Fuzzy attribute logic: syntactic entailment and completeness

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Abstract

Fuzzy attribute implications are particular if-then rules which can be extracted from data tables with fuzzy attributes. We develop fuzzy attribute logic (FAL), its syntax, semantics, and prove a classical as well as Pavelka-style completeness theorem for FAL. Also, we present relationships to completeness of sets of attribute implications w.r.t. data tables. Because of the limited scope of the paper, we omit examples.

Keywords: fuzzy logic, fuzzy attribute, if-then rule, provability degree, completeness

1. Introduction

The present paper describes a logical calculus for reasoning with rules of the form "if A then B" ($A \Rightarrow B$), where A and B are fuzzy collections of attributes, with the meaning: if an object has all the attributes of A then it has also all attributes of B. Our motivation to look at such rules from the viewpoint of fuzzy logic is that attributes are often fuzzy (an attribute can apply to an object in degrees) rather then bivalent (an object either has an attribute or not). Attribute implications in fuzzy setting were for the first time studied in [10]. The main difference between our approach and that of [10] is that (i) we use, as an additional parameter, a unary connective * which enables us to control various interpretations of fuzzy attribute implications; (ii) [10] deals only with the semantic entailment and thus does not contain any results on completeness. This paper is a complement to papers [2, 4, 6] in which we investigate semantic consequence from sets of fuzzy attribute implications, problems connected with attribute scaling, and efficient generating of non-redundant bases from object-attribute data tables with fuzzy attributes. Our paper elaborates on syntactic consequence (provability), shows that FAL is complete, and shows some relationships to completeness of attribute implications w.r.t. data.

2. Preliminaries

As a set of truth degrees equipped with suitable operations (truth functions of logical connectives) we use a complete residuated lattice with truth-stressing hedge (shortly, a hedge). A complete residuated lattice with a hedge is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, ^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$; for each $a, b, c \in L$; hedge * satisfies (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, (iv) $a^{**} = a^*$, for all $a, b \in L$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true", see [7, 8]. Properties (i)–(iv) have natural interpretations, e.g. (iii) can be read: "if $a \rightarrow b$ is very true and if a is very true, then b is very true", etc.

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), Gödel: ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = b$ else), Goguen (product): ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = \frac{b}{a}$ else). In applications, we usually need a finite linearly ordered $\mathbf{L}$. For instance, we can take a finite subset $L \subseteq [0, 1]$ that is closed under Łukasiewicz or Gödel operations. If we take $L = \{0, 1\}$, we obtain this way the two-element Boolean algebra (structure of truth degrees of classical logic). Two boundary cases of hedges are (i) identity, i.e. $a^* = a$ ($a \in L$); (ii) globalization [11]: $a^* = 1$ if $a = 1$, $a^* = 0$ else.

Having $\mathbf{L}$ as our structure of truth degrees, we define usual notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A\colon U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. For $a \in L$ and $A \in \mathbf{L}^U$, we define $\mathbf{L}$-sets $a \otimes A$ (a-multiple of A) and $a \rightarrow A$ (a-shift of A) by $(a \otimes A)(u) = a \otimes A(u)$, $(a \rightarrow A)(u) = a \rightarrow A(u)$ ($u \in U$). Given $A, B \in \mathbf{L}^U$, we define a subsethood degree $S(A, B) = \bigwedge_{u \in U} \big(A(u) \rightarrow B(u)\big)$, which generalizes the classical subsethood relation $\subseteq$. Described verbally,

$S(A,B)$ represents the degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. Observe that $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$. In the following we use well-known properties of residuated lattices and fuzzy structures which can be found in monographs [1, 7].

3. Fuzzy attribute logic

3.1. Basic notions

Fuzzy attribute implication (*over attributes* Y) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ (A and B are fuzzy sets of attributes). Fuzzy attribute implications are the basic formulas of fuzzy attribute logic. The intended meaning of $A \Rightarrow B$ is: "if it is (very) true that an object has all attributes from A, then it has also all attributes from B". The notions "being very true", "to have an attribute", and logical connective "if-then" are determined by the chosen $\mathbf{L}$.

A fuzzy attribute implication does not have any kind of "validity" on its own—it is a syntactic notion. In order to consider validity, we must introduce an interpretation of fuzzy attribute implications. Thus, for an $\mathbf{L}$-set $M \in \mathbf{L}^Y$ of attributes, we define a *degree* $||A \Rightarrow B||_M \in L$ *to which* $A \Rightarrow B$ *is valid in* M by

$$||A \Rightarrow B||_M = S(A,M)^* \to S(B,M). \tag{1}$$

The truth degree $||A \Rightarrow B||_M$ can be understood as follows: if M represents presence of attributes observed by a user, i.e. the degree $M(y)$ means that "the user observers the attribute $y \in Y$ to be present in degree $M(y)$", then $||A \Rightarrow B||_M$ is the truth degree of "if all attributes from A are present then all attributes of B are present as well", i.e. the truth degree to which $A \Rightarrow B$ is valid (under the user's observations).

3.2. Semantic entailment

Let T be a set of fuzzy attribute implications. $M \in \mathbf{L}^Y$ is called a *model* of T if $||A \Rightarrow B||_M = 1$ for each $A \Rightarrow B \in T$. The set of all models of T is denoted by $\mathrm{Mod}(T)$. A degree $||A \Rightarrow B||_T \in L$ to which $A \Rightarrow B$ *semantically follows* from T is defined by

$$||A \Rightarrow B||_T = \textstyle\bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M. \tag{2}$$

A set T of fuzzy attribute implications is said to be *semantically closed* if $||A \Rightarrow B||_T = 1$ iff $A \Rightarrow B \in T$.

The following assertion shows that the fuzzy attribute implications which are fully true in a given $M \in \mathbf{L}^Y$ (i.e., $||\cdots||_M = 1$) are in fact the most important ones, because an arbitrary (partial) truth degree $||\cdots||_M$ can be expressed by means of that implications.

Theorem 1 (i): $c \to S(B,M) = S(c \otimes B, M)$;
(ii): $c \leq ||A \Rightarrow B||_M$ iff $||A \Rightarrow c \otimes B||_M = 1$;
(iii): $||A \Rightarrow B||_T = \bigvee\{c \in L \mid ||A \Rightarrow c \otimes B||_T = 1\}$.

Proof. (i): We have

$$\begin{aligned} c \to S(B,M) &= c \to \textstyle\bigwedge_{y \in Y}(B(y) \to M(y)) = \\ &= \textstyle\bigwedge_{y \in Y}(c \to (B(y) \to M(y))) = \\ &= \textstyle\bigwedge_{y \in Y}((c \otimes B(y)) \to M(y)) = \\ &= \textstyle\bigwedge_{y \in Y}((c \otimes B)(y) \to M(y)) = S(c \otimes B, M). \end{aligned}$$

(ii): Using (i), $c \leq ||A \Rightarrow B||_M$ iff $c \leq S(A,M)^* \to S(B,M)$ iff $S(A,M)^* \leq c \to S(B,M)$ iff $S(A,M)^* \leq S(c \otimes B, M)$ iff $||A \Rightarrow c \otimes B||_M = 1$.

(iii): Using (ii), we have

$$\begin{aligned} ||A \Rightarrow B||_T &= \textstyle\bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M = \\ &= \textstyle\bigvee\{c \in L \mid c \leq ||A \Rightarrow B||_M \text{ for each } M \in \mathrm{Mod}(T)\} = \\ &= \textstyle\bigvee\{c \in L \mid ||A \Rightarrow c \otimes B||_T = 1\}, \end{aligned}$$

proving the claim. □

3.3. Syntactic entailment

The main obstacle to exploring degrees of consequence $||\cdots||_T$ is that there is no obvious (and efficient) algorithm to compute $||\cdots||_T$. For instance, if $\mathbf{L}$ is a k-element structure of truth degrees and if Y is an n-element set of attributes, then computing $||\cdots||_T$ requires k^n steps if we use the naive algorithm (iteration over all possible models of T). This is not acceptable in practical applications. In this section, we present a syntactic consequence $|\cdots|_T$ which does not involve models. In subsequent sections, we prove that $|\cdots|_T$ fully characterizes $||\cdots||_T$.

First, we need the following *deduction rules*:

(Ax) infer $A \Rightarrow S(B,A) \otimes B$,
(Wea) from $A \Rightarrow B$ infer $A \cup C \Rightarrow B$,
(Cut) from $A \Rightarrow e \otimes B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow e^* \otimes D$

for each $A, B, C, D \in \mathbf{L}^Y$, and $e \in L$. Rules (Ax)–(Cut) should be understood as rules describing what fuzzy attribute implications can be inferred (in one elementary step) from another fuzzy attribute implications. (Ax) is a nullary rule (axiom) which says that each $A \Rightarrow S(B,A) \otimes B$ is inferred in one step. (Wea) and (Cut) generalize well-known rules of the weakening and cut. Our rules are inspired by Armstrong-like axioms, see [9] for a good overview.

A fuzzy attribute implication $A \Rightarrow B$ is called *provable* from a set T of fuzzy attribute implications, written $T \vdash A \Rightarrow B$, if there is a sequence $\varphi_1, \ldots, \varphi_n$ of fuzzy attribute implications such that φ_n is $A \Rightarrow B$ and for each φ_i we either have $\varphi_i \in T$ or φ_i is inferred (in one step) from the preceding formulas (i.e., $\varphi_1, \ldots, \varphi_{i-1}$) using one of the deduction rules (Ax)–(Cut). For a set T of fuzzy attribute implications and for $A \Rightarrow B$ we define a *degree* $|A \Rightarrow B|_T \in L$ *to which* $A \Rightarrow B$ *is provable from* T by

$$|A \Rightarrow B|_T = \bigvee\{c \in L \,|\, T \vdash A \Rightarrow c \otimes B\}. \quad (3)$$

A set T of fuzzy attribute implications is said to be *syntactically closed* if $T \vdash A \Rightarrow B$ iff $A \Rightarrow B \in T$. In the sequel we show that assuming $\mathbf{L}$ and Y to be finite, the syntactic closedness coincides with the semantic one.

Theorem 2 *If T is semantically closed then T is syntactically closed.*

Proof. (Ax): Take $A \Rightarrow S(B,A) \otimes B$. For each $M \in \mathbf{L}^Y$ we have $S(B,A) \otimes S(A,M)^* \leq S(B,A) \otimes S(A,M) \leq S(B,M)$. Taking into account (i) of Theorem 1, we have $S(A,M)^* \leq S(B,A) \rightarrow S(B,M) = S(S(B,A) \otimes B, M)$ for each $M \in \mathbf{L}^Y$. That is, $||A \Rightarrow S(B,A) \otimes B||_M = 1$ for each $M \in \mathrm{Mod}(T)$, i.e. $||A \Rightarrow S(B,A) \otimes B||_T = 1$. Since T is semantically closed, we have $A \Rightarrow S(B,A) \otimes B \in T$.

(Wea): Let $A \Rightarrow B \in T$. So, $||A \Rightarrow B||_M = 1$ for each $M \in \mathrm{Mod}(T)$, i.e. $S(A,M)^* \leq S(B,M)$. Now $S(A \cup C, M)^* = (S(A,M) \wedge S(C,M))^* \leq S(A,M)^* \leq S(B,M)$. Hence, $||A \cup C \Rightarrow B||_M = 1$ which yields $A \cup C \Rightarrow B \in T$.

(Cut): Let $A \Rightarrow e \otimes B \in T$ and $B \cup C \Rightarrow D \in T$. For any $M \in \mathrm{Mod}(T)$, $||A \Rightarrow e \otimes B||_M = 1$ and $||B \cup C \Rightarrow D||_M = 1$. Thus, (i) of Theorem 1 yields $S(A,M)^* \leq S(e \otimes B, M) = e \rightarrow S(B,M)$, i.e. $e \otimes S(A,M)^* \leq S(B,M)$. Further,

$$\begin{aligned} e^* \otimes S(A \cup C, M)^* &= e^* \otimes (S(A,M) \wedge S(C,M))^* \leq \\ &\leq ((e \otimes S(A,M)^*) \wedge S(C,M))^* \leq \\ &\leq (S(B,M) \wedge S(C,M))^* = S(B \cup C, M)^*. \end{aligned}$$

Now since $S(B \cup C, M)^* \leq S(D,M)$, we have $e^* \otimes S(A \cup C, M)^* \leq S(D,M)$, which gives $S(A \cup C, M)^* \leq e^* \rightarrow S(D,M) = S(e^* \otimes D, M)$. Hence, $||A \cup C \Rightarrow e^* \otimes D||_M = 1$, i.e. $A \cup C \Rightarrow e^* \otimes D \in T$. ☐

Theorem 3 *Let $\mathbf{L}$ and Y be finite. If T is syntactically closed then T is semantically closed.*

Proof. We are going to show that if $||A \Rightarrow B||_T = 1$ then $A \Rightarrow B$ belongs to T. Assuming $A \Rightarrow B \notin T$, it suffices to find a model $M \in \mathrm{Mod}(T)$ such that $||A \Rightarrow B||_M \neq 1$.

First, for each $A \in \mathbf{L}^Y$ define $A^+ \in \mathbf{L}^Y$ by

$$A^+ = \bigcup\{A' \,|\, A \Rightarrow A' \in T\}. \quad (4)$$

Observe that for any $A \in \mathbf{L}^Y$ we have $A \Rightarrow A^+ \in T$. Indeed, for $A \Rightarrow B \in T$ and $A \Rightarrow C \in T$ we have

$$\begin{aligned} &B \cup C \Rightarrow B \cup C \in T && \text{[by (Ax)]},\\ &A \cup B \cup C \Rightarrow B \cup C \in T && \text{[by (Wea)]},\\ &A \cup C \Rightarrow B \cup C \in T && \text{[by (Cut) using } A \Rightarrow B \in T],\\ &A \Rightarrow B \cup C \in T && \text{[by (Cut) using } A \Rightarrow C \in T]. \end{aligned}$$

Since $\mathbf{L}^Y$ is finite, the above idea can be repeated so that after finitely many steps we get $A \Rightarrow A^+ \in T$. Further, we show $A^+ \in \mathrm{Mod}(T)$. For each $C \Rightarrow D \in T$, we have

$$\begin{aligned} &A^+ \Rightarrow S(C,A^+) \otimes C \in T && \text{[by (Ax)]},\\ &A \Rightarrow S(C,A^+) \otimes C \in T && \text{[by (Cut) using } A \Rightarrow A^+ \in T],\\ &A \Rightarrow S(C,A^+)^* \otimes D \in T && \text{[by (Cut) using } C \Rightarrow D \in T]. \end{aligned}$$

Hence, $A \Rightarrow S(C,A^+)^* \otimes D \in T$ gives $S(C,A^+)^* \otimes D \subseteq A^+$ by definition of A^+. That is, $S(C,A^+)^* \otimes D(y) \leq A^+(y)$ for each $y \in Y$, i.e. $S(C,A^+)^* \leq S(D,A^+)$, proving $||C \Rightarrow D||_{A^+} = 1$. As a consequence, A^+ is a model of T.

Finally, take $A \Rightarrow B \notin T$. We have $||A \Rightarrow B||_{A^+} = S(A,A^+)^* \rightarrow S(B,A^+) = S(B,A^+)$ since $S(A,A^+)^* = 1$. Clearly, we have $S(B,A^+) \neq 1$ because $S(B,A^+) = 1$ gives $A^+ \Rightarrow B \in T$ by (Ax) which together with $A \Rightarrow A^+ \in T$ would yield $A \Rightarrow B \in T$ by (Cut), a contradiction. To sum up, $A^+ \in \mathrm{Mod}(T)$ and $||A \Rightarrow B||_{A^+} \neq 1$. ☐

Corollary 4 *Let $\mathbf{L}$ and Y be finite. Then T is semantically closed iff T is syntactically closed.* ☐

3.4. Completeness of FAL

First, we present a completeness theorem which characterizes the semantic entailment of the fully true implications. Then we show that FAL has a completeness in Pavelka-style [7], i.e. $|A \Rightarrow B|_T$ equals $||A \Rightarrow B||_T$.

Theorem 5 (completeness) *Let $\mathbf{L}$ and Y be finite. Let T be a set of fuzzy attribute implications. Then*

$$T \vdash A \Rightarrow B \quad \textit{iff} \quad ||A \Rightarrow B||_T = 1.$$

Proof. Follows from Corollary 4. ☐

Theorem 6 (graded completeness) *Let $\mathbf{L}$ and Y be finite. Then for every set T of fuzzy attribute implications and $A \Rightarrow B$ we have $|A \Rightarrow B|_T = ||A \Rightarrow B||_T$.*

Proof. Consequence of Theorem 1 (iii) and Theorem 5. ☐

3.5. Completeness w.r.t. a data table

In the previous sections, we were interested in completeness of FAL, i.e. a desirable property of our logical calculus saying that $A \Rightarrow B$ is provable from a set T of attribute implications iff it is true in all models of T (plus the same in degrees). In FAL, and in logic in general, completeness is used still in another sense, see [2, 4, 6]. Let X and Y be sets of objects and attributes, respectively, $I \in \mathbf{L}^{X \times Y}$ be a fuzzy relation between X and Y with $I(x,y)$ being interpreted as a degree to which object $x \in X$ has attribute $y \in Y$. $\langle X,Y,I\rangle$ will be called a data table with fuzzy attributes. For $\mathbf{L}$-sets $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$ define $\mathbf{L}$-sets $A^\uparrow \in \mathbf{L}^Y$ and $B^\downarrow \in \mathbf{L}^X$ by $A^\uparrow(y) = \bigwedge_{x \in X}\big(A(x)^* \rightarrow I(x,y)\big)$, and $B^\downarrow(x) = \bigwedge_{y \in Y}\big(B(y) \rightarrow I(x,y)\big)$. The structure $\mathcal{B}(X^*,Y,I) = \{\langle A,B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \,|\, A^\uparrow = B, B^\downarrow = A\}$ (all fixed points of $^\uparrow$ and $^\downarrow$) is called a fuzzy concept lattice and its elements, called formal fuzzy concepts, are interpreted as interesting clusters in $\langle X,Y,I\rangle$, see [1, 5]. The set $\mathrm{Int}(X^*,Y,I) = \{B \in \mathbf{L}^Y \mid \langle A,B\rangle \in \mathcal{B}(X^*,Y,I)$ for some $A \in \mathbf{L}^X\}$ is useful in FAL. A degree $||A \Rightarrow B||_{\mathcal{M}}$ to which $A \Rightarrow B$ is true in a system $\mathcal{M} \subseteq \mathbf{L}^Y$ is defined by $||A \Rightarrow B||_{\mathcal{M}} = \bigwedge_{M\in\mathcal{M}} ||A \Rightarrow B||_M$. A degree $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ to which $A \Rightarrow B$ is true in $\langle X,Y,I\rangle$ is defined by $||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\{I_x\,|\,x\in X\}}$ with I_x being the row of $\langle X,Y,I\rangle$ corresponding to object x, i.e. $I_x \in \mathbf{L}^Y$ with $I_x(y) = I(x,y)$ for $y \in Y$.

Now, a set T of attribute implications is said to be complete w.r.t. data table $\langle X,Y,I\rangle$ if $||A \Rightarrow B||_T = ||A \Rightarrow B||_{\langle X,Y,I\rangle}$ for each $A \Rightarrow B$ (degree to which $A \Rightarrow B$ semantically follows from T is just the degree to which $A \Rightarrow B$ is true in $\langle X,Y,I\rangle$, see [2, 4, 6]). In what follows we answer some natural questions arising in this context. First, since we are primarily interested in implications which are fully true in $\langle X,Y,I\rangle$, the following notion seems to be of interest. Call T *1-complete w.r.t.* $\langle X,Y,I\rangle$ if we have that $||A \Rightarrow B||_T = 1$ iff $||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1$ for each $A \Rightarrow B$. Interestingly, we have

Theorem 7 *T is 1-complete w.r.t. $\langle X,Y,I\rangle$ iff T is complete w.r.t. $\langle X,Y,I\rangle$.*

Proof. We present only a sketch of the proof (we use results of [2, 6]): Clearly, it suffices to show the "⇒"-part. Let T be 1-complete. As shown in [2], T is complete w.r.t. $\langle X,Y,I\rangle$ iff $||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)} = ||A \Rightarrow B||_{\mathrm{Mod}(T)}$ which we now verify. Since $||A \Rightarrow S(B,A^{\downarrow\uparrow}) \otimes B||_{\mathrm{Int}(X^*,Y,I)} = 1$, 1-completeness of T gives $||A \Rightarrow S(B,A^{\downarrow\uparrow}) \otimes B||_{\mathrm{Mod}(T)} = 1$. Since $||A \Rightarrow S(B,A^{\downarrow\uparrow}) \otimes B||_{\mathrm{Mod}(T)} = \cdots = S(B,A^{\downarrow\uparrow}) \to ||A \Rightarrow B||_{\mathrm{Mod}(T)}$, we have $||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)} = S(B,A^{\downarrow\uparrow}) \leq ||A \Rightarrow B||_{\mathrm{Mod}(T)}$. To establish $||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)} \geq ||A \Rightarrow B||_{\mathrm{Mod}(T)}$, just check that $\mathrm{Int}(X^*,Y,I) \subseteq \mathrm{Mod}(T)$. □

Next, put $\mathrm{Fml}(X,Y,I) = \{A \Rightarrow B \mid ||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1\}$, i.e. $\mathrm{Fml}(X,Y,I)$ is the set of all implications fully true in $\langle X,Y,I\rangle$. Therefore, Theorem 7 yields that T is complete iff $\mathrm{Fml}(X,Y,I) = \{A \Rightarrow B \mid ||A \Rightarrow B||_T = 1\}$. The following theorem answers some natural questions (proof omitted due to lack of space).

Theorem 8 $\mathrm{Fml}(X,Y,I)$ *is (syntactically/semantically) closed.* $\mathrm{Fml}(X,Y,I)$ *is complete w.r.t.* $\langle X,Y,I\rangle$. *If T is complete w.r.t. $\langle X,Y,I\rangle$ then $T \vdash A \Rightarrow B$ iff $||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1$ (T proves just all attribute implications which are true in $\langle X,Y,I\rangle$ in degree 1).* □

If T is complete w.r.t. $\langle X,Y,I\rangle$, it need not be closed. The following theorem shows that a closed T is always complete w.r.t. some data table.

Theorem 9 *If T is (syntactically/semantically) closed then T is complete w.r.t. some $\langle X,Y,I\rangle$ such that* $T = \mathrm{Fml}(X,Y,I)$.

Proof. Let T be closed. Consider a data table $\langle \mathrm{Mod}(T),Y,I_T\rangle$ with I_T defined by $I_T(M,y) = M(y)$ for each $M \in \mathrm{Mod}(T)$ and $y \in Y$. In oder to show that T is complete w.r.t. $\langle \mathrm{Mod}(T),Y,I_T\rangle$, it suffices to check that $\mathrm{Mod}(T) = \mathrm{Int}(\mathrm{Mod}(T)^*,Y,I_T)$. This can be done as follows. First, verify that $\mathrm{Mod}(T)$ is an $\mathbf{L}^*$-closure system [3]. Second, using the fact that the corresponding $\mathbf{L}^*$-closure operator C is given by $(C(A))(y) = \bigwedge_{M\in\mathrm{Mod}(T)} \big(S(A,M)^* \to M(y)\big)$ for each $A \in \mathbf{L}^Y$, see [3], verify that $C(A) = A^{\downarrow\uparrow}$ (easy). From this we get that $\mathrm{Mod}(T) = \mathrm{Int}(\mathrm{Mod}(T)^*,Y,I_T)$. Finally, since T is complete w.r.t. $\langle \mathrm{Mod}(T),Y,I_T\rangle$, Theorem 8 yields that T proves all implications from $\mathrm{Fml}(X,Y,I)$ and due to completeness of T again, we have $\mathrm{Fml}(X,Y,I) = T$. □

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