_i},\overline{e_i})\\
&\quad = a^{\left(\sum_{i=1}^{m}\sum_{j=1}^{n}|p_{ij}|_{x_i}\right)}\otimes\bigotimes_{i=1}^{m}\Pi(a_i,b_i,p_{i1},\ldots,p_{in},\overline{s_i'},\overline{t_i'},\overline{e})\\
&\quad\leqslant a^{\sum_{j=1}^{n}|p_{1j}|_{x_1}}\otimes\bigotimes_{i=1}^{m}\Pi(a_i,b_i,p_{i1},\ldots,p_{in},\overline{s_i'},\overline{t_i'},\overline{e})\\
&\quad\leqslant a^{\sum_{j=1}^{n}|p_{1j}|_{x_1}}\otimes\Pi(f^{\mathbf{M}}(a_1,\ldots,a_m),f^{\mathbf{M}}(b_1,\ldots,b_m),\\
&\qquad f(p_{11},\ldots,p_{m1}),\ldots,f(p_{1n},\ldots,p_{mn}),\overline{s_1},\overline{t_1},\overline{e}).
\end{aligned}$$

Clearly, $f(p_{11},\ldots,p_{m1}),\ldots,f(p_{1n},\ldots,p_{mn})$ are terms and $\sum_{j=1}^{n}|p_{1j}|_{x_1}$ is the number of occurrences of the variable x_1 in these terms. This observation yields the compatibility of $f^{\mathbf{M}}$ with θ^*. Together with the foregoing results, $\theta^*\in\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ and $\theta^*(c,d)\geqslant a$. Consequently, $\theta(a/\langle c,d\rangle)\subseteq\theta^*$.

"$\theta(a/\langle c,d\rangle)\supseteq\theta^*$": Take a set $X=\{x,y_1,\ldots,y_k\}$ of variables, terms $p_1,\ldots,p_n\in T(X)$ and arbitrary elements $s_1,t_1,\ldots,s_n,t_n\in M$ abbreviated by $\overline{s},\overline{t}$ such that $\left\{s_j,t_j\right\}=\{c,d\}$, $1\leqslant j\leqslant n$ and $b,b',e_1,\ldots,e_k\in M$, where $\overline{e}$ is an abbreviation for $e_1,\ldots,e_k$. It is easy to see that

$$a^{\sum_{i=1}^{n}|p_i|_x}=a^{|p_1|_x}\otimes\cdots\otimes a^{|p_n|_x}\leqslant\theta(a/\langle c,d\rangle)(c,d)^{|p_1|_x}\otimes\cdots\otimes\theta(a/\langle c,d\rangle)(c,d)^{|p_n|_x}.$$

Theorem 3.2 yields $\theta(a/\langle c,d\rangle)(c,d)^{|p|_x}\leqslant\theta(a/\langle c,d\rangle)\left(p^{\mathbf{M}}(c,\overline{e}),p^{\mathbf{M}}(d,\overline{e})\right)$ for every term $p(x,\overline{y})$ and any $b,b',e_1,\ldots,e_k\in M$. Thus, we have

$$\begin{aligned}
&a^{\sum_{i=1}^{n}|p_i|_x}\otimes\Pi(b,b',p_1,\ldots,p_n,\overline{s},\overline{t},\overline{e})\\
&\quad = a^{\sum_{i=1}^{n}|p_i|_x}\otimes b\approx^{\mathbf{M}}p_1^{\mathbf{M}}(s_1,\overline{e})\otimes\bigotimes_{i=1}^{n-1}\left(p_i^{\mathbf{M}}(t_i,\overline{e})\approx^{\mathbf{M}}p_{i+1}^{\mathbf{M}}(s_{i+1},\overline{e})\right)\otimes p_n^{\mathbf{M}}(t_n,\overline{e})\approx^{\mathbf{M}}b'\\
&\quad\leqslant a^{\sum_{i=1}^{n}|p_i|_x}\otimes\theta(a/\langle c,d\rangle)\left(b,p_1^{\mathbf{M}}(s_1,\overline{e})\right)\otimes\bigotimes_{i=1}^{n-1}\left(\theta(a/\langle c,d\rangle)\left(p_i^{\mathbf{M}}(t_i,\overline{e}),p_{i+1}^{\mathbf{M}}(s_{i+1},\overline{e})\right)\right)\\
&\qquad\otimes\theta(a/\langle c,d\rangle)\left(p_n^{\mathbf{M}}(t_n,\overline{e}),b'\right)\\
&\quad\leqslant\theta(a/\langle c,d\rangle)\left(b,p_1^{\mathbf{M}}(s_1,\overline{e})\right)\otimes\theta(a/\langle c,d\rangle)(c,d)^{|p_1|_x}\\
&\qquad\otimes\bigotimes_{i=1}^{n-1}\left(\theta(a/\langle c,d\rangle)\left(p_i^{\mathbf{M}}(t_i,\overline{e}),p_{i+1}^{\mathbf{M}}(s_{i+1},\overline{e})\right)\right.\\
&\qquad\left.\otimes\theta(a/\langle c,d\rangle)(c,d)^{|p_{i+1}|_x}\right)\otimes\theta(a/\langle c,d\rangle)\left(p_n^{\mathbf{M}}(t_n,\overline{e}),b'\right)\\
&\quad\leqslant\theta(a/\langle c,d\rangle)\left(b,p_1^{\mathbf{M}}(s_1,\overline{e})\right)\otimes\theta(a/\langle c,d\rangle)\left(p_1^{\mathbf{M}}(s_1,\overline{e}),p_1^{\mathbf{M}}(t_1,\overline{e})\right)
\end{aligned}$$

$$\otimes \bigotimes_{i=1}^{n-1} \left(\theta(a/\langle c,d\rangle)\left(p_i^{\mathbf{M}}(t_i,\overline{e}),\, p_{i+1}^{\mathbf{M}}(s_{i+1},\overline{e})\right)\right.$$
$$\left.\otimes\theta(a/\langle c,d\rangle)\left(p_{i+1}^{\mathbf{M}}(s_{i+1},\overline{e}),\, p_{i+1}^{\mathbf{M}}(t_{i+1},\overline{e})\right)\right) \otimes \theta(a/\langle c,d\rangle)\left(p_n^{\mathbf{M}}(t_n,\overline{e}), b'\right).$$

Hence, we can apply the transitivity of $\theta(a/\langle c,d\rangle)$ repeatedly to obtain

$$a^{\sum_{i=1}^{n}|p_i|_x} \otimes \Pi(b, b', p_1, \ldots, p_n, \overline{s}, \overline{t}, \overline{e}) \leqslant \theta(a/\langle c,d\rangle)(b,b').$$

Since $X = \{x, y_1, \ldots, y_k\}$, $p_1, \ldots, p_n \in T(X)$ and $b, b', \overline{s}, s_1, t_1, \ldots, s_n, t_n \in M$, $e_1, \ldots, e_k \in M$ have been chosen arbitrarily, we have proven the desired inequality.

Altogether, $\theta^* = \theta(a/\langle c,d\rangle)$. □

A kernel of an **L**-equivalence θ (i.e. the 1-cut ${}^1\theta$) is always a bivalent equivalence relation. This property enables us to define a factor **L**-algebra $\mathbf{M}/\theta$ of an **L**-algebra **M** modulo a congruence $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$.

Definition 3.5. Let θ be a congruence on an **L**-algebra **M** of type F. An **L**-algebra $\mathbf{M}/\theta = \langle M/{}^1\theta, \approx^{\mathbf{M}/\theta}, F^{\mathbf{M}/\theta}\rangle$ of type F is called a *factor* **L**-*algebra of* **M** *modulo* θ, if

(i) $f^{\mathbf{M}/\theta}([a_1]_\theta, \ldots, [a_n]_\theta) = \left[f^{\mathbf{M}}(a_1, \ldots, a_n)\right]_\theta$ for any n-ary $f^{\mathbf{M}/\theta} \in F^{\mathbf{M}/\theta}$ and $a_1, \ldots, a_n \in M$,
(ii) $\left([a]_\theta \approx^{\mathbf{M}/\theta} [b]_\theta\right) = \theta(a,b)$ for every $a, b \in M$,

where $[a]_\theta = \left\{a' \mid \theta(a,a') = 1\right\}$ for all $a \in M$. For brevity, the support $M/{}^1\theta$ shall be denoted by M/θ.

Remark 3.8. A factor **L**-algebra is well defined. For $c \in [a]_\theta$, $d \in [b]_\theta$ we have $\theta(a,c) = 1$, $\theta(b,d) = 1$. Using transitivity and symmetry of θ we obtain

$$\theta(a,b) = \theta(a,b) \otimes \theta(a,c) \leqslant \theta(b,c) = \theta(b,c) \otimes \theta(b,d) \leqslant \theta(c,d).$$

Similarly, $\theta(c,d) \leqslant \theta(a,b)$, thus $\approx^{\mathbf{M}/\theta}$ is well-defined **L**-relation. Moreover $\approx^{\mathbf{M}/\theta}$ is reflexive, symmetric, and transitive. For $[a]_\theta \approx^{\mathbf{M}/\theta} [b]_\theta = 1$ we have $\theta(a,b) = 1$, that is $a \in [b]_\theta$ and thus $[a]_\theta = [b]_\theta$. For each n-ary function $f^{\mathbf{M}/\theta} \in F^{\mathbf{M}/\theta}$ and $a_1, b_1, \ldots, a_n, b_n \in M$, we have

$$\begin{aligned}
&[a_1]_\theta \approx^{\mathbf{M}/\theta} [b_1]_\theta \otimes \cdots \otimes [a_n]_\theta \approx^{\mathbf{M}/\theta} [b_n]_\theta \\
&\quad = \theta(a_1,b_1) \otimes \cdots \otimes \theta(a_n,b_n) \leqslant \theta\left(f^{\mathbf{M}}(a_1,\ldots,a_n), f^{\mathbf{M}}(b_1,\ldots,b_n)\right) \\
&\quad = \left[f^{\mathbf{M}}(a_1,\ldots,a_n)\right]_\theta \approx^{\mathbf{M}/\theta} \left[f^{\mathbf{M}}(b_1,\ldots,b_n)\right]_\theta \\
&\quad = f^{\mathbf{M}/\theta}([a_1]_\theta,\ldots,[a_n]_\theta) \approx^{\mathbf{M}/\theta} f^{\mathbf{M}/\theta}([b_1]_\theta,\ldots,[b_n]_\theta).
\end{aligned}$$

Hence, every function $f^{\mathbf{M}/\theta} \in F^{\mathbf{M}/\theta}$ is well defined (it suffices to apply the above inequality for $[a_1]_\theta \approx^{\mathbf{M}/\theta} [b_1]_\theta = 1, \ldots, [a_n]_\theta \approx^{\mathbf{M}/\theta} [b_n]_\theta = 1$) and compatible with the **L**-equality $\approx^{\mathbf{M}}$.

Remark 3.9. An important role in the definition of a factor **L**-algebra is played by the 1-cut ${}^1\theta$ of a congruence θ on **M**. It is important to note that in general, a congruence θ on an **L**-algebra is not determined by ${}^1\theta$, not even if θ is an $\mathbf{L}$-equality (in fact, all $\mathbf{L}$-equalities on $\mathbf{M}$ have a common 1-cut of the form $\{\langle a, a\rangle \mid a \in M\}$). Furthermore, one can easily verify that for a congruence $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$, ${}^1\theta$ is an ordinary congruence relation on $\langle M, F^{\mathbf{M}}\rangle$. Therefore, the functional part of $\mathbf{M}/\theta$ coincides with $\langle M, F^{\mathbf{M}}\rangle / {}^1\theta$. Clearly, this issue is completely degenerate in the ordinary case since if $\mathbf{L} = \mathbf{2}$ then θ and ${}^1\theta$ coincide (modulo the relationship between crisp fuzzy relations and the corresponding ordinary relations).

Before discussing morphisms between $\mathbf{L}$-algebras, i.e. structure-preserving mappings, we consider separately mappings preserving $\mathbf{L}$-equalities. On the verbal level, such mappings are not allowed to map similar elements to dissimilar ones. For the following concepts, see also [20,21].

Definition 3.6. Let M be a set equipped with an $\mathbf{L}$-equality $\approx^M$. The couple $\langle M, \approx^M\rangle$ is called a *set with* $\mathbf{L}$*-equality*. For two sets with $\mathbf{L}$-equalities $\langle M, \approx^M\rangle$ and $\langle N, \approx^N\rangle$, a mapping $h: M \to N$ is called an $\approx$*-morphism* if

$$a_1 \approx^M a_2 \leqslant h(a_1) \approx^N h(a_2) \tag{9}$$

holds for all $a_1, a_2 \in M$. For an $\approx$*-morphism* $h: M \to N$ denote by θ_h a binary $\mathbf{L}$-relation on M for which we have $\theta_h(a_1, a_2) = h(a_1) \approx^N h(a_2)$. θ_h is called a *kernel of the* $\approx$*-morphism* h.

Remark 3.10. If $h: M \to N$ is an $\approx$-morphism between $\langle M, \approx^M\rangle$ and $\langle N, \approx^N\rangle$, we usually denote this fact explicitly by $h: \langle M, \approx^M\rangle \to \langle N, \approx^N\rangle$.

Lemma 3.3. *Let* $h: M \to N$ *be a mapping between* $\langle M, \approx^M\rangle$ *and* $\langle N, \approx^N\rangle$. *Then* h *is an* $\approx$*-morphism if and only if* $\approx^M \subseteq \theta_h$.

Proof. Follows directly from Definition 3.6. □

Lemma 3.4. *Let us have* $\approx$*-morphisms* $g: \langle M, \approx^M\rangle \to \langle M', \approx^{M'}\rangle$ *and* $h: \langle M', \approx^{M'}\rangle \to \langle M'', \approx^{M''}\rangle$. *Then*

(i) *the composed mapping* $(g \circ h): M \to M''$ *is an* $\approx$*-morphism.*
(ii) *If* g *is a bijection, then* g^{-1} *is an* $\approx$*-morphism iff* $a_1 \approx^M a_2 = g(a_1) \approx^{M'} g(a_2)$ *for all* $a_1, a_2 \in M$.

Proof. (i): The mapping $g \circ h$, i.e. $(g \circ h)(a) = h(g(a))$ for every $a \in M$, is evidently an $\approx$-morphism.
(ii): "⇒": Suppose g is a bijection and g^{-1} is an $\approx$-morphism. Then

$$a_1 \approx^M a_2 \leqslant g(a_1) \approx^{M'} g(a_2) \leqslant g^{-1}(g(a_1)) \approx^M g^{-1}(g(a_2)) = a_1 \approx^M a_2,$$

which implies the requested condition $a_1 \approx^M a_2 = g(a_1) \approx^{M'} g(a_2)$.
"⇐": If g is a bijection satisfying $a_1 \approx^M a_2 = g(a_1) \approx^{M'} g(a_2)$ for all $a_1, a_2 \in M$, then g^{-1} is evidently an $\approx$-morphism. □

Lemma 3.5. *Let* $h\colon \langle M, \approx^M\rangle \to \langle N, \approx^N\rangle$ *be an* $\approx$*-morphism. Then*

(i) θ_h *is an* **L**-*equivalence*;
(ii) θ_h *is an* **L**-*equality iff* h *is injective*.

Proof. The proof is easy (it follows e.g. from [5, Lemma 3.114] which contains a graded version of the assertion), see also [13, part I]. □

In the sequel, a morphism of **L**-algebras is defined as a mapping that preserves both their operations $f^{\mathbf{M}}$ and their fuzzy equality $\approx^{\mathbf{M}}$. All properties well known from ordinary case generalize in full scope.

Definition 3.7. Let **M** and **N** be **L**-algebras of type F. A mapping $h\colon M \to N$ is called a *morphism* (or *homomorphism*) of **M** to **N** if

(i) $h(f^{\mathbf{M}}(a_1, \ldots, a_n)) = f^{\mathbf{N}}(h(a_1), \ldots, h(a_n))$
for every n-ary $f \in F$ and arbitrary $a_1, \ldots, a_n \in M$;
(ii) $a \approx^{\mathbf{M}} b \leqslant h(a) \approx^{\mathbf{N}} h(b)$ for all $a, b \in M$.

The fact that $h\colon M \to N$ is a morphism will be usually denoted by $h\colon \mathbf{M} \to \mathbf{N}$. Moreover,

- an injective morphism is called a *monomorphism*,
- a morphism such that

$$a \approx^{\mathbf{M}} b = h(a) \approx^{\mathbf{N}} h(b) \quad \text{for all } a, b \in M \tag{10}$$

  is called an *embedding*,
- a surjective morphism is called an *epimorphism*,
- an epimorphism which is an embedding is called an *isomorphism*,
- a morphism $h\colon \mathbf{M} \to \mathbf{M}$ is called an *endomorphism*,
- an isomorphism $h\colon \mathbf{M} \to \mathbf{M}$ is called an *automorphism*.

For an epimorphism $h\colon \mathbf{M} \to \mathbf{N}$, the **L**-algebra **N** is called an *image of* **M**. We say **M** *is isomorphic to* **N**, written $\mathbf{M} \cong \mathbf{N}$, if there exists an isomorphism $h\colon \mathbf{M} \to \mathbf{N}$. Let id_M denote the *identical mapping* on M (we use $\mathrm{id}_{\mathbf{M}}$ to emphasize that M is a universe of **M**).

Remark 3.11. Note that condition $(a \approx^{\mathbf{M}} b) \leqslant \big(h(a) \approx^{\mathbf{N}} h(b)\big)$ can be verbally described as "if a and b are similar, then $h(a)$ and $h(b)$ are similar". Namely, $(a \approx^{\mathbf{M}} b) \leqslant \big(h(a) \approx^{\mathbf{N}} h(b)\big)$ is true for all $a, b \in M$ iff formula $(\forall x)(\forall y)\,(x \approx y \Rightarrow h(x) \approx h(y))$ is fully true (i.e. has truth degree 1) in an appropriate structure encompassing both **M** and **N**. Condition $h(f^{\mathbf{M}}(a_1, \ldots, a_n)) = f^{\mathbf{N}}(h(a_1), \ldots, h(a_n))$ is equivalent to saying that h is an (ordinary) morphism of the functional parts $\langle M, F^{\mathbf{M}}\rangle$ and $\langle N, F^{\mathbf{N}}\rangle$.

Theorem 3.4. *Let* $h\colon \mathbf{M} \to \mathbf{N}$ *be a morphism. Then we have*

(i) *if* $M' \in \mathrm{Sub}(\mathbf{M})$ *then* $h(M') \in \mathrm{Sub}(\mathbf{N})$,
(ii) *if* $N' \in \mathrm{Sub}(\mathbf{N})$ *then* $h^{-1}(N') = \{a \in M \mid h(a) \in N'\} \in \mathrm{Sub}(\mathbf{M})$,
(iii) *if* $h\colon \mathbf{M} \to \mathbf{N}$ *is an embedding then* $\mathbf{M} \cong h(\mathbf{M})$, *where* $h(\mathbf{M})$ *is a subalgebra of* **N** *induced by* $h(M)$.

Proof. (i): We apply Remark 3.3: As noted in Remark 3.11, h is a morphism of $\langle M, F^{\mathbf{M}}\rangle$ to $\langle N, F^{\mathbf{N}}\rangle$. Furthermore, by definition, $h(M') \in \mathrm{Sub}(\mathbf{N})$ (i.e. $h(M')$ is a subuniverse of $\mathbf{N}$) iff $h(M') \in \mathrm{Sub}(\mathrm{ske}(\mathbf{N}))$ (i.e. $h(M')$ is a subuniverse of the skeleton of $\mathbf{N}$). The assertion therefore follows from the ordinary case since it is well known that if h is a morphism of $\langle M, F^{\mathbf{M}}\rangle$ to $\langle N, F^{\mathbf{N}}\rangle$ then $h(M')$ is a subuniverse of ske($\mathbf{N}$).

(ii): Follows from the ordinary case as (i).

(iii): Suppose $h\colon \mathbf{M} \to \mathbf{N}$ is an embedding. Statement (i) yields that $h(M) \in \mathrm{Sub}(\mathbf{N})$. Thus, $g\colon \mathbf{M} \to h(\mathbf{M})$, where $g(a) = h(a)$ for every $a \in M$ is a surjective embedding, i.e. $\mathbf{M} \cong h(\mathbf{M})$. □

Theorem 3.5. *Suppose* $g : \mathbf{M} \to \mathbf{M}'$ *and* $h : \mathbf{M}' \to \mathbf{M}''$ *are morphisms. The composition* $g \circ h$ *is a morphism* $(g \circ h)\colon \mathbf{M} \to \mathbf{M}''$.

Proof. Lemma 3.4 yields that h is an $\approx$-morphism. The rest follows from the ordinary case. □

Theorem 3.6. *Let* $h\colon \mathbf{M} \to \mathbf{N}$ *be a morphism. Then* $\theta_h \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$.

Proof. From Lemmas 3.3 and 3.5 it follows that $a \approx^{\mathbf{M}} b \leqslant \theta_h(a, b)$ for all $a, b \in M$ and $\theta_h \in \mathrm{Eq}_{\mathbf{L}}(M)$. Now it is sufficient to show that θ_h is also compatible with functions $f^{\mathbf{M}} \in F^{\mathbf{M}}$. For n-ary $f^{\mathbf{M}} \in F^{\mathbf{M}}$ and arbitrary $a_1, b_1, \ldots, a_n, b_n \in M$, the following inequality holds:

$$\begin{aligned}\bigotimes_{i=1}^{n} \theta_h(a_i, b_i) = \bigotimes_{i=1}^{n} h(a_i) \approx^{\mathbf{N}} h(b_i) &\leqslant f^{\mathbf{N}}(h(a_1), \ldots, h(a_n)) \approx^{\mathbf{N}} f^{\mathbf{N}}(h(b_1), \ldots, h(b_n)) \\ &= h\left(f^{\mathbf{M}}(a_1, \ldots, a_n)\right) \approx^{\mathbf{N}} h\left(f^{\mathbf{M}}(b_1, \ldots, b_n)\right) \\ &= \theta_h\left(f^{\mathbf{M}}(a_1, \ldots, a_n), f^{\mathbf{M}}(b_1, \ldots, b_n)\right).\end{aligned}$$

Thus, $\theta_h \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. □

Definition 3.8. For every $\mathbf{L}$-algebra $\mathbf{M}$ and $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ a mapping $h_\theta\colon M \to M/\theta$, where $h_\theta(a) = [a]_\theta$ for all $a \in M$, is called a *natural mapping*.

Theorem 3.7. *A natural mapping* h_θ *from an* $\mathbf{L}$*-algebra* $\mathbf{M}$ *to a factor* $\mathbf{L}$*-algebra* $\mathbf{M}/\theta$ *is an epimorphism.*

Proof. For any $a, b \in M$ we have $a \approx^{\mathbf{M}} b \leqslant \theta(a, b) = [a]_\theta \approx^{\mathbf{M}/\theta} [b]_\theta = h_\theta(a) \approx^{\mathbf{M}/\theta} h_\theta(b)$. Furthermore, for n-ary $f^{\mathbf{M}} \in F^{\mathbf{M}}$ and arbitrary $a_1, \ldots, a_n \in M$ we have

$$\begin{aligned}h_\theta\left(f^{\mathbf{M}}(a_1, \ldots, a_n)\right) = \left[f^{\mathbf{M}}(a_1, \ldots, a_n)\right]_\theta &= f^{\mathbf{M}/\theta}([a_1]_\theta, \ldots, [a_n]_\theta) \\ &= f^{\mathbf{M}/\theta}(h_\theta(a_1), \ldots, h_\theta(a_n)).\end{aligned}$$

Surjectivity of h_θ is evident. □

Definition 3.9. Suppose $\mathbf{M}$ is an $\mathbf{L}$-algebra and $\phi, \theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$, $\theta \subseteq \phi$. Then we let ϕ/θ denote an $\mathbf{L}$-relation on M/θ defined by $(\phi/\theta)([a]_\theta, [b]_\theta) = \phi(a, b)$ for all $a, b \in M$. For $N \subseteq M$ we put $N^\theta = \{a \in M \mid [a]_\theta \cap N \neq \emptyset\}$ (that is, N^θ is a union of congruence classes being incident with N),

$\mathbf{N}^\theta = \left[N^\theta\right]_{\mathbf{M}}$ (recall that $\mathbf{N}^\theta$ is a subalgebra of $\mathbf{M}$ generated by N^θ) and let $\theta|N$ denote the restriction of θ to N.

Lemma 3.6. *Let* $\mathbf{N}$ *be a subalgebra of* $\mathbf{M}$, $\theta \subseteq \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. *Then*

(i) N^θ *is the universe of* $\mathbf{N}^\theta$,
(ii) $\theta|N \in \mathrm{Con}_{\mathbf{L}}(\mathbf{N})$.

Proof. (i): Take any n-ary $f^{\mathbf{N}} \in F^{\mathbf{N}}$ and $a_1, \ldots, a_n \in N^\theta$. By Definition 3.9 there are elements $b_1, \ldots, b_n \in N$ for which we have $a_1 \in [b_1]_\theta, \ldots, a_n \in [b_n]_\theta$, in other words $\theta(a_1, b_1) = \cdots = \theta(a_n, b_n) = 1$. Hence, it follows that $\theta\left(f^{\mathbf{M}}(a_1, \ldots, a_n), f^{\mathbf{M}}(b_1, \ldots, b_n)\right) = 1$. Since the result $f^{\mathbf{M}}(b_1, \ldots, b_n)$ is in N, we have $f^{\mathbf{M}}(a_1, \ldots, a_n) \in N^\theta$.

(ii) is easy to check. □

Theorem 3.8 (*Morphism theorems*). *Let* $\mathbf{M}, \mathbf{N}$ *be* $\mathbf{L}$*-algebras of type* F.

(i) *Let* $h : \mathbf{M} \to \mathbf{N}$ *be an epimorphism. Then there is an isomorphism* $g : \mathbf{M}/\theta_h \to \mathbf{N}$ *such that* $h_{\theta_h} \circ g = h$.
(ii) *Let* $\phi, \theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M}), \theta \subseteq \phi$. *Then the mapping* $h\colon (M/\theta)/(\phi/\theta) \to M/\phi$ *defined by* $h\left(\left[[a]_\theta\right]_{\phi/\theta}\right) = [a]_\phi$ *is an isomorphism.*
(iii) *Let* $\mathbf{N}$ *be a subalgebra of* $\mathbf{M}$, $\theta \subseteq \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. *Then* $\mathbf{N}/(\theta|N) \cong \mathbf{N}^\theta/(\theta|N^\theta)$.

Proof. The assertion is a consequence of morphism theorems for general fuzzy structures, see [5, Chapter 3]. □

Theorem 3.9. *Let an* $\mathbf{L}$*-algebra* $\mathbf{M}$ *and* $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ *be given. Then the mapping* $h : [\theta, M \times M] \to \mathrm{Con}_{\mathbf{L}}(\mathbf{M}/\theta)$ *defined by* $h(\phi) = \phi/\theta$ *is a lattice isomorphism.*

Proof. For every $\phi_1, \phi_2 \in [\theta, M \times M]$ we have $\phi_1 \subseteq \phi_2$ iff $\phi_1/\theta \subseteq \phi_2/\theta$ iff $h(\phi_1) \subseteq h(\phi_2)$. Now it is sufficient to prove that h is a bijective mapping. Suppose that $(\phi_1/\theta)([a]_\theta, [b]_\theta) = (\phi_2/\theta)([a]_\theta, [b]_\theta)$ for every $a, b \in M$. Then also $\phi_1(a, b) = \phi_2(a, b)$ for all $a, b \in M$. Thus, the mapping h is injective. To show surjectivity, first take any $\varphi \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M}/\theta)$ and define an $\mathbf{L}$-relation ϕ on $\mathbf{M}$ by $\phi(a, b) = \varphi([a]_\theta, [b]_\theta)$. We will show that ϕ is an $\mathbf{L}$-equivalence on M for which $\theta \subseteq \phi$ and $h(\phi) = \varphi$.

First, we check conditions (i), (ii) of Definition 3.3. Clearly, ϕ is an $\mathbf{L}$-equivalence relation and for every $a, b \in M$ we have

$$a \approx^{\mathbf{M}} b \leqslant [a]_\theta \approx^{\mathbf{M}/\theta} [b]_\theta \leqslant \varphi([a]_\theta, [b]_\theta) = \phi(a, b).$$

To check condition (iii), take any n-ary $f^{\mathbf{M}} \in F^{\mathbf{M}}$ and $a_1, b_1, \ldots, a_n, b_n \in M$. Now it follows that

$$\begin{aligned}\bigotimes_{i=1}^{n} \phi(a_i, b_i) = \bigotimes_{i=1}^{n} \varphi([a_i]_\theta, [b_i]_\theta) &\leqslant \varphi\left(f^{\mathbf{M}/\theta}([a_1]_\theta, \ldots, [a_n]_\theta), f^{\mathbf{M}/\theta}([b_1]_\theta, \ldots, [b_n]_\theta)\right)\\ &= \varphi\left(\left[f^{\mathbf{M}}(a_1, \ldots, a_n)\right]_\theta, \left[f^{\mathbf{M}}(b_1, \ldots, b_n)\right]_\theta\right)\\ &= \phi\left(f^{\mathbf{M}}(a_1, \ldots, a_n), f^{\mathbf{M}}(b_1, \ldots, b_n)\right).\end{aligned}$$

Thus, $\phi \in [\theta, M \times M] \subseteq \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. Clearly, $h(\phi) = \phi/\theta = \varphi$. To sum up, h is a surjective mapping. Hence, h is a lattice isomorphism. □

Let us close this section with several clarifying remarks on the approach developed in [24,28] and its relationship to algebras with fuzzy equalities. The concepts under investigation are that of fuzzy congruence relations on universal algebras, and fuzzy factor algebras. There are several distinctions with respect to our approach.

First, in [24,28], $\mathbf{M}$ is supposed to be an ordinary universal algebra. This can be considered a particular case of ours for $\approx^{\mathbf{M}}$ (fuzzy equality) being the crisp identity. As demonstrated in Section 3.1, allowing a general fuzzy equality enables us to model existing examples where the fuzzy equality naturally accompanies the universe set of the algebra in question. Moreover, taking a general fuzzy equality, we obtain the usual relationship between congruence relations (as fuzzy relations) and morphisms which is not possible under the approach of [24,28], i.e. ordinary algebras and congruences (as fuzzy relations).

Second, the set of truth degrees used in [24,28] is supposed to be the real unit interval [0, 1]. That is, it is supposed to be infinite, linearly ordered, and having numerical character. None of these features was justified by the authors. On the one hand [0, 1] is the most common structure used in applications of fuzzy sets. On the other hand, in the study of algebraic concepts, one should try to use a general framework if possible. We find our framework assuming a complete residuated lattice $\mathbf{L}$ instead of [0, 1] a natural one in that it captures the basic logical requirements concerning the structure of truth degrees and, at the same time, is sufficiently powerful to formulate nontrivial results. Restricting oneself to [0, 1], one may loose insight into some important structural concepts like that of a subdirect product. As will be demonstrated later on, the well-known representation theorem is not true in fuzzy setting if we take $L = [0, 1]$ but is true, for example, if L is a finite chain.

Third, the authors in [24,28] use min (minimum) as $\otimes$. The idempotency property of minimum has, however, nontrivial consequences, see e.g. Remark 3.5. Therefore, taking min may be regarded as an unjustified simplification. Moreover, $\otimes$ explicitly appears in the compatibility condition $(a_1 \approx^{\mathbf{M}} b_1) \otimes \cdots \otimes (a_n \approx^{\mathbf{M}} b_n) \leqslant (f^{\mathbf{M}}(a_1, \ldots, a_n) \approx^{\mathbf{M}} f^{\mathbf{M}}(b_1, \ldots, b_n))$ of a congruence relation. Taking min for $\otimes$, the left-hand side depends only on the least $(a_i \approx^{\mathbf{M}} b_i)$, which might be regarded as not very intuitive.

In [28], the author presents a factorization of an ordinary algebra $\mathbf{M} = \langle M, F^{\mathbf{M}} \rangle$ by a fuzzy congruence relation. The resulting structure is an ordinary algebra whose universe consists of congruence classes which are special $\mathbf{L}$-sets in the universe M. The concept of "fuzzy factorization" can be generalized for $\mathbf{L}$-algebras as well. However, it can be shown that such a construction is already covered by the factorization presented in Definition 3.5. The following definition presents the generalization of a fuzzy factorization for $\mathbf{L}$-algebras.

Definition 3.10. Let $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ be an $\mathbf{L}$-algebra of type F, $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. Let us assume $M_\theta = \{a\theta \mid a \in M\}$, where $a\theta \in \mathbf{L}^M$, $(a\theta)(b) = \theta(a, b)$ for all $a, b \in M$. Then $\mathbf{M}_\theta = \langle M_\theta, \approx^{\mathbf{M}}_\theta, F^{\mathbf{M}}_\theta \rangle$, where

(i) $f^{\mathbf{M}_\theta}(a_1\theta, \ldots, a_n\theta) = \left(f^{\mathbf{M}}(a_1, \ldots, a_n)\right)\theta$ for any n-ary $f^{\mathbf{M}_\theta} \in F^{\mathbf{M}_\theta}$ and $a_1\theta, \ldots, a_n\theta \in M_\theta$,
(ii) $(a\theta \approx^{\mathbf{M}_\theta} b\theta) = \theta(a, b)$ for every $a\theta, b\theta \in M_\theta$,

is called a *fuzzy factor* $\mathbf{L}$*-algebra of* $\mathbf{M}$ *modulo* θ.

Remark 3.12. (1) Every fuzzy factor **L**-algebra is well defined. Indeed, we can take advantage of the fact that $a\theta = b\theta$ iff $\theta(a, b) = 1$. The compatibility of $\approx_\theta^{\mathbf{M}}$ with operations follows directly from the compatibility of $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$.

(2) Let us have an **L**-algebra $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ with the crisp **L**-equality $\approx^{\mathbf{M}}$. For every $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$, the functional part $\langle M_\theta, F_\theta^{\mathbf{M}}\rangle$ of the fuzzy factorization $\mathbf{M}_\theta$ coincides with the notion of a fuzzy quotient (factor) algebra as it has been presented in [28], however, we use arbitrary complete residuated lattices as structures of truth values.

(3) Consider two **L**-algebras **M**, **N** the functional parts of which coincide, but $\approx^{\mathbf{M}} \neq \approx^{\mathbf{N}}$. In this case, **M** is not isomorphic to **N** although the functional parts are isomorphic in the sense of the isomorphism of ordinary algebras. On the other hand, $\left\langle M_{\approx^{\mathbf{M}}}, F_{\approx^{\mathbf{M}}}^{\mathbf{M}}\right\rangle \cong \left\langle N_{\approx^{\mathbf{N}}}, F_{\approx^{\mathbf{N}}}^{\mathbf{N}}\right\rangle$ in the sense of [28], which is quite unnatural, because both $\approx^{\mathbf{M}}, \approx^{\mathbf{N}}$ represent different similarities.

(4) Let us have an **L**-algebra $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$. $\approx^{\mathbf{M}}$ is a fuzzy congruence on $\langle M, F^{\mathbf{M}}\rangle$ in the sense of [28]. Therefore, we can consider the factor algebra $\left\langle M_{\approx^{\mathbf{M}}}, F_{\approx^{\mathbf{M}}}^{\mathbf{M}}\right\rangle$ (factorization of $\langle M, F^{\mathbf{M}}\rangle$ by $\approx^{\mathbf{M}}$ in the sense of [28]). Now, $\left\langle M_{\approx^{\mathbf{M}}}, F_{\approx^{\mathbf{M}}}^{\mathbf{M}}\right\rangle$ encompasses the same information as $\langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ (observe that the **L**-equality $\approx^{\mathbf{M}}$ can be reconstructed from $M_{\approx^{\mathbf{M}}}$, and vice versa, since $M_{\approx^{\mathbf{M}}} = \{a\approx^{\mathbf{M}} \mid a \in M\}$ and $a \approx^{\mathbf{M}} b = (a\approx^{\mathbf{M}})(b)$). However, we still cannot formalize algebras with fuzzy equalities $\langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ as fuzzy factor algebras $\left\langle M_{\approx^{\mathbf{M}}}, F_{\approx^{\mathbf{M}}}^{\mathbf{M}}\right\rangle$. Namely, the ordinary algebraic constructions, e.g. morphisms, do not preserve similarity. Therefore, such constructions, when applied to fuzzy factor algebras, can lead to a "loss of similarity information". For instance, there are morphisms from $\left\langle M_{\approx^{\mathbf{M}}}, F_{\approx^{\mathbf{M}}}^{\mathbf{M}}\right\rangle$ which are not morphisms from $\langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$, etc.

The following theorem shows that the notion of fuzzy factor **L**-algebras is covered by factor **L**-algebras and therefore unnecessary.

Theorem 3.10. *For every* **L**-*algebra* $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ *and* $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$, *we have* $\mathbf{M}_\theta \cong \mathbf{M}/\theta$.

Proof. Take a mapping $h : M_\theta \to M/\theta$, where $h(a\theta) = [a]_\theta$ for every $a\theta \in M_\theta$. Evidently, h is an embedding since $a\theta \approx^{\mathbf{M}_\theta} b\theta = \theta(a, b) = [a]_\theta \approx^{\mathbf{M}/\theta} [b]_\theta$. For any n-ary $f^{\mathbf{M}_\theta} \in F^{\mathbf{M}_\theta}$, $a_1, \ldots, a_n \in M$ we have

$$\begin{aligned} h\left(f^{\mathbf{M}_\theta}(a_1\theta, \ldots, a_n\theta)\right) &= h\left(f^{\mathbf{M}}(a_1, \ldots, a_n)\theta\right) = \left[f^{\mathbf{M}}(a_1, \ldots, a_n)\right]_\theta \\ &= f^{\mathbf{M}/\theta}\left([a_1]_\theta, \ldots, [a_n]_\theta\right) = f^{\mathbf{M}/\theta}\left(h(a_1\theta), \ldots, h(a_n\theta)\right). \end{aligned}$$

Clearly, h is surjective. Hence, $\mathbf{M}_\theta \cong \mathbf{M}/\theta$. □

Finally, let us note that even if we confine ourselves to the particular case of [0, 1], min as conjunction, and crisp identity as the fuzzy equality, there is only a small overlap between our results and the results obtained in [24,28].

3.3. Direct and subdirect products

Direct product is a basic construction yielding larger **L**-algebras from collections of input **L**-algebras. Subdirect product can be thought of as a derived construction since it is a special subalgebra of a direct product. This section introduces basic properties of both constructions. It is worth to add that some results presented in this section hold true only for certain subclasses of residuated lattices as the structures of truth values.

Definition 3.11. Let I be an index set. An **L**-algebra $\prod_{i\in I}\mathbf{M}_i$ is called a *direct product of a family* $\{\mathbf{M}_i \mid i \in I\}$ *of* **L**-*algebras* $\mathbf{M}_i = \langle M_i, \approx^{\mathbf{M}_i}, F^{\mathbf{M}_i}\rangle$ *of type F* if $M = \prod_{i\in I} M_i$ and for every n-ary function $f^{\prod_{i\in I}\mathbf{M}_i} \in F^{\prod_{i\in I}\mathbf{M}_i}$ and $a_1, \ldots, a_n \in M$ we have

$$f^{\prod_{i\in I}\mathbf{M}_i}(a_1, \ldots, a_n)(i) = f^{\mathbf{M}_i}(a_1(i), \ldots, a_n(i)), \tag{11}$$

for all $i \in I$ and the **L**-equality $\approx^{\prod_{i\in I}\mathbf{M}_i}$ is defined by

$$(a \approx^{\prod_{i\in I}\mathbf{M}_i} b) = \bigwedge_{i\in I}\left(a(i) \approx^{\mathbf{M}_i} b(i)\right) \tag{12}$$

for all $a, b \in M$.

Remark 3.13. A direct product as defined above is a well-defined **L**-algebra. Suppose $\prod_{i\in I}\mathbf{M}_i$ is a direct product of a family $\{\mathbf{M}_i \mid i \in I\}$ of **L**-algebras. **L**-relation $\approx^{\prod_{i\in I}\mathbf{M}_i}$ is evidently reflexive and symmetric. Now for every $a, b, c \in \prod_{i\in I} M_i$ we have

$$\begin{aligned}
&\left(a \approx^{\prod_{i\in I}\mathbf{M}_i} b\right) \otimes \left(b \approx^{\prod_{i\in I}\mathbf{M}_i} c\right)\\
&\quad= \left(\bigwedge_{i\in I} a(i) \approx^{\mathbf{M}_i} b(i)\right) \otimes \left(\bigwedge_{j\in I} b(j) \approx^{\mathbf{M}_j} c(j)\right) \leqslant \bigwedge_{i,j\in I}\left(a(i) \approx^{\mathbf{M}_i} b(i) \otimes b(j) \approx^{\mathbf{M}_j} c(j)\right)\\
&\quad\leqslant \bigwedge_{i\in I}\left(a(i) \approx^{\mathbf{M}_i} b(i) \otimes b(i) \approx^{\mathbf{M}_i} c(i)\right) \leqslant \bigwedge_{i\in I}\left(a(i) \approx^{\mathbf{M}_i} c(i)\right) = a \approx^{\prod_{i\in I}\mathbf{M}_i} c.
\end{aligned}$$

Hence, $\approx^{\prod_{i\in I}\mathbf{M}_i}$ is transitive. For $a \approx^{\prod_{i\in I}\mathbf{M}_i} b = 1$ we have $a(j) \approx^{\mathbf{M}_j} b(j) = 1$ for every $j \in I$, thus $a \approx^{\prod_{i\in I}\mathbf{M}_i} b = 1$ implies $a = b$. Altogether, $\approx^{\prod_{i\in I}\mathbf{M}_i}$ is an **L**-equality relation. The compatibility of functions with $\approx^{\prod_{i\in I}\mathbf{M}_i}$ follows from the definition. Indeed, for every n-ary $f^{\prod_{i\in I}\mathbf{M}_i} \in F^{\prod_{i\in I}\mathbf{M}_i}$ and arbitrary $a_1, b_1, \ldots, a_n, b_n \in \prod_{i\in I} M_i$, we have

$$\begin{aligned}
&\left(a_1 \approx^{\prod_{i\in I}\mathbf{M}_i} b_1\right) \otimes \cdots \otimes \left(a_n \approx^{\prod_{i\in I}\mathbf{M}_i} b_n\right)\\
&\quad= \bigotimes_{j=1}^{n} \bigwedge_{i\in I} a_j(i) \approx^{\mathbf{M}_i} b_j(i) \leqslant \bigwedge_{i\in I} \bigotimes_{j=1}^{n} a_j(i) \approx^{\mathbf{M}_i} b_j(i)\\
&\quad\leqslant \bigwedge_{i\in I} f^{\mathbf{M}_i}(a_1(i), \ldots, a_n(i)) \approx^{\mathbf{M}_i} f^{\mathbf{M}_i}(b_1(i), \ldots, b_n(i))\\
&\quad= \left(f^{\prod_{i\in I}\mathbf{M}_i}(a_1, \ldots, a_n) \approx^{\prod_{i\in I}\mathbf{M}_i} f^{\prod_{i\in I}\mathbf{M}_i}(b_1, \ldots, b_n)\right).
\end{aligned}$$

Hence, $\approx^{\prod_{i\in I}\mathbf{M}_i}$ is an **L**-equality compatible with all functions of $\prod_{i\in I}\mathbf{M}_i$.

Definition 3.12. Let $\prod_{i\in I} \mathbf{M}_i$ be a direct product of a family $\{\mathbf{M}_i \mid i \in I\}$ of $\mathbf{L}$-algebras. For every $j \in I$ a mapping $\pi_j : \prod_{i\in I} M_i \to M_j$, where $\pi_j(a) = a(j)$ is called a *projection map on the* jth *coordinate* of $\prod_{i\in I} \mathbf{M}_i$ or shortly jth *projection of* $\prod_{i\in I} \mathbf{M}_i$.

Lemma 3.7. (i) *For a direct product* $\prod_{i\in I} \mathbf{M}_i$ *of a family* $\{\mathbf{M}_i \mid i \in I\}$ *of* $\mathbf{L}$*-algebras, the jth projection* π_j *is an epimorphism for every* $j \in I$. (ii) *For every family of morphisms* $\{h_i : \mathbf{M} \to \mathbf{M}_i \mid i \in I\}$ *there is a uniquely determined morphism* $h: \mathbf{M} \to \prod_{i\in I}\mathbf{M}_i$ *such that* $h \circ \pi_i = h_i$ *for every* $i \in I$.

Proof. (i): Take any $j \in I$. For arbitrary $a \in M_j$ we have $\pi_j(b) = a$ for every $b \in \prod_{i\in I} M_i$, where $b(j) = a$. Thus, π_j is a surjective mapping. Moreover,

$$a \approx^{\prod_{i\in I} \mathbf{M}_i} b = \bigwedge_{i\in I} a(i) \approx^{\mathbf{M}_i} b(i) \leqslant a(j) \approx^{\mathbf{M}_j} b(j) = \pi_j(a) \approx^{\mathbf{M}_j} \pi_j(b).$$

The rest follows from the ordinary case.

(ii): Let us have a mapping $h: M \to \prod_{i\in I} M_i$ defined by $h(a)(i) = h_i(a)$ for every $i \in I, a \in M$. We have $a \approx^{\mathbf{M}} b \leqslant h_i(a) \approx^{\mathbf{M}_i} h_i(b)$ since h_i is supposed to be a morphism for every $i \in I$. Thus,

$$a \approx^{\mathbf{M}} b \leqslant \bigwedge_{i\in I} h_i(a) \approx^{\mathbf{M}_i} h_i(b) = h(a) \approx^{\prod_{i\in I}\mathbf{M}_i} h(b).$$

The rest of the proof follows from the ordinary case. □

Lemma 3.8. *For a direct product* $\mathbf{M} = \mathbf{M}_1 \times \mathbf{M}_2$, *we have*

(i) $\theta_{\pi_1} \wedge \theta_{\pi_2} = \approx^{\mathbf{M}}$,
(ii) $\theta_{\pi_1} \vee \theta_{\pi_2} = M \times M$,
(iii) $\theta_{\pi_1} \circ \theta_{\pi_2} = \theta_{\pi_2} \circ \theta_{\pi_1}$.

Proof. For (i), we have $\left(\theta_{\pi_1} \wedge \theta_{\pi_2}\right)(a, b) = \theta_{\pi_1}(a, b) \wedge \theta_{\pi_2}(a, b)$. That is $\left(\theta_{\pi_1} \wedge \theta_{\pi_2}\right)(a, b) = a(1) \approx^{\mathbf{M}_1} b(1) \wedge a(2) \approx^{\mathbf{M}_2} b(2)$, but this is exactly the definition of $\approx^{\prod_{i\in I}\mathbf{M}_i}$ for $I = \{1, 2\}$. Thus, (i) holds.

(ii): For any $a, b \in M_1 \times M_2$, we have

$$\theta_{\pi_1}(a, \langle a(1), b(2)\rangle) = 1, \text{ and } \theta_{\pi_2}(\langle a(1), b(2)\rangle, b) = 1. \tag{13}$$

By transitivity, for each congruence $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ containing both θ_{π_1} and θ_{π_2} we have $\theta(a, b) = 1$, whence $\theta_{\pi_1} \vee \theta_{\pi_2} = M \times M$ proving (ii).

(iii): Clearly, $(\theta_{\pi_1} \circ \theta_{\pi_2})(a, b) = 1$ for every $a, b \in M$. Thus, $\theta_{\pi_1} \circ \theta_{\pi_2} = \theta_{\pi_1} \vee \theta_{\pi_2}$ from which we get (iii). □

Definition 3.13. $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ is called a *factor congruence*, if there is a congruence $\theta^* \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ such that

(i) $\theta \wedge \theta^* = \approx^{\mathbf{M}}$,
(ii) $[a]_\theta \cap [b]_{\theta^*} \neq \emptyset$ for all $a, b \in M$.

The pair $\langle \theta, \theta^* \rangle$ is called a *pair of factor congruences on* $\mathbf{M}$.

Remark 3.14. The proof of Lemma 3.8 yields that for every $\mathbf{M} = \mathbf{M}_1 \times \mathbf{M}_2$ the couple $\langle \theta_{\pi_1}, \theta_{\pi_2} \rangle$ is a pair of factor congruences on $\mathbf{M}$. Moreover, condition (ii) of Definition 3.13 implies that for every $a, b \in M$ there is some $c \in M$ such that $\theta^*(a, c) \otimes \theta(c, b) = 1$. Hence, from (ii) it follows that $\theta \circ \theta^* = \theta^* \circ \theta = \theta \vee \theta^* = M \times M$, this is easy to check.

Theorem 3.11. *If* $\langle \theta, \theta^* \rangle$ *is a pair of factor congruences on an* $\mathbf{L}$*-algebra* $\mathbf{M}$, *then* $\mathbf{M} \cong \mathbf{M}/\theta \times \mathbf{M}/\theta^*$.

Proof. Put $h(a) = \langle [a]_\theta, [a]_{\theta^*} \rangle$ for all $a \in M$. For every $a, b \in M$, we have

$$\begin{aligned} a \approx^{\mathbf{M}} b &= (\theta \wedge \theta^*)(a, b) = \theta(a, b) \wedge \theta^*(a, b) = [a]_\theta \approx^{\mathbf{M}/\theta} [b]_\theta \wedge [a]_{\theta^*} \approx^{\mathbf{M}/\theta^*} [b]_{\theta^*} \\ &= \langle [a]_\theta, [a]_{\theta^*} \rangle \approx^{\mathbf{M}/\theta \times \mathbf{M}/\theta^*} \langle [b]_\theta, [b]_{\theta^*} \rangle = h(a) \approx^{\mathbf{M}/\theta \times \mathbf{M}/\theta^*} h(b). \end{aligned}$$

Condition (ii) of Definition 3.13 yields that for every $a, b \in M$ there is some $c \in M$ such that $c \in [a]_\theta \cap [b]_{\theta^*}$. That is, $c \in [a]_\theta$ and $c \in [b]_{\theta^*}$, i.e. $[a]_\theta = [c]_\theta$, $[b]_{\theta^*} = [c]_{\theta^*}$. Therefore, we obtain $h(c) = \langle [c]_\theta, [c]_{\theta^*} \rangle = \langle [a]_\theta, [b]_{\theta^*} \rangle$, i.e. h is surjective. Finally, we have to check the compatibility with functions. Take any n-ary $f^{\mathbf{M}} \in F^{\mathbf{M}}$ and arbitrary $a_1, \ldots, a_n \in M$. The rest follows from the ordinary case. □

Remark 3.15. (1) The previous theorem yields that every $\mathbf{L}$-algebra is isomorphic to a direct product. Namely, $\langle \approx^{\mathbf{M}}, M \times M \rangle$ is a pair of factor congruences on $\mathbf{M}$, whence $\mathbf{M} \cong \mathbf{M}/{\approx^{\mathbf{M}}} \times \mathbf{M}/(M \times M)$.

(2) In [28], instead of (ii) of Definition 3.13, the author requires that $\theta \circ \theta^* = M \times M$ (denote this condition by (ii′)). As we know from Remark 3.14, (ii) implies (ii′). In the ordinary case, both (ii) and (ii′) are equivalent. This, however, might not be the case in general since we may have $\theta \circ \theta^*(a, b) = 1$, i.e. $\bigvee_{c \in M} (\theta(a, c) \otimes \theta^*(b, c)) = 1$, although there is no $c \in M$ with $\theta(a, c) = 1$ and $\theta^*(b, c) = 1$, i.e. $c \in [a]_\theta$ and $c \in [b]_{\theta^*}$. Therefore, (ii) is indeed stronger than (ii′) in general. This fact reflects itself in Theorem 4.2 of [28] which is a generalization of the well-known theorem on factor congruences. Contrary to our approach, Theorem 4.2 of [28] needs an additional assumption.

(3) An $\mathbf{L}$-algebra $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ is said to be *trivial* if $M = \{a\}$. Since the universe of a trivial $\mathbf{L}$-algebra contains only one element, all mappings $\{a\}^n \to \{a\}$ are trivially compatible with $\approx^{\mathbf{M}}$. In the case of trivial $\mathbf{L}$-algebras, the $\mathbf{L}$-equality $\approx^{\mathbf{M}}$ does not represent any constraint for functions $f^{\mathbf{M}}$ and vice versa. An $\mathbf{L}$-algebra that is not trivial is called *nontrivial*.

Definition 3.14. An $\mathbf{L}$-algebra $\mathbf{M}$ is said to be *directly indecomposable* if $\mathbf{M}$ is not isomorphic to a direct product of two nontrivial $\mathbf{L}$-algebras.

Theorem 3.12. *An* $\mathbf{L}$*-algebra is directly indecomposable iff* $\langle \approx^{\mathbf{M}}, M \times M \rangle$ *is the only one pair of its factor congruences. Moreover, every finite* $\mathbf{L}$*-algebra is isomorphic to a direct product of directly indecomposable* $\mathbf{L}$*-algebras.*

Proof. The assertion follows by Theorem 3.12 and Lemma 3.8 using the same arguments as in the ordinary case. □

Definition 3.15. If $a, b \in M$ and $h : \langle M, \approx^M \rangle \to \langle N, \approx^N \rangle$ is a mapping we say h *separates* a *and* b, if $h(a) \approx^n h(b) < 1$. A family of mappings $\{h_i : \langle M, \approx^M \rangle \to \langle N_i, \approx^{N_i} \rangle \mid i \in I\}$ *separates points* iff for every $a, b \in M$ with $a \approx^M b < 1$ there is an index $j \in I$ such that $h_j : \langle M, \approx^M \rangle \to \langle N_j, \approx^{N_j} \rangle$ separates a and b.

Lemma 3.9. *For a family of morphisms* $\{h_i : \mathbf{M} \to \mathbf{M}_i \mid i \in I\}$, *the following conditions are equivalent*:

(i) *the family* $\{h_i : \mathbf{M} \to \mathbf{M}_i \mid i \in I\}$ *separates points*,
(ii) *a mapping* $h : \mathbf{M} \to \prod_{i \in I} \mathbf{M}_i$, *where* $h(a)(i) = h_i(a)$ *for every* $i \in I$, $a \in M$ *is an injective morphism*,
(iii) ${}^1\left(\bigcap_{i \in I} \theta_{h_i}\right) = \{\langle a, a \rangle \mid a \in M\}$.

Proof. "(i)⇒(ii)": Lemma 3.7 yields that h is a morphism. For $a \neq b$ we have $a \approx^{\mathbf{M}} b < 1$. Thus, if $\{h_i : \mathbf{M} \to \mathbf{M}_i \mid i \in I\}$ separates points, then there is some $i_0 \in I$ such that $h_{i_0}(a) \approx^{\mathbf{M}_i} h_{i_0}(b) < 1$, which implies $h(a) \approx^{\prod_{i \in I} \mathbf{M}_i} h(b) < 1$, i.e. h is injective.

"(ii)⇒(iii)": Evidently, $\left(\bigcap_{i \in I} \theta_{h_i}\right)(a, a) = 1$, thus $\{\langle a, a \rangle \mid a \in M\} \subseteq {}^1\left(\bigcap_{i \in I} \theta_{h_i}\right)$. Let us assume that (ii) holds. Then for $a \neq b$ we obtain

$$1 > h(a) \approx^{\prod_{i \in I} \mathbf{M}_i} h(b) = \bigwedge_{i \in I} h_i(a) \approx^{\mathbf{M}_i} h_i(b) = \bigwedge_{i \in I} \theta_{h_i}(a, b) = \left(\bigcap_{i \in I} \theta_{h_i}\right)(a, b),$$

that is $\langle a, b \rangle \notin {}^1\left(\bigcap_{i \in I} \theta_{h_i}\right)$. Hence, (ii) implies (iii).

"(iii)⇒(i)": Suppose that $\left(\bigcap_{i \in I} \theta_{h_i}\right)(a, b) < 1$ holds for $a \neq b$. Then $\bigwedge_{i \in I} h_i(a) \approx^{\mathbf{M}_i} h_i(b) < 1$. Thus, there is $i_0 \in I$ such that $h_{i_0}(a) \approx^{\mathbf{M}_{i_0}} h_{i_0}(b) < 1$. Since $a \neq b$ have been chosen arbitrarily, $\{h_i : \mathbf{M} \to \mathbf{M}_i \mid i \in I\}$ separates points. □

Lemma 3.10. *Let* $\{h_i : \mathbf{M} \to \mathbf{M}_i \mid i \in I\}$ *be a family of morphisms and let* $h : \mathbf{M} \to \prod_{i \in I} \mathbf{M}_i$ *denote a morphism, where* $h(a)(i) = h_i(a)$ *for every* $i \in I$, $a \in M$. *The morphism* h *is an embedding iff*

$$\left(\bigcap_{i \in I} \theta_{h_i}\right)(a, b) = a \approx^{\mathbf{M}} b.$$

Proof. Recall that $h(a) \approx^{\prod_{i \in I} \mathbf{M}_i} h(b) = \left(\bigcap_{i \in I} \theta_{h_i}\right)(a, b)$ for all $a, b \in M$. The rest is evident. □

In bivalent case, every algebra can be represented by a subdirect product of subdirectly irreducible algebras. In fuzzy case, this is not true in general. In the subsequent development, we introduce a sufficient condition for subdirect representation. However, unlike the bivalent case, we will also show that there are **L**-algebras, which are not subdirectly representable.

Definition 3.16. Let **M** be an **L**-algebra of type F. The **L**-algebra **M** is said to be a *subdirect product of a family* $\{\mathbf{M}_i \mid i \in I\}$ *of* **L**-*algebras of type* F if

(i) **M** is a subalgebra of $\prod_{i \in I} \mathbf{M}_i$,
(ii) $\pi_i(M) = M_i$ for every $i \in I$.

An embedding $h : \mathbf{M} \to \prod_{i\in I}\mathbf{M}_i$ is called *subdirect* if $h(\mathbf{M})$ is a subdirect product of the family $\{\mathbf{M}_i \mid i \in I\}$.

Lemma 3.11. *If* $\theta_i \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ *for every* $i \in I$ *and* $\bigcap_{i\in I}\theta_i = {\approx^{\mathbf{M}}}$, *then the mapping* $g : \mathbf{M} \to \prod_{\theta\in I}\mathbf{M}/\theta_i$, *where* $g(a)(i) = [a]_{\theta_i}$ *is a subdirect embedding. If* $h : \mathbf{M} \to \prod_{i\in I}\mathbf{M}_i$ *is a subdirect embedding, then there is a family of congruences* $\{\theta_i \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M}) \mid i \in I\}$ *such that* $\bigcap_{i\in I}\theta_i = {\approx^{\mathbf{M}}}$ *and* $\mathbf{M}_i \cong \mathbf{M}/\theta_i$ *for every* $i \in I$.

Proof. Since g defined by $g(a)(i) = [a]_{\theta_i}$ is a morphism, using Lemma 3.10 we can deduce that g is an embedding. Moreover, $g(M)(i) = \{[a]_{\theta_i} \mid a \in M\} = M/\theta_i$ for every $i \in I$, thus g is a subdirect embedding.

Furthermore, let $h: \mathbf{M} \to \prod_{i\in I}\mathbf{M}_i$ be a subdirect embedding. Put $\theta_i = \theta_{h_i}$ for every $i \in I$ (recall that $h_i: M \to M_i$, where $h_i(a) = h(a)(i)$ is a surjective morphism). Now by applying Lemma 3.10 one can conclude that $\bigcap_{i\in I}\theta_i = {\approx^{\mathbf{M}}}$. Since every h_i is surjective, (i) of Theorem 3.8 yields that $\mathbf{M}_i \cong \mathbf{M}/\theta_i$ for every $i \in I$. □

Definition 3.17. An $\mathbf{L}$-algebra $\mathbf{M}$ is *subdirectly irreducible* if for every subdirect embedding $h: \mathbf{M} \to \prod_{i\in I}\mathbf{M}_i$ there is an index $j \in I$ such that $h \circ \pi_j : \mathbf{M} \to \mathbf{M}_j$ is an isomorphism.

Theorem 3.13. *An* $\mathbf{L}$-*algebra* $\mathbf{M}$ *is subdirectly irreducible iff* $\mathbf{M}$ *is either trivial, or there is a least congruence in* $\mathrm{Con}_{\mathbf{L}}(\mathbf{M}) - \{\approx^{\mathbf{M}}\}$. *In the latter case* $\bigcap\left\{\mathrm{Con}_{\mathbf{L}}(\mathbf{M}) - \{\approx^{\mathbf{M}}\}\right\}$ *is a principal congruence and* $\langle \mathrm{Con}_{\mathbf{L}}(\mathbf{M}), \subseteq\rangle$ *contains exactly one atom.*

Proof. "⇒": Suppose, by contradiction, that $\mathbf{M}$ is not a trivial $\mathbf{L}$-algebra and $\mathrm{Con}_{\mathbf{L}}(\mathbf{M}) - \{\approx^{\mathbf{M}}\}$ does not have the least element. Then $\bigcap\left\{\mathrm{Con}_{\mathbf{L}}(\mathbf{M}) - \{\approx^{\mathbf{M}}\}\right\} = {\approx^{\mathbf{M}}}$. Put $I = \mathrm{Con}_{\mathbf{L}}(\mathbf{M}) - \{\approx^{\mathbf{M}}\}$. Using Lemma 3.11, there is a subdirect embedding $h: \mathbf{M} \to \prod_{\theta\in I}\mathbf{M}/\theta$, where $h(a)(\theta) = [a]_\theta$ for every $a \in M$ and $\theta \in I$. Moreover, for every congruence $\theta \in I$ we have $\theta \supset {\approx^{\mathbf{M}}}$. Thus, there are $a, b \in M$ such that $a \approx^{\mathbf{M}} b < \theta(a, b)$, i.e.

$$a \approx^{\mathbf{M}} b < \theta(a, b) = [a]_\theta \approx^{\mathbf{M}/\theta} [b]_\theta = h_\theta(a) \approx^{\mathbf{M}/\theta} h_\theta(b) = (h \circ \pi_\theta)(a) \approx^{\mathbf{M}/\theta} (h \circ \pi_\theta)(b).$$

Hence, for every $\theta \in I$ the mapping $h \circ \pi_\theta: \mathbf{M} \to \mathbf{M}$ is not an isomorphism. Consequently, the $\mathbf{L}$-algebra $\mathbf{M}$ is not subdirectly irreducible.

"⇐": A trivial $\mathbf{L}$-algebra $\mathbf{M}$ is subdirectly irreducible. Indeed, for every subdirect embedding $h: \mathbf{M} \to \prod_{i\in I}\mathbf{M}_i$, each $\mathbf{L}$-algebra $\mathbf{M}_i$ must be trivial, because $\pi_i(h(M)) = \pi_i(h(\{a\})) = M_i$ holds for every $i \in I$ by virtue of the assumption. Hence, $h \circ \pi_i : \mathbf{M} \to \mathbf{M}_i$ is an isomorphism for all $i \in I$. Altogether, $\mathbf{M}$ is subdirectly irreducible. So suppose $\mathbf{M}$ is nontrivial and let $\theta = \bigcap\left\{\mathrm{Con}_{\mathbf{L}}(\mathbf{M}) - \{\approx^{\mathbf{M}}\}\right\} > {\approx^{\mathbf{M}}}$, i.e. θ is the least congruence in $\mathrm{Con}_{\mathbf{L}}(\mathbf{M}) - \{\approx^{\mathbf{M}}\}$. Then there are $a, b \in M$ such that $\theta(a, b) > a \approx^{\mathbf{M}} b$. Let $h : \mathbf{M} \to \prod_{i\in I}\mathbf{M}_i$ be a subdirect embedding. Now we have, $a \approx^{\mathbf{M}} b = \bigwedge_{i\in I} h(a)(i) \approx^{\mathbf{M}_i} h(b)(i) < \theta(a, b)$. Therefore, there is an index $i \in I$ such that $h(a)(i) \approx^{\mathbf{M}_i} h(b)(i) \not\geq \theta(a, b)$, that is, $(h \circ \pi_i)(a) \approx^{\mathbf{M}_i} (h \circ \pi_i)(b) \not\geq \theta(a, b)$, so $\theta_{h\circ\pi_i} \not\supseteq \theta$. Since θ is the least congruence in $\mathrm{Con}_{\mathbf{L}}(\mathbf{M}) - \{\approx^{\mathbf{M}}\}$, we readily obtain $\theta_{h\circ\pi_i} = {\approx^{\mathbf{M}}}$. Thus, $h \circ \pi_i : \mathbf{M} \to \mathbf{M}_i$ is an isomorphism and the $\mathbf{L}$-algebra $\mathbf{M}$ is subdirectly irreducible.

If θ is the least congruence in $\mathrm{Con}_{\mathbf{L}}(\mathbf{M}) - \{\approx^{\mathbf{M}}\}$, then there are elements $a, b \in M$ such that $\theta(a,b) > a \approx^{\mathbf{M}} b$. Obviously, we have $\theta(\theta(a,b)/\langle a,b\rangle) \subseteq \theta$. The converse inclusion holds since $\theta(\theta(a,b)/\langle a,b\rangle) > a \approx^{\mathbf{M}} b$ and θ is the least congruence with $\theta(a,b) > a \approx^{\mathbf{M}} b$. Thus, we have $\theta(\theta(a,b)/\langle a,b\rangle) = \theta$, i.e. θ is a principal congruence. $\square$

Theorem 3.14 (*Representation theorem*). *Let* $\mathbf{M}$ *be a nontrivial* $\mathbf{L}$*-algebra such that for every distinct* $a, b \in M$ *there exists* $\theta_{a,b} \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$, *where* $\theta_{a,b}(a,b) = a \approx^{\mathbf{M}} b$, *and* $\theta_{a,b}$ *is* $\wedge$*-irreducible in* $\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. *Then* $\mathbf{M}$ *is isomorphic to a subdirect product of a family of subdirectly irreducible* $\mathbf{L}$*-algebras.*

Proof. Let $\theta_{a,b}$ be $\wedge$-irreducible in $\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. As a result, $\left[\theta_{a,b}, M\times M\right] - \left\{\theta_{a,b}\right\}$ has the least element. Now from Theorems 3.9 and 3.13 it follows that $\mathbf{M}/\theta_{a,b}$ is subdirectly irreducible. We have $\bigcap_{\langle a,b\rangle\in I}\theta_{a,b} = \approx^{\mathbf{M}}$, where $I = \{\langle a,b\rangle \mid a,b\in M \text{ and } a\neq b\}$. Thus, Lemma 3.11 yields that there is a subdirect embedding $h\colon \mathbf{M}\to\prod_{\langle a,b\rangle\in I}\mathbf{M}/\theta_{a,b}$, where $h(c)(\langle a,b\rangle) = [c]_{\theta_{a,b}}$ for all $a,b,c\in M, a\neq b$. Hence, $\mathbf{M}$ is isomorphic to a subdirect product of subdirectly irreducible $\mathbf{L}$-algebras. $\square$

The previous representation theorem is sort of abstract. It only delimits a condition under which an $\mathbf{L}$-algebra $\mathbf{M}$ has a subdirect representation. In the following we will focus on the existence of suitable $\wedge$-irreducible elements in $\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. First, we prove a technical lemma.

Lemma 3.12. *Let* $\{\theta_i \mid i\in I\}\subseteq \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ *be a directed system of congruences, i.e. for any finite number* $\theta_{i_1},\ldots,\theta_{i_k}\in\{\theta_i\mid i\in I\}$, *there is* θ_i, $i\in I$ *such that* $\theta_{i_j}\subseteq\theta_i$, *for all* $j=1,\ldots,k$. *Then* $\bigcup_{i\in I}\theta_i = \bigvee_{i\in I}\theta_i$ *(supremum in* $\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$*).*

Proof. Clearly, $\bigcup_{i\in I}\theta_i\subseteq\bigvee_{i\in I}\theta_i$. We have to show "$\supseteq$":

$$
\begin{aligned}
\left(\bigvee_{i\in I}\theta_i\right)(a,b) &= \bigvee_{\substack{i_1,\ldots,i_k\in I\\ c_1,\ldots,c_{k-1}\in M}} \left(\theta_{i_1}(a,c_1)\otimes\cdots\otimes\theta_{i_k}(c_{k-1},b)\right)\\
&\leqslant \bigvee_{\substack{i\in I\\ c_1,\ldots,c_{k-1}\in M}} (\theta_i(a,c_1)\otimes\cdots\otimes\theta_i(c_{k-1},b)) \leqslant \bigvee_{i\in I}\theta_i(a,b) = \left(\bigcup_{i\in I}\theta_i\right)(a,b)
\end{aligned}
$$

for every $a,b\in M$. $\square$

Lemma 3.13. *Suppose* $\mathbf{M}$ *is a nontrivial* $\mathbf{L}$*-algebra. Then for every distinct elements* $a,b\in M$:

(i) *there is a maximal* $\theta_{a,b}\in\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ *such that* $\theta_{a,b}(a,b) = a\approx^{\mathbf{M}} b$;
(ii) *if* $a\approx^{\mathbf{M}} b$ is $\wedge$*-irreducible in* $\mathbf{L}$ *then* $\theta_{a,b}$ *is* $\wedge$*-irreducible in* $\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$.

Proof. (i): Let $I_{a,b} = \left\{\theta\in\mathrm{Con}_{\mathbf{L}}(\mathbf{M})\mid \theta(a,b) = a\approx^{\mathbf{M}} b\right\}$. It is easy to observe that $\left\langle I_{a,b},\subseteq\right\rangle$ is a partially ordered set, $I_{a,b}\subseteq\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. Moreover, $\approx^{\mathbf{M}}\in I_{a,b}$ implies $I_{a,b}\neq\emptyset$. For every chain $I\subseteq I_{a,b}$ of congruences we have $\left(\bigvee_{i\in I}\theta_i\right)(a,b) = \left(\bigcup_{i\in I}\theta_i\right)(a,b) = a\approx^{\mathbf{M}} b$ due to Lemma 3.12. That is, $\bigvee_{i\in I}\theta_i\in I_{a,b}$. Hence, every chain $I\subseteq I_{a,b}$ is bounded from above. Now using Zorn Lemma it readily follows that there is a maximal element $\theta_{a,b}\in I_{a,b}$.

(ii): Put $J = \left[\theta_{a,b}, M\times M\right] - \left\{\theta_{a,b}\right\}$ and suppose $a \approx^{\mathbf{M}} b$ to be an $\wedge$-irreducible element of $\mathbf{L}$. By maximality of $\theta_{a,b}$, $\theta(a,b) > \theta_{a,b}(a,b)$ for each $\theta \in J$. Thus, it follows that

$$\left(\bigcap J\right)(a,b) = \bigwedge_{\theta\in J} \theta(a,b) > a \approx^{\mathbf{M}} b.$$

As a consequence, $\bigcap J \in J$, i.e. $\theta_{a,b}$ is $\wedge$-irreducible in $\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. □

If the lattice part of $\mathbf{L}$ is a finite chain, then every element $1 \neq a \in L$ is $\wedge$-irreducible. Thus, for finite linearly ordered residuated lattices we can use Theorem 3.14 and Lemma 3.13 to obtain the following consequence.

Corollary 3.1. *If* $\mathbf{L}$ *is a finite chain then every nontrivial* $\mathbf{L}$*-algebra is isomorphic to a subdirect product of subdirectly irreducible* $\mathbf{L}$*-algebras.*

Remark 3.16. (1) There is a more general criterion for subdirect representation, which is, however, more technical than the one given by Lemma 3.13. Let $\mathbf{M}$ be an $\mathbf{L}$-algebra. If for every distinct $b, b' \in M$ we have

$$b \approx^{\mathbf{M}} b' < \left(\bigcap_{a\,>\,b\,\approx^{\mathbf{M}} b'} \theta(a/\langle b, b'\rangle)\right)(b, b'), \tag{14}$$

then $\mathbf{M}$ is isomorphic to a subdirect product of subdirectly irreducible $\mathbf{L}$-algebras. Indeed, take a maximal $\theta_{b,b'} \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ with $\theta_{b,b'}(b,b') = b \approx^{\mathbf{M}} b'$. Clearly, $\theta_{b,b'}$ is $\wedge$-irreducible in $\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ due to (14). Namely, $\theta_{b,b'} \vee \left(\bigcap_{a\,>\,b\,\approx^{\mathbf{M}} b'} \theta(a/\langle b, b'\rangle)\right)$ is the least congruence in $\left[\theta_{b,b'}, M\times M\right] - \left\{\theta_{b,b'}\right\}$. Now apply Theorem 3.14. It is easily seen that if $b \approx^{\mathbf{M}} b'$ is $\wedge$-irreducible in $\mathbf{L}$, then (14) holds trivially.

(2) The subdirect representation does not pass for every $\mathbf{L}$. In other words, for certain structures of truth values, there are still $\mathbf{L}$-algebras which are not isomorphic to a subdirect product of subdirectly irreducible $\mathbf{L}$-algebras. An example follows.

Example 3.4. Take a complete residuated lattice $\mathbf{L}$ on the rational unit interval [0, 1]. Let us have an $\mathbf{L}$-algebra $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, \emptyset\rangle$ of empty type, where $M = \{a, b\}$, and $a \approx^{\mathbf{M}} b = 0$. It is easy to see that every reflexive and symmetric binary $\mathbf{L}$-relation θ on M is a congruence on $\mathbf{M}$ since M is a two-element set with $a \approx^{\mathbf{M}} b = 0$, and $F^{\mathbf{M}} = \emptyset$. Obviously, every congruence $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ is uniquely determined by the truth degree $\theta(a,b) \in L$. Thus, there is a one-to-one correspondence between congruences from $\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ and truth degrees from L. Moreover, $\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ is isomorphic to the lattice part of $\mathbf{L}$. For $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$, Theorem 3.9 yields $\mathrm{Con}_{\mathbf{L}}(\mathbf{M}/\theta) \cong [\theta, M\times M] \cong [c, 1]$, where $\theta(a,b) = c$. Since $[c,1] - \{c\}$ does not have the least element, $\mathbf{M}/\theta$ is subdirectly reducible. As a consequence, if $h\colon \mathbf{M} \to \prod_{i\in I}\mathbf{M}_i$ is a subdirect embedding, then every $\mathbf{M}_i \cong \mathbf{M}/\theta_{h\circ\pi_i}$ is subdirectly reducible, i.e. $\mathbf{M}$ is not isomorphic to a subdirect product of subdirectly irreducible $\mathbf{L}$-algebras.

3.4. Direct unions

In general, there is no obvious way to naturally equip the union $\bigcup_{i\in I} M_i$ of universes of $\mathbf{L}$-algebras $\mathbf{M}_i$ with functions so that it became an $\mathbf{L}$-algebra. In some cases, it is possible to define functions on $\bigcup_{i\in I} M_i$

such that $\bigcup_{i\in I} M_i$ equipped with such functions and a suitable **L**-equality turns into an **L**-algebra. For instance, when $\{\mathbf{M}_i \mid i \in I\}$ is a directed family of **L**-algebras, we can define a direct union of $\{\mathbf{M}_i \mid i \in I\}$ which is a well-defined **L**-algebra.

Definition 3.18. A partially ordered index set $\langle I, \leqslant\rangle$ is called *directed*, if $I \neq \emptyset$ and for every $i, j \in I$ there is $k \in I$ such that $i, j \leqslant k$. A family $\{\mathbf{M}_i \mid i \in I\}$ of **L**-algebras of type F, where $\langle I, \leqslant\rangle$ is a directed index set and $\mathbf{M}_i \in \mathrm{Sub}(\mathbf{M}_j)$ whenever $i \leqslant j$ is called a *directed family of* **L***-algebras.*

For a directed family $\{\mathbf{M}_i \mid i \in I\}$ of **L**-algebras we define $\bigcup_{i\in I} \mathbf{M}_i = \left\langle \bigcup_{i\in I} M_i, \approx^{\bigcup_{i\in I}\mathbf{M}_i}, F^{\bigcup_{i\in I}\mathbf{M}_i}\right\rangle$ such that for every n-ary $f \in F$, and arbitrary $a_1, \ldots, a_n \in \bigcup_{i\in I} M_i$ we put

$$f^{\bigcup_{i\in I}\mathbf{M}_i}(a_1, \ldots, a_n) = f^{\mathbf{M}_j}(a_1, \ldots, a_n), \tag{15}$$

where $a_1 \in M_{i_1}, \ldots, a_n \in M_{i_n}$, $j \in I$, and $i_1, \ldots, i_n \leqslant j$. For every elements $a, b \in \bigcup_{i\in I} M_i$ such that $a \in M_i$, $b \in M_j$ we define the degree of equality $a \approx^{\bigcup_{i\in I}\mathbf{M}_i} b$ as follows:

$$(a \approx^{\bigcup_{i\in I}\mathbf{M}_i} b) = (a \approx^{\mathbf{M}_k} b), \tag{16}$$

where $k \in I$, $i, j \leqslant k$. $\bigcup_{i\in I} \mathbf{M}_i$ is called a *direct union* of $\{\mathbf{M}_i \mid i \in I\}$.

Remark 3.17. The direct union $\bigcup_{i\in I} \mathbf{M}_i$ of a directed family $\{\mathbf{M}_i \mid i \in I\}$ is a well-defined **L**-algebra. First, observe that for finitely many $i_1, \ldots, i_n \in I$ there is always an index $j \in I$ such that $i_1, \ldots, i_n \leqslant j$ (this follows from the definition of directed index set). Moreover, $\{\mathbf{M}_i \mid i \in I\}$ is a directed family so we have $\mathbf{M}_{i_1}, \ldots, \mathbf{M}_{i_n} \in \mathrm{Sub}(\mathbf{M}_j)$, i.e. $f^{\bigcup_{i\in I}\mathbf{M}_i}(a_1, \ldots, a_n)$ always exists. The fact that $\bigcup_{i\in I} M_i$ equipped with functions $f^{\bigcup_{i\in I}\mathbf{M}_i} \in F^{\bigcup_{i\in I}\mathbf{M}_i}$ is an algebra follows from the ordinary case.

The **L**-relation $\approx^{\bigcup_{i\in I}\mathbf{M}_i}$ is well defined. Indeed, it suffices to check that $a \approx^{\bigcup_{i\in I}\mathbf{M}_i} b$ is independent on the choice of indices $i, j, k \in I$ used in (16). Let $a \in M_i$, $a \in M_{i'}$, $b \in M_j$, $b \in M_{j'}$. Take $k \geqslant i, j$ and $k' \geqslant i', j'$. Since $\langle I, \leqslant\rangle$ is directed, there is an index $l \in I$ such that $l \geqslant k, k'$. As a consequence, $\mathbf{M}_k, \mathbf{M}_{k'} \in \mathrm{Sub}(\mathbf{M}_l)$. Thus, it follows that $(a \approx^{\mathbf{M}_k} b) = (a \approx^{\mathbf{M}_l} b) = (a \approx^{\mathbf{M}_{k'}} b)$. Therefore, $a \approx^{\bigcup_{i\in I}\mathbf{M}_i} b$ is independent on the choice of $i, j, k \in I$. Furthermore, for $a \in \bigcup_{i\in I} M_i$ we have $a \in M_j$ for some $j \in I$. Since $a \approx^{\mathbf{M}_j} a = 1$, it follows that $a \approx^{\bigcup_{i\in I}\mathbf{M}_i} a = 1$ (reflexivity). Symmetry follows from the symmetry of all $\approx^{\mathbf{M}_i}$'s. For every $a, b, c \in M$ there is an index $j \in I$ such that $a, b, c \in M_j$, thus

$$(a \approx^{\bigcup_{i\in I}\mathbf{M}_i} b) \otimes (b \approx^{\bigcup_{i\in I}\mathbf{M}_i} c) = (a \approx^{\mathbf{M}_j} b) \otimes (b \approx^{\mathbf{M}_j} c) \leqslant (a \approx^{\mathbf{M}_j} c) = (a \approx^{\bigcup_{i\in I}\mathbf{M}_i} c),$$

i.e. $\approx^{\bigcup_{i\in I}\mathbf{M}_i}$ is transitive. Moreover, if $a \approx^{\bigcup_{i\in I}\mathbf{M}_i} b = 1$ then for $j \in I$ such that $a, b \in M_j$ we have $(a \approx^{\mathbf{M}_j} b) = 1$. Thus, $(a \approx^{\bigcup_{i\in I}\mathbf{M}_i} b) = 1$ implies $a = b$. Altogether, $\approx^{\bigcup_{i\in I}\mathbf{M}_i}$ is an **L**-equality.

It suffices to verify, that every $f^{\bigcup_{i\in I}\mathbf{M}_i} \in F^{\bigcup_{i\in I}\mathbf{M}_i}$ is compatible with $\approx^{\bigcup_{i\in I}\mathbf{M}_i}$. For every n-ary $f \in F$ and arbitrary $a_1, b_1, \ldots, a_n, b_n \in \bigcup_{i\in I} M_i$ there is an index $j \in I$ such that $a_1, b_1, \ldots, a_n, b_n \in M_j$. Thus,

$$\begin{aligned}
&(a_1 \approx^{\bigcup_{i\in I}\mathbf{M}_i} b_1) \otimes \cdots \otimes (a_n \approx^{\bigcup_{i\in I}\mathbf{M}_i} b_n)\\
&\quad = (a_1 \approx^{\mathbf{M}_j} b_1) \otimes \cdots \otimes (a_n \approx^{\mathbf{M}_j} b_n)\\
&\quad \leqslant f^{\mathbf{M}_j}(a_1, \ldots, a_n) \approx^{\mathbf{M}_j} f^{\mathbf{M}_j}(b_1, \ldots, b_n)\\
&\quad = f^{\bigcup_{i\in I}\mathbf{M}_i}(a_1, \ldots, a_n) \approx^{\bigcup_{i\in I}\mathbf{M}_i} f^{\bigcup_{i\in I}\mathbf{M}_i}(b_1, \ldots, b_n).
\end{aligned}$$

Hence, $\bigcup_{i\in I} \mathbf{M}_i$ is a well-defined **L**-algebra.

The following theorem shows that every **L**-algebra can be constructed from its finitely generated subalgebras.

Theorem 3.15. *Every* **L**-*algebra is isomorphic to a direct union of finitely generated* **L**-*algebras*.

Proof. Suppose an **L**-algebra $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ is given. Consider an index set $I_{\mathbf{M}} = \{M' \subseteq M \mid M'$ is finite$\}$. $I_{\mathbf{M}}$ can be partially ordered using the set inclusion $\subseteq$. It is also evident that for every $M', M'' \in I_{\mathbf{M}}$ we have $M' \cup M'' \in I_{\mathbf{M}}$. Moreover, $[M']_{\mathbf{M}}$ is a subalgebra of $[M'']_{\mathbf{M}}$, whenever $M' \subseteq M''$. Hence, $\{[M']_{\mathbf{M}} \mid M' \in I_{\mathbf{M}}\}$ is a directed family of finitely generated **L**-algebras.

Now we can claim $\mathbf{M} \cong \bigcup \{[M']_{\mathbf{M}} \mid M' \in I_{\mathbf{M}}\}$. Indeed, since every $a \in M$ is contained in $[\{a\}]_{\mathbf{M}}$, it is possible to define a mapping $h: \mathbf{M} \to \bigcup \{[M']_{\mathbf{M}} \mid M' \in I_{\mathbf{M}}\}$ by $h(a) = a$ $(a \in M)$. For all $a, b \in M$ we have

$$(a \approx^{\mathbf{M}} b) = \left(a \approx^{[\{a,b\}]_{\mathbf{M}}} b\right) = \left(h(a) \approx^{\bigcup\{[M']_{\mathbf{M}} \mid M' \in I_{\mathbf{M}}\}} h(b)\right)$$

i.e. h is an $\approx$-morphism. For any n-ary $f \in F$ and $a_1, \ldots, a_n \in M$ we have,

$$\begin{aligned} h\left(f^{\mathbf{M}}(a_1, \ldots, a_n)\right) &= h\left(f^{[\{a_1,\ldots,a_n\}]_{\mathbf{M}}}(a_1, \ldots, a_n)\right) \\ &= f^{[\{a_1,\ldots,a_n\}]_{\mathbf{M}}}(a_1, \ldots, a_n) = f^{\bigcup\{[M']_{\mathbf{M}} \mid M' \in I_{\mathbf{M}}\}}(h(a_1), \ldots, h(a_n)). \end{aligned}$$

Since for every finite $M' \subseteq M$ the **L**-algebra $[M']_{\mathbf{M}}$ is a subalgebra of **M**, the mapping h is a surjective embedding, proving the claim. □

Remark 3.18. (1) Clearly, every **L**-algebra **M** is isomorphic to a trivial direct union. Namely, $\mathbf{M} \cong \bigcup_{i \in I} \mathbf{M}_i$, where $I = \{1\}$ and $\mathbf{M}_1$ is **M**.

(2) If $\{\mathbf{M}_i \mid i \in I\}$ is a directed family of **L**-algebras, where I is finite, then evidently I has the greatest element, let us denote it by k. Clearly, for every $i \in I$ we have $\mathbf{M}_i \in \mathrm{Sub}(\mathbf{M}_k)$, that is $\bigcup_{i \in I} \mathbf{M}_i \cong \mathbf{M}_k$.

In certain sense, the direct union can be thought of as a derived construction since it is isomorphic to a factorization of certain subalgebra of a direct product. The following lemma with be used further in Section 3.5.

Lemma 3.14. *Let* $\{\mathbf{M}_i \mid i \in I\}$ *be a directed family of* **L**-*algebras of type* F. *Then* $\bigcup_{i \in I} \mathbf{M}_i \cong \mathbf{N}/\theta$, *where*

(i) $\mathbf{N} \in \mathrm{Sub}(\prod_{i \in I} \mathbf{M}_i)$, *where* $N = \{a \in \prod_{i \in I} M_i \mid$ *there is* $k \in I$ *such that* $a(k) = a(i)$ *for all* $i \geqslant k\}$,
(ii) $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{N})$ *such that* $\theta(a, b) = \bigvee_{i \in I} \bigwedge_{k \geqslant i} a(k) \approx^{\mathbf{M}_k} b(k)$.

Proof. For the sake of brevity, let $a \in N_{(j)}$ denote the fact that $a(j) = a(i)$ for all $i \geqslant j$. Clearly, if $a \in N_{(j)}$, then $a \in N_{(l)}$ for every $l \geqslant j$. First, we will check that N is a nonempty subuniverse of $\prod_{i \in I} \mathbf{M}_i$. Evidently, $N \neq \emptyset$. Furthermore, for any n-ary $f \in F$ and arbitrary elements $a_1, \ldots, a_n \in N$ there is an index $j \in I$, such that $a_1 \in N_{(j)}, \ldots, a_n \in N_{(j)}$. Since $\mathbf{M}_j \in \mathrm{Sub}(\mathbf{M}_i)$ for all $i \geqslant j$, it follows that

$$\begin{aligned} f^{\prod_{i \in I} \mathbf{M}_i}(a_1, \ldots, a_n)(j) &= f^{\mathbf{M}_j}(a_1(j), \ldots, a_n(j)) = f^{\mathbf{M}_i}(a_1(i), \ldots, a_n(i)) \\ &= f^{\prod_{i \in I} \mathbf{M}_i}(a_1, \ldots, a_n)(i) \end{aligned}$$

for every $i \geqslant j$, i.e. $f^{\prod_{i \in I} \mathbf{M}_i}(a_1, \ldots, a_n) \in N_{(j)}$. Hence, $\mathbf{N} \in \mathrm{Sub}(\prod_{i \in I} \mathbf{M}_i)$.

In the rest of the proof, we will take advantage of the following claim: for any $a \in N_{(j)}, b \in N_{(j')}$ we have

$$\theta(a, b) = a(m) \approx^{\mathbf{M}_m} b(m) \quad \text{for every } m \geqslant j, j'. \tag{17}$$

"$\leqslant$" of (17): Take $m \geqslant j, j'$, where $a \in N_{(j)}, b \in N_{(j')}$. For arbitrary index $i \in I$ there is some $m' \in I$ such that $i, m \leqslant m'$. Moreover, from $a(m') = a(m), b(m') = b(m), \mathbf{M}_m \in \mathrm{Sub}(\mathbf{M}_{m'})$ it follows that

$$\bigwedge_{k \geqslant i} a(k) \approx^{\mathbf{M}_k} b(k) \leqslant a(m') \approx^{\mathbf{M}_{m'}} b(m') = a(m) \approx^{\mathbf{M}_m} b(m).$$

Hence, we have $\theta(a, b) = \bigvee_{i \in I} \bigwedge_{k \geqslant i} a(k) \approx^{\mathbf{M}_k} b(k) \leqslant a(m) \approx^{\mathbf{M}_m} b(m)$.

"$\geqslant$" of (17): Let us have $a \in N_{(j)}, b \in N_{(j')}$ and let $m \geqslant j, j'$. We can take $i = m$. Since $\mathbf{M}_m \in \mathrm{Sub}(\mathbf{M}_k)$ for all $k \geqslant m$, we readily obtain $a(m) \approx^{\mathbf{M}_m} b(m) = \bigwedge_{k \geqslant m} a(k) \approx^{\mathbf{M}_k} b(k)$. That is,

$$a(m) \approx^{\mathbf{M}_m} b(m) \leqslant \bigvee_{i \in I} \bigwedge_{k \geqslant i} a(k) \approx^{\mathbf{M}_k} b(k) = \theta(a, b).$$

Altogether, claim (17) holds true.

Now it is easy to check that $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{N})$. Reflexivity and symmetry of θ are obvious. For every $a, b, c \in N$, we have $\theta(a, b) = a(m) \approx^{\mathbf{M}_m} b(m), \theta(b, c) = b(m') \approx^{\mathbf{M}_{m'}} c(m')$ for certain $m, m' \in I$ due to (17). Evidently, $a, b \in N_{(m)}, b, c \in N_{(m')}$. Thus, for $m'' \geqslant m, m'$ it follows that $a, b, c \in N_{(m'')}$. Hence,

$$\begin{aligned}\theta(a, b) \otimes \theta(b, c) &= \left(a(m) \approx^{\mathbf{M}_m} b(m)\right) \otimes \left(b(m') \approx^{\mathbf{M}_{m'}} c(m')\right) \\ &= \left(a(m'') \approx^{\mathbf{M}_{m''}} b(m'')\right) \otimes \left(b(m'') \approx^{\mathbf{M}_{m''}} c(m'')\right) \leqslant a(m'') \approx^{\mathbf{M}_{m''}} c(m'') = \theta(a, c),\end{aligned}$$

i.e. θ is transitive. In an analogous way it is possible to prove that every n-ary $f^{\mathbf{N}} \in F^{\mathbf{N}}$ is compatible with θ. For $a_1, b_1 \in N_{(m_1)}, \ldots, a_n, b_n \in N_{(m_n)}$, and $m \in I$ such that $m \geqslant m_1, \ldots, m_n$ we have

$$\begin{aligned}\bigotimes_{i=1}^{n} \left(a_i(m_i) \approx^{\mathbf{M}_{m_i}} b_i(m_i)\right) &= \bigotimes_{i=1}^{n} \left(a_i(m) \approx^{\mathbf{M}_m} b_i(m)\right) \\ &\leqslant f^{\mathbf{M}_m}(a_1(m), \ldots, a_n(m)) \approx^{\mathbf{M}_m} f^{\mathbf{M}_m}(b_1(m), \ldots, b_n(m)) \\ &= f^{\mathbf{N}}(a_1, \ldots, a_n)(m) \approx^{\mathbf{M}_m} f^{\mathbf{N}}(b_1, \ldots, b_n)(m) \\ &= \theta\left(f^{\mathbf{N}}(a_1, \ldots, a_n), f^{\mathbf{N}}(b_1, \ldots, b_n)\right).\end{aligned}$$

That is, $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{N})$.

Let us introduce mappings $h_i : M_i \to N/\theta$ $(i \in I)$ defined by $h_i(a) = [a']_\theta$, where $a' \in N$ such that for every $j \geqslant i$ we have $a'(j) = a$. Every h_i is a well-defined mapping since for $a', a'' \in N$, where $a'(j) = a''(j) = a$ for all $j \geqslant i$ we have $\theta(a', a'') = 1$, i.e. $[a']_\theta = [a'']_\theta$. Furthermore, for any n-ary $f^{\mathbf{M}_i} \in F^{\mathbf{M}_i}$ and arbitrary $a_1, \ldots, a_n \in M_i$ it is obvious that

$$\begin{aligned}f^{\mathbf{N}/\theta}(h_i(a_1), \ldots, h_i(a_n)) &= f^{\mathbf{N}/\theta}\left([a'_1]_\theta, \ldots, [a'_n]_\theta\right) = \left[f^{\mathbf{N}}(a'_1, \ldots, a'_n)\right]_\theta \\ &= h_i\left(f^{\mathbf{M}_i}(a_1, \ldots, a_n)\right).\end{aligned}$$

Hence, every $h_i : M_i \to N/\theta$ is compatible with operations. Moreover, h_i is a restriction of h_j on M_i whenever $i \leqslant j$. Now it is possible to define a mapping $h: \bigcup_{i \in I} M_i \to N/\theta$ by $h(a) = h_i(a)$ for all $a \in M_i, i \in I$. Clearly, $f^{\mathbf{N}/\theta}(h(a_1), \ldots, h(a_n)) = h\left(f^{\mathbf{M}_i}(a_1, \ldots, a_n)\right)$. Furthermore, for $a, b \in \bigcup_{i \in I} M_i$,

$a \in M_i, b \in M_j$ we can take $k \geqslant i, j$, thus $a \approx^{\bigcup_{i\in I}\mathbf{M}_i} b = a \approx^{\mathbf{M}_k} b = a'(k) \approx^{\mathbf{M}_k} b'(k)$, where $a', b' \in N$ such that for every $l \geqslant k$ we have $a'(l) = a$, $b'(l) = b$, i.e. $a', b' \in N_{(k)}$. Moreover, we can apply (17) to obtain

$$a'(k) \approx^{\mathbf{M}_k} b'(k) = \theta(a', b') = \left[a'\right]_\theta \approx^{\mathbf{N}/\theta} \left[b'\right]_\theta = h_k(a) \approx^{\mathbf{N}/\theta} h_k(b)$$
$$= h_i(a) \approx^{\mathbf{N}/\theta} h_j(b) = h(a) \approx^{\mathbf{N}/\theta} h(b).$$

Consequently, h is an embedding. Furthermore, for any $[b]_\theta \in N/\theta$ there is an index $i \in I$ such that $b(j) = a$ for some $a \in M_i$ and every $j \geqslant i$. Thus, $h(a) = h_i(a) = [b]_\theta$, i.e. h is surjective. Altogether, $\bigcup_{i\in I} \mathbf{M}_i \cong \mathbf{N}/\theta$. □

3.5. Class operators, varieties, free L-algebras

The notion of a variety of **L**-algebras was introduced in [6]. This section elaborates on the notion of a variety and related notions in more detail.

Definition 3.19. For a class $\mathcal{K}$ of **L**-algebras of the same type we define the following operators:

$$H(\mathcal{K}) = \{h(\mathbf{M}) \mid \mathbf{M} \in \mathcal{K}, h \text{ is a morphism}\},$$
$$I(\mathcal{K}) = \{\mathbf{M} \mid \mathbf{M} \text{ is isomorphic to some } \mathbf{N} \in \mathcal{K}\},$$
$$S(\mathcal{K}) = \{\mathbf{M} \mid \mathbf{M} \text{ is a subalgebra of some } \mathbf{N} \in \mathcal{K}\},$$
$$P(\mathcal{K}) = \{\mathbf{M} \mid \mathbf{M} \text{ is a direct product of a family } P \subseteq \mathcal{K}\},$$
$$P_S(\mathcal{K}) = \{\mathbf{M} \mid \mathbf{M} \text{ is a subdirect product of a family } P \subseteq \mathcal{K}\},$$
$$U(\mathcal{K}) = \{\mathbf{M} \mid \mathbf{M} \text{ is a direct union of a directed family } U \subseteq \mathcal{K}\}.$$

That is, $H(\mathcal{K})$ is the class of all *homomorphic images* of **L**-algebras from $\mathcal{K}$. $I(\mathcal{K})$ is the class of all **L**-algebras *isomorphic* to some $\mathbf{N} \in \mathcal{K}$. $S(\mathcal{K})$ is the class of all *subalgebras* of **L**-algebras from $\mathcal{K}$. $P(\mathcal{K})$ and $P_S(\mathcal{K})$ denote the classes of all *direct* and *subdirect products* of families of **L**-algebras from $\mathcal{K}$, respectively. $U(\mathcal{K})$ denotes the class of all *direct unions* of directed families of **L**-algebras from $\mathcal{K}$.

A class $\mathcal{K}$ of **L**-algebras of type F is said to be *closed under the operator* O if $O(\mathcal{K}) \subseteq \mathcal{K}$. Class operators may be composed. That is, we may have HS, HHPHS, and so on. An operator O is said to be *idempotent* iff $O = OO$. If O_1, O_2 are two operators on classes of **L**-algebras of the same type we put $O_1 \leqslant O_2$ iff $O_1(\mathcal{K}) \subseteq O_2(\mathcal{K})$ for every class $\mathcal{K}$ of **L**-algebras of type F. It is easy to see that $\leqslant$ is a partial order on operators on classes of **L**-algebras of the same type.

Lemma 3.15. *We have* $SH \leqslant HS$, $PS \leqslant SP$, $PH \leqslant HP$. *Furthermore*, H, S, *and* IP *are idempotent.*

Proof. Suppose $\mathbf{M} \in SH(\mathcal{K})$. Then for some $\mathbf{M}' \in \mathcal{K}$ and some epimorphism $h : \mathbf{M}' \to \mathbf{N}$, $\mathbf{M}$ is a subalgebra of $\mathbf{N}$. Theorem 3.4 yields that the inverse image $h^{-1}(\mathbf{M})$ is a subalgebra of $\mathbf{M}'$. Moreover, $h(h^{-1}(\mathbf{M})) = \mathbf{M}$. Hence, it follows that $\mathbf{M} \in HS(\mathcal{K})$.

If $\mathbf{M} \in PS(\mathcal{K})$, then $\mathbf{M} = \prod_{i\in I} \mathbf{M}_i$ for suitable subalgebras $\mathbf{M}_i$ of $\mathbf{M}'_i \in \mathcal{K}$, $i \in I$. Since $\prod_{i\in I} \mathbf{M}_i$ is a subalgebra of $\prod_{i\in I} \mathbf{M}'_i$, we have $\mathbf{M} \in SP(\mathcal{K})$.

Let $\mathbf{M} \in PH(\mathcal{K})$. Then there are $\mathbf{L}$-algebras $\mathbf{M}'_i$ and epimorphisms $h_i : \mathbf{M}'_i \to \mathbf{M}_i$, $i \in I$ such that $\mathbf{M} = \prod_{i\in I}\mathbf{M}_i$. A mapping $h\colon \prod_{i\in I}\mathbf{M}'_i \to \prod_{i\in I}\mathbf{M}_i$ defined by $h(b)(i) = h_i(b(i))$ for all $b \in \prod_{i\in I} M'_i$ is an epimorphism. Indeed, for $a, b \in \prod_{i\in I} M'_i$ we have,

$$
\begin{aligned}
a \approx^{\prod_{i\in I}\mathbf{M}'_i} b &= \bigwedge_{i\in I} a(i) \approx^{\mathbf{M}'_i} b(i) \leqslant \bigwedge_{i\in I} h_i(a(i)) \approx^{\mathbf{M}_i} h_i(b(i)) \\
&= \bigwedge_{i\in I} h(a)(i) \approx^{\mathbf{M}_i} h(b)(i) = h(a) \approx^{\prod_{i\in I}\mathbf{M}_i} h(b).
\end{aligned}
$$

For any n-ary $f^{\prod_{i\in I}\mathbf{M}'_i} \in F^{\prod_{i\in I}\mathbf{M}'_i}$, $a_1, \ldots, a_n \in \prod_{i\in I} M'_i$, the fact

$$
h\left(f^{\prod_{i\in I}\mathbf{M}'_i}(a_1,\ldots,a_n)\right) = f^{\prod_{i\in I}\mathbf{M}_i}\,(h(a_1),\ldots,h(a_n))
$$

follows from the ordinary case. Moreover, h is evidently a surjective mapping. Hence, $\mathbf{M} \in HP(\mathcal{K})$.

The idempotency of H, S is evident. Let $\mathbf{M} \in IPIP(\mathcal{K})$, i.e. $\mathbf{M} \cong \prod_{i\in I}\mathbf{M}'_i$ and $\mathbf{M}'_i \cong \prod_{j\in J_i}\mathbf{N}_{ij}$ for all $i \in I$, where every $\mathbf{N}_{ij}$ belongs to $\mathcal{K}$. Thus, there are isomorphisms $h : \mathbf{M} \to \prod_{i\in I}\mathbf{M}'_i$ and $h_i : \mathbf{M}'_i \to \prod_{j\in J_i}\mathbf{N}_{ij}$, $i \in I$. Put $K = \{\langle i, j\rangle \mid i \in I,\ j \in J_i\}$. Now, a mapping $g : \mathbf{M} \to \prod_{\langle i,j\rangle\in K}\mathbf{N}_{ij}$ defined by $g(a)(\langle i, j\rangle) = h_i\,(h(a)(i))\,(j)$ is an isomorphism. For $a, b \in M$ we have

$$
\begin{aligned}
a \approx^{\mathbf{M}} b &= h(a) \approx^{\prod_{i\in I}\mathbf{M}'_i} h(b) = \bigwedge_{i\in I} h(a)(i) \approx^{\mathbf{M}'_i} h(b)(i) = \bigwedge_{i\in I} h_i(h(a)(i)) \approx^{\prod_{j\in J_i}\mathbf{N}_{ij}} h_i(h(b)(i)) \\
&= \bigwedge_{i\in I}\bigwedge_{j\in J_i} h_i(h(a)(i))(j) \approx^{\mathbf{N}_{ij}} h_i(h(b)(i))(j) = \bigwedge_{i\in I}\bigwedge_{j\in J_i} g(a)(\langle i, j\rangle) \approx^{\mathbf{N}_{ij}} g(b)(\langle i, j\rangle) \\
&= \bigwedge_{\langle i,j\rangle\in K} g(a)(\langle i, j\rangle) \approx^{\mathbf{N}_{ij}} g(b)(\langle i, j\rangle) = g(a) \approx^{\prod_{\langle i,j\rangle\in K}\mathbf{N}_{ij}} g(b).
\end{aligned}
$$

The rest follows from the ordinary case. Hence, IP is idempotent. □

Definition 3.20. A nonempty class $\mathcal{K}$ of $\mathbf{L}$-algebras of type F is called a *variety* if it is closed under homomorphic images, subalgebras and direct products, that is, if $H(\mathcal{K}) \subseteq \mathcal{K}$, $S(\mathcal{K}) \subseteq \mathcal{K}$ and $P(\mathcal{K}) \subseteq \mathcal{K}$. If $\mathcal{K}$ is a class of $\mathbf{L}$-algebras of the same type let $V(\mathcal{K})$ denote the least variety containing $\mathcal{K}$. We say that $V(\mathcal{K})$ is a *variety generated by* $\mathcal{K}$.

Theorem 3.16. *For the class operators V, HSP we have $V = HSP$.*

Proof. Use Lemma 3.15 and the same arguments as in the ordinary case, see [11]. □

Definition 3.21. Let $\mathcal{K}$ be a class of $\mathbf{L}$-algebras of the same type. Let $\mathbf{M}$ be an $\mathbf{L}$-algebra generated by $M' \subseteq M$, i.e. $\mathbf{M} = [M']_{\mathbf{M}}$. If for each $\mathbf{N} \in \mathcal{K}$ and every $\approx$-morphism $h\colon \left\langle M', \approx^{M'}\right\rangle \to \langle N, \approx^{N}\rangle$ there exists a homomorphic extension $h^\sharp\colon \mathbf{M} \to \mathbf{N}$ of h (that is $h^\sharp$ is an morphism and $h^\sharp(a) = h(a)$ for every $a \in M'$), we say that $\mathbf{M}$ has a *universal mapping property* (UMP) *for $\mathcal{K}$ over M'*. In this case M' is said to be a set of free generators of $\mathbf{M}$ over $\mathcal{K}$.

Theorem 3.17. *Suppose that $\mathbf{M}$ has UMP for $\mathcal{K}$ over M'. Let $\mathbf{N} \in \mathcal{K}$. Then for every $\approx$-morphism $h\colon M' \to N$ there exists a unique morphism $h^\sharp\colon \mathbf{M} \to \mathbf{N}$ extending h.*

Proof. Suppose there are two homomorphic extensions, $h^\sharp, h' : \mathbf{M} \to \mathbf{N}$. Since $\mathbf{M}$ is generated by M', for every $a \in M$ there is a term $t \in T(X)$, $|X| = |M'|$ such that $t^{\mathbf{M}}(a_1, \ldots, a_k) = a$ for some $a_1, \ldots, a_k \in M'$. Thus,

$$h^\sharp(a) = h^\sharp\left(t^{\mathbf{M}}(a_1, \ldots, a_k)\right) = t^{\mathbf{N}}\left(h^\sharp(a_1), \ldots, h^\sharp(a_k)\right) = t^{\mathbf{N}}\left(h'(a_1), \ldots, h'(a_k)\right)$$
$$= h'\left(t^{\mathbf{M}}(a_1, \ldots, a_k)\right) = h'(a).$$

Hence, h admits a unique homomorphic extension $h^\sharp$. □

Definition 3.22. Given a type F and a set of variables X, if $T(X) \neq \emptyset$ then the *term* $\mathbf{L}$*-algebra of type* F *over* X, written $\mathbf{T}(X)$, has the universe $T(X)$; functions are defined by

$$f^{\mathbf{T}(X)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n) \tag{18}$$

for any n-ary $f^{\mathbf{T}(X)} \in F^{\mathbf{T}(X)}$ and $t_1, \ldots, t_n \in T(X)$; the $\mathbf{L}$-equality $\approx^{\mathbf{T}(X)}$ is defined by

$$s \approx^{\mathbf{T}(X)} t = \begin{cases} 1 & \text{for } t = s, \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

for all $s, t \in T(X)$. The set X is called a set of *free generators of* $\mathbf{T}(X)$.

Theorem 3.18. *For a type* F, *a set* X *of variables, and a class* $\mathcal{K}$ *of* $\mathbf{L}$*-algebras of type* F, *if* $\mathbf{T}(X)$ *exists, then it has the universal mapping property for* $\mathcal{K}$ *over* X.

Proof. The homomorphic extension $h^\sharp : \mathbf{T}(X) \to \mathbf{M}$ can be defined inductively by $h^\sharp(x) = h(x)$ for every variable $x \in X$ and $h^\sharp(f(t_1, \ldots, t_n)) = f^{\mathbf{M}}\left(h^\sharp(t_1), \ldots, h^\sharp(t_n)\right)$ for every n-ary $f \in F$ and terms $t_1, \ldots, t_n \in T(X)$. Since $\approx^{\mathbf{T}(X)}$ is a crisp identity relation, $h^\sharp$ is evidently an $\approx$-morphism. The rest is evident. □

Definition 3.23. Let $\mathcal{K}$ be a class of $\mathbf{L}$-algebras of the same type, X be a set of variables. Put

$$\Phi_{\mathcal{K}}(X) = \{\phi \in \mathrm{Con}_{\mathbf{L}}(\mathbf{T}(X)) \mid \mathbf{T}(X)/\phi \in IS(\mathcal{K})\}, \tag{20}$$

i.e. $\Phi_{\mathcal{K}}(X)$ is the set of all congruences ϕ on $\mathbf{T}(X)$ such that the factor $\mathbf{L}$-algebra $\mathbf{T}(X)/\phi$ is isomorphic to some subalgebra of some $\mathbf{M} \in \mathcal{K}$. Let

$$\theta_{\mathcal{K}}(X) = \bigcap \Phi_{\mathcal{K}}(X) \tag{21}$$

denote the intersection of all congruences from $\Phi_{\mathcal{K}}(X)$. Then

$$\mathbf{F}_{\mathcal{K}}(\overline{X}) = \mathbf{T}(X)/\theta_{\mathcal{K}}(X), \tag{22}$$

where $\overline{X} = \left\{[x]_{\theta_{\mathcal{K}}(X)} \mid x \in X\right\}$ is the set of generators of $\mathbf{T}(X)/\theta_{\mathcal{K}}(X)$, is called a $\mathcal{K}$*-free* $\mathbf{L}$*-algebra over* $\overline{X}$. For $x \in X$ we write $\overline{x}$ instead of $[x]_{\theta_{\mathcal{K}}(X)}$.

Theorem 3.19. *For a type* F, *a set of variables* X, *and a class* $\mathcal{K}$ *of* $\mathbf{L}$*-algebras of type* F, *if* $\mathbf{T}(X)$ *exists, then* $\mathbf{F}_{\mathcal{K}}(\overline{X})$ *has the universal mapping property for* $\mathcal{K}$ *over* $\overline{X}$.

Proof. Let $\mathbf{M} \in \mathcal{K}$ and take a mapping $g : \left\langle \overline{X}, \approx^{\mathbf{F}_{\mathcal{K}}(\overline{X})} \right\rangle \to \langle M, \approx^{\mathbf{M}} \rangle$. Furthermore, let $n : \mathbf{T}(X) \to \mathbf{F}_{\mathcal{K}}(\overline{X})$ denote the natural morphism, that is $n(t) = [t]_{\theta_{\mathcal{K}}(X)}$ for all $t \in T(X)$. Suppose n_X is a restriction of n on X, i.e. n_X is a mapping $n_X : X \to \overline{X}$. Thus, the universal mapping property of $\mathbf{T}(X)$ implies that there is a morphism $k : \mathbf{T}(X) \to \mathbf{M}$ extending $n_X \circ g$. Moreover, from (i) of Theorem 3.8 it follows that $\mathbf{T}(X)/\theta_k$ is isomorphic to a subalgebra of $\mathbf{M} \in \mathcal{K}$, thus $\theta_k \in \Phi_{\mathcal{K}}(X)$, $\theta_{\mathcal{K}}(X) \subseteq \theta_k$. It is immediate that g is an $\approx$-morphism, since the following inequality holds for every $\overline{x}, \overline{y} \in \overline{X}$:

$$\overline{x} \approx^{\mathbf{F}_{\mathcal{K}}(\overline{X})} \overline{y} = \theta_{\mathcal{K}}(X)(x, y) \leqslant \theta_k(x, y) = g(n_X(x)) \approx^{\mathbf{M}} g(n_X(y)) = g(\overline{x}) \approx^{\mathbf{M}} g(\overline{y}).$$

Now using (ii) of Theorem 3.8 we obtain $\mathbf{F}_{\mathcal{K}}(\overline{X})/(\theta_k/\theta_{\mathcal{K}}(X)) \cong \mathbf{T}(X)/\theta_k$. Hence, there are mappings h, i and j,

$$\mathbf{F}_{\mathcal{K}}(\overline{X}) \xrightarrow{h} \mathbf{F}_{\mathcal{K}}(\overline{X})/(\theta_k/\theta_{\mathcal{K}}(X)) \xrightarrow{i} \mathbf{T}(X)/\theta_k \xrightarrow{j} \mathbf{M},$$

where $h : \mathbf{F}_{\mathcal{K}}(\overline{X}) \to \mathbf{F}_{\mathcal{K}}(\overline{X})/(\theta_k/\theta_{\mathcal{K}}(X))$ is a natural mapping, i is an isomorphism between $\mathbf{F}_{\mathcal{K}}(\overline{X})/(\theta_k/\theta_{\mathcal{K}}(X))$ and $\mathbf{T}(X)/\theta_k$. Finally, using (i) of Theorem 3.8, $j : \mathbf{T}(X)/\theta_k \to \mathbf{M}$ is defined by $j([t]_{\theta_k}) = k(t)$ for every $t \in T(X)$. Thus,

$$\begin{aligned}(h \circ i \circ j)(\overline{x}) &= j\,(i\,(h(\overline{x}))) = j\left(i\left([\overline{x}]_{\theta_k/\theta_{\mathcal{K}}(X)}\right)\right) = j\left([x]_{\theta_k}\right) = k(x) = g\,(n_X(x)) \\ &= g\left([x]_{\theta_{\mathcal{K}}(X)}\right) = g(\overline{x}).\end{aligned}$$

Altogether $h \circ i \circ j$ is a homomorphic extension of g. □

Theorem 3.20. *Suppose* $\mathbf{T}(X)$ *exists. Then for* $\mathcal{K} \neq \emptyset$ *we have* $\mathbf{F}_{\mathcal{K}}(\overline{X}) \in ISP(\mathcal{K})$. *Thus if* $\mathcal{K}$ *is closed under* I, S, P, *in particular if* $\mathcal{K}$ *is a variety, then* $\mathbf{F}_{\mathcal{K}}(\overline{X}) \in \mathcal{K}$.

Proof. Theorem 3.9 and Lemma 3.11 yield that for $\mathbf{F}_{\mathcal{K}}(\overline{X}) = \mathbf{T}(X)/\theta_{\mathcal{K}}(X)$ there is a subdirect embedding $h : \mathbf{T}(X)/\theta_{\mathcal{K}}(X) \to \prod_{\theta \in \Phi_{\mathcal{K}}(X)} \mathbf{T}(X)/\theta$. Using $P_S(\mathcal{K}) \subseteq SP(\mathcal{K})$ and Lemma 3.15 we obtain

$$\begin{aligned}\mathbf{F}_{\mathcal{K}}(\overline{X}) &\in IP_S\left(\{\mathbf{T}(X)/\theta \mid \theta \in \Phi_{\mathcal{K}}(X)\}\right) \\ &\subseteq IP_S\left(\{\mathbf{T}(X)/\theta \mid \theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{T}(X)) \text{ and } \mathbf{T}(X)/\theta \in IS(\mathcal{K})\}\right) \\ &\subseteq IP_S IS(\mathcal{K}) \subseteq ISPIS(\mathcal{K}) \subseteq ISIPS(\mathcal{K}) \subseteq ISISP(\mathcal{K}) \subseteq ISP(\mathcal{K})\end{aligned}$$

completing the proof. □

Theorem 3.21. *Let* $\mathcal{K}$ *denote a class of* $\mathbf{L}$*-algebras of the same type,* $\mathbf{M} \in \mathcal{K}$. *Then for some sufficiently large set of variables* X *we have* $\mathbf{M} \in H(\mathbf{F}_{\mathcal{K}}(\overline{X}))$.

Proof. Choose a set of variables X such that $|M| \leqslant |X|$. Every surjective mapping $v : \overline{X} \to M$ has a homomorphic extension $h : \mathbf{F}_{\mathcal{K}}(\overline{X}) \to \mathbf{M}$. Hence, we have $\mathbf{M} \in H(\mathbf{F}_{\mathcal{K}}(\overline{X}))$. □

Thus, we have the following consequence.

Corollary 3.2. $\mathcal{K} \subseteq H\left(\{\mathbf{F}_{\mathcal{K}}(\overline{X}) \mid X \textit{ is a set of variables}\}\right)$.

Theorem 3.22. *Every variety $\mathcal{K}$ of* **L***-algebras is closed under direct unions. Moreover,* $\mathcal{K} = HSP(\{\mathbf{F}_{\mathcal{K}}(\overline{X})\})$ *for X being a denumerable set of variables.*

Proof. Let $\{\mathbf{M}_i \in \mathcal{K} \mid i \in I\}$ be a directed family of **L**-algebras. Lemma 3.14 yields that $\bigcup_{i\in I} \mathbf{M}_i \cong \mathbf{N}/\theta$ for a suitable **L**-algebra $\mathbf{N} \in \mathrm{Sub}(\prod_{i\in I}\mathbf{M}_i)$ and $\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{N})$. Since $\mathcal{K} = HSP(\mathcal{K})$, we have $\mathbf{N} \in SP(\mathcal{K}) = \mathcal{K}$ and $\mathbf{N}/\theta \in H(\{\mathbf{N}\}) \subseteq \mathcal{K}$, i.e. $\bigcup_{i\in I} \mathbf{M}_i \in I(\mathcal{K}) = \mathcal{K}$. Thus, every variety is closed under the formation of direct unions.

Let us have $\mathbf{M} \in \mathcal{K}$. Every finitely generated **L**-algebra $[M']_{\mathbf{M}}$, is an image of $\mathbf{F}_{\mathcal{K}}(\overline{X})$, where X is a denumerable set of variables. Indeed, let $h : \overline{X} \to [M']_{\mathbf{M}}$ be any mapping such that $h(\overline{X}) = M'$. This mapping can be extended to a morphism $h^\sharp : \mathbf{F}_{\mathcal{K}}(\overline{X}) \to [M']_{\mathbf{M}}$. Take an element $a \in [M']_{\mathbf{M}}$. Since $[M']_{\mathbf{M}}$ is finitely generated, we can express $a = t^{\mathbf{M}}(a_1, \ldots, a_n)$, where $M' = \{a_1, \ldots, a_n\}$, $t \in T(X)$. Thus for $\overline{x}_{i_1}, \ldots, \overline{x}_{i_n} \in \overline{X}$ such that $h^\sharp(\overline{x}_{i_1}) = a_1, \ldots, h^\sharp(\overline{x}_{i_n}) = a_n$ we have

$$h^\sharp\left(\left[t(x_{i_1}, \ldots, x_{i_n})\right]_{\theta_{\mathcal{K}}(X)}\right) = h^\sharp\left(t^{\mathbf{F}_{\mathcal{K}}(\overline{X})}(\overline{x}_{i_1}, \ldots, \overline{x}_{i_n})\right) = t^{\mathbf{M}}\left(h^\sharp(\overline{x}_{i_1}), \ldots, h^\sharp(\overline{x}_{i_n})\right) = a.$$

Hence, $h^\sharp : \mathbf{F}_{\mathcal{K}}(\overline{X}) \to [M']_{\mathbf{M}}$ is a surjective morphism. Consequently, $[M']_{\mathbf{M}} \in H(\{\mathbf{F}_{\mathcal{K}}(\overline{X})\}) \subseteq HSP(\{\mathbf{F}_{\mathcal{K}}(\overline{X})\})$. Using Theorem 3.15 it follows that

$$\mathbf{M} \cong \bigcup\left\{[M']_{\mathbf{M}} \in HSP(\{\mathbf{F}_{\mathcal{K}}(\overline{X})\}) \mid M' \text{ is a finite subset of } M\right\}$$

for every $\mathbf{M} \in \mathcal{K}$. Since $HSP(\{\mathbf{F}_{\mathcal{K}}(\overline{X})\})$ is a variety, i.e. it is closed under direct unions, we have $\mathbf{M} \in HSP(\{\mathbf{F}_{\mathcal{K}}(\overline{X})\})$, i.e. $\mathcal{K} \subseteq HSP(\{\mathbf{F}_{\mathcal{K}}(\overline{X})\})$. The converse inclusion follows from Theorem 3.20. □

In [6], a characterization of varieties of **L**-algebras is presented in a way analogous to a well-known result by Birkhoff [10]. Given an **L**-set Σ of identities, one may consider the class $\mathrm{Mod}(\Sigma)$ (models of Σ) of all **L**-algebras **M** such that for each identity $t \approx t'$, the degree $\|t \approx t'\|_{\mathbf{M}}$ to which $t \approx t'$ is true in **M** is at least $\Sigma(t \approx t')$. The following is the main result of [6]:

Theorem 3.23. *A class $\mathcal{K}$ of* **L***-algebras of the same type is a variety iff there is an* **L***-set Σ of identities such that* $\mathcal{K} = \mathrm{Mod}(\Sigma)$.

4. Conclusions and further research

In the paper, we presented examples, fundamental notions, and several results related to algebras with fuzzy equalities. The presented examples demonstrate that algebras with fuzzy equalities are natural "fuzzy structures" which appear in the study of mathematical fuzzy logic (namely, they are the semantical structures for equational fragment of first-order fuzzy logic) and also as abstractions of natural examples in "fuzzy mathematics" (e.g., functions mapping similar to similar, closure operators, concept lattices). The presented results provide us with a new look on classical results from universal algebra not only since they show that the classical results have generalizations which are not artificial and come from natural examples but also since they explicitly show the assumptions necessary for the results to be true. The latter is particularly apparent if the results in fuzzy setting need restrictive assumptions as in the case of subdirect representation theorem.

The follow-up paper [30] deals with advanced topics in algebras with fuzzy equalities (direct limits, reduced products, etc.). Note also that results of this paper and those of [30] are substantially used in our further study of the equational fragment of first-order fuzzy logic; two papers continuing [4,6] are under submission.

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