FUZZY ATTRIBUTE LOGIC: ATTRIBUTE IMPLICATIONS, THEIR VALIDITY, ENTAILMENT, AND NON-REDUNDANT BASIS*

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ABSTRACT: We present results related to if-then rules and extraction of if-then rules from data tables with fuzzy attibutes. We provide basic definitions and results related to validity of if-then rules, semantic entailment, non-redundant bases, automatic generation of non-redundant bases, and relationships to concept lattices over data with fuzzy attributes.

Keywords: data table, fuzzy logic, fuzzy attribute, if-then rule, non-redundant basis

1 INTRODUCTION

If-then rules (implications) are perhaps the most popular patterns being extracted from object-attribute data tables. In our paper, we are interested in if-then rules generated from data with fuzzy attributes: rows and columns of data table correspond to objects $x \in X$ and attributes $y \in Y$, respectively. Table entries $I(x,y)$ are truth degrees to which object x has attribute y. We are interested in rules of the form "if A then B" ($A \Rightarrow B$), where A and B are collections of attributes, with the meaning: if an object has all the attributes of A then it has also all attributes of B. In crisp case, these rules, called attribute implications, were thoroughly investigated, see e.g. [8] for the first paper and [7] for further information and references. Our aim is basically to look at such rules from the point of view of fuzzy logic. Attribute implications in fuzzy setting were for the first time studied in [11]. The main difference between our approach and that of [11] is that we use, as an additional parameter, a unary connective *, and a different approach to pseudointents (see later) which enables us to obtain a complete and non-redundant basis of attribute implications which, by a suitable setting of *, is also minimal w.r.t. its size. We comment in detail on the results of [11] in a forthcoming paper. Our paper is accompanied by [4] and [6] in which we describe an algorithm for computing a complete and non-redundant set of attribute implications from data table with fuzzy attributes, and some relationship between implications from data with fuzzy attributes and the corresponding data with scaled (transformed) crisp attributes.

In the present paper, we present theoretical results related to if-then rules and extraction of if-then rules from data tables with fuzzy attibutes. We provide basic definitions and results related to validity of if-then rules, semantic entailment, non-redundant bases, automatic generation of non-redundant bases, and relationships to concept lattices over data with fuzzy attributes.

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2 PRELIMINARIES

As a set of truth degrees equipped with suitable operations (truth functions of logical connectives) we use a so-called complete residuated lattice with truth-stressing hedge. A complete residuated lattice with truth-stressing hedge (shortly, a hedge) is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, ^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

for each $a,b,c \in L$; hedge * satisfies

$$1^* \leq 1, \tag{2}$$
$$a^* \leq a, \tag{3}$$
$$(a \rightarrow b)^* \leq a^* \rightarrow b^*, \tag{4}$$
$$a^{**} = a^*, \tag{5}$$

for each $a,b \in L$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true", see [9, 10]. Properties (2)–(5) have natural interpretations, e.g. (3) can be read: "if a is very true, then a is true", (4) can be read: "if $a \rightarrow b$ is very true and if a is very true, then b is very true", etc.

A common choice of $\mathbf{L}$ is a structure with $L = [0,1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are:

$$\text{Łukasiewicz:} \quad \begin{array}{rcl} a \otimes b &=& \max(a+b-1,0), \\ a \rightarrow b &=& \min(1-a+b,1), \end{array} \tag{6}$$

$$\text{Gödel:} \quad \begin{array}{rcl} a \otimes b &=& \min(a,b), \\ a \rightarrow b &=& \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases} \end{array} \tag{7}$$

$$\text{Goguen (product):} \quad \begin{array}{rcl} a \otimes b &=& a \cdot b, \\ a \rightarrow b &=& \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases} \end{array} \tag{8}$$

In applications, we usually need a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of L and restrictions of Gödel operations on $[0,1]$ to L. A special case of both of these chains is the Boolean algebra with $L=\{0,1\}$ (structure of truth degrees of classical logic).

Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^*=a$ ($a\in L$); (ii) globalization [12]:

$$a^* = \begin{cases} 1 & \text{if } a=1, \\ 0 & \text{otherwise.} \end{cases} \qquad (9)$$

Having $\mathbf{L}$ as our structure of truth degrees, we define usual notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A\colon U\to L$, $A(u)$ being interpreted as "the degree to which u belongs to A". If $U=\{u_1,\dots,u_n\}$ then A can be denoted by $A=\{{}^{a_1}/u_1,\dots,{}^{a_n}/u_n\}$ meaning that $A(u_i)$ equals a_i for each $i=1,\dots,n$. For brevity, we introduce the following convention: we write $\{\dots,u,\dots\}$ instead of $\{\dots,{}^{1}/u,\dots\}$, and we also omit elements of U whose membership degree is zero. For example, we write $\{u,{}^{0.5}/v\}$ instead of $\{{}^{1}/u,{}^{0.5}/v,{}^{0}/w\}$, etc. Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, intersection of $\mathbf{L}$-sets $A,B\in\mathbf{L}^U$ is an $\mathbf{L}$-set $A\cap B$ in U such that $(A\cap B)(u)=A(u)\wedge B(u)$ for each $u\in U$; $A^*(u)=A(u)^*$ for each $u\in U$, etc. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in $X\times Y$.

Given $A,B\in\mathbf{L}^U$, we define a subsethood degree

$$S(A,B)=\textstyle\bigwedge_{u\in U}\bigl(A(u)\to B(u)\bigr), \qquad (10)$$

which generalizes the classical subsethood relation $\subseteq$ (note that unlike $\subseteq$, S is a binary $\mathbf{L}$-relation on $\mathbf{L}^U$). Described verbally, $S(A,B)$ represents a degree to which A is a subset of B. In particular, we write $A\subseteq B$ iff $S(A,B)=1$. As a consequence, we have $A\subseteq B$ iff $A(u)\leq B(u)$ for each $u\in U$. In the following we use well-known properties of residuated lattices and fuzzy structures which can be found in monographs [1, 9].

3 FUZZY ATTRIBUTE IMPLICATIONS

3.1 Definition, validity, and basic properties

Fuzzy attribute implication (*over attributes Y*) is an expression $A\Rightarrow B$, where $A,B\in\mathbf{L}^Y$ (A and B are fuzzy sets of attributes). The intended meaning of $A\Rightarrow B$ is: "if it is (very) true that an object has all attributes from A, then it has also all attributes from B". The notions "being very true", "to have an attribute", and logical connective "if-then" are determined by the chosen $\mathbf{L}$.

For an $\mathbf{L}$-set $M\in\mathbf{L}^Y$ of attributes, we define a *degree* $||A\Rightarrow B||_M\in L$ *to which* $A\Rightarrow B$ *is valid in* M:

$$||A\Rightarrow B||_M = S(A,M)^*\to S(B,M). \qquad (11)$$

If M is the fuzzy set of all attributes of an object x, then $||A\Rightarrow B||_M$ is the truth degree to which $A\Rightarrow B$ holds for x.

Fuzzy attribute implications can be used to describe dependencies in data tables with fuzzy attributes. Let X and Y be sets of objects and attributes, respectively, I be an $\mathbf{L}$-relation between X and Y, i.e. I is a mapping $I\colon X\times Y\to L$. $\langle X,Y,I\rangle$ is called a *data table with fuzzy attributes.* $\langle X,Y,I\rangle$ represents a table which assigns to each $x\in X$ and each $y\in Y$ a truth degree $I(x,y)\in L$ to which object x has attribute y. Another way of looking at $\langle X,Y,I\rangle$ is the following: fix $y\in Y$, and focus on the column of $\langle X,Y,I\rangle$ corresponding with y. This column can be seen as an $\mathbf{L}$-set $X_y\in\mathbf{L}^X$ of objects so that $X_y(x)=I(x,y)$ for each $x\in X$. X_y can be called a *fuzzy attribute*; an object $x\in X$ has the fuzzy attribute X_y in degree to which x belongs to X_y. If both X and Y are finite, $\langle X,Y,I\rangle$ can be visualized as in Fig. 1.

I	$\cdots$	y	$\cdots$
$\vdots$		$\vdots$	
x	$\cdots$	$I(x,y)$	$\cdots$
$\vdots$		$\vdots$	

Figure 1: Data table with fuzzy attributes

For $A\in\mathbf{L}^X$, $B\in\mathbf{L}^Y$ (i.e. A is a fuzzy set of objects, B is a fuzzy set of attributes), we define fuzzy sets $A^{\uparrow}\in\mathbf{L}^Y$ (fuzzy set of attributes), $B^{\downarrow}\in\mathbf{L}^X$ (fuzzy set of objects) by

$$A^{\uparrow}(y) \;=\; \textstyle\bigwedge_{x\in X}\bigl(A(x)^*\to I(x,y)\bigr), \qquad (12)$$

$$B^{\downarrow}(x) \;=\; \textstyle\bigwedge_{y\in Y}\bigl(B(y)\to I(x,y)\bigr). \qquad (13)$$

We put

$$\mathcal{B}(X^*,Y,I)=\{\langle A,B\rangle\in\mathbf{L}^X\times\mathbf{L}^Y\,|\,A^{\uparrow}=B,\,B^{\downarrow}=A\}$$

and define for $\langle A_1,B_1\rangle,\langle A_2,B_2\rangle\in\mathcal{B}(X^*,Y,I)$ a binary relation $\leq$ by $\langle A_1,B_1\rangle\leq\langle A_2,B_2\rangle$ iff $A_1\subseteq A_2$ (or, iff $B_2\subseteq B_1$; both ways are equivalent). Operators ${}^{\downarrow},{}^{\uparrow}$ form so-called Galois connection with hedge, see [5]. The structure $\langle\mathcal{B}(X^*,Y,I),\leq\rangle$ is called a *fuzzy concept lattice* induced by $\langle X,Y,I\rangle$. The elements $\langle A,B\rangle$ of $\mathcal{B}(X^*,Y,I)$ are naturally interpreted as concepts (clusters) hidden in the input data represented by I. Namely, $A^{\uparrow}=B$ and $B^{\downarrow}=A$ say that B is the collection of all attributes shared by all objects from A, and A is the collection of all objects sharing all attributes from B. Note that these conditions represent exactly the definition of a concept as developed in the so-called Port-Royal logic; A and B are called the *extent* and the *intent* of the concept $\langle A,B\rangle$, respectively, and represent the collection of all objects and all attributes covered by the particular concept. Furthermore, $\leq$ models the natural subconcept-superconcept hierarchy—concept $\langle A_1,B_1\rangle$ is a subconcept of $\langle A_2,B_2\rangle$ iff each object from A_1 belongs to A_2 (dually for attributes).

Now we define a validity degree of fuzzy attribute implications in data tables and intents of fuzzy concept lattices. First, for a set $\mathcal{M}\subseteq\mathbf{L}^Y$ (i.e. $\mathcal{M}$ is an ordinary set of $\mathbf{L}$-sets) we define a degree $||A\Rightarrow B||_{\mathcal{M}}\in L$ to which $A\Rightarrow B$ holds in $\mathcal{M}$ by

$$||A\Rightarrow B||_{\mathcal{M}}=\textstyle\bigwedge_{M\in\mathcal{M}}||A\Rightarrow B||_M. \qquad (14)$$

Having $\langle X,Y,I\rangle$, let $I_x\in\mathbf{L}^Y$ ($x\in X$) be an $\mathbf{L}$-set of attributes such that $I_x(y)=I(x,y)$ for each $y\in Y$. Described verbally, I_x is the $\mathbf{L}$-set of all attributes of $x\in X$ in $\langle X,Y,I\rangle$, I_x corresponds to a row labeled x. Clearly, we have $I_x=\{{}^{1}/x\}^{\uparrow}$ for each $x\in X$.

A *degree* $||A\Rightarrow B||_{\langle X,Y,I\rangle}\in L$ *to which* $A\Rightarrow B$ *holds in* (*each row of*) $\langle X,Y,I\rangle$ is defined by

$$||A\Rightarrow B||_{\langle X,Y,I\rangle}=||A\Rightarrow B||_{\mathcal{M}},\text{ where }\mathcal{M}=\{I_x\,|\,x\in X\}. \qquad (15)$$

Denote

$$\mathrm{Int}(X^*,Y,I) = \{B \in \mathbf{L}^Y \mid \langle A,B\rangle \in \mathcal{B}(X^*,Y,I) \text{ for some } A\}$$

the set of all intents of concepts of $\mathcal{B}(X^*,Y,I)$. Thus, we can consider a *degree* $||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)} \in L$ *to which* $A \Rightarrow B$ *holds in the system of all intents of* $\mathcal{B}(X^*,Y,I)$. Not that since $M \in \mathbf{L}^Y$ is an intent of some concept of $\mathcal{B}(X^*,Y,I)$ iff $M = M^{\downarrow\uparrow}$, we have $\mathrm{Int}(X^*,Y,I) = \{M \in \mathbf{L}^Y \mid M = M^{\downarrow\uparrow}\}$.

For convenience, we introduce the following notation. Let $\langle^{\Uparrow},^{\Downarrow}\rangle$ denote $\langle^{\uparrow},^{\downarrow}\rangle$ defined by (12) and (13) with * being identity. That is, for each hedge * and the induced mappings $\langle^{\uparrow},^{\downarrow}\rangle$ we have $A^{\uparrow} = (A^*)^{\Uparrow} = A^{*\Uparrow}$ and $B^{\downarrow} = B^{\Downarrow}$ ($A \in \mathbf{L}^X$, $B \in \mathbf{L}^Y$).

The following assertions connect the degree to which $A \Rightarrow B$ holds in $\langle X,Y,I\rangle$, the degree to which $A \Rightarrow B$ holds in intents of $\mathcal{B}(X^*,Y,I)$, and the degree of subsethood of B in $A^{\downarrow\uparrow}$.

Lemma 1 $||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)}$.

Proof. "$\leq$": Observe that $||A \Rightarrow B||_{\langle X,Y,I\rangle} \leq ||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)}$ iff $||A \Rightarrow B||_{\langle X,Y,I\rangle} \leq ||A \Rightarrow B||_M$ for each $M \in \mathrm{Int}(X^*,Y,I)$ iff

$$\textstyle\bigwedge_{x\in X}\big(S(A,I_x)^* \to S(B,I_x)\big) \leq S(A,M)^* \to S(B,M)$$

for each $M \in \mathrm{Int}(X^*,Y,I)$ iff

$$\textstyle\bigwedge_{x\in X}\big(A^{\Downarrow}(x)^* \to B^{\Downarrow}(x)\big) \leq S(A,M)^* \to S(B,M)$$

for each $M \in \mathrm{Int}(X^*,Y,I)$ iff

$$\textstyle\bigwedge_{x\in X}\big((A^{\Downarrow *}(x) \to B^{\Downarrow}(x)\big) \leq S(A,M)^* \to S(B,M)$$

for each $M \in \mathrm{Int}(X^*,Y,I)$ iff

$$S(A^{\Downarrow *},B^{\Downarrow}) \leq S(A,M)^* \to S(B,M)$$

for each $M \in \mathrm{Int}(X^*,Y,I)$ iff

$$S(A,M)^* \otimes S(A^{\Downarrow *},B^{\Downarrow}) \leq S(B,M) \tag{16}$$

for each $M \in \mathrm{Int}(X^*,Y,I)$. Thus, it suffices to prove (16) for each $M \in \mathbf{L}^Y$ satisfying $M = M^{\Downarrow *\Uparrow}$. We have

$$\begin{aligned}S(A,M)^* \otimes S(A^{\Downarrow *},B^{\Downarrow}) &\leq S(M^{\Downarrow},A^{\Downarrow})^* \otimes S(A^{\Downarrow *},B^{\Downarrow}) \leq\\ &\leq S(M^{\Downarrow *},A^{\Downarrow *}) \otimes S(A^{\Downarrow *},B^{\Downarrow}) \leq S(M^{\Downarrow *},B^{\Downarrow}) \leq\\ &\leq S(B^{\Downarrow\Uparrow},M^{\Downarrow *\Uparrow}) = S(B^{\Downarrow\Uparrow},M) \leq S(B,M),\end{aligned}$$

showing (16).

"$\geq$": We prove the claim by showing that each I_x ($x \in X$) is a concept intent. Take $x \in X$. We have

$$I_x = \{{}^1/x\}^{\uparrow} = \{{}^1/x\}^{*\Uparrow} = \{{}^1/x\}^{*\Uparrow\Downarrow *\Uparrow} = {I_x}^{\Downarrow *\Uparrow} = {I_x}^{\downarrow\uparrow}.$$

Hence, $I_x \in \mathrm{Int}(X^*,Y,I)$. □

Lemma 2 $||A \Rightarrow B||_{\langle X,Y,I\rangle} = S(B,A^{\downarrow\uparrow})$.

Proof. We have

$$\begin{aligned}&||A \Rightarrow B||_{\langle X,Y,I\rangle} =\\ &\quad= \textstyle\bigwedge_{x\in X}\big(S(A,I_x)^* \to S(B,I_x)\big) =\\ &\quad= \textstyle\bigwedge_{x\in X}\big(A^{\Downarrow}(x)^* \to B^{\Downarrow}(x)\big) =\\ &\quad= \textstyle\bigwedge_{x\in X}\big(A^{\Downarrow *}(x) \to B^{\Downarrow}(x)\big) =\\ &\quad= \textstyle\bigwedge_{x\in X}\big(A^{\Downarrow *}(x) \to \bigwedge_{y\in Y}\big(B(y) \to I(x,y)\big)\big) =\\ &\quad= \textstyle\bigwedge_{y\in Y}\bigwedge_{x\in X}\big(A^{\Downarrow *}(x) \to \big(B(y) \to I(x,y)\big)\big) =\\ &\quad= \textstyle\bigwedge_{y\in Y}\bigwedge_{x\in X}\big(B(y) \to \big(A^{\Downarrow *}(x) \to I(x,y)\big)\big) =\\ &\quad= \textstyle\bigwedge_{y\in Y}\big(B(y) \to \bigwedge_{x\in X}\big(A^{\Downarrow *}(x) \to I(x,y)\big)\big) =\\ &\quad= \textstyle\bigwedge_{y\in Y}\big(B(y) \to A^{\Downarrow *\Uparrow}(y)\big) = S(B,A^{\Downarrow *\Uparrow}) = S(B,A^{\downarrow\uparrow}),\end{aligned}$$

proving the claim. □

Theorem 3 *Let* $\langle X,Y,I\rangle$ *be a data table with fuzzy attributes. Then*

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)} = S(B,A^{\downarrow\uparrow}) \tag{17}$$

for each fuzzy attribute implication $A \Rightarrow B$.

Proof. Consequence of Lemma 1 and Lemma 2. □

3.2 Semantic entailment and bases

Let $\langle X,Y,I\rangle$ be a data table with fuzzy attributes and let T be a set of fuzzy attribute implications. $M \in \mathbf{L}^Y$ is called a *model* of T if $||A \Rightarrow B||_M = 1$ for each $A \Rightarrow B \in T$. The set of all models of T is denoted by $\mathrm{Mod}(T)$, i.e.

$$\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid M \text{ is a model of } T\}. \tag{18}$$

A degree $||A \Rightarrow B||_T \in L$ to which $A \Rightarrow B$ *semantically follows* from T is defined by

$$||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{Mod}(T)}. \tag{19}$$

T is called *complete* (in $\langle X,Y,I\rangle$) if $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\langle X,Y,I\rangle}$ for each $A \Rightarrow B$. If T is complete and no proper subset of T is complete, then T is called a *non-redundant basis* (of $\langle X,Y,I\rangle$). Note that both the notions of a complete set and a non-redundant basis refer to a given data table with fuzzy attributes.

Complete sets of fuzzy attribute implications, and their special case of non-redundant bases, are interesting from several viewpoints. The following assertion shows that the models of a complete set of fuzzy attribute implications are exactly the intents of the corresponding fuzzy concept lattice.

Theorem 4 T *is complete iff* $\mathrm{Mod}(T) = \mathrm{Int}(X^*,Y,I)$.

Proof. See [4]. □

We are interested in finding non-redundant bases. The motivation is twofold. First, a non-redundant basis T is a minimal set of implications which conveys, via the notion of semantic entailment, information about validity of attribute implications in $\langle X,Y,I\rangle$. In particular, attribute implications which are true (in degree 1) in $\langle X,Y,I\rangle$ are exactly those which follow (in degree 1) from T. Second, non-redundant bases are promising candidates for being the minimal complete sets of attribute implications which describe the concept intents (and consequently, the whole fuzzy concept lattice).

3.3 Non-redundant bases

We present particular non-redundant bases which can be obtained using the so-called systems of pseudo-intents. In the classical case, the notion of a pseudo-intent was introduced in [8]. Unlike the classical case, the pseudo-intents of data tables with fuzzy attributes are not uniquely given in general and lack several properties which are automatically available in the classical case. In the sequel we present more general approach to pseudo-intents than that one introduced in [8] (the classical results are now a special case of ours for $\mathbf{L}$ being the two-element Boolean algebra).

Definition 5 *Given $\langle X,Y,I\rangle$, $\mathcal{P} \subseteq \mathbf{L}^Y$ (a system of fuzzy sets of attributes) is called a* system of pseudo-intents *of $\langle X,Y,I\rangle$ if for each $P \in \mathbf{L}^Y$ we have:*

$$P \in \mathcal{P} \quad \text{iff} \quad P \neq P^{\downarrow\uparrow} \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P = 1 \text{ for each } Q \in \mathcal{P} \text{ with } Q \neq P.$$

It is easily seen that if $\mathbf{L}$ is a complete residuated lattice with globalization then $\mathcal{P}$ is a system of pseudo-intents of $\langle X,Y,I\rangle$ if for each $P \in \mathbf{L}^Y$ we have:

$$P \in \mathcal{P} \quad \text{iff} \quad P \neq P^{\downarrow\uparrow} \text{ and } Q^{\downarrow\uparrow} \subseteq P \text{ for each } Q \in \mathcal{P} \text{ with } Q \subset P.$$

In addition to that, in case of finite $\mathbf{L}$, for each data table with finite set of attributes there is exactly one system of pseudo-intents which can be described recursively in much the same way as in the classical case [7, 8]:

Theorem 6 *Let $\mathbf{L}$ be a finite residuated lattice with globalization. Then for each $\langle X,Y,I\rangle$ with finite Y there is a unique system of pseudo-intents $\mathcal{P}$ of $\langle X,Y,I\rangle$ and*

$$\mathcal{P} = \{P \in \mathbf{L}^Y \mid P \neq P^{\downarrow\uparrow} \text{ and } Q^{\downarrow\uparrow} \subseteq P \text{ holds for each } Q \in \mathcal{P} \text{ such that } Q \subset P\}.$$

Proof. Easy by induction and therefore omitted. □

Neither the uniqueness of $\mathcal{P}$ nor the existence of $\mathcal{P}$ is assured for general $\mathbf{L}$ and $\langle X,Y,I\rangle$ as we will see later on.

Until otherwise mentioned, we assume a fuzzy data table $\langle X,Y,I\rangle$ is given and we let $\mathcal{P}$ denote a system of pseudo-intents of $\langle X,Y,I\rangle$ (which is supposed to exist). In the sequel we are going to show that a set

$$T = \{P \Rightarrow P^{\downarrow\uparrow} \,|\, P \in \mathcal{P}\} \tag{20}$$

is a non-redundant basis of $\langle X,Y,I\rangle$.

Lemma 7 $||A \Rightarrow A^{\downarrow\uparrow}||_M = 1$ *for each $A \in \mathbf{L}^Y$ and $M = M^{\downarrow\uparrow}$.*

Proof. Let $M = M^{\downarrow\uparrow}$. We have

$$S(A,M)^* \leq S(M^{\Downarrow},A^{\Downarrow})^* \leq S(M^{\Downarrow*},A^{\Downarrow*}) \leq \leq S(A^{\Downarrow*\Uparrow},M^{\Downarrow*\Uparrow}) = S(A^{\downarrow\uparrow},M^{\downarrow\uparrow}) = S(A^{\downarrow\uparrow},M).$$

Thus, $S(A,M)^* \to S(A^{\downarrow\uparrow},M) = 1$, i.e. $||A \Rightarrow A^{\downarrow\uparrow}||_M = 1$. □

Lemma 8 $\mathrm{Mod}(T) \subseteq \mathrm{Int}(X^*,Y,I)$ *for $T = \{P \Rightarrow P^{\downarrow\uparrow} \,|\, P \in \mathcal{P}\}$.*

Proof. It suffices to show that each model $M \in \mathrm{Mod}(T)$ is an intent of $\mathrm{Int}(X^*,Y,I)$. By contradiction, let $M \in \mathrm{Mod}(T)$ and assume $M \notin \mathrm{Int}(X^*,Y,I)$. That is, $M \neq M^{\downarrow\uparrow}$.

Since $M \in \mathrm{Mod}(T)$, we have $||Q \Rightarrow Q^{\downarrow\uparrow}||_M = 1$ ($Q \in \mathcal{P}$). Therefore, $M \in \mathcal{P}$ by definition of $\mathcal{P}$, i.e. $M \Rightarrow M^{\downarrow\uparrow}$ belongs to T. We have

$$||M \Rightarrow M^{\downarrow\uparrow}||_M = S(M,M)^* \to S(M^{\downarrow\uparrow},M) = = 1^* \to S(M^{\downarrow\uparrow},M) = S(M^{\downarrow\uparrow},M) \neq 1,$$

which contradicts $M \in \mathrm{Mod}(T)$. □

Theorem 9 $T = \{P \Rightarrow P^{\downarrow\uparrow} \,|\, P \in \mathcal{P}\}$ *is complete.*

Proof. We show that $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)}$ for each fuzzy attribute implication $A \Rightarrow B$. The completeness of T will then be a consequence of Lemma 1. By Lemma 7, we have that each intent $M \in \mathrm{Int}(X^*,Y,I)$ is a model of T, proving the $\leq$-part. The $\geq$-part follows from Lemma 8. □

Remark For general complete residuated lattices with hedges taken as structures of truth degrees, there are finite data tables with fuzzy attributes for which there does not exist any system of pseudo-intents not even if we use a globalization as the truth-stressing hedge. For instance, let $\mathbf{L}$ be the standard Łukasiewicz algebra (defined on the real unit interval) with globalization and let $X = \{x\}$, $Y = \{y\}$, and $I(x,y) = 0$. It is easily seen that $\mathrm{Int}(X^*,Y,I) = \{\{\},\{y\}\}$.

By contradiction, let there be a system of pseudo-intents $\mathcal{P}$. Put $T = \{P \Rightarrow P^{\downarrow\uparrow} \,|\, P \in \mathcal{P}\}$. Theorem 9 and Theorem 4 give that $\mathrm{Mod}(T) = \{\{\},\{y\}\}$. Thus, for $a \in (0,1)$ there must be $c \in [0,1]$ such that $\{{}^{c}\!/y\} \in \mathcal{P}$ and $||\{{}^{c}\!/y\} \Rightarrow \{{}^{c}\!/y\}^{\downarrow\uparrow}||_{\{{}^{a}\!/y\}} \neq 1$. Since * is a globalization, it follows that $c \in (0,a]$. Take $b \in (0,c)$. By repeating the above idea, there is $d \in (0,b]$ such that $\{{}^{d}\!/y\} \in \mathcal{P}$ and $||\{{}^{d}\!/y\} \Rightarrow \{{}^{d}\!/y\}^{\downarrow\uparrow}||_{\{{}^{b}\!/y\}} \neq 1$. Hence, the system of pseudo-intents $\mathcal{P}$ contains $\{{}^{c}\!/y\}$ and $\{{}^{d}\!/y\}$ with $0 < d < c < 1$. Thus, $\{{}^{d}\!/y\} \subset \{{}^{c}\!/y\}$, however, $\{{}^{d}\!/y\}^{\downarrow\uparrow} = \{y\} \not\subseteq \{{}^{c}\!/y\}$ which violates the fact that $\mathcal{P}$ is a system of pseudo-intents. Thus, the general notion of a system of pseudo-intents lacks a property of the existence of $\mathcal{P}$ known from the classical case.

The following theorem shows that T given by (20) is also a non-redundant basis.

Theorem 10 $T = \{P \Rightarrow P^{\downarrow\uparrow} \,|\, P \in \mathcal{P}\}$ *is a non-redundant basis.*

Proof. By Theorem 9, T is complete. Now we are going to show the non-redundancy. Take $T' \subset T$. Clearly, there must by $P \in \mathcal{P}$ such that $P \Rightarrow P^{\downarrow\uparrow}$ does not belong to T'. In addition to that, we have $||Q \Rightarrow Q^{\downarrow\uparrow}||_P = 1$ ($Q \in \mathcal{P}$, $Q \neq P$) by Definition 5, i.e. $P \in \mathrm{Mod}(T')$. On the other hand, $P \notin \mathrm{Mod}(T)$ since $||P \Rightarrow P^{\downarrow\uparrow}||_P = S(P^{\downarrow\uparrow},P) \neq 1$. That is,

$$||P \Rightarrow P^{\downarrow\uparrow}||_{\langle X,Y,I\rangle} = ||P \Rightarrow P^{\downarrow\uparrow}||_T \neq ||P \Rightarrow P^{\downarrow\uparrow}||_{T'},$$

i.e. T' is not complete, showing the non-redundancy of T. □

3.4 Minimality of non-redundant bases

The non-redundancy of a complete set of fuzzy attributes implications says that no implication can be removed from such a set without loosing the completeness. However, the non-redundancy itself does not ensure that a set of fuzzy attribute implications in question is minimal in terms of its size.

In this section we first show that there are data tables with fuzzy attributes that have multiple systems of pseudo-intents (and thus multiple non-redundant bases) with various numbers of elements. Later on, we prove that in certain cases the bases defined by (20) are minimal.

Example Suppose $\mathbf{L}$ with $L = \{0, 0.5, 1\}$ is a three-element Gödel chain with truth-stressing hedge * being the identity. Consider a data table $\langle X,Y,I\rangle$, where $X = \{x\}$, $Y = \{y,z\}$, and $I(x,y) = I(x,z) = 0$. The following systems of $\mathbf{L}$-sets of attributes are the systems of pseudo-intents of $\langle X,Y,I\rangle$:

$$\mathcal{P}_1 = \{\{z\},\{{}^{0.5}\!/y,{}^{0.5}\!/z\},\{y\}\}, \quad \mathcal{P}_3 = \{\{y\},\{{}^{0.5}\!/z\}\},$$
$$\mathcal{P}_2 = \{\{z\},\{{}^{0.5}\!/y\}\}, \quad \mathcal{P}_4 = \{\{{}^{0.5}\!/y\},\{{}^{0.5}\!/z\}\}.$$

As one can see, $\mathcal{P}_1$ has three elements while the other systems only have two. Hence, in this particular case there are multiple systems of pseudo-intents which vary in their sizes.

If both Y and $\mathbf{L}$ are finite, and * is the globalization, we can show that the non-redundant basis determined by the unique system of pseudo-intents (see Theorem 6) has always the minimal size. First, we present a technical lemma.

Lemma 11 *Let $P,Q \in \mathcal{P} \cup \mathrm{Int}(X^*,Y,I)$ such that*

$$S(P,Q)^* \leq S(P^{\downarrow\uparrow}, P\cap Q), \quad (21)$$
$$S(Q,P)^* \leq S(Q^{\downarrow\uparrow}, P\cap Q). \quad (22)$$

Then $P\cap Q \in \mathrm{Int}(X^*,Y,I)$.

Proof. Put $T' = T - \{P \Rightarrow P^{\downarrow\uparrow}, Q \Rightarrow Q^{\downarrow\uparrow}\}$, where T is a set of fuzzy attribute implications defined by (20). Definition 5 and Lemma 7 yield $P,Q \in \mathrm{Mod}(T')$. Hence, for each $A \Rightarrow B \in T'$ we have $S(A,P)^* \leq S(B,P)$ and $S(A,Q)^* \leq S(B,Q)$. Thus,

$$\begin{aligned} S(A,P\cap Q)^* &= (S(A,P)\wedge S(A,Q))^* \leq \\ &\leq S(A,P)^* \wedge S(A,Q)^* \leq S(B,P)\wedge S(B,Q) = \\ &= S(B,P\cap Q), \end{aligned}$$

which immediately gives that $P\cap Q$ is a model of T'. Taking into account Lemma 8, it is sufficient to show that $P\cap Q$ is a model of $\{P \Rightarrow P^{\downarrow\uparrow}, Q \Rightarrow Q^{\downarrow\uparrow}\}$. By (21) and (22), we have

$$S(P,P\cap Q)^* = S(P,Q)^* \leq S(P^{\downarrow\uparrow}, P\cap Q)$$

and

$$S(Q,P\cap Q)^* = S(Q,P)^* \leq S(Q^{\downarrow\uparrow}, P\cap Q),$$

i.e. $||P \Rightarrow P^{\downarrow\uparrow}||_{P\cap Q} = 1$ and $||Q \Rightarrow Q^{\downarrow\uparrow}||_{P\cap Q} = 1$. □

Remarks (i) If P,Q are (pseudo) intents such that $S(P,Q)^* = S(Q,P)^* = 0$ then $P\cap Q$ is an intent on account of Lemma 11.

(ii) Let * be the globalization. By using (i), we get that for (pseudo) intents P,Q with $P \not\subseteq Q$ and $Q \not\subseteq P$, $P\cap Q$ is an intent.

Theorem 12 *Let $\mathbf{L}$ be a finite residuated lattice with * being the globalization. Let T' be a complete in $\langle X,Y,I\rangle$, where Y is finite. Then $|T| \leq |T'|$, where $T = \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$.*

Proof. The system $\mathcal{P}$ of pseudo-intents of such $\langle X,Y,I\rangle$ is uniquely given, see Theorem 6. We prove the claim by showing that for each $P \in \mathcal{P}$, T' contains an implication $A \Rightarrow B$ such that $A^{\downarrow\uparrow} = P^{\downarrow\uparrow}$.

Take $P \in \mathcal{P}$. Since $P \neq P^{\downarrow\uparrow}$ and T' is complete, Theorem 4 yields that T' contains $A \Rightarrow B$ such that $||A \Rightarrow B||_P \neq 1$. That is, we have $A \subseteq P$ and $B \not\subseteq P$ because * is the globalization. The completeness of T' together with (17) yield $S(B,A^{\downarrow\uparrow}) = 1$, i.e. $B \subseteq A^{\downarrow\uparrow}$. Thus, from $B \subseteq A^{\downarrow\uparrow}$ and $B \not\subseteq P$ it follows that $A^{\downarrow\uparrow} \not\subseteq P$. As a consequence, $A^{\downarrow\uparrow}\cap P$ is not an intent, because $A \subseteq P$ and $A^{\downarrow\uparrow} \not\subseteq P$ yield $A \subseteq A^{\downarrow\uparrow}\cap P \subset A^{\downarrow\uparrow}$, i.e. the intersection $A^{\downarrow\uparrow}\cap P$ is not closed under $^{\downarrow\uparrow}$.

Now we claim that $P \subseteq A^{\downarrow\uparrow}$. By contradiction, $P \not\subseteq A^{\downarrow\uparrow}$ and $A^{\downarrow\uparrow} \not\subseteq P$ would give $P\cap A^{\downarrow\uparrow} \in \mathrm{Int}(X^*,Y,I)$ by Lemma 11 which would violate $A^{\downarrow\uparrow}\cap P \notin \mathrm{Int}(X^*,Y,I)$ as observed lately.

Therefore, $A \subseteq P$ gives $A^{\downarrow\uparrow} \subseteq P^{\downarrow\uparrow}$ while $P \subseteq A^{\downarrow\uparrow}$ gives $P^{\downarrow\uparrow} \subseteq A^{\downarrow\uparrow\downarrow\uparrow} = A^{\downarrow\uparrow}$, showing that $A^{\downarrow\uparrow} = P^{\downarrow\uparrow}$. □

4 EXAMPLES AND EXPERIMENTS

The main obstacle to extract fuzzy concepts from data tables with fuzzy attributes is that large data tables and fine scales of truth degrees usually lead to large amounts of concepts which are then not graspable by our mind (it is unlikely to benefit from thousands of concepts, because we would have serious problems just trying to read them). It is then interesting to describe concepts as models of (possibly smaller) sets of fuzzy attribute implications. This section contains an illustrative example and some remarks.

	size		distance	
	small (s)	large (l)	far (f)	near (n)
Mercury (Me)	1	0	0	1
Venus (Ve)	1	0	0	1
Earth (Ea)	1	0	0	1
Mars (Ma)	1	0	0.5	1
Jupiter (Ju)	0	1	1	0.5
Saturn (Sa)	0	1	1	0.5
Uranus (Ur)	0.5	0.5	1	0
Neptune (Ne)	0.5	0.5	1	0
Pluto (Pl)	1	0	1	0

Table 1: Data table with fuzzy attributes

	extent									intent			
no.	Me	Ve	Ea	Ma	Ju	Sa	Ur	Ne	Pl	s	l	f	n
1.	0	0	0	0	0	0	0	0	0	1	1	1	1
2.	0	0	0	0.5	0	0	0.5	0.5	1	1	0	1	0
3.	0.5	0.5	0.5	1	0	0	0	0	0	1	0	0.5	1
4.	0.5	0.5	0.5	1	0	0	0.5	0.5	1	1	0	0.5	0
5.	1	1	1	1	0	0	0	0	0	1	0	0	1
6.	1	1	1	1	0	0	0.5	0.5	1	1	0	0	0
7.	0	0	0	0.5	0.5	0.5	1	1	0.5	0.5	0.5	1	0
8.	0	0	0	0.5	0.5	0.5	1	1	1	0.5	0	1	0
9.	0.5	0.5	0.5	1	0.5	0.5	1	1	1	0.5	0	0.5	0
10.	1	1	1	1	0.5	0.5	1	1	1	0.5	0	0	0
11.	0	0	0	0	1	1	0.5	0.5	0	0	1	1	0.5
12.	0	0	0	0.5	1	1	1	1	0.5	0	0.5	1	0
13.	0	0	0	0.5	1	1	1	1	1	0	0	1	0
14.	0.5	0.5	0.5	1	1	1	0.5	0.5	0.5	0	0	0.5	0.5
15.	0.5	0.5	0.5	1	1	1	1	1	1	0	0	0.5	0
16.	1	1	1	1	1	1	0.5	0.5	0.5	0	0	0	0.5
17.	1	1	1	1	1	1	1	1	1	0	0	0	0

Table 2: Extracted concepts

Let $\mathbf{L}$ be a three-element Łukasiewicz chain such that $\mathbf{L}$ consists of $L = \{0, 0.5, 1\}$ $(0 < 0.5 < 1)$ endowed with $\otimes, \rightarrow$ defined by (6), and let * be the globalization. The data table is given by Table 1. The set X of object consists of objects "Mercury", "Venus", ..., Y contains four attributes: size of the planet (small / large), distance from the sun (far / near). The corresponding fuzzy concepts (clusters) extracted from this data table are identified in Table 2, where each row represents a single concept. The subconcept-superconcept hierarchy (fuzzy concept lattice) is depicted in Fig. 2.

The system $\mathcal{P}$ of pseudo-intents is the following

$$\mathcal{P} = \{\{s, {}^{0.5}/l, f\}, \{{}^{0.5}/s, {}^{0.5}/n\}, \{l,f\}, \{{}^{0.5}/l\}, \{f, {}^{0.5}/n\}, \{n\}\}.$$

Hence, the minimal non-redundant basis T defined by (20) consists of the following fuzzy attribute implications:

$$\begin{aligned} &\{s, {}^{0.5}/l, f\} \Rightarrow \{s,l,f,n\}, \quad &&\{{}^{0.5}/s, {}^{0.5}/n\} \Rightarrow \{s,n\}, \\ &\{l,f\} \Rightarrow \{l,f,{}^{0.5}/n\}, &&\{{}^{0.5}/l\} \Rightarrow \{{}^{0.5}/l, f\}, \\ &\{f, {}^{0.5}/n\} \Rightarrow \{l,f,{}^{0.5}/n\}, &&\{n\} \Rightarrow \{s,n\}. \end{aligned}$$

Note that minimal non-redundant bases are not given uniquely in general, see [4]. The implications of T represent information which completely describes all concepts. For instance, $\{n\} \Rightarrow \{s,n\}$ can be read: "each near planet is small",

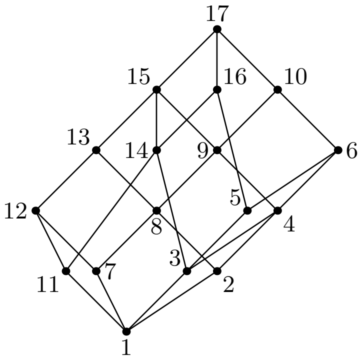

Figure 2: Fuzzy concept lattice

	3 attributes			5 attributes			7 attributes		
objs.	$\mathcal{B}$	$\mathcal{P}$	ratio	$\mathcal{B}$	$\mathcal{P}$	ratio	$\mathcal{B}$	$\mathcal{P}$	ratio
5	12	9	0.750	17	19	1.118	25	26	1.040
10	26	12	0.462	69	29	0.420	130	51	0.392
15	40	12	0.300	139	38	0.273	358	77	0.215
20	52	12	0.231	216	47	0.218	591	102	0.173
25	66	12	0.182	322	54	0.168	853	129	0.151
30	71	12	0.169	421	59	0.140	1326	166	0.125
35	76	11	0.145	505	64	0.127	2115	207	0.098
40	81	11	0.136	575	68	0.118	2356	234	0.099

Table 3: Average sizes of bases (five truth degrees)

$\{{}^{0.5}\!/l\} \Rightarrow \{{}^{0.5}\!/l, f\}$ can be read: "if x is large in degree 0.5, then x is far". A finer reading of fuzzy attribute implications depends on the interpretation of the truth degrees, on the chosen structure of truth degrees, and of course, on the particular application.

To sum up, in this particular example, six fuzzy attribute implications were sufficient to determine 17 concepts of a data table with fuzzy attributes of size 9×4 with three truth degrees. Experimental results have shown that the number of implications is usually (considerably) smaller than the number of concepts. However, the number of implications varies depending on density of the input data table (sparse tables can lead to relatively small amounts of concepts but large amounts of implications). Table 3 contains a summary of average number of (pseudo) intents of randomly generated data tables with 3, 5, or 7 attributes and with 5 up to 40 objects (columns labeled "$\mathcal{B}$" contain average number of concepts, columns labeled "$\mathcal{P}$" contain average size of the minimal bases, columns labeled "ratio" contain quotient of the previous values).

5 CONCLUSIONS AND REMARKS

Our paper presents basic results on attribute implications from data tables with fuzzy attributes. We concentrated on the results directly related to description of non-redundant bases which are important for automatic extraction of attribute implications from data. In addition to that, we showed basic relationships to concept lattices over data with fuzzy attributes. The main research goals for the future are the following:

- Similarity issues. The main goal is to exploit similarity in order to decrease the number of extracted rules by "putting similar together", in a similar manner as in [1, Section 5.2].
- Study of pseudointents for other hedges than globalization (for globalization, we are able to efficiently compute a minimal non-redundant basis).
- Relationship to association rules and fuzzy association rules. From the point of view of association rules, we did not consider the so-called support of a rule. This needs to be taken into account. The expected benefit is less rules and higher value of rules. On the other hand, we expect some loss of theoretical insight.
- Experiments. So far, we run experiments with small real data or with randomly generated larger data. We intend to run experiments with larger real data.

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