Arch. Math. Logic 45, 3–51 (2006)
Digital Object Identifier (DOI):
10.1007/s00153-005-0287-x

Mathematical Logic

Radim Bělohlávek · Vilém Vychodil

Fuzzy Horn logic I

proof theory

Received: 20 January 2004 / Revised version: 22 November 2004 /
Published online: 6 July 2005 – © Springer-Verlag 2005

Abstract. The paper presents generalizations of results on so-called Horn logic, well-known in universal algebra, to the setting of fuzzy logic. The theories we consider consist of formulas which are implications between identities (equations) with premises weighted by truth degrees. We adopt Pavelka style: theories are fuzzy sets of formulas and we consider degrees of provability of formulas from theories. Our basic structure of truth degrees is a complete residuated lattice. We derive a Pavelka-style completeness theorem (degree of provability equals degree of truth) from which we get some particular cases by imposing restrictions on the formulas under consideration. As a particular case, we obtain completeness of fuzzy equational logic.

1. Introduction and preliminaries

Since the inception of fuzzy approach in 1960s, fuzzy logic in narrow sense, i.e. logical calculi for reasoning in presence of vagueness, has been substantially developed. An overview can be found in [20, 22, 29] (but note that many new results were published after these monographs). In addition to results on general first-order fuzzy logic, there also appeared results on logical calculi resulting from the general first-order case by natural restrictions. As an example, monadic fuzzy logic was studied in [24]; there are several papers on systems related to fuzzy logic programming, e.g. [34]. The present paper studies an equational fragment of first-order fuzzy logic and is a continuation of [5]. The equational fragment results by restricting relation symbols to a single one, namely, to the symbol of equality, and considering only particular formulas, namely, (generalized) implications between identities. Structures of the equational fragment are the so-called algebras with fuzzy equalities which generalize ordinary algebras and can be thought of as systems of functions mapping similar elements to similar ones. The main aim of this paper is to study theories and provability in our equational fragment. This generalizes [5] where the formulas under consideration were identities. Identities are, in fact, implications with empty premises which is a special case of formulas considered in this

R. Bělohlávek: Deptartment of Computer Science, Palacký University, Tomkova 40, 77900, Olomouc, Czech Republic. e-mail: radim.belohlavek@upol.cz

V. Vychodil: Department Computer Science, Palacký University, Tomkova 40, 77900, Olomouc, Czech Republic. e-mail: vilem.vychodil@upol.cz

Key words or phrases: Fuzzy logic – Equational logic – Horn logic – Implication – Degree of provability

paper. In the ordinary (crisp) case, implications between identities have been studied in universal algebra. There are numerous results on properties of implicationally defined classes of algebras [2, 12, 17, 30] and proofs from implicational theories, e.g. [32]. A survey on implications in the context of universal algebra can be found in [38].

Recall that by an *implication* it is usually meant a generalized formula

$$\bigwedge_{i\in I} t_i \approx t_i' \Rightarrow t \approx t'$$

where $t_i, t_i' (i \in I), t, t' \in T(X)$ are terms over a type F and a set X of variables. Putting $P = \{\langle t_i, t_i'\rangle;\ i \in I\}$, an implication can be denoted by $P \Rightarrow (t \approx t')$ with P interpreted as a set of premises. In general, P is not required to be finite. Model classes of implications with possible infinite number of premises are closed under isomorphic images, subalgebras, and direct products. They are known as sur-reflective classes. Simpler sets of premises naturally lead to further closure properties. The following summary gives an overview of some of the classes of implications defined by restrictions on their premises.

(i) *Implications:* P is arbitrary, model classes (sur-reflective classes) are classes closed under subalgebras, and direct products.
(ii) *Finitary implications:* P may be infinite but X is finite. Model classes (semivarieties) are classes closed under subalgebras, direct products, and direct unions.
(iii) *Horn clauses:* P is finite, model classes (quasivarieties) are classes closed under subalgebras, direct products, and direct limits. Alternatively, quasivarieties can be characterized as classes closed under subalgebras and reduced products or as classes closed under subalgebras, direct products, and ultraproducts.
(iv) *Equation implications:* P is either singleton, i.e. $P = \{\langle s, s'\rangle\}$ for certain $s, s' \in T(X)$, or $P = \emptyset$. Equation implications are special Horn clauses.
(v) *Identities:* $P = \emptyset$, model classes (varieties) are closed under homomorphic images, subalgebras, and direct products.

In order to generalize the concept of an implication to fuzzy setting, we need to recall basic notions from fuzzy sets and fuzzy logic. We use complete residuated lattices as the structures of truth degrees. Recall that a (*complete*) *residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that (i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, (ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, (iii) $\langle \otimes, \rightarrow\rangle$ is an *adjoint pair,* i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ is valid for each $a, b, c \in L$ (so-called adjointness property). Complete residuated lattices were introduced into the context of fuzzy logic by Goguen [18, 19]. Note that particular types of residuated lattices (distinguishable by identities) include Boolean algebras, Heyting algebras, MV-algebras, Gödel algebras, product algebras, and more generally, BL-algebras, see [6, 22, 25, 29]. In the sequel, $\mathbf{L}$ always denotes a complete residuated lattice (not necessarily linear).

An $\mathbf{L}$*-set* A (*fuzzy set with truth degrees in* $\mathbf{L}$) in a universe U is any mapping $A : U \rightarrow L$, $A(u) \in L$ being interpreted as the truth value of "element u belongs to

A". Let $\mathbf{L}^U$ denote the set of all $\mathbf{L}$-sets in U. A mapping $\emptyset_U : U \to L$ with $\emptyset_U(\mathrm{u}) = 0$ ($\mathrm{u} \in U$) is called an empty $\mathbf{L}$-set in U. For every $\mathbf{L}$-set $A : U \to L$, the support of A is an ordinary set $\mathrm{Supp}(A)$ defined by $\mathrm{Supp}(A) = \{\mathrm{u} \in U \mid A(\mathrm{u}) > 0\}$. $\mathbf{L}$-set A is called *finite* if $\mathrm{Supp}(A)$ is finite. For $\mathbf{L}$-sets A and B in U, the *subsethood degree* $S(A, B)$ of A in B is defined by

$$S(A, B) = \bigwedge\nolimits_{\mathrm{u} \in U} \big(A(\mathrm{u}) \to B(\mathrm{u})\big). \tag{1}$$

We write $A \subseteq B$ (A is a subset of B) iff $S(A, B) = 1$, i.e. iff for each $\mathrm{u} \in U$, $A(\mathrm{u}) \leq B(\mathrm{u})$. As usual, operations on $\mathbf{L}$ extend componentwise to operations with $\mathbf{L}$-sets. A *binary* $\mathbf{L}$*-relation* R (*binary fuzzy relation with truth degrees in* $\mathbf{L}$) on U is an $\mathbf{L}$-set in $U \times U$, i.e. a mapping $R : U \times U \to L$. An $\mathbf{L}$*-equivalence* (or *similarity*) on U is a binary $\mathbf{L}$-relation on U satisfying $E(\mathrm{u}, \mathrm{u}) = 1$ (reflexivity), $E(\mathrm{u}, \mathrm{v}) = E(\mathrm{v}, \mathrm{u})$ (symmetry), $E(\mathrm{u}, \mathrm{v}) \otimes E(\mathrm{v}, \mathrm{w}) \leq E(\mathrm{u}, \mathrm{w})$ (transitivity) for all $\mathrm{u}, \mathrm{v}, \mathrm{w} \in U$. An $\mathbf{L}$-equivalence on U for which $E(\mathrm{u}, \mathrm{v}) = 1$ implies $\mathrm{u} = \mathrm{v}$ is called an $\mathbf{L}$*-equality.* Function $f : U^n \to U$ is *compatible* with an $\mathbf{L}$-equivalence E on U if $E(\mathrm{u}_1, \mathrm{v}_1) \otimes \cdots \otimes E(\mathrm{u}_n, \mathrm{v}_n) \leq E\big(f(\mathrm{u}_1, \ldots, \mathrm{u}_n), f(\mathrm{v}_1, \ldots, \mathrm{v}_n)\big)$ for all $\mathrm{u}_1, \mathrm{v}_1 \ldots \mathrm{u}_n, \mathrm{v}_n \in U$. Note that compatibility says that pairwise similar elements are mapped to similar elements. An $\mathbf{L}$-set A is called *crisp* if $A(\mathrm{u}) \in \{0, 1\}$ for each $\mathrm{u} \in U$. Following common usage, we sometimes identify crisp $\mathbf{L}$-sets with the corresponding ordinary sets. Further details on fuzzy sets and fuzzy relations can be found e.g. in [6, 22, 29].

A collection F of function symbols, each with its arity will be called a *type*. Given a complete residuated lattice $\mathbf{L}$, the *language* of our $\mathbf{L}$-Horn logic consists of (at least denumerable) set X of variables, a type F, a binary predicate symbol $\approx$ standing for (fuzzy) equality, a set $\{\overline{a};\ a \in L\}$ of symbols of truth values (however, for the sake of convenience and since there is no danger of misunderstanding, we identify $\overline{a}$ with a), and symbols of logical connectives $\Rightarrow$ (implication), $\mathbin{\text{⩓}}$ (conjunction) and $\bigwedge\!\!\!\bigwedge$ (generalized conjunction). The set $T(X)$ of all terms over F and X is defined as usual. Terms are denoted by $p, q, \ldots, t$, possibly with indices. The set of all variables occurring in t is denoted by $\mathrm{var}(t)$.

An *algebra with* $\mathbf{L}$*-equality* (shortly an $\mathbf{L}$*-algebra*) of type F is a triplet $\mathbf{M} = \big\langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\big\rangle$, where $\big\langle M, F^{\mathbf{M}}\big\rangle$ is an (ordinary) algebra of type F and $\approx^{\mathbf{M}}$ is an $\mathbf{L}$-equality on M such that each $f^{\mathbf{M}} \in F^{\mathbf{M}}$ is compatible with $\approx^{\mathbf{M}}$. Unless stated otherwise, we use a fixed type F. An $\mathbf{L}$-relation θ on M such that (i) θ is an $\mathbf{L}$-equivalence relation on M, (ii) $\approx^{\mathbf{M}} \subseteq \theta$, (iii) all functions $f^{\mathbf{M}} \in F^{\mathbf{M}}$ are compatible with θ, is called a *congruence on* $\mathbf{M}$. Congruences on an $\mathbf{L}$-algebra form a complete lattice [9]. For a congruence θ on an $\mathbf{L}$-algebra $\mathbf{M}$, an $\mathbf{L}$-algebra $\mathbf{M}/\theta = \big\langle M/\theta, \approx^{\mathbf{M}/\theta}, F^{\mathbf{M}/\theta}\big\rangle$, where (i) $\big\langle M/\theta, F^{\mathbf{M}/\theta}\big\rangle$ is an ordinary factor algebra of $\big\langle \mathbf{M}, F^{\mathbf{M}}\big\rangle$ modulo $\{\langle \mathrm{a}, \mathrm{b}\rangle \mid \theta(\mathrm{a}, \mathrm{b}) = 1\}$ and (ii) $[\mathrm{a}]_\theta \approx^{\mathbf{M}/\theta} [\mathrm{b}]_\theta = \theta(\mathrm{a}, \mathrm{b})$ for all $\mathrm{a}, \mathrm{b} \in M$, is called a *factor* $\mathbf{L}$*-algebra of* $\mathbf{M}$ *modulo* θ. Let $\mathbf{M}, \mathbf{N}$ be $\mathbf{L}$-algebras of the same type. A mapping $h : M \to N$ satisfying $\mathrm{a} \approx^{\mathbf{M}} \mathrm{b} \leq h(\mathrm{a}) \approx^{\mathbf{N}} h(\mathrm{b})$ is called an $\approx$*-morphism.* An $\approx$-morphism $h : M \to N$ is called a *morphism* (*of* $\mathbf{L}$*-algebras*) if h is a morphism between ordinary algebras $\big\langle \mathbf{M}, F^{\mathbf{M}}\big\rangle$ and $\big\langle \mathbf{N}, F^{\mathbf{N}}\big\rangle$. A morphism $h : \mathbf{M} \to \mathbf{M}$ is called an endomorphism.

The *term* **L**-*algebra* is an **L**-algebra $\mathbf{T}(X) = \langle T(X), \approx^{\mathbf{T}(X)}, F^{\mathbf{T}(X)} \rangle$ where $\langle T(X), F^{\mathbf{T}(X)} \rangle$ is the ordinary term algebra and $\approx^{\mathbf{T}(X)}$ is a crisp **L**-equality.

The paper is organized as follows. Section 2 defines implications between identities in fuzzy setting and their semantics. In Section 3 and Section 4 we introduce and study an algebraic counterpart to semantic closure of implicational theories. Section 5 presents a completeness result. A special case of implications with crisp premises is dealt with in Section 6. Section 7 shows that fuzzy equational logic as presented in [5] results from fuzzy Horn logic as a special case (empty premises).

Note also that in [10], a follow up to the present paper, we characterize model classes of implicational theories.

2. Implications between identities

We are going to introduce the concept of an implication between identities in fuzzy setting. We start with an approach as general as possible which, nevertheless, reflects natural requirements. Moreover, we wish to get particular cases which are of interest by imposing suitable restrictive conditions on the general concept of an implication. Before going to the definition of implications and their semantics, consider the following comments.

First, in evaluating an implication $\varphi \Rightarrow \psi$ in a standard way (call it the first way), one takes truth degrees $\|\varphi\|$ and $\|\psi\|$, and a truth function $\rightarrow$ of implication and defines the truth degree $\|\varphi \Rightarrow \psi\|$ of $\varphi \Rightarrow \psi$ to be $\|\varphi\| \rightarrow \|\psi\|$. There is, however, a second way to look at evaluating implications in bivalent case. Namely, one takes φ and tests whether it is true ($\|\varphi\|$ equals 1). If not, $\|\varphi \Rightarrow \psi\|$ equals 1; if yes, $\|\varphi \Rightarrow \psi\|$ equals $\|\psi\|$. Note that in bivalent case, both of the ways yield the same truth degree of $\varphi \Rightarrow \psi$. While the first way can be directly adopted for fuzzy setting (one just allows $\|\varphi\|$ and $\|\psi\|$ to take also intermediate truth values and takes a suitable "fuzzy implication" $\rightarrow$), the second way is not so straightforward. A general way to go is the following: Pick a threshold $a \in L$ such that φ is considered sufficiently true iff its truth degree is at least a. In evaluating $\varphi \Rightarrow \psi$, one takes $\|\varphi\|$ and compares it to a. If $\|\varphi\|$ does not exceed the threshold (i.e. $a \not\leq \|\varphi\|$), set $\|\varphi \Rightarrow \psi\|$ to 1; otherwise (i.e. $a \leq \|\varphi\|$) set $\|\varphi \Rightarrow \psi\|$ to $\|\psi\|$. In bivalent case, there is only one nontrivial choice of a, namely, $a = 1$ which yields the usual interpretation of implications. In fuzzy setting, however, the choice of a is not unique.

Second, interestingly enough, both of the above ways of interpreting implications are particular cases of a more general approach via a unary connective $\triangle$ representing a truth-stressing hedge like "very true" (i.e. $\triangle\varphi$ reads "φ is very true"), and a corresponding unary function * on L which interprets $\triangle$ (see [22, 23]): Consider a formula $\triangle(\overline{a} \Rightarrow \varphi) \Rightarrow \psi$ with $\overline{a}$ a truth constant interpreted by $a \in L$. If * is the identity on L and $a = 1$, the truth degree of $\triangle(\overline{a} \Rightarrow \varphi) \Rightarrow \psi$ is just the truth degree defined in the first way. If * is the globalization (i.e. $a^* = 1$ for $a = 1$ and $a^* = 0$ otherwise), the truth degree of $\triangle(\overline{a} \Rightarrow \varphi) \Rightarrow \psi$ is just the truth degree defined in the second way with a being the threshold.

Third, as we are interested in implications between identities with possibly several identities in the premise connected in a conjunctive manner and want to allow each identity to have its own threshold, we deal with formulas $\triangle(\bigwedge_{i\in I}(\overline{a_i} \Rightarrow \varphi_i)) \Rightarrow \psi$ where φ_i's and ψ are identities, a_i's are the thresholds of φ_i's, and $\bigwedge$ is a "general conjunction" interpreted by infimum in L. For convenience and simplicity, we write only $\bigwedge_{i\in I}\langle\varphi_i, a\rangle \Rightarrow \psi$ instead of $\triangle(\bigwedge_{i\in I}(\overline{a_i} \Rightarrow \varphi_i)) \Rightarrow \psi$. That is, we omit $\triangle$ (since its placement is fixed), we write $\langle\varphi_i, \overline{a_i}\rangle$ instead of $\overline{a_i} \Rightarrow \varphi_i$, and do not distinguish between constants $\overline{a_i}$ for truth degrees and truth degrees a_i themselves. Note that the second convention follows the identification of so-called weighted formulas $\langle\varphi, \overline{a}\rangle$ of fuzzy logic with evaluated syntax [29, 31] with ordinary formulas $\overline{a} \Rightarrow \varphi$, see e.g. [22, Section 3.3]. Doing so, we deal with formulas which can be thought of as implications $P \Rightarrow \psi$ where P is a fuzzy set of formulas φ_i with $P(\varphi_i) = a_i$. For our purpose, this is a convenient and sufficiently general way.

* * *

To sum up, we wish to consider implications with weighted premises. Doing so, we will distinguish several types of families $\mathcal{P}$ of premises P (e.g. P finite). For reasons that will appear later we assume that the families $\mathcal{P}$ of premises P are closed under *endomorphic images*. An endomorphic image of a binary fuzzy relation P on $T(X)$ has to be defined to respect the fact that for two terms $t, t' \in T(X)$ there can be $r, r', s, s' \in T(X)$ such that $h(r) = h(s) = t$, $h(r') = h(s') = t'$, but $P(r, r') \neq P(s, s')$. This kind of ambiguity can be avoided using the general suprema.

Definition 1. *For $P \in \mathbf{L}^{T(X)\times T(X)}$ and an endomorphism $h\colon \mathbf{T}(X) \to \mathbf{T}(X)$, we define an **endomorphic image** $h(P) \in \mathbf{L}^{T(X)\times T(X)}$ of P by*

$$h(P)(t, t') = \bigvee_{\substack{h(s)=t \\ h(s')=t'}} P(s, s') \tag{2}$$

*for every terms $t, t' \in T(X)$. Let $\emptyset \neq \mathcal{P} \subseteq \mathbf{L}^{T(X)\times T(X)}$. The family $\mathcal{P}$ is called a **proper family of premises** of type F (in variables X) if for every $P \in \mathcal{P}$ and every endomorphism h on $\mathbf{T}(X)$ we have $h(P) \in \mathcal{P}$. Then, each $P \in \mathcal{P}$ is called an **$\mathbf{L}$-set of premises** of type F (in variables X).*

For a crisp $P \subseteq T(X) \times T(X)$, the notion of the endomorphic image as defined above coincides with the classical one: $h(P) = \{\langle h(t), h(t')\rangle \mid \langle t, t'\rangle \in P\}$.

Example 1. The following are examples of proper families of premises.

(a) If $\mathcal{P} = \mathbf{L}^{T(X)\times T(X)}$, then $\mathcal{P}$ is a proper family of premises trivially. This family does not represent any constraints on premises. Implications with such premises will represent the *most general type* of formulas used in our investigation.

(b) For $P \in \mathbf{L}^{T(X)\times T(X)}$, put $\mathrm{var}(P) = \bigcup\{\mathrm{var}(t) \cup \mathrm{var}(t') \mid P(t, t') > 0\}$, i.e. $\mathrm{var}(P)$ is a set of variables occurring in identities which belong to P in some nonzero degree. Let $\mathcal{P} = \{P \in \mathbf{L}^{T(X)\times T(X)} \mid \mathrm{var}(P) \text{ is finite}\}$. Then $\mathcal{P}$ is a proper family of premises, so-called *finitary premises*.

(c) A family $\mathcal{P} = \{P \in \mathbf{L}^{T(X)\times T(X)} \mid P \text{ is finite}\}$ is a special subfamily of that of (b). It is easy to observe that endomorphic image of every finite $\mathbf{L}$-relation is finite. Trivially, $\mathrm{var}(P)$ is finite. $\mathcal{P}$ of this form is called a proper family of all *finite premises.*

(d) Families of premises defined in (a)–(c) have their *crisp variants*. It follows immediately from the fact that if $P \in \mathbf{L}^{T(X)\times T(X)}$ is crisp, then $h(P)$ is crisp as well. Hence, the following families
$\mathcal{P}_1 = \{P \in \mathbf{L}^{T(X)\times T(X)} \mid P \text{ is crisp}\}$,
$\mathcal{P}_2 = \{P \in \mathbf{L}^{T(X)\times T(X)} \mid \mathrm{var}(P) \text{ is finite and } P \text{ is crisp}\}$,
$\mathcal{P}_3 = \{P \in \mathbf{L}^{T(X)\times T(X)} \mid P \text{ is finite and crisp}\}$,
are proper families of premises. In fact, they can be used to determine families of implications with crisp premises. A calculus for such formulas is introduced in Section 6.

(e) There are approaches [32] which have used also implications with exactly one premise. In this case $\mathcal{P} = \big\{P \in \mathbf{L}^{T(X)\times T(X)} \mid P \text{ is a singleton}\big\}$ and $\mathcal{P}$ is trivially a proper family of premises. This concept can be generalized to families with at most n premises.

(f) $\mathcal{P} = \{\emptyset\}$ is a proper family of premises trivially. Implications with empty premises represent just *identities*. As we will see in Section 7, putting $\mathcal{P} = \{\emptyset\}$, our results yield the well-known results on equational logic [11] and our previous results on fuzzy equational logic [5] as a special case.

Now we can define the notion of a (weighted) $\mathcal{P}$-implication.

Definition 2. *Suppose* $\mathbf{L}$ *is a complete residuated lattice,* X *a set of variables,* F *is a type,* $\mathcal{P}$ *is a proper family of premises of type* F *in variables* X*. A* ***$\mathcal{P}$-implication*** *is an expression of the form*

$$\textstyle\bigwedge_{P(s,s')>0}\big\langle s \approx s',\, P(s,s')\big\rangle \Rightarrow (t \approx t'), \tag{3}$$

where $P \in \mathcal{P}$ *and* $t, t' \in T(X)$*. For a* $\mathcal{P}$*-implication* φ *and a truth value* $a \in L$*, the couple* $\langle \varphi, a\rangle$ *is called a* ***weighted $\mathcal{P}$-implication****.*

Remark 1. (1) $\mathcal{P}$-implications are in general “infinite expressions”. Following the discussion in the beginning of Section 2, we consider (3) a shorthand for

$$\triangle\Big(\textstyle\bigwedge_{P(s,s')>0}\big(\overline{P(s,s')} \Rightarrow (s \approx s')\big)\Big) \Rightarrow (t \approx t'), \tag{4}$$

a formula in language containing a unary connective $\triangle$, symbols $\overline{a}$ for truth constants $a \in L$, and a “generalized conjunction” $\bigwedge$. (4) says “if it is very true that all identities $s \approx s'$ from P are true then $t \approx t'$ is true”, which is the intended meaning of (3).

(2) In the following, we will freely use the fact that an $\mathbf{L}$-set Σ of $\mathcal{P}$-implications can be thought of as an ordinary set I_Σ of weighted $\mathcal{P}$-implications, where $I_\Sigma = \{\langle\varphi, a\rangle \mid \Sigma(\varphi) = a\}$.

(3) If $\mathcal{P}$ is a proper family of finitary (finite) premises, a $\mathcal{P}$-implication will be called a *finitary implication* (*Horn clause*). Similarly for weighted $\mathcal{P}$-implications.

(4) For simplicity, a $\mathcal{P}$-implication (3) will be denoted by $P \Rightarrow (t \approx t')$. A Horn clause $P \Rightarrow (t \approx t')$ will be occasionally denoted by

$$\big\langle t_1 \approx t_1', a_1\big\rangle \wedge \big\langle t_2 \approx t_2', a_2\big\rangle \wedge \cdots \wedge \big\langle t_n \approx t_n', a_n\big\rangle \Rightarrow t \approx t', \tag{5}$$

where $P(t_i, t_i') = a_i$ for $i = 1, \ldots, n$, and $\mathrm{Supp}(P) \subseteq \{\langle t_1, t_1'\rangle, \ldots, \langle t_n, t_n'\rangle\}$. Identities and weighted identities are denoted by $t \approx t'$ and $\langle t \approx t', a\rangle$.

We are going to introduce *semantics* of $\mathcal{P}$-implications, i.e. to define a truth degree $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ of $P \Rightarrow (t \approx t')$ in an $\mathbf{L}$-algebra $\mathbf{M}$ under a valuation v. Given terms $t, t' \in T(X)$, a degree $\|t \approx t'\|_{\mathbf{M},v}$ to which the identity $t \approx t'$ is true in $\mathbf{M}$ under a valuation $v\colon X \to M$ is defined in the usual manner, i.e. by $\|t \approx t'\|_{\mathbf{M},v} = \|t\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|t'\|_{\mathbf{M},v}$. The truth degree $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ of (3) is defined in a straightforward way, taking into account that (3) is a shorthand for (4). Connective $\triangle$ will be interpreted by a unary operation * on L, called a truth stresser, which is similar to to that of Hájek's $\otimes$-*truth stressing hedge*, see [23] (a particular case of a truth stressing hedge was introduced by Baaz [1], see also [22]).

Definition 3. *Let* $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ *be a complete residuated lattice. A unary operation* $^*\colon L \to L$ *satisfying*

$$1^* = 1, \tag{6}$$

$$a^* \leq a, \tag{7}$$

$$(a \rightarrow b)^* \leq a^* \rightarrow b^*, \tag{8}$$

for every $a, b \in L$*, is called a* ***truth stresser*** for $\mathbf{L}$. *A residuated lattice* $\mathbf{L}$ *equipped with a truth stresser* * *for* $\mathbf{L}$ *will be denoted by* $\mathbf{L}^*$.

Later on, we will formulate additional constraints for *. Conditions (6)–(8) represent three of the four basic conditions required in [23]. The fourth condition $(a \vee b)^* \leq a^* \vee b^*$ presented in [23] is not necessary e.g. to establish a characterization of semantic consequence. On the other hand, an analogy to $(a \vee b)^* \leq a^* \vee b^*$ is important when considering implications with crisp premises, see Section 6.

Example 2. Let $\mathbf{L}$ be a complete residuated lattice.

(a) Let * be the identity on L, i.e. $a^* = a$ ($a \in L$). Then * is a truth stresser.
(b) Let * be defined by

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Then * is a truth stresser. We follow Takeuti and Titani [33] and call it *globalization*. Note that in [33], globalization is defined on complete Heyting algebras. Note also that if we restrict ourselves to linearly ordered residuated lattices, globalization is axiomatically defined by (6), (7), and $a^* \vee (a^* \rightarrow 0) = 1$, see [22, Section 2.4].
(c) Denote the two above truth stressers by ^{*_1} (identity) and ^{*_2} (globalization). Trivially, for each truth stresser * we have $a^{*_1} = 0 \leq a^* \leq a = a^{*_2}$ for $a < 1$ and $1 = 1^{*_1} = 1^* = 1^{*_2}$. Therefore, truth stressers are bounded by ^{*_1} and ^{*_2}.

A general definition of the truth degree of an implication follows.

Definition 4. *Let* $P \Rightarrow (t \approx t')$ *be a* $\mathcal{P}$*-implication and let* * *be a truth stresser for a complete residuated lattice* $\mathbf{L}$. *For an* $\mathbf{L}$*-algebra* $\mathbf{M}$ *and a valuation* $v\colon X \to M$, *we define the* truth degree $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ of $P \Rightarrow (t \approx t')$ in $\mathbf{M}$ under v with respect to $*$ *by*

$$\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v} = \|P\|_{\mathbf{M},v} \to \|t \approx t'\|_{\mathbf{M},v}, \tag{10}$$

where

$$\|P\|_{\mathbf{M},v} = \left(\bigwedge\nolimits_{s,s' \in T(X)} \left(P(s,s') \to \|s \approx s'\|_{\mathbf{M},v}\right)\right)^{*}. \tag{11}$$

For a weighted $\mathcal{P}$*-implication* $\langle P \Rightarrow (t \approx t'), a\rangle$, $a \in L$, *we define a* truth degree $\|\langle P \Rightarrow (t \approx t'), a\rangle\|_{\mathbf{M},v}$ of $\langle P \Rightarrow (t \approx t'), a\rangle$ in $\mathbf{M}$ under v w.r.t. $*$ *by*

$$a \to \|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}. \tag{12}$$

Therefore, we consider $*$ as a parameter controlling the interpretation of (3). For the boundary truth stressers of Example 2, we have the following. First, for $*$ being the identity we get

$$\begin{aligned}&\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}\\ &\quad = \left(\bigwedge\nolimits_{s,s' \in T(X)} \left(P(s,s') \to \|s \approx s'\|_{\mathbf{M},v}\right)\right) \to \|t \approx t'\|_{\mathbf{M},v}.\end{aligned} \tag{13}$$

Second, for $*$ being globalization we get

$$\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v} = \begin{cases} \|t \approx t'\|_{\mathbf{M},v} & \text{if } P(s,s') \leq \|s \approx s'\|_{\mathbf{M},v} \\ & \text{for all } s, s' \in T(X), \\ 1 & \text{otherwise.} \end{cases} \tag{14}$$

Denoting by a and b the truth degree of $P \Rightarrow (t \approx t')$ as defined by (13) and (14), respectively, it is easily seen that $a \leq b$. (13) and (14) are thus the boundary cases of (10). As we will see, both types of semantics are reasonable (cf. also discussion in the beginning of Section 2). Note that in the bivalent case, both (13) and (14) coincide.

Remark 2. (1) We will use only one structure of truth values and one truth stresser at a time, so there is no danger of confusion if the degree is denoted simply by $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$. Sometimes, we will use $\|P \Rightarrow (t \approx t')\|^{\mathbf{L}^*}_{\mathbf{M},v}$ to point out $\mathbf{L}$ and $*$ explicitly.

(2) It is easily seen, that (12) is equal to $\|P\|_{\mathbf{M},v} \to (a \to \|t \approx t'\|_{\mathbf{M},v})$. This corresponds well to the intuitive meaning of a weighted implication and also justifies a possible notation

$$\langle t_1 \approx t'_1, a_1\rangle \mathbin{\&} \langle t_2 \approx t'_2, a_2\rangle \mathbin{\&} \cdots \mathbin{\&} \langle t_n \approx t'_n, a_n\rangle \Rightarrow \langle t \approx t', a\rangle$$

for a weighted Horn clause

$$\langle\langle t_1 \approx t'_1, a_1\rangle \mathbin{\&} \langle t_2 \approx t'_2, a_2\rangle \mathbin{\&} \cdots \mathbin{\&} \langle t_n \approx t'_n, a_n\rangle \Rightarrow t \approx t', a\rangle.$$

(3) Evidently, $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v} \geq b$ iff $\|\langle P \Rightarrow (t \approx t'), b\rangle\|_{\mathbf{M},v} = 1$.

(4) Suppose we are given an $\mathbf{L}$-algebra $\mathbf{M}$, and a class $\mathcal{K}$ of $\mathbf{L}$-algebras of the same type. Truth degrees of $P \Rightarrow (t \approx t')$ in $\mathbf{M}$, and $\mathcal{K}$ are defined by

$$\|P \Rightarrow (t \approx t')\|_{\mathbf{M}} = \bigwedge_{v:X\to M} \|P \Rightarrow (t \approx t')\|_{\mathbf{M},v},$$
$$\|P \Rightarrow (t \approx t')\|_{\mathcal{K}} = \bigwedge_{\mathbf{M}\in\mathcal{K}} \|P \Rightarrow (t \approx t')\|_{\mathbf{M}}.$$

Example 3. The apparatus of weighted implications can be seen as a tool in formal specification in presence of vagueness. In particular, applications of weighted implications lie mainly in the field of so-called humanistic systems, where the description of a system behavior is influenced by human judgment or perceptions, and is therefore inherently vague. In the following, we present a way to describe approximate knowledge about a simple function-based system.

We deal with the problem of human perception of colors and the related problem of color mixture. Needles to say, the problems in question are hardly graspable by bivalent logic (crisp structures) since the notions of "color similarity" and "color indistinguishability" that naturally appear in the problem domain are vague. The color perception itself is a complex neuro-chemical process with a psychological feedback. Denote the set of all colors by M. Equip M with an $\mathbf{L}$-equality relation $\approx^M$ the meaning of which is to represent similarity of colors from M. Note that $\approx^M$ is a nontrivial $\mathbf{L}$-relation for which all properties of an $\mathbf{L}$-equality seem to be fully justified.

The (additive) mixture of colors can be thought of as an operation on M. Thus, suppose we have a language $F = \{f\}$, where f is a binary function symbol and a term $f(t, t')$ of type F represents a color resulting by the mixture of colors represented by terms t, t'. It is a well-known fact [16] that assuming sufficiently high light intensities, if x is indistinguishable from x', and y is indistinguishable from y', then $f(x, y)$ is indistinguishable from $f(x', y')$. This rule immediately translates into a compatibility condition for $f^{\mathbf{M}}$.

To sum up, an $\mathbf{L}$-algebra $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, f^{\mathbf{M}}\rangle$ of type F seems to be a suitable semantical structure enabling us to study color mixture. Weighted implications can be used to define additional constraints on our perception of color mixture. For instance, the weighted $\mathcal{P}$-implication

$$\langle x \approx x', a\rangle \curlywedge \langle f(x, y) \approx f(x', y'), b\rangle \Rightarrow \langle y \approx y', c\rangle \tag{15}$$

can be read as: "if colors x, x' are similar in degree a and if mixtures $f(x, y)$, $f(x', y')$ are similar in degree b, then colors y, y' are similar (at least) in degree c".

The following definition introduces three additional conditions.

Definition 5. *A truth stresser* * *for* $\mathbf{L}$ *is called an* ***implicational truth stresser*** *if it satisfies*

$$a^{**} = a^*, \tag{16}$$
$$a^* \otimes a^* = a^*, \tag{17}$$
$$\bigwedge_{i\in I} a_i^* = \left(\bigwedge_{i\in I} a_i\right)^*, \tag{18}$$

for every $a \in L$, $a_i \in L$ $(i \in I)$, *and each* I.

Lemma 1. *The following are properties of truth stressers:*

(i) (7) *implies* $0^* = 0$,
(ii) (18) *implies* (6),
(iii) (18) *implies monotony of* * ($a \leq b$ *implies* $a^* \leq b^*$ *for every* $a, b \in L$),
(iv) (6) *and* (8) *imply monotony of* *,
(v) *monotony of* * *and* (8) *imply* $a^* \otimes b^* \leq (a \otimes b)^*$ *for every* $a, b \in L$.

Proof. (i): $0^* \leq 0$, i.e. $0^* = 0$.
(ii): Put $I = \emptyset$ and apply (18) (and convention saying that infimum of $\emptyset$ is 1.
(iii): For $a \leq b$, we have $a = a \wedge b$, thus $a^* = (a \wedge b)^*$. Applying (18) for $I = \{1, 2\}$ we obtain $a^* = a^* \wedge b^*$, whence $a^* \leq b^*$.
(iv): $a \leq b$ implies $a \to b = 1$, thus $(a \to b)^* = 1^* = 1$ by (6). Then $a^* \to b^* = 1$ by (8), thus $a^* \leq b^*$.
(v): $a \leq b \to (a \otimes b)$ by adjointness, using monotony it follows that $a^* \leq (b \to (a \otimes b))^*$ and so $a^* \leq b^* \to (a \otimes b)^*$ by (8), whence $a^* \otimes b^* \leq (a \otimes b)^*$ by adjointness. □

Remark 3. (1) Using (8), (6), and condition (29) presented in Section 4, we can prove a weaker form of condition (18). Namely, ${a_1}^* \wedge {a_2}^* = (a_1 \wedge a_2)^*$, which can be extended for finitely many a_i's. But as it will appear later, this weaker form of (18) is not sufficient in our development.
(2) Every implicational truth stresser * for $\mathbf{L}$ is an *interior operator* on $\langle L, \leq \rangle$, i.e. * verifies $a^* \leq a$, monotony ($a \leq b$ implies $a^* \leq b^*$), and $a^* = a^{**}$.
(3) The "$\leq$"-part of (18) is one of the definitional conditions of Takeuti and Titani's globalization [33].

Example 4. The following are examples of implicational truth stressers.

(a) Suppose $\mathbf{L}$ is a complete Heyting algebra (i.e. $a \otimes b = a \wedge b$), putting $a^* = a$, we get an implicational truth-stresser. Conversely, if $a^* = a$ is an implicational truth stresser on $\mathbf{L}$ then $\mathbf{L}$ is a Heyting algebra.
(b) For every complete residuated lattice $\mathbf{L}$, the truth stresser * defined by (9) is an implicational truth stresser. Obviously, (16), (17) are satisfied. For every index set I and $a_i \in L$ ($i \in I$) we have $\bigwedge_{i \in I} a_i^* = 1$ iff for every $i \in I$ we have $a_i^* = 1$ iff $a_i = 1$ iff $\bigwedge_{i \in I} a_i = 1$ iff $\left(\bigwedge_{i \in I} a_i\right)^* = 1$. Consequently, $\bigwedge_{i \in I} a_i^* = 0$ iff $\left(\bigwedge_{i \in I} a_i\right)^* = 0$. Hence, (18) holds as well.

There are, of course, other nontrivial implicational truth stressers and thus other interesting "semantics of implications". An example is given by the following theorem.

Theorem 1. *Let* $\mathbf{L}$ *be a complete BL-chain (i.e. a linearly ordered BL-algebra, see [22]) satisfying*

$$a \otimes \bigwedge_{i \in I} b_i = \bigwedge_{i \in I} (a \otimes b_i) \tag{19}$$

for every $a \in L$, $\{b_i \in L \mid i \in I\}$. *Then the unary operation* * *defined by*

$$a^* = \bigwedge_{n \in \mathbb{N}_0} a^n \tag{20}$$

for all $a \in L$ is an implicational truth stresser. Moreover, a^* is the greatest idempotent less or equal to a.

Proof. As (6) is a consequence of (18), we have to check conditions (7), (8), (16)–(18).

(7): For $n = 1$ we have $a^1 \leq a$, thus $a^* = \bigwedge_{n\in\mathbb{N}_0} a^n \leq a$.

(8): We have

$$(a \to b)^* = \bigwedge_{n\in\mathbb{N}_0}(a \to b)^n \leq \bigwedge_{n\in\mathbb{N}_0} (a^n \to b^n) \leq \bigwedge_{n\in\mathbb{N}_0} a^n \to \bigwedge_{n\in\mathbb{N}_0} b^n = a^* \to b^*.$$

(16): Due to (19), the following equality holds:

$$\begin{aligned} a^{**} &= \bigwedge_{m\in\mathbb{N}_0} \Big(\bigwedge_{n\in\mathbb{N}_0} a^n\Big)^m = \bigwedge_{m\in\mathbb{N}_0} \bigotimes_{i=1}^m \bigwedge_{n_i\in\mathbb{N}_0} a^{n_i} \\ &= \bigwedge_{m\in\mathbb{N}_0} \bigwedge_{n_1,\ldots,n_m\in\mathbb{N}_0} \bigotimes_{i=1}^m a^{n_i} = \bigwedge_{m\in\mathbb{N}_0} \bigwedge_{n_1,\ldots,n_m\in\mathbb{N}_0} a^{\sum_{i=1}^m n_i} \\ &= \bigwedge_{m\in\mathbb{N}_0} \bigwedge_{n\in\mathbb{N}_0} a^n = \bigwedge_{n\in\mathbb{N}_0} a^n = a^*. \end{aligned}$$

(17): Analogously as for (16), we have

$$a^* \otimes a^* = \big(\bigwedge_{n\in\mathbb{N}_0} a^n\big) \otimes \big(\bigwedge_{m\in\mathbb{N}_0} a^m\big) = \bigwedge_{m,n\in\mathbb{N}_0} a^{n+m} = \bigwedge_{k\in\mathbb{N}_0} a^k = a^*.$$

(18): We have to show $\bigwedge_{i\in I} \bigwedge_{n\in\mathbb{N}_0} a_i^n = \bigwedge_{n\in\mathbb{N}_0}\big(\bigwedge_{i\in I} a_i\big)^n$. The "$\geq$"-part is evident. For the "$\leq$"-part, observe that since $a_i^n \leq a_{i_1} \otimes \cdots \otimes a_{i_n}$ for $a_i = \min\{a_{i_1}, \ldots, a_{i_n}\}$ holds for each n (recall that $\mathbf{L}$ is supposed to be a chain), we have

$$\bigwedge_{i\in I} \bigwedge_{n\in\mathbb{N}_0} a_i^n \leq \bigwedge_{i\in I} a_i^n \leq \bigwedge_{i_1,\ldots,i_n\in I} (a_{i_1} \otimes \cdots \otimes a_{i_n}) = \big(\bigwedge_{i\in I} a_i\big)^n.$$

Hence, the required inequality follows immediately.

It remains to show that a^* is the greatest idempotent which is less or equal to a. Indeed, let $b \in L$ be an idempotent such that $b \leq a$. Then $b = \bigwedge_{n\in\mathbb{N}_0} b^n \leq \bigwedge_{n\in\mathbb{N}_0} a^n = a^*$. □

Remark 4. Since prelinearity implies $a \otimes (b_1 \wedge b_2) = (a \otimes b_1) \wedge (a \otimes b_2)$, any finite prelinear $\mathbf{L}$ satisfies (19). Thus, every finite BL-chain satisfies (19). If $\mathbf{L} = \langle [0, 1], \max, \min, \otimes, \to, 0, 1\rangle$ is a residuated lattice with $\otimes$ being a continuous t-norm, then $\mathbf{L}$ is a BL-algebra and (19) is a consequence of right-continuity of $\otimes$, see [6, 22].

3. Semantic consequence

In ordinary equational logic [11], logical notions have their algebraic counterparts. A set of identities is handled as a binary relation on $T(X)$. Semantically closed sets then correspond to fully invariant congruence relations on $\mathbf{T}(X)$. Fuzzy equational logic [5] naturally develops these notions in fuzzy setting. A situation in (fuzzy) Horn logic is more complex, $\mathbf{L}$-sets of $\mathcal{P}$-implications will be represented by so-called $\mathcal{P}$-indexed systems of $\mathbf{L}$-relations.

Definition 6. *Let $\mathcal{P}$ be a proper family of premises. A system*

$$\mathcal{S} = \left\{ S_P \in \mathbf{L}^{T(X)\times T(X)} \mid P \in \mathcal{P} \right\}$$

is called a ***$\mathcal{P}$-indexed system of L-relations***. *For a $\mathcal{P}$-indexed system $\mathcal{S}$ of $\mathbf{L}$-relations we define an $\mathbf{L}$-set $\Sigma_{\mathcal{S}}$ of $\mathcal{P}$-implications by*

$$\Sigma_{\mathcal{S}}(P \Rightarrow (t \approx t')) = S_P(t, t'), \quad \textit{for every } P \in \mathcal{P} \textit{ and } t, t' \in T(X).$$

For every $\mathbf{L}$-set Σ of $\mathcal{P}$-implications we define a $\mathcal{P}$-indexed system $\mathcal{S}_\Sigma$ of $\mathbf{L}$-relations as follows

$$\mathcal{S}_\Sigma = \left\{ S_P \in \mathbf{L}^{T(X)\times T(X)} \mid S_P(t, t') = \Sigma(P \Rightarrow (t \approx t')) \right. \\ \left. \textit{for every } P \in \mathcal{P} \textit{ and } t, t' \in T(X) \right\}.$$

For convenience, if Σ is an $\mathbf{L}$-set of $\mathcal{P}$-implications, then by Σ_P we denote the corresponding $\mathbf{L}$-relation $S_P \in \mathcal{S}_\Sigma$.

Remark 5. (1) It is immediate that $\Sigma_{\mathcal{S}_\Sigma} = \Sigma$, $\mathcal{S}_{\Sigma_{\mathcal{S}}} = \mathcal{S}$. That is, there is an obvious bijective correspondence between $\mathcal{P}$-indexed systems $\mathcal{S}$ of $\mathbf{L}$-relations and $\mathbf{L}$-sets Σ of $\mathcal{P}$-implications, and we can go from $\mathcal{S}$ to the corresponding Σ and *vice versa*.

(2) For $\mathcal{P}$-indexed systems $\mathcal{S}$, $\mathcal{S}'$ of $\mathbf{L}$-relations, we put $\mathcal{S} \leq \mathcal{S}'$ iff for every $P \in \mathcal{P}$ we have $S_P \subseteq S'_P$. Consequently, $\mathcal{S} = \mathcal{S}'$ iff $S_P = S'_P$ for every $P \in \mathcal{P}$.

(3) An $\mathbf{L}$-set of identities Σ can be thought of as an $\mathbf{L}$-set of $\mathcal{P}$-implications for $\mathcal{P} = \{\emptyset\}$. Thus, the corresponding $\{\emptyset\}$-indexed system $\mathcal{S}_\Sigma$ consists a single $\mathbf{L}$-relation denoted by $\Sigma_\emptyset$.

(4) S_P denotes elements of a system $\mathcal{S}$. Notice that we use S to denote the subsethood degree, see (1). There is no danger of confusion here since S_P is a fuzzy relation on terms while S is a fuzzy relation on fuzzy sets. Moreover, elements of $\mathcal{S}$ are always used with subscripts (S_P, S_i, etc.).

Definition 7. *Suppose $\mathbf{L}^*$ is a complete residuated lattice with implicational truth stresser *. Let Σ be an $\mathbf{L}$-set of $\mathcal{P}$-implications. If the corresponding $\mathcal{S}_\Sigma$ satisfies*

$$P \subseteq \Sigma_P, \tag{21}$$

$$\Sigma_P(t, t') \leq \Sigma_{h(P)}(h(t), h(t')), \tag{22}$$

$$S(P, \Sigma_Q)^* \leq S(\Sigma_P, \Sigma_Q), \tag{23}$$

for every $P, Q \in \mathcal{P}$, $t, t' \in T(X)$, and every endomorphism h on $\mathbf{T}(X)$, then $\mathcal{S}_\Sigma$ is called an ***$\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of L-relations***. *Moreover, if Σ_P is a congruence for every $P \in \mathcal{P}$, then $\mathcal{S}_\Sigma$ is called an* ***$\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of congruences***.

Remark 6. (1) Let us comment on the meaning of (21)–(23). Note first that for Σ, $\Sigma_P(t \approx t') = \Sigma(P \Rightarrow (t \approx t'))$. Now, think of $\Sigma(P \Rightarrow (t \approx t'))$ as of a degree to which $P \Rightarrow (t \approx t')$ is true in some $\mathbf{L}$-algebra $\mathbf{M}$, i.e. Σ is a theory of $\mathbf{M}$. Then (21) says that the validity degree of $P \Rightarrow (t \approx t')$ is at least as high as the degree to which $\langle t, t' \rangle$ belongs to P. Roughly speaking, if $\langle t, t' \rangle$ is in P then

$P \Rightarrow (t \approx t')$ is always true which is an obvious property. (22) says, roughly speaking, that if $P \Rightarrow (t \approx t')$ is true then $h(P) \Rightarrow (h(t) \approx h(t'))$ is true, i.e., thinking of endomorphisms as of substitutions, (22) says that validity is preserved under substitutions. Finally, (23) says that a mapping sending P to Σ_P obeys one of characteristic properties of a so-called $\mathbf{L}^*$-closure operator [8]. An $\mathbf{L}^*$-closure operator is a mapping $\mathrm{C} : \mathbf{L}^U \to \mathbf{L}^U$ satisfying $A \subseteq \mathrm{C}(A)$, $S(A, B)^* \leq S(\mathrm{C}(A), \mathrm{C}(B))$, $\mathrm{C}(A) = \mathrm{C}(\mathrm{C}(A))$. It can be shown that if * is an implicational truth stresser, then $\mathrm{C}\colon \mathbf{L}^U \to \mathbf{L}^U$ is an $\mathbf{L}^*$-closure operator iff C satisfies $A \subseteq \mathrm{C}(A)$ and $S(A, \mathrm{C}(B))^* \leq S(\mathrm{C}(A), \mathrm{C}(B))$ [8], whence (21) and (23). Note that $\mathbf{L}^*$-closure operators satisfy stronger conditions than the usual fuzzy closure operators, see e.g. [3, 4]. Note also that our mapping sending P to Σ_P is in fact only a partial $\mathbf{L}^*$-closure operator since it is a mapping of $\mathcal{P}$ to $\mathbf{L}^{T(X)\times T(X)}$, not of $\mathbf{L}^{T(X)\times T(X)}$ to $\mathbf{L}^{T(X)\times T(X)}$ (for instance, in case of crisp premises, $\mathcal{P} = \mathbf{2}^{T(X)\times T(X)}$). Nevertheless, (21)–(23) say that sending P to Σ_P has closure properties and that Σ is closed under substitutions.

(2) Since $\mathcal{P}$ is supposed to be proper, $h(P) \in \mathcal{P}$ for every $P \in \mathcal{P}$ as required by Definition 1. That is, condition (22) is defined correctly.

(3) Condition (23) has equivalent formulations. For instance, (23) holds for every $P, Q \in \mathcal{P}$ iff

$$\Sigma_P(t, t') \otimes \Big(\bigwedge\nolimits_{s,s' \in T(X)} \big(P(s, s') \to \Sigma_Q(s, s')\big)\Big)^* \leq \Sigma_Q(t, t') \quad (24)$$

for all $t, t' \in T(X)$. This equivalent formulation will be used later on.

(4) Suppose $\Sigma_\emptyset$ is a congruence. Then if (22) holds for $P = \emptyset$, we can claim $\Sigma_\emptyset(t, t') \leq \Sigma_{h(\emptyset)}\big(h(t), h(t')\big) = \Sigma_\emptyset\big(h(t), h(t')\big)$. Hence, $\Sigma_\emptyset$ is a *fully invariant congruence* [5].

(5) For particular proper families of premises, some of the conditions (21)–(23) will simplify or hold trivially. We will take advantage of this fact in Section 6 and Section 7.

Conditions (22), (23) can be replaced equivalently by a single condition as shown below.

Lemma 2. *Suppose we have a $\mathcal{P}$-indexed system $\mathcal{S}_\Sigma$ of $\mathbf{L}$-relations. Then $\mathcal{S}_\Sigma$ is an $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of $\mathbf{L}$-relations if and only if $\mathcal{S}_\Sigma$ satisfies* (21) *together with*

$$\begin{aligned}&\Sigma_P(t, t')\\ &\leq \Big[\bigwedge\nolimits_{s,s' \in T(X)} \big(P(s, s') \to \Sigma_Q\big(h(s), h(s')\big)\big)\Big]^* \to \Sigma_Q\big(h(t), h(t')\big)\end{aligned} \quad (25)$$

for every $P, Q \in \mathcal{P}$, $t, t' \in T(X)$, and every endomorphism h on $\mathbf{T}(X)$.

Proof. "$\Rightarrow$": Let us suppose (21), (22), and (23) hold. Take $P, Q \in \mathcal{P}$, $t, t' \in T(X)$, and an endomorphism $h : \mathbf{T}(X) \to \mathbf{T}(X)$. Using (2) and the adjointness property it follows that

$$h(P)(r, r') \to \Sigma_Q(r, r') = \Big(\bigvee\nolimits_{\substack{h(s)=r\\ h(s')=r'}} P(s, s')\Big) \to \Sigma_Q(r, r')$$

$$= \bigwedge\nolimits_{\substack{h(s)=r\\ h(s')=r'}} \big(P(s, s') \to \Sigma_Q(r, r')\big) = \bigwedge\nolimits_{\substack{h(s)=r\\ h(s')=r'}} \big(P(s, s') \to \Sigma_Q\big(h(s), h(s')\big)\big),$$

for every $r, r' \in T(X)$. Thus, using (22), (24) we obtain

$$\begin{aligned}
\Sigma_P(t,t') &\leq \Sigma_{h(P)}\big(h(t), h(t')\big) \\
&\leq \big[\textstyle\bigwedge_{r,r' \in T(X)} \big(h(P)(r,r') \to \Sigma_Q(r,r')\big)\big]^* \to \Sigma_Q\big(h(t), h(t')\big) \\
&= \Big[\textstyle\bigwedge_{r,r' \in T(X)} \bigwedge_{\substack{h(s)=r \\ h(s')=r'}} \big(P(s,s') \to \Sigma_Q\big(h(s), h(s')\big)\big)\Big]^* \to \Sigma_Q\big(h(t), h(t')\big) \\
&= \big[\textstyle\bigwedge_{s,s' \in T(X)} \big(P(s,s') \to \Sigma_Q\big(h(s), h(s')\big)\big)\big]^* \to \Sigma_Q\big(h(t), h(t')\big).
\end{aligned}$$

Hence, the inequality (25) is satisfied.

"$\Leftarrow$": Assume that conditions (21), (25) hold.

(22): Take any endomorphism $h: \mathbf{T}(X) \to \mathbf{T}(X)$ and put $Q = h(P)$. From Definition 1 it follows that $Q \in \mathcal{P}$. Now, using (25) we obtain

$$\begin{aligned}
&\Sigma_P(t,t') \\
&\leq \big[\textstyle\bigwedge_{r,r' \in T(X)} \big(P(r,r') \to \Sigma_{h(P)}\big(h(r), h(r')\big)\big)\big]^* \to \Sigma_{h(P)}\big(h(t), h(t')\big).
\end{aligned}$$

Moreover, applying (2), (21), we have

$$P(r,r') \leq \textstyle\bigvee_{\substack{h(s)=h(r) \\ h(s')=h(r')}} P(s,s') = h(P)\big(h(r), h(r')\big) \leq \Sigma_{h(P)}\big(h(r), h(r')\big),$$

i.e. $P(r,r') \to \Sigma_{h(P)}\big(h(r), h(r')\big) = 1$ for all $r, r' \in T(X)$. Now, (6) gives

$$\Sigma_P\ (t,t') \leq 1^* \to \Sigma_{h(P)}\big(h(t), h(t')\big) = \Sigma_{h(P)}\big(h(t), h(t')\big),$$

proving (22).

(23): Applying (25) to h being the identical morphism (i.e. $h(t) = t$) we get (24), a condition equivalent to (23). $\square$

Remark 7. In ordinary case, the concept of a semantically closed set of implications corresponds to so-called *fully invariant closure operators* and *fully invariant closure systems* in $T(X) \times T(X)$, see [38]. Recall that the condition of full invariance of a closure operator cl means that $h(cl(P)) \subseteq cl(h(P))$ for every $P \subseteq T(X) \times T(X)$. In case of implications with finite premises, semantically closed sets of implications correspond to algebraic (i.e. finitely generated) fully invariant closure operators.

In fuzzy setting, the above-described relationship is not so straightforward. That is why we use the notion of a $\mathcal{P}$-indexed system of $\mathbf{L}$-relations and postulate additional conditions (these conditions are "present" in the concept of a fully invariant closure system of relations in the ordinary case). It is an open problem whether there is some nicer algebraic characterization of our $\mathcal{P}$-indexed systems of $\mathbf{L}$-relations.

Theorem 2. *Let $\mathcal{P} = \mathbf{L}^{T(X) \times T(X)}$ and let $\mathcal{S}_\Sigma$ be an $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of $\mathbf{L}$-relations. Then an operator cl on $\mathbf{L}^{T(X) \times T(X)}$ defined by $cl(P) = \Sigma_P$ is an $\mathbf{L}^*$-closure operator and*

$$h\big(cl(P)\big) \subseteq cl\big(h(P)\big) \qquad (26)$$

holds for every $P \in \mathbf{L}^{T(X) \times T(X)}$, and every endomorphism $h: \mathbf{T}(X) \to \mathbf{T}(X)$.

Proof. We check (26) since the rest is obvious. Let $P \in \mathbf{L}^{T(X)\times T(X)}$ and let h be an endomorphism on $\mathbf{T}(X)$. We have

$$\begin{aligned} h(\Sigma_P)(t,t') &= \bigvee_{\substack{h(s)=t\\ h(s')=t'}} \Sigma_P(s,s') \le \bigvee_{\substack{h(s)=t\\ h(s')=t'}} \Sigma_{h(P)}(h(s),h(s')) \\ &= \bigvee_{\substack{h(s)=t\\ h(s')=t'}} \Sigma_{h(P)}(t,t') = \Sigma_{h(P)}(t,t') \end{aligned}$$

for all $t, t' \in T(X)$. Thus, $h\big(cl(P)\big) \subseteq cl\big(h(P)\big)$. □

Theorem 3. *Let $\mathcal{P} = \mathbf{L}^{T(X)\times T(X)}$ and suppose * is an implicational truth stresser. Let cl be an $\mathbf{L}^*$-closure operator on $\mathbf{L}^{T(X)\times T(X)}$ satisfying* (26) *for every $P \in \mathbf{L}^{T(X)\times T(X)}$ and every endomorphism $h : \mathbf{T}(X) \to \mathbf{T}(X)$. Then the $\mathcal{P}$-indexed system $\mathcal{S}_\Sigma = \{\Sigma_P \mid \Sigma_P = cl(P)\}$ is an $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of $\mathbf{L}$-relations.*

Proof. (22): Using (26) we get

$$\begin{aligned} \Sigma_P(t,t') = cl(P)(t,t') &\le \bigvee_{\substack{h(s)=h(t)\\ h(s')=h(t')}} cl(P)(s,s') = h\big(cl(P)\big)(h(t),h(t')) \\ &\le cl\big(h(P)\big)(h(t),h(t')) = \Sigma_{h(P)}(h(t),h(t')) \end{aligned}$$

for all terms $t, t' \in T(X)$. The rest follows easily. □

Remark 8. Note that the correspondences described by Theorem 2 and Theorem 3 are in fact mutually inverse. Therefore, for $\mathcal{P} = \mathbf{L}^{T(X)\times T(X)}$ (no restriction on premises) there is a natural bijective correspondence between $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of binary $\mathbf{L}$-relations and fully invariant $\mathbf{L}^*$-closure operators in $T(X)$, generalizing the ordinary case.

We are going to show that $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed systems of congruences are in one-to-one correspondence with $\mathbf{L}^*$-implicational $\mathcal{P}$-theories, i.e. theories of classes of $\mathbf{L}$-algebras with respect to a given implicational truth stresser.

Definition 8. *Let * be an implicational truth stresser and let $\mathcal{P}$ be a proper family of premises of type F. For a class $\mathcal{K}$ of $\mathbf{L}$-algebras of type F we define an $\mathbf{L}$-set of $\mathcal{P}$-implications* Impl($\mathcal{K}$) *by*

$$\big(\mathrm{Impl}(\mathcal{K})\big)\big(P \Rightarrow (t \approx t')\big) = \big\| P \Rightarrow (t \approx t') \big\|_{\mathcal{K}}$$

for every $P \in \mathcal{P}$, $t, t' \in T(X)$ (the truth degree $\| P \Rightarrow (t \approx t') \|_{\mathcal{K}}$ is w.r.t. *).

*An $\mathbf{L}$-set Σ of $\mathcal{P}$-implications is said to be an **$\mathbf{L}^*$-implicational $\mathcal{P}$-theory**, if there is a class $\mathcal{K}$ of $\mathbf{L}$-algebras of the same type such that* $\Sigma = \mathrm{Impl}(\mathcal{K})$.

Theorem 4. *Let Σ be an $\mathbf{L}^*$-implicational $\mathcal{P}$-theory. Then Σ_P is a congruence on $\mathbf{T}(X)$ for every $P \in \mathcal{P}$.*

Proof. Since Σ is an $\mathbf{L}^*$-implicational $\mathcal{P}$-theory, there is a class $\mathcal{K}$ of $\mathbf{L}$-algebras such that $\Sigma = \mathrm{Impl}(\mathcal{K})$. That is, we have

$$\Sigma\big(P \Rightarrow (t \approx t')\big) = \big\|P \Rightarrow (t \approx t')\big\|_{\mathcal{K}} = \bigwedge_{\mathbf{M}\in\mathcal{K}} \bigwedge_{v:X\to M} \big\|P \Rightarrow (t \approx t')\big\|_{\mathbf{M},v}.$$

Denote $\Sigma\big(P \Rightarrow (t \approx t')\big)$ simply by $\wedge_{\mathbf{M},v} \big\|P \Rightarrow (t \approx t')\big\|_{\mathbf{M},v}$.

Reflexivity and symmetry of Σ_P follow from reflexivity and symmetry of every **L**-equality. For transitivity, let $P \in \mathcal{P}$ and $t, t', t'' \in T(X)$. Using (17), properties of $\approx^{\mathbf{M}}$'s, and the isotony of $\to$ in the second argument, we have

$$\begin{aligned}
&\Sigma_P(t,t') \otimes \Sigma_P(t',t'')\\
&\quad= \big(\textstyle\bigwedge_{\mathbf{M},v} \|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}\big) \otimes \big(\bigwedge_{\mathbf{M},v} \|P \Rightarrow (t' \approx t'')\|_{\mathbf{M},v}\big)\\
&\quad\le \textstyle\bigwedge_{\mathbf{M},v}\big(\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v} \otimes \|P \Rightarrow (t' \approx t'')\|_{\mathbf{M},v}\big)\\
&\quad= \textstyle\bigwedge_{\mathbf{M},v}\big((\|P\|_{\mathbf{M},v} \to \|t \approx t'\|_{\mathbf{M},v}) \otimes (\|P\|_{\mathbf{M},v} \to \|t' \approx t''\|_{\mathbf{M},v})\big)\\
&\quad\le \textstyle\bigwedge_{\mathbf{M},v}\big((\|P\|_{\mathbf{M},v} \otimes \|P\|_{\mathbf{M},v}) \to (\|t \approx t'\|_{\mathbf{M},v} \otimes \|t' \approx t''\|_{\mathbf{M},v})\big)\\
&\quad= \textstyle\bigwedge_{\mathbf{M},v}\big(\|P\|_{\mathbf{M},v} \to (\|t \approx t'\|_{\mathbf{M},v} \otimes \|t' \approx t''\|_{\mathbf{M},v})\big) \le \Sigma_P(t,t'').
\end{aligned}$$

Hence, Σ_P is transitive.

It suffices to check compatibility with operations since $\approx^{\mathbf{T}(X)} \subseteq \Sigma_P$ holds trivially. Take $P \in \mathcal{P}$, an n-ary $f \in F$, and terms $t_1, t_1', \ldots, t_n, t_n'$. Since

$$\begin{aligned}
\textstyle\bigotimes_{i=1}^n \|t_i \approx t_i'\|_{\mathbf{M},v} &= \textstyle\bigotimes_{i=1}^n \|t_i\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|t_i'\|_{\mathbf{M},v}\\
&\le f^{\mathbf{M}}\big(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v}\big) \approx^{\mathbf{M}} f^{\mathbf{M}}\big(\|t_1'\|_{\mathbf{M},v}, \ldots, \|t_n'\|_{\mathbf{M},v}\big)\\
&= \|f(t_1,\ldots,t_n)\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|f(t_1',\ldots,t_n')\|_{\mathbf{M},v}\\
&= \|f(t_1,\ldots,t_n) \approx f(t_1',\ldots,t_n')\|_{\mathbf{M},v},
\end{aligned}$$

we get

$$\begin{aligned}
\Sigma_P(t_1,t_1') \otimes \cdots \otimes \Sigma_P(t_n,t_n') &= \textstyle\bigotimes_{i=1}^n \bigwedge_{\mathbf{M},v} \|P \Rightarrow (t_i \approx t_i')\|_{\mathbf{M},v}\\
&\le \textstyle\bigwedge_{\mathbf{M},v} \bigotimes_{i=1}^n \|P \Rightarrow (t_i \approx t_i')\|_{\mathbf{M},v}\\
&= \textstyle\bigwedge_{\mathbf{M},v} \bigotimes_{i=1}^n \big(\|P\|_{\mathbf{M},v} \to \|t_i \approx t_i'\|_{\mathbf{M},v}\big)\\
&\le \textstyle\bigwedge_{\mathbf{M},v}\big((\|P\|_{\mathbf{M},v})^n \to \bigotimes_{i=1}^n \|t_i \approx t_i'\|_{\mathbf{M},v}\big)\\
&\le \textstyle\bigwedge_{\mathbf{M},v}\big(\|P\|_{\mathbf{M},v} \to \|f(t_1,\ldots,t_n) \approx f(t_1',\ldots,t_n')\|_{\mathbf{M},v}\big)\\
&= \Sigma_P\big(f(t_1,\ldots,t_n), f(t_1',\ldots,t_n')\big).
\end{aligned}$$

Hence, Σ_P is compatible. Altogether, Σ_P is a congruence for every $P \in \mathcal{P}$. □

Remark 9. In the proof of Theorem 4 we used (17), i.e. $a^* \otimes a^* = a^*$, which is required for implicational truth stressers. Without postulating (17), we are not able to prove that Σ_P is transitive and compatible. When every Σ_P is a congruence, we can easily define a class of models as factorizations of $\mathbf{T}(X)$, see Theorem 6.

We need the following technical assertion.

Lemma 3. *Suppose $\mathcal{P}$ is a proper set of premises and * is an implicational truth stresser. Then for every $P, Q \in \mathcal{P}$, for every $\mathbf{L}$-algebra $\mathbf{M}$, and for every valuation $v\colon X \to M$, we have*

$$\begin{aligned}&\left[\bigwedge_{s,s'\in T(X)}\left(P(s,s') \to \bigwedge_{\mathbf{M},v}\|Q \Rightarrow (s\approx s')\|_{\mathbf{M},v}\right)\right]^*\\ &\leq \bigwedge_{\mathbf{M},v}\left(\|Q\|_{\mathbf{M},v} \to \|P\|_{\mathbf{M},v}\right).\end{aligned} \tag{27}$$

Proof. First, observe that

$$\begin{aligned}&\bigwedge_{s,s'\in T(X)}\left(P(s,s') \to \bigwedge_{\mathbf{M},v}\|Q \Rightarrow (s\approx s')\|_{\mathbf{M},v}\right)\\ &\quad= \bigwedge_{s,s'\in T(X)}\left(P(s,s') \to \bigwedge_{\mathbf{M},v}\left[\|Q\|_{\mathbf{M},v} \to \|s\approx s'\|_{\mathbf{M},v}\right]\right)\\ &\quad= \bigwedge_{s,s'\in T(X)}\bigwedge_{\mathbf{M},v}\left(P(s,s') \to \left[\|Q\|_{\mathbf{M},v} \to \|s\approx s'\|_{\mathbf{M},v}\right]\right)\\ &\quad= \bigwedge_{\mathbf{M},v}\bigwedge_{s,s'\in T(X)}\left(\|Q\|_{\mathbf{M},v} \to \left[P(s,s') \to \|s\approx s'\|_{\mathbf{M},v}\right]\right)\\ &\quad= \bigwedge_{\mathbf{M},v}\left(\|Q\|_{\mathbf{M},v} \to \bigwedge_{s,s'\in T(X)}\left[P(s,s') \to \|s\approx s'\|_{\mathbf{M},v}\right]\right).\end{aligned}$$

Now using (16), (8), and (18), we have

$$\begin{aligned}&\left[\bigwedge_{s,s'\in T(X)}\left(P(s,s') \to \bigwedge_{\mathbf{M},v}\|Q \Rightarrow (s\approx s')\|_{\mathbf{M},v}\right)\right]^*\\ &\quad= \left[\bigwedge_{\mathbf{M},v}\left(\|Q\|_{\mathbf{M},v} \to \bigwedge_{s,s'\in T(X)}\left[P(s,s') \to \|s\approx s'\|_{\mathbf{M},v}\right]\right)\right]^*\\ &\quad= \bigwedge_{\mathbf{M},v}\left(\|Q\|_{\mathbf{M},v} \to \bigwedge_{s,s'\in T(X)}\left[P(s,s') \to \|s\approx s'\|_{\mathbf{M},v}\right]\right)^*\\ &\quad\leq \bigwedge_{\mathbf{M},v}\left(\left(\|Q\|_{\mathbf{M},v}\right)^* \to \left(\bigwedge_{s,s'\in T(X)}\left[P(s,s') \to \|s\approx s'\|_{\mathbf{M},v}\right]\right)^*\right)\\ &\quad= \bigwedge_{\mathbf{M},v}\left(\|Q\|_{\mathbf{M},v} \to \left(\bigwedge_{s,s'\in T(X)}\left[P(s,s') \to \|s\approx s'\|_{\mathbf{M},v}\right]\right)^*\right)\\ &\quad= \bigwedge_{\mathbf{M},v}\left(\|Q\|_{\mathbf{M},v} \to \|P\|_{\mathbf{M},v}\right),\end{aligned}$$

proving (27). □

Theorem 5. *Let Σ be an $\mathbf{L}^*$-implicational $\mathcal{P}$-theory. Then the corresponding $\mathcal{S}_\Sigma$ is an $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of congruences.*

Proof. Let us have a class $\mathcal{K}$ of $\mathbf{L}$-algebras such that $\Sigma = \mathrm{Impl}(\mathcal{K})$. We have to check (21)–(23) of $\mathcal{S}_\Sigma$. The rest follows from Theorem 4.

(21): Suppose we have $P \in \mathcal{P}$ and terms $t, t' \in T(X)$. For every $\mathbf{M} \in \mathcal{K}$ and a valuation $v\colon X \to M$, we have

$$\begin{aligned}P(t,t') &\leq \left(P(t,t') \to \|t\approx t'\|_{\mathbf{M},v}\right) \to \|t\approx t'\|_{\mathbf{M},v}\\ &\leq \bigwedge_{s,s'\in T(X)}\left(P(s,s') \to \|s\approx s'\|_{\mathbf{M},v}\right) \to \|t\approx t'\|_{\mathbf{M},v}\\ &\leq \left(\bigwedge_{s,s'\in T(X)}\left(P(s,s') \to \|s\approx s'\|_{\mathbf{M},v}\right)\right)^* \to \|t\approx t'\|_{\mathbf{M},v}\\ &= \|P \Rightarrow (t\approx t')\|_{\mathbf{M},v},\end{aligned}$$

showing that $P(t,t') \leq \bigwedge_{\mathbf{M},v}\|P \Rightarrow (t\approx t')\|_{\mathbf{M},v} = \Sigma_P(t,t')$, i.e. $P \subseteq \Sigma_P$.

(22): For all terms $r, r' \in T(X)$, a truth degree $\|h(r) \approx h(r')\|_{\mathbf{M},v}$ equals to $\|r \approx r'\|_{\mathbf{M},w}$ for some valuation $w\colon X \to M$. Moreover, w is determined uniquely by the endomorphism h. Hence, it follows that

$$\Sigma_P(t,t') = \bigwedge\nolimits_{\mathbf{M},v}\big[\big(\bigwedge\nolimits_{s,s' \in T(X)} \big(P(s,s') \to \|s \approx s'\|_{\mathbf{M},v}\big)\big)^{*} \to \|t \approx t'\|_{\mathbf{M},v}\big]$$
$$\leq \bigwedge\nolimits_{\mathbf{M},v}\big[\big(\bigwedge\nolimits_{s,s' \in T(X)} \big(P(s,s') \to \|h(s) \approx h(s')\|_{\mathbf{M},v}\big)\big)^{*} \to \|h(t) \approx h(t')\|_{\mathbf{M},v}\big]$$

for any endomorphism h on $\mathbf{T}(X)$. Furthermore,

$$\begin{aligned}
&\bigwedge\nolimits_{s,s' \in T(X)} \big(P(s,s') \to \|h(s) \approx h(s')\|_{\mathbf{M},v}\big) \\
&= \bigwedge\nolimits_{s,s' \in T(X)} \bigwedge\nolimits_{\substack{h(r)=h(s)\\ h(r')=h(s')}} \big(P(r,r') \to \|h(s) \approx h(s')\|_{\mathbf{M},v}\big) \\
&= \bigwedge\nolimits_{s,s' \in T(X)} \Big(\Big(\bigvee\nolimits_{\substack{h(r)=h(s)\\ h(r')=h(s')}} P(r,r')\Big) \to \|h(s) \approx h(s')\|_{\mathbf{M},v}\Big) \\
&= \bigwedge\nolimits_{s,s' \in T(X)} \big(h(P)(h(s),h(s')) \to \|h(s) \approx h(s')\|_{\mathbf{M},v}\big) \\
&= \bigwedge\nolimits_{s,s' \in T(X)} \big(h(P)(s,s') \to \|s \approx s'\|_{\mathbf{M},v}\big).
\end{aligned}$$

Putting both facts together, we obtain

$$\begin{aligned}
\Sigma_P(t,t') &\leq \bigwedge\nolimits_{\mathbf{M},v}\big[\big(\bigwedge\nolimits_{s,s' \in T(X)} \big(P(s,s') \to \|h(s) \approx h(s')\|_{\mathbf{M},v}\big)\big)^{*} \\
&\qquad \to \|h(t) \approx h(t')\|_{\mathbf{M},v}\big] \\
&= \bigwedge\nolimits_{\mathbf{M},v}\big[\big(\bigwedge\nolimits_{s,s' \in T(X)} \big(h(P)(s,s') \to \|s \approx s'\|_{\mathbf{M},v}\big)\big)^{*} \\
&\qquad \to \|h(t) \approx h(t')\|_{\mathbf{M},v}\big] \\
&= \bigwedge\nolimits_{\mathbf{M},v} \|h(P) \Rightarrow (h(t) \approx h(t'))\|_{\mathbf{M},v} = \Sigma_{h(P)}(h(t),h(t')).
\end{aligned}$$

(23): By Lemma 3,

$$\begin{aligned}
&\Sigma_P(t,t') \otimes \big(\bigwedge\nolimits_{s,s' \in T(X)} \big(P(s,s') \to \Sigma_Q(s,s')\big)\big)^{*} \\
&= \big(\bigwedge\nolimits_{\mathbf{M},v} \big(\|P\|_{\mathbf{M},v} \to \|t \approx t'\|_{\mathbf{M},v}\big)\big) \\
&\otimes \big[\bigwedge\nolimits_{s,s' \in T(X)} \big(P(s,s') \to \bigwedge\nolimits_{\mathbf{M},v} \|Q \Rightarrow (s \approx s')\|_{\mathbf{M},v}\big)\big]^{*} \\
&\leq \big(\bigwedge\nolimits_{\mathbf{M},v} \big(\|P\|_{\mathbf{M},v} \to \|t \approx t'\|_{\mathbf{M},v}\big)\big) \otimes \big(\bigwedge\nolimits_{\mathbf{M},v} \big(\|Q\|_{\mathbf{M},v} \to \|P\|_{\mathbf{M},v}\big)\big) \\
&\leq \bigwedge\nolimits_{\mathbf{M},v}\big[\big(\|P\|_{\mathbf{M},v} \to \|t \approx t'\|_{\mathbf{M},v}\big) \otimes \big(\|Q\|_{\mathbf{M},v} \to \|P\|_{\mathbf{M},v}\big)\big] \\
&\leq \bigwedge\nolimits_{\mathbf{M},v} \big(\|Q\|_{\mathbf{M},v} \to \|t \approx t'\|_{\mathbf{M},v}\big) = \Sigma_Q(t,t').
\end{aligned}$$

The previous inequality is equivalent to (23). The proof is complete. □

Now we turn our attention to the converse problem. Having given an $\mathbf{L}^{*}$-implicational $\mathcal{P}$-indexed system of congruences $\mathcal{S}_\Sigma$, we construct a suitable class of $\mathbf{L}$-algebras $\mathcal{K}$, such that $\Sigma = \mathrm{Impl}(\mathcal{K})$.

Theorem 6. *Let $\mathcal{S}_\Sigma$ be an* $\mathbf{L}^*$*-implicational* $\mathcal{P}$*-indexed system of congruences. Then* Σ *is an* $\mathbf{L}^*$*-implicational* $\mathcal{P}$*-theory. Namely,* $\Sigma = \mathrm{Impl}(\mathcal{K})$ *for a class* $\mathcal{K}$ *defined by* $\mathcal{K} = \{\mathbf{T}(X)/\Sigma_P \mid P \in \mathcal{P}\}$.

Proof. We have to check that $\Sigma_P(t, t') = \|P \Rightarrow (t \approx t')\|_{\mathcal{K}}$, where

$$\|P \Rightarrow (t \approx t')\|_{\mathcal{K}} = \bigwedge_{\substack{Q \in \mathcal{P} \\ v\colon X \to T(X)/\Sigma_Q}} \|P \Rightarrow (t \approx t')\|_{\mathbf{T}(X)/\Sigma_Q, v}.$$

"$\leq$": Take $\mathbf{T}(X)/\Sigma_Q \in \mathcal{K}$ and a valuation $v : X \to T(X)/\Sigma_Q$. There is an endomorphism h on $\mathbf{T}(X)$, such that $[h(s)]_{\Sigma_Q} = \|s\|_{\mathbf{T}(X)/\Sigma_Q, v}$ for every $s \in T(X)$. Consequently,

$$\begin{aligned}\|s \approx s'\|_{\mathbf{T}(X)/\Sigma_Q, v} &= \|s\|_{\mathbf{T}(X)/\Sigma_Q, v} \approx^{\mathbf{T}(X)/\Sigma_Q} \|s'\|_{\mathbf{T}(X)/\Sigma_Q, v} \\ &= [h(s)]_{\Sigma_Q} \approx^{\mathbf{T}(X)/\Sigma_Q} [h(s')]_{\Sigma_Q} = \Sigma_Q\big(h(s), h(s')\big)\end{aligned}$$

for every $s, s' \in T(X)$. Using (25) we get

$$\begin{aligned}\Sigma_P(t, t') &\leq \Big[\bigwedge_{s, s' \in T(X)} \big(P(s, s') \to \Sigma_Q\big(h(s), h(s')\big)\big)\Big]^* \to \Sigma_Q\big(h(t), h(t')\big) \\ &= \Big[\bigwedge_{s, s' \in T(X)} \big(P(s, s') \to \|s \approx s'\|_{\mathbf{T}(X)/\Sigma_Q, v}\big)\Big]^* \to \|t \approx t'\|_{\mathbf{T}(X)/\Sigma_Q, v} \\ &= \|P \Rightarrow (t \approx t')\|_{\mathbf{T}(X)/\Sigma_Q, v},\end{aligned}$$

proving the "$\leq$" inequality.

"$\geq$": For every $P \in \mathcal{P}$ and a valuation $v\colon X \to T(X)/\Sigma_P$, where $v(x) = [x]_{\Sigma_P}$ for all $x \in X$, we have $\|t\|_{\mathbf{T}(X)/\Sigma_P, v} = [t]_{\Sigma_P}$ for every $t \in T(X)$ (easy proof by structural induction). Thus, it follows that

$$\begin{aligned}\Sigma_P(s, s') &= [s]_{\Sigma_P} \approx^{\mathbf{T}(X)/\Sigma_P} [s']_{\Sigma_P} \\ &= \|s\|_{\mathbf{T}(X)/\Sigma_P, v} \approx^{\mathbf{T}(X)/\Sigma_P} \|s'\|_{\mathbf{T}(X)/\Sigma_P, v} = \|s \approx s'\|_{\mathbf{T}(X)/\Sigma_P, v}\end{aligned}$$

for every $s, s' \in T(X)$. Now we can use property (6) of * and (21) to obtain the desired inequality:

$$\begin{aligned}\Sigma_P(t, t') &= \|t \approx t'\|_{\mathbf{T}(X)/\Sigma_P, v} = 1^* \to \|t \approx t'\|_{\mathbf{T}(X)/\Sigma_P, v} \\ &= \Big[\bigwedge_{s, s' \in T(X)} \big(P(s, s') \to \Sigma_P(s, s')\big)\Big]^* \to \|t \approx t'\|_{\mathbf{T}(X)/\Sigma_P, v} \\ &= \Big[\bigwedge_{s, s' \in T(X)} \big(P(s, s') \to \|s \approx s'\|_{\mathbf{T}(X)/\Sigma_P, v}\big)\Big]^* \to \|t \approx t'\|_{\mathbf{T}(X)/\Sigma_P, v} \\ &= \|P \Rightarrow (t \approx t')\|_{\mathbf{T}(X)/\Sigma_P, v} \geq \bigwedge_{\substack{Q \in \mathcal{P} \\ v\colon X \to T(X)/\Sigma_Q}} \|P \Rightarrow (t \approx t')\|_{\mathbf{T}(X)/\Sigma_Q, v}.\end{aligned}$$

□

* * *

We close this section with a definition of a *truth degree* of a *semantic consequence*. Moreover, we will express this logical notion algebraically. First, we introduce the notion of a *semantic closure* of an $\mathbf{L}$-set of $\mathcal{P}$-implications. As it will appear later, a degree of a semantic consequence is equal to a membership degree of a formula in a suitable semantic closure.

A semantic closure of Σ is defined as an $\mathbf{L}^*$-implicational closure of the corresponding $\mathcal{S}_\Sigma$. Theorem 7 shows that such closure always exists. First, we check that a system of all $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed systems of congruences is a *closure system* itself. Then every $\mathcal{P}$-indexed system of $\mathbf{L}$-relations (corresponding to Σ) can be closed to the least $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of congruences containing $\mathcal{S}_\Sigma$.

Recall that for $\mathcal{P}$-indexed systems of $\mathbf{L}$-relations $\mathcal{S} = \{S_P \mid P \in \mathcal{P}\}$, $\mathcal{S}' = \{S'_P \mid P \in \mathcal{P}\}$, we put $\mathcal{S} \leq \mathcal{S}'$ iff $S_P \subseteq S'_P$ for all $P \in \mathcal{P}$. Let $\mathcal{N} = \{\mathcal{S}_i \mid i \in I\}$ be a system of $\mathcal{P}$-indexed systems $\mathcal{S}_i$ of $\mathbf{L}$-relations. That is, each $\mathcal{S}_i \in \mathcal{N}$ is a $\mathcal{P}$-indexed system $\mathcal{S}_i = \{S_{i,P} \in \mathbf{L}^{T(X)\times T(X)} \mid P \in \mathcal{P}\}$. We can define a $\mathcal{P}$-indexed system $\bigcap_{i\in I} \mathcal{S}_i = \{(\bigcap_{i\in I} \mathcal{S}_i)_P \mid P \in \mathcal{P}\}$ of $\mathbf{L}$-relations as the intersection $\bigcap \mathcal{N}$, i.e.

$$\left(\bigcap\nolimits_{i\in I} \mathcal{S}_i\right)_P = \bigcap\nolimits_{i\in I} S_{i,P}.$$

for every $P \in \mathcal{P}$. Hence, $\left(\bigcap_{i\in I} \mathcal{S}_i\right)_P(t,t') = \bigwedge_{i\in I} S_{i,P}(t,t')$ for all $t,t' \in T(X)$ and $P \in \mathcal{P}$.

Theorem 7. *Let $\mathcal{S}^*$ denote a system of all $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed systems of congruences. Then $\mathcal{S}^*$ is a closure system.*

Proof. We have to check, that $\mathcal{S}^*$ is nonempty and closed under arbitrary intersections. Clearly,

$$\mathcal{S}_{\max} = \left\{S_P \mid S_P(t,t') = 1 \text{ for every } P \in \mathcal{P},\ t',t \in T(X)\right\}$$

belongs to $\mathcal{S}^*$ and so $\mathcal{S}^*$ is nonempty.

Let $\mathcal{N} = \{\mathcal{S}_i \in \mathcal{S}^* \mid i \in I\}$. We have to show that $\bigcap \mathcal{N} = \bigcap_{i\in I} \mathcal{S}_i \in \mathcal{S}^*$. It is easy to show that each $\left(\bigcap_{i\in I} \mathcal{S}_i\right)_P = \bigcap_{i\in I} S_{i,P}$ $(P \in \mathcal{P})$ is a congruence. Hence, it remains to check that $\bigcap_{i\in I} \mathcal{S}_i$ satisfies conditions (21)–(23).

(21): Let $P \in \mathcal{P}$, $t,t' \in T(X)$. Since $P(t,t') \leq S_{i,P}(t,t')$, $P(t,t') \leq \bigwedge_{i\in I} S_{i,P}(t,t') = \left(\bigcap_{i\in I} \mathcal{S}_i\right)_P(t,t')$.

(22): From $S_{i,P}(t,t') \leq S_{i,h(P)}\big(h(t),h(t')\big)$ $(i \in I)$, we get

$$\begin{aligned}\left(\bigcap\nolimits_{i\in I} \mathcal{S}_i\right)_P(t,t') &= \bigwedge\nolimits_{i\in I} S_{i,P}(t,t') \leq \bigwedge\nolimits_{i\in I} S_{i,h(P)}\big(h(t),h(t')\big)\\ &= \left(\bigcap\nolimits_{i\in I} \mathcal{S}_i\right)_{h(P)}\big(h(t),h(t')\big).\end{aligned}$$

(23): Take any $P, Q \in \mathcal{P}$, and terms $t,t' \in T(X)$. Taking into account (18) and (23) of $\mathcal{S}_i$, it follows that

$$\begin{aligned}&\left(\bigcap\nolimits_{i\in I} \mathcal{S}_i\right)_P(t,t') \otimes \left[\bigwedge\nolimits_{s,s'\in T(X)} \left(P(s,s') \to \left(\bigcap\nolimits_{i\in I} \mathcal{S}_i\right)_Q(s,s')\right)\right]^*\\ &= \bigwedge\nolimits_{i\in I} S_{i,P}(t,t') \otimes \left[\bigwedge\nolimits_{s,s'\in T(X)} \left(P(s,s') \to \bigwedge\nolimits_{j\in I} S_{j,Q}(s,s')\right)\right]^*\\ &\leq \bigwedge\nolimits_{i\in I}\left(S_{i,P}(t,t') \otimes \left[\bigwedge\nolimits_{s,s'\in T(X)} \left(P(s,s') \to S_{i,Q}(s,s')\right)\right]^*\right) \leq S_{k,Q}(t,t')\end{aligned}$$

for every $k \in I$. Hence,

$$\begin{aligned}&\left(\bigcap\nolimits_{i\in I} \mathcal{S}_i\right)_P(t,t') \otimes \left[\bigwedge\nolimits_{s,s'\in T(X)} \left(P(s,s') \to \left(\bigcap\nolimits_{i\in I} \mathcal{S}_i\right)_Q(s,s')\right)\right]^*\\ &\leq \bigwedge\nolimits_{k\in I} S_{k,Q}(t,t') = \left(\bigcap\nolimits_{i\in I} \mathcal{S}_i\right)_Q(t,t'),\end{aligned}$$

verifying (24) which is equivalent to (23). □

Corollary 1. *For every* $\mathbf{L}$*-set* Σ *of* $\mathcal{P}$*-implications, and every implicational truth stresser* **, there exists the least* $\mathbf{L}^*$*-implicational* $\mathcal{P}$*-indexed system of congruences* $\mathcal{S}_{\Sigma^{\models}} = \{\Sigma_P^{\models} \mid P \in \mathcal{P}\}$*, such that* $\Sigma_P \subseteq \Sigma_P^{\models}$ *for every* $P \in \mathcal{P}$. □

Definition 9. *Suppose* Σ *is an* $\mathbf{L}$*-set of* $\mathcal{P}$*-implications, and* * *is an implicational truth stresser. Then the associated* $\mathbf{L}^*$*-implicational* $\mathcal{P}$*-indexed system of congruences* $\mathcal{S}_{\Sigma^{\models}} = \{\Sigma_P^{\models} \mid P \in \mathcal{P}\}$ *(see Corollary 1) is called a* ***semantic closure*** of $\mathcal{S}_\Sigma$*. Analogously, the corresponding* $\mathbf{L}$*-set* $\Sigma^{\models}$ *of* $\mathcal{P}$*-implications is called a* semantic closure of Σ.

As usual, let $\mathrm{Mod}(\Sigma)$ *denote the class of all models of* Σ*, i.e.*

$$\mathrm{Mod}(\Sigma) = \left\{\mathbf{M} \mid \Sigma(\varphi) \leq \|\varphi\|_{\mathbf{M}} \textit{ for every } \mathcal{P}\textit{-implication } \varphi\right\}.$$

For every $\mathcal{P}$*-implication* φ *we define a* ***truth degree*** $\|\varphi\|_\Sigma$ to which φ ***semantically follows*** from Σ w.r.t. * *by*

$$\|\varphi\|_\Sigma = \|\varphi\|_{\mathrm{Mod}(\Sigma)}\,.$$

Theorem 8. *Suppose* Σ *is an* $\mathbf{L}$*-set of* $\mathcal{P}$*-implications, and* * *is an implicational truth stresser. Then*

$$\left\|P \Rightarrow (t \approx t')\right\|_\Sigma = \Sigma_P^{\models}(t, t')$$

for every $P \in \mathcal{P}$*, and for all terms* $t, t' \in T(X)$.

Proof. "$\leq$": Previous results yield that Σ can be closed to $\Sigma^{\models}$. Since $\Sigma \subseteq \Sigma^{\models}$, we have $\mathrm{Mod}(\Sigma^{\models}) \subseteq \mathrm{Mod}(\Sigma)$. Furthermore, $\mathcal{S}_{\Sigma^{\models}}$ is an $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of congruences, thus Theorem 6 yields $\left\{\mathbf{T}(X)/\Sigma_Q^{\models} \mid Q \in \mathcal{P}\right\} \subseteq \mathrm{Mod}(\Sigma^{\models})$ since $\Sigma_P^{\models}(t, t') \leq \left\|P \Rightarrow (t \approx t')\right\|_{\mathbf{T}(X)/\Sigma_Q^{\models}}$ for every $P, Q \in \mathcal{P}$, and $t, t' \in T(X)$. Thus, we have

$$\begin{aligned}
\left\|P \Rightarrow (t \approx t')\right\|_\Sigma &= \bigwedge\nolimits_{\mathbf{M} \in \mathrm{Mod}(\Sigma)} \left\|P \Rightarrow (t \approx t')\right\|_{\mathbf{M}} \\
&\leq \bigwedge\nolimits_{\mathbf{M} \in \mathrm{Mod}(\Sigma^{\models})} \left\|P \Rightarrow (t \approx t')\right\|_{\mathbf{M}} \\
&\leq \bigwedge\nolimits_{Q \in \mathcal{P}} \left\|P \Rightarrow (t \approx t')\right\|_{\mathbf{T}(X)/\Sigma_Q^{\models}} = \Sigma_P^{\models}(t, t').
\end{aligned}$$

"$\geq$": Take $\mathbf{M} \in \mathrm{Mod}(\Sigma)$. That is, for every $\mathcal{P}$-implication $P \Rightarrow (t \approx t')$ we have $\Sigma(P \Rightarrow (t \approx t')) \leq \left\|P \Rightarrow (t \approx t')\right\|_{\mathbf{M}}$, whence $\Sigma \subseteq \mathrm{Impl}(\{\mathbf{M}\})$. Moreover, $\mathrm{Impl}(\{\mathbf{M}\})$ is an $\mathbf{L}^*$-implicational $\mathcal{P}$-theory (defined by the one element class $\mathcal{K} = \{\mathbf{M}\}$), and so $\mathcal{S}_{\mathrm{Impl}(\{\mathbf{M}\})}$ is an $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of congruences (due to Theorem 5), containing $\mathcal{S}_\Sigma$. As $\mathcal{S}_{\Sigma^{\models}}$ is the least one containing $\mathcal{S}_\Sigma$, we get $\mathcal{S}_{\Sigma^{\models}} \leq \mathcal{S}_{\mathrm{Impl}(\{\mathbf{M}\})}$. As a consequence,

$$\Sigma_P^{\models}(t, t') \leq \big(\mathrm{Impl}(\{\mathbf{M}\})\big)(P \Rightarrow (t \approx t')) = \left\|P \Rightarrow (t \approx t')\right\|_{\mathbf{M}}.$$

Since $\mathbf{M} \in \mathrm{Mod}(\Sigma)$ is arbitrary, we obtain

$$\Sigma_P^{\models}(t, t') \leq \bigwedge\nolimits_{\mathbf{M} \in \mathrm{Mod}(\Sigma)} \left\|P \Rightarrow (t \approx t')\right\|_{\mathbf{M}} = \left\|P \Rightarrow (t \approx t')\right\|_\Sigma$$

proving "$\geq$". Altogether, we have $\left\|P \Rightarrow (t \approx t')\right\|_\Sigma = \Sigma_P^{\models}(t, t')$ for every $P \Rightarrow (t \approx t')$. □

Remark 10. Theorem 8 provides an algebraic characterization of semantic consequence for arbitrary residuated lattice **L** together with an implicational truth stresser for **L** and for any proper family of premises $\mathcal{P}$. In Section 5 we describe the notion of a syntactic consequence (provability) and we establish a general completeness theorem. However, we need to postulate two more restrictive conditions on truth stressers. These conditions will be introduced in Section 4.

4. Implicational and Horn theories

In this section, we investigate a relationship between $\mathbf{L}^*$-implicational $\mathcal{P}$-theories and their special subtheories determined by a restriction on $\mathcal{P}$, namely by a restriction on finiteness of every $P \in \mathcal{P}$. From the point of view of axiomatization, finiteness is an important property. As a consequence, we will not have to generalize the notions of a deduction rule (having finitely many hypotheses) and a weighted proof (being a finite sequence).

In the case of infinitely many premises, things are much more complicated. First, restrictions on * necessary to prove the completeness theorem would be very strong. In addition to that, there would be deduction rules using infinitely many hypotheses, so when we want such rules to be involved in proofs, we have to generalize the notion of a proof itself (to be an infinitely branching tree of a finite depth). But we will not go into this mainly because it does not seem to be any natural motivation there.

This section is aimed to investigate the relationship between theories with general $\mathcal{P}$-implications and restrictions on finite premises. An important thing to stress is that we need to postulate two additional conditions for implicational truth stressers to be able to achieve a result which is similar to the well-known relationship between ordinary implicational theories and ordinary Horn theories.

Definition 10. *For every proper family of premises $\mathcal{P}$, let $\mathcal{P}_{\mathrm{Fin}}$ denote a restricted proper family of premises defined by*

$$\mathcal{P}_{\mathrm{Fin}} = \{P \mid P \in \mathcal{P} \text{ and } P \text{ is finite}\}.$$

$\mathcal{P}_{\mathrm{Fin}}$ is called a ***Horn restriction*** of $\mathcal{P}$. *Let Σ be an $\mathbf{L}^*$-implicational $\mathcal{P}$-theory. Then an $\mathbf{L}$-set Σ_{Fin} of $\mathcal{P}_{\mathrm{Fin}}$-implications, where $\Sigma_{\mathrm{Fin}}(P \Rightarrow (t \approx t')) = \Sigma(P \Rightarrow (t \approx t'))$ for every $P \in \mathcal{P}_{\mathrm{Fin}}$, $t, t' \in T(X)$ is called an* ***$\mathbf{L}^*$-implicational Horn subtheory*** of Σ.

Remark 11. (1) For an $\mathbf{L}^*$-implicational $\mathcal{P}$-theory Σ, the corresponding $\mathbf{L}^*$-implicational Horn subtheory Σ_{Fin} is an $\mathbf{L}^*$-implicational $\mathcal{P}_{\mathrm{Fin}}$-theory.
(2) In case of finite premises, condition (24) simplifies to

$$\Sigma_P(t, t') \otimes \left(\bigwedge\nolimits_{i=1}^{n} \left(P(t_i, t_i') \to \Sigma_Q(t_i, t_i')\right)\right)^* \leq \Sigma_Q(t, t'), \qquad (28)$$

where $\mathrm{Supp}(P) \subseteq \left\{\langle t_1, t_1'\rangle, \ldots, \langle t_n, t_n'\rangle\right\}$.

Example 5. The following are examples of Horn restrictions.

(a) Let us have an $\mathbf{L}^*$-implicational $\mathcal{P}$-theory Σ, where $\mathcal{P} = \mathbf{L}^{T(X)\times T(X)}$. Then $\mathcal{P}_{\mathrm{Fin}}$ is a proper family of all finite premises and Horn subtheory Σ_{Fin} is an $\mathbf{L}$-set of Horn clauses.

(b) In case of crisp premises, i.e. $\mathcal{P} = \{P \in \mathbf{L}^{T(X)\times T(X)} \mid P \text{ is crisp}\}$, the corresponding $\mathcal{P}_{\mathrm{Fin}}$ is a proper family of all crisp finite premises. Hence, for an $\mathbf{L}^*$-implicational $\mathcal{P}$-theory Σ, the Horn subtheory consists of Horn clauses with crisp premises.

(c) In ordinary case, for every implicational theory Σ, we can consider a Horn theory [38] being a restriction of Σ on finite premises. In our approach for $\mathbf{L} = \mathbf{2}$, this is exactly the Horn subtheory of Σ.

(d) There are also other nontrivial examples of Horn subtheories. For every $a \in L$, let $\mathcal{P}_a$ denote

$$\mathcal{P}_a = \{P \in \mathbf{L}^{T(X)\times T(X)} \mid P(t,t') > 0 \text{ implies } P(t,t') \geq a \text{ for all } t,t' \in T(X)\}.$$

It is easy to see that $\mathcal{P}_a$ is a proper family of premises since for every $P \in \mathcal{P}$, $h(P)(t,t')$ is either zero or $h(P)(t,t') \geq a$. In fact, we have $\mathcal{P}_0 = \mathbf{L}^{T(X)\times T(X)}$, and $\mathcal{P}_1$ denotes the proper family of crisp premises. For every $\mathbf{L}^*$-implicational $\mathcal{P}_a$-theory, we can consider the corresponding $(\mathcal{P}_a)_{\mathrm{Fin}}$-subtheory.

For every Σ, Σ_{Fin} is a restriction of Σ on implications with finite premises. Conversely, we can ask, whether for a given $\mathcal{P}$ and an $\mathbf{L}^*$-implicational $\mathcal{P}_{\mathrm{Fin}}$-theory Γ there is an $\mathbf{L}^*$-implicational $\mathcal{P}$-theory Σ such that $\Gamma = \Sigma_{\mathrm{Fin}}$. Under some additional assumptions, such theory always exists.

Definition 11. *An implicational truth stresser* * *for a complete residuated lattice* $\mathbf{L}$ *is called a* ***Horn truth stresser*** *if it satisfies*

$$a \otimes b^* = a \wedge b^*, \tag{29}$$

$$\left(a \to \bigvee_{i\in I} b_i\right)^* = \bigvee_{i\in I} (a \to b_i)^*, \tag{30}$$

for every nonempty index set I *and* $a, b, b_i \in L$.

Remark 12. (1) Monotony of * yields that $(a \to \bigvee_{i\in I} b_i)^* \geq \bigvee_{i\in I} (a \to b_i)^*$ is always true. Therefore, (30) is equivalent to $(a \to \bigvee_{i\in I} b_i)^* \leq \bigvee_{i\in I} (a \to b_i)^*$.

(2) Note that (29) is not too restrictive. For instance, it is well-known fact that (29) is implied by *divisibility* (recall that b^* is an idempotent), e.g. it holds in every BL-algebra, see [6].

(3) Condition (30) is similar to left-continuity of $\to$ in the second argument: $a \to \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \to b_i)$. However, our condition seems to be more restrictive. For instance, in the standard Łukasiewicz algebra $\mathbf{Ł} = \langle [0,1], \max, \min, \otimes, \to, 0, 1\rangle$, where $a \otimes b = \max(a + b - 1, 0)$, $a \to b = \min(1 - a + b, 1)$, the operation $\to$ is continuous in the second argument. But for * defined by (9), condition (30) is not satisfied. Take $I = [0, 0.5)$, $b_i = i$ for every $i \in I$, so for $a = 0.5$ we have

$$(0.5 \to 0.5)^* = 1^* = 1 \not\leq 0 = \bigvee_{i\in[0,0.5)} 0^* = \bigvee_{i\in[0,0.5)} (0.5 \to i)^*.$$

For some special proper families of premises, condition (30) simplifies, see Section 6.

(4) In finite chains, condition (30) is always satisfied (obvious).

Any finite **L**-relation $P' \in \mathbf{L}^{T(X)\times T(X)}$, where $P'(t,t') > 0$ implies $P'(t,t') = P(t,t')$ for every terms $t, t' \in T(X)$, is called a *finite restriction* of $P \in \mathcal{P}$. In the sequel, a *set of all finite restrictions* of $P \in \mathcal{P}$ will be denoted by $\mathrm{Fin}(P)$. Note that for $\emptyset_{T(X)\times T(X)}$, we have $\emptyset_{T(X)\times T(X)} \in \mathrm{Fin}(P)$, since both conditions hold trivially. We say that $\mathcal{P}$ is *closed under finite restrictions* if $\mathrm{Fin}(P) \subseteq \mathcal{P}$ for each $P \in \mathcal{P}$.

Theorem 9. *Let* $\mathbf{L}^*$ *be a complete residuated lattice with a Horn truth stresser* **, let* Γ *be an* $\mathbf{L}^*$*-implicational* $\mathcal{P}_{\mathrm{Fin}}$*-theory, where* $\mathcal{P}$ *is a proper family of premises closed under finite restrictions. Then putting*

$$\Sigma_P = \bigcup\nolimits_{P'\in \mathrm{Fin}(P)} \Gamma_{P'}, \qquad \textit{for } P \in \mathcal{P}, \tag{31}$$

Σ *is an* $\mathbf{L}^*$*-implicational* $\mathcal{P}$*-theory. Moreover,* Σ *is the least* $\mathbf{L}^*$*-implicational* $\mathcal{P}$*-theory with* $\Gamma = \Sigma_{\mathrm{Fin}}$.

Proof. Using Theorem 6, we need to check conditions (21)–(23) of $\mathcal{S}_\Sigma$, the fact that every Σ_P is a congruence, and the fact that Σ is the least one with $\Gamma = \Sigma_{\mathrm{Fin}}$.

Σ_P is a congruence: This is easy to see since the system of congruences $\{\Gamma_{P'} \mid P' \in \mathrm{Fin}(P)\}$ is directed. Indeed, for every $P'_1, P'_2 \in \mathrm{Fin}(P)$, we have $P'_1, P'_2 \subseteq P'_1 \cup P'_2$, and $P'_1 \cup P'_2 \in \mathrm{Fin}(P)$, thus by (21) of $\mathcal{S}_\Gamma$, it follows that $P'_1, P'_2 \subseteq \Gamma_{P'_1\cup P'_2}$, whence, by (23) $\Gamma_{P'_1}, \Gamma_{P'_2} \subseteq \Gamma_{P'_1\cup P'_2}$. This idea generalizes easily for finitely many congruences, i.e. $\{\Gamma_{P'} \mid P' \in \mathrm{Fin}(P)\}$ is a directed system of congruences. Hence, $\Sigma_P = \bigcup_{P'\in \mathrm{Fin}(P)} \Gamma_{P'}$ is a congruence for every $P \in \mathcal{P}$ (see [9]).

(21): Since $P = \bigcup_{P'\in \mathrm{Fin}(P)} P'$, condition (21) of $\mathcal{S}_\Gamma$ gives

$$P = \bigcup\nolimits_{P'\in \mathrm{Fin}(P)} P' \subseteq \bigcup\nolimits_{P'\in \mathrm{Fin}(P)} \Gamma_{P'} = \Sigma_P,$$

verifying (21) for every $P \in \mathcal{P}$.

(22): For $P' \in \mathrm{Fin}(P)$, $P \in \mathcal{P}$, and an endomorphism h on $\mathbf{T}(X)$ we have $h(P') \subseteq h(P)$ and $h(P')$ is finite. Thus,

$$\begin{aligned}\Sigma_P(t,t') &= \bigvee\nolimits_{P'\in \mathrm{Fin}(P)} \Gamma_{P'}(t,t') \le \bigvee\nolimits_{P'\in \mathrm{Fin}(P)} \Gamma_{h(P')}\big(h(t), h(t')\big)\\ &\le \bigvee\nolimits_{P''\in \mathrm{Fin}(h(P))} \Gamma_{P''}\big(h(t), h(t')\big) = \Sigma_{h(P)}\big(h(t), h(t')\big)\end{aligned}$$

proves (22) for $\mathcal{S}_\Sigma$.

(23): Let $P' \in \mathrm{Fin}(P)$ with $\mathrm{Supp}\big(P'\big) = \big\{\langle t_i, t'_i\rangle \mid i = 1, \ldots, k\big\}$, $P \in \mathcal{P}$, and $t, t' \in T(X)$. Since $\mathcal{S}_\Gamma$ satisfies (23) equivalently formulated by (24), we have

$$\Gamma_{P'}(t,t') \otimes \Big(\bigwedge\nolimits_{i=1}^{k} \big(P'(t_i,t'_i) \to \Gamma_{Q'}(t_i,t'_i)\big)\Big)^* \le \Gamma_{Q'}(t,t')$$

for every $Q' \in \mathrm{Fin}(Q)$, $Q \in \mathcal{P}$. Hence,

$$\begin{aligned}&\Gamma_{P'}(t,t') \otimes \bigvee\nolimits_{Q'\in \mathrm{Fin}(Q)} \Big(\bigwedge\nolimits_{i=1}^{k} \big(P'(t_i,t'_i) \to \Gamma_{Q'}(t_i,t'_i)\big)\Big)^*\\ &\le \bigvee\nolimits_{Q'\in \mathrm{Fin}(Q)} \Gamma_{Q'}(t,t') = \Sigma_Q(t,t').\end{aligned}$$

Using (18), (29) we get

$$\Gamma_{P'}(t,t') \otimes \bigvee_{Q' \in \mathrm{Fin}(Q)} \bigotimes_{i=1}^{k} \left(P'(t_i,t'_i) \to \Gamma_{Q'}(t_i,t'_i)\right)^* \leq \Sigma_Q(t,t'). \quad (32)$$

Since $\left\{\Gamma_{Q'} \mid Q' \in \mathrm{Fin}(Q)\right\}$ is a directed system, we have

$$\begin{aligned}
&\bigotimes_{i=1}^{k} \bigvee_{Q' \in \mathrm{Fin}(Q)} \left(P'(t_i,t'_i) \to \Gamma_{Q'}(t_i,t'_i)\right)^* \\
&= \bigvee_{Q'_1,\ldots,Q'_k \in \mathrm{Fin}(Q)} \bigotimes_{i=1}^{k} \left(P'(t_i,t'_i) \to \Gamma_{Q'_i}(t_i,t'_i)\right)^* \\
&= \bigvee_{Q' \in \mathrm{Fin}(Q)} \bigotimes_{i=1}^{k} \left(P'(t_i,t'_i) \to \Gamma_{Q'}(t_i,t'_i)\right)^*.
\end{aligned}$$

Using (18), (29), (30), (32), and the previous equality we further get

$$\begin{aligned}
&\Gamma_{P'}(t,t') \otimes \left(\bigwedge_{i=1}^{k} \left(P'(t_i,t'_i) \to \Sigma_Q(t_i,t'_i)\right)\right)^* \\
&= \Gamma_{P'}(t,t') \otimes \bigotimes_{i=1}^{k} \left(P'(t_i,t'_i) \to \Sigma_Q(t_i,t'_i)\right)^* \\
&= \Gamma_{P'}(t,t') \otimes \bigotimes_{i=1}^{k} \left(P'(t_i,t'_i) \to \bigvee_{Q' \in \mathrm{Fin}(Q)} \Gamma_{Q'}(t_i,t'_i)\right)^* \\
&= \Gamma_{P'}(t,t') \otimes \bigotimes_{i=1}^{k} \bigvee_{Q' \in \mathrm{Fin}(Q)} \left(P'(t_i,t'_i) \to \Gamma_{Q'}(t_i,t'_i)\right)^* \\
&= \Gamma_{P'}(t,t') \otimes \bigvee_{Q' \in \mathrm{Fin}(Q)} \bigotimes_{i=1}^{k} \left(P'(t_i,t'_i) \to \Gamma_{Q'}(t_i,t'_i)\right)^* \leq \Sigma_Q(t,t').
\end{aligned}$$

Note that (30) can be used properly because $\mathrm{Fin}(Q)$ is always a nonempty set (e.g. $\emptyset_{T(X)\times T(X)} \in \mathrm{Fin}(Q)$). Moreover, for every $P' \in \mathrm{Fin}(P)$, we have $P'(s,s') = P(s,s')$ for every $\langle s,s'\rangle \in \mathrm{Supp}(P')$. Thus, using monotony of *, we have

$$\begin{aligned}
&\Gamma_{P'}(t,t') \otimes \left(\bigwedge_{s,s' \in T(X)} \left(P(s,s') \to \Sigma_Q(s,s')\right)\right)^* \\
&\leq \Gamma_{P'}(t,t') \otimes \left(\bigwedge_{i=1}^{k} \left(P(t_i,t'_i) \to \Sigma_Q(t_i,t'_i)\right)\right)^* \leq \Sigma_Q(t,t')
\end{aligned}$$

for any $P' \in \mathrm{Fin}(P)$. Hence, we have

$$\begin{aligned}
&\Sigma_P(t,t') \otimes \left(\bigwedge_{s,s' \in T(X)} \left(P(s,s') \to \Sigma_Q(s,s')\right)\right)^* \\
&= \bigvee_{P' \in \mathrm{Fin}(P)} \Gamma_{P'}(t,t') \otimes \left(\bigwedge_{s,s' \in T(X)} \left(P(s,s') \to \Sigma_Q(s,s')\right)\right)^* \leq \Sigma_Q(t,t'),
\end{aligned}$$

verifying (23). Altogether, $\mathcal{S}_\Sigma$ is an $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of congruences, thus the corresponding Σ is an $\mathbf{L}^*$-implicational $\mathcal{P}$-theory by Theorem 6.

Σ is the least one with $\Gamma = \Sigma_{\mathrm{Fin}}$: First, we check $\Gamma = \Sigma_{\mathrm{Fin}}$. Take any $P \in \mathcal{P}_{\mathrm{Fin}}$. For every $P' \in \mathrm{Fin}(P)$, using (21), (23), we have $P' \subseteq \Gamma_P$, and consequently $\Gamma_{P'} \subseteq \Gamma_P$. On the other hand, $\Gamma_P \subseteq \bigcup_{P' \in \mathrm{Fin}(P)} \Gamma_{P'}$ holds trivially due to the fact $P \in \mathrm{Fin}(P)$. Hence, $\Sigma_P = \bigcup_{P' \in \mathrm{Fin}(P)} \Gamma_{P'} = \Gamma_P$ for every $P \in \mathcal{P}_{\mathrm{Fin}}$, i.e. $\Gamma = \Sigma_{\mathrm{Fin}}$.

Suppose Σ' is an $\mathbf{L}^*$-implicational $\mathcal{P}$-theory such that $\Gamma = \Sigma'_{\mathrm{Fin}}$. Take any $P \in \mathcal{P}$. We will show that $\Sigma_P \subseteq \Sigma'_P$. If $P \in \mathcal{P}_{\mathrm{Fin}}$, we are done, since $\Sigma_P = \Gamma_P = \Sigma'_P$ as we have shown in the previous paragraph. For $P \notin \mathcal{P}_{\mathrm{Fin}}$, and arbitrary $P' \in \mathrm{Fin}(P)$, it follows that $P' \subseteq P$. Using (21) and (23) for $\mathcal{S}_{\Sigma'}$, we have $P' \subseteq \Sigma'_P$, and $\Sigma'_{P'} \subseteq \Sigma'_P$. Thus, $\Sigma_{P'} = \Gamma_{P'} = \Sigma'_{P'} \subseteq \Sigma'_P$ for all $P' \in \mathrm{Fin}(P)$. Hence, $\Sigma_P = \bigcup_{P' \in \mathrm{Fin}(P)} \Gamma_{P'} \subseteq \Sigma'_P$. That is, Σ is the least $\mathbf{L}^*$-implicational $\mathcal{P}$-theory for which $\Gamma = \Sigma_{\mathrm{Fin}}$. □

If $\mathbf{L}$ is finite, we can avoid the usage of (30):

Theorem 10. *Let* $\mathbf{L}^*$ *be a finite residuated lattice with an implicational truth stresser* * *satisfying* (29)*, let* Γ *be an* $\mathbf{L}^*$*-implicational* $\mathcal{P}_{\mathrm{Fin}}$*-theory, where* $\mathcal{P}$ *is a proper family of premises closed under finite restrictions. Let* Σ_P $(P \in \mathcal{P})$ *be defined by* (31)*. Then* Σ *is an* $\mathbf{L}^*$*-implicational* $\mathcal{P}$*-theory. Moreover,* Σ *is the least* $\mathbf{L}^*$*-implicational* $\mathcal{P}$*-theory with* $\Gamma = \Sigma_{\mathrm{Fin}}$.

Proof. We show only the critical part where (30) is used. Since $\mathbf{L}$ is finite and $\{\Gamma_{Q'} \mid Q' \in \mathrm{Fin}(Q)\}$ is directed due to (21) and (23), for every $i = 1, \ldots, k$ there is $Q_i \in \mathrm{Fin}(Q)$ such that $\bigvee_{Q' \in \mathrm{Fin}(Q)} \Gamma_{Q'}(t_i, t_i') = \Gamma_{Q_i}(t_i, t_i')$. Thus,

$$\begin{aligned}
&\left(\bigwedge\nolimits_{i=1}^{k} \left(P'(t_i, t_i') \to \Sigma_Q(t_i, t_i')\right)\right)^* \\
&= \bigotimes\nolimits_{i=1}^{k} \left(P'(t_i, t_i') \to \bigvee\nolimits_{Q' \in \mathrm{Fin}(Q)} \Gamma_{Q'}(t_i, t_i')\right)^* \\
&= \bigotimes\nolimits_{i=1}^{k} \left(P'(t_i, t_i') \to \Gamma_{Q_i}(t_i, t_i')\right)^* \\
&\leq \bigotimes\nolimits_{i=1}^{k} \bigvee\nolimits_{Q' \in \mathrm{Fin}(Q)} \left(P'(t_i, t_i') \to \Gamma_{Q'}(t_i, t_i')\right)^*.
\end{aligned}$$

The rest follows from the proof of Theorem 9. □

Remark 13. (1) For all Horn restrictions of $\mathcal{P}$ presented in Example 5, the condition of Theorem 9 holds, i.e. for all $P \in \mathcal{P}$, every finite restriction of P belongs to $\mathcal{P}$. This fact is easy to observe. Thus, the connection between $\mathbf{L}^*$-implicational $\mathcal{P}_{\mathrm{Fin}}$-theories and $\mathbf{L}^*$-implicational $\mathcal{P}$-theories is established for all widely used families of premises.

(2) In the ordinary case [38], fully invariant algebraic closure systems of congruences form the algebraic counterparts of Horn theories. Theorem 9 shows a condition analogous to algebraicity in our framework. Thus, we could define algebraic $\mathcal{P}$-indexed systems of congruences as those $\mathcal{S}_\Sigma$, where Σ satisfies (31). For $\mathcal{P} = \mathbf{L}^{T(X)\times T(X)}$, this could yield fully invariant algebraic $\mathbf{L}^*$-closure systems of congruences. For $\mathbf{L} = \mathbf{2}$, this would further yield fully invariant algebraic closure systems of congruences—the classical case.

5. Completeness theorem

This section introduces a general completeness theorem which can be used to obtain completeness theorems of particular calculi depending on what proper family of premises $\mathcal{P}$ and what Horn truth stresser * is given. The choice of $\mathcal{P}$ and * is of crucial importance. For some combinations of $\mathcal{P}$ and * the theorem yields simplified deduction schemes or some restrictive requirements for * get weaker, e.g. (30). As an extreme example, for $\mathcal{P} = \{\emptyset\}$ we get *fuzzy equational logic*—the truth stresser * can be omitted and the axiomatic system reduces to the one presented in [5].

The completeness theorem will be proven as usual. First, we define the notion of a *deductive closure* of an $\mathbf{L}$-set of $\mathcal{P}$-implications and then we show that the deductive closure equals the semantic closure. After this, we propose a system of *weighted deduction rules* and the notion of a *provability degree.* Finally, we will prove the completeness in Pavelka style.

In the sequel, $\mathbf{L}^*$ denotes a complete residuated lattice $\mathbf{L}$ equipped with a Horn truth stresser *. $\mathcal{P}$ denotes a proper family of premises for which each $P \in \mathcal{P}$ is *finite*. $\mathcal{P}$-implications will be called Horn clauses (or $\mathcal{P}$-Horn clauses to denote $\mathcal{P}$ explicitly). By Γ we denote an $\mathbf{L}$-set of Horn clauses.

Definition 12. *As usual, let $t(x/r)$ denote the result of substituting term r for variable x in term t. Furthermore, for $P \in \mathcal{P}$, let $P(x/r) \in \mathbf{L}^{T(X)\times T(X)}$ denote a binary $\mathbf{L}$-relation defined by*

$$\big(P(x/r)\big)(t,t') = \bigvee_{\substack{s(x/r)=t\\ s'(x/r)=t'}} P(s,s')$$

for all terms $t, t' \in T(X)$.

Remark 14. (1) Evidently, every substitution (x/r) can be expressed as an endomorphism h on $\mathbf{T}(X)$, which is a homomorphic extension of a mapping $g : X \to T(X)$, where $g(x) = r$, and $g(y) = y$ for $y \in X, y \neq x$. Thus, we have $t(x/r) = h(t)$ for every term $t \in T(X)$. Consequently, we have $P(x/r) = h(P) \in \mathcal{P}$.

(2) We denote *substitutions* also by $\tau, \tau_1, \tau_2, \ldots,$ and so on. Moreover, instead of $(\cdots((t\tau_1)\tau_2)\cdots)\tau_n$, we simply write $t\tau_1\tau_2\cdots\tau_n$. Similarly, we denote $(\cdots((P\tau_1)\tau_2)\cdots)\tau_n$ by $P\tau_1\tau_2\cdots\tau_n$.

The following lemma is easy to verify.

Lemma 4. *Suppose $\mathcal{P}$ is a proper family of premises, and let us have substitutions $\tau_1, \ldots, \tau_k$. Let τ denote $\tau_1\cdots\tau_k$. We have,*

$$P\tau(t,t') = \bigvee_{\substack{s\tau=t\\ s'\tau=t'}} P(s,s')$$

for all terms $t, t' \in T(X)$. □

Definition 13. *Suppose Γ is an $\mathbf{L}$-set of $\mathcal{P}$-Horn clauses. A* ***deductive closure*** of *$\mathcal{S}_\Gamma$ is the least $\mathcal{P}$-indexed system $\mathcal{S}_{\Gamma^\vdash}$ of $\mathbf{L}$-relations satisfying*

$$\Gamma_P \subseteq \Gamma_P^\vdash, \tag{33}$$

$$\Gamma_P^\vdash(t,t) = 1, \tag{34}$$

$$\Gamma_P^\vdash(t,t') \leq \Gamma_P^\vdash(t',t), \tag{35}$$

$$\Gamma_P^\vdash(t,t') \otimes \Gamma_P^\vdash(t',t'') \leq \Gamma_P^\vdash(t,t''), \tag{36}$$

$$\Gamma_P^\vdash(t,t') \leq \Gamma_P^\vdash(s,s'), \tag{37}$$

$$P(t,t') \leq \Gamma_P^\vdash(t,t'), \tag{38}$$

$$\Gamma_P^\vdash(t,t') \leq \Gamma_{P(x/r)}^\vdash\big(t(x/r), t'(x/r)\big), \tag{39}$$

$$\Gamma_P^\vdash(t,t') \otimes \bigotimes_{i=1}^{k} \big(P(t_i,t_i') \to \Gamma_Q^\vdash(t_i,t_i')\big)^* \leq \Gamma_Q^\vdash(t,t'), \tag{40}$$

for every $P, Q \in \mathcal{P}$, $\mathrm{Supp}(P) = \big\{\langle t_i, t_i'\rangle \mid i = 1, \ldots, k\big\}$, $x \in X$, terms $t, t', t'', r, s, s' \in T(X)$, where s contains t as a subterm and s' results from s by replacing one occurrence of t by t'. The corresponding $\mathbf{L}$-set $\Gamma^\vdash$ of $\mathcal{P}$-Horn clauses is called a deductive closure of Γ.

Remark 15. (1) Observe that for a Horn truth stresser * and finite $P \in \mathcal{P}$, condition (40) coincides with (28). Indeed, this is a consequence of (18) and (29).
(2) The deductive closure $\Gamma^{\vdash}$ of an $\mathbf{L}$-set Γ of Horn clauses always exists since the system of all $\mathcal{P}$-indexed systems of $\mathbf{L}$-relations satisfying conditions (33)–(40) is nonempty and closed under arbitrary intersections. The proof is analogous to that of Theorem 7 and therefore omitted.

Theorem 11. *Let* Γ *be an* $\mathbf{L}$*-set of* $\mathcal{P}$*-Horn clauses. Then* $\Gamma^{\models} = \Gamma^{\vdash}$.

Proof. "$\subseteq$": It is sufficient to show that $\mathcal{S}_{\Gamma^{\vdash}}$ is an $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of congruences. Then $\Gamma^{\models} \subseteq \Gamma^{\vdash}$ since $\Gamma^{\models}$ is the least $\mathbf{L}^*$-implicational $\mathcal{P}$-indexed system of congruences containing Γ.

First, we check that every $\Gamma^{\vdash}_P$ ($P \in \mathcal{P}$) is a congruence on $\mathbf{T}(X)$. Reflexivity, symmetry, and transitivity of $\Gamma^{\vdash}_P$ follows from (34)–(36). For compatibility with operations, we repeatedly use (37). Namely, for $t_1, s_1, \dots, t_n, s_n \in T(X)$ and any n-ary $f \in F$ (37) yields

$$\begin{aligned}&\Gamma^{\vdash}_P(t_i, s_i)\\ \leq\;\; &\Gamma^{\vdash}_P\big(f(s_1,\dots,s_{i-1},t_i,t_{i+1},\dots,t_n),\, f(s_1,\dots,s_{i-1},s_i,t_{i+1},\dots,t_n)\big)\end{aligned}$$

for every $i = 1,\dots,n$. Hence, by (36) and monotony of $\otimes$,

$$\begin{aligned}&\Gamma^{\vdash}_P(t_1, s_1) \otimes \cdots \otimes \Gamma^{\vdash}_P(t_n, s_n)\\ &\quad\leq \textstyle\bigotimes_{i=1}^{n} \Gamma^{\vdash}_P\big(f(s_1,\dots,s_{i-1},t_i,\dots,t_n),\, f(s_1,\dots,s_i,t_{i+1},\dots,t_n)\big)\\ &\quad\leq \Gamma^{\vdash}_P\big(f(t_1,\dots,t_n),\, f(s_1,\dots,s_n)\big)\end{aligned}$$

which is the desired compatibility with operations. Clearly, $\approx^{\mathbf{T}(X)} \subseteq \Gamma^{\vdash}_P$. Thus, every $\Gamma^{\vdash}_P$ is a congruence on $\mathbf{T}(X)$.

Now we check (21)–(23).

(21): Trivial, because of (38).

(22): Take $t, t' \in T(X)$, $P \in \mathcal{P}$, where $\mathrm{Supp}(P) = \{\langle t_i, t'_i\rangle \mid i = 1,\dots,k\}$. Since P is finite, a set of variables $X' = \mathrm{var}(P) \cup \mathrm{var}(t) \cup \mathrm{var}(t') = \{x_1,\dots,x_n\}$ is finite as well. For arbitrary endomorphism h on $\mathbf{T}(X)$ we can take a set variables $Y = \{y_1,\dots,y_n\}$ such that $y_i \notin X'$, $y_i \notin \mathrm{var}(h(x))$ for every $i = 1,\dots,n$, and every variable $x \in X'$. That is, every y_i is different from variables occurring in terms $t_1, t'_1, \dots, t_n, t'_n, t, t'$, and y_i does not occur in any endomorphic image $h(x)$ of a variable $x \in X'$.

Put $\tau = (x_1/y_1)\cdots(x_n/y_n)(y_1/h(x_1))\cdots(y_n/h(x_n))$ and observe that for a term $r \in T(X')$ we have $r\tau = h(r)$. Applying this fact to $t, t' \in T(X')$ and using (39) we obtain

$$\begin{aligned}\Gamma^{\vdash}_P(t, t') &\leq \Gamma^{\vdash}_{P(x_1/y_1)}\big(t(x_1/y_1),\, t'(x_1/y_1)\big)\\ &\leq \Gamma^{\vdash}_{P(x_1/y_1)(x_2/y_2)}\big(t(x_1/y_1)(x_2/y_2),\, t'(x_1/y_1)(x_2/y_2)\big) \leq \cdots\\ &\leq \Gamma^{\vdash}_{P\tau}(t\tau, t'\tau) = \Gamma^{\vdash}_{P\tau}\big(h(t), h(t')\big).\end{aligned}$$

Clearly, now it suffices to show that $h(P) = P\tau$. Observe that $P(r, r') > 0$ implies $r, r' \in T(X')$ (X' consists of all the variables occurring in couples of terms which belong to P in some nonzero degree). Thus, using Lemma 4, and the fact $r\tau = h(r)$ for $r \in T(X')$, it follows that

$$P\tau(s,s') = \bigvee_{\substack{r\tau=s\\ r'\tau=s'\\ r,r'\in T(X')}} P(r,r') = \bigvee_{\substack{h(r)=s\\ h(r')=s'\\ r,r'\in T(X')}} P(r,r') = \big(h(P)\big)(s,s')$$

for every $s, s' \in T(X)$. Hence, $h(P) = P\tau$, thus $\Gamma^{\vdash}_{P}(t,t') \leq \Gamma^{\vdash}_{h(P)}\big(h(t), h(t')\big)$, proving (22).

(23): For finite $P \in \mathcal{P}$, we can express (23) equivalently by (28). Hence, using (18), (29), and (40) we have

$$\begin{aligned}
&\Gamma^{\vdash}_{P}(t,t') \otimes \Big(\bigwedge\nolimits_{i=1}^{k} \big(P(t_i,t_i') \to \Gamma^{\vdash}_{Q}(t_i,t_i')\big)\Big)^{*}\\
&\quad= \Gamma^{\vdash}_{P}(t,t') \otimes \bigwedge\nolimits_{i=1}^{k} \big(P(t_i,t_i') \to \Gamma^{\vdash}_{Q}(t_i,t_i')\big)^{*}\\
&\quad= \Gamma^{\vdash}_{P}(t,t') \otimes \bigotimes\nolimits_{i=1}^{k} \big(P(t_i,t_i') \to \Gamma^{\vdash}_{Q}(t_i,t_i')\big)^{*} \leq \Gamma^{\vdash}_{Q}(t,t')
\end{aligned}$$

for every $P, Q \in \mathcal{P}$, $\mathrm{Supp}(P) = \big\{\langle t_i, t_i'\rangle \mid i = 1,\ldots,k\big\}$ and every terms $t, t' \in T(X)$. Altogether, $\mathcal{S}_{\Gamma^{\vdash}}$ is an $\mathbf{L}^{*}$-implicational $\mathcal{P}$-indexed system of congruences, showing "$\subseteq$".

"$\supseteq$": We check that $\mathcal{S}_{\Gamma^{\models}}$ satisfies conditions (33)–(40). Since $\mathcal{S}_{\Gamma^{\vdash}}$ is the least $\mathcal{P}$-indexed system of $\mathbf{L}$-relations satisfying (33)–(40), we obtain $\Gamma^{\vdash} \subseteq \Gamma^{\models}$.

(33) holds trivially. Since every $\Gamma^{\models}_{P}$ ($P \in \mathcal{P}$) is a congruence, conditions (34)–(36) are satisfied obviously.

(37): This property will be proven using the compatibility of every $\Gamma^{\models}_{P}$ with operations. Let us have terms $t, t', s, s' \in T(X)$, where s has an occurrence of t as a subterm and s' is a term resulting from s by substitution of t by t'. If $s = f(t_1,\ldots,t_{k-1},t,t_{k+1},\ldots,t_n)$ and $s' = f(t_1,\ldots,t_{k-1},t',t_{k+1},\ldots,t_n)$, compatibility of $\Gamma^{\models}_{P}$ with $f \in F$ and $\Gamma^{\models}_{P}(t_i,t_i) = 1$ yield

$$\begin{aligned}
\Gamma^{\models}_{P}(t,t') &= \bigotimes\nolimits_{i=1}^{k-1} \Gamma^{\models}_{P}(t_i,t_i) \otimes \Gamma^{\models}_{P}(t,t') \otimes \bigotimes\nolimits_{j=k+1}^{n} \Gamma^{\models}_{P}(t_j,t_j)\\
&\leq \Gamma^{\models}_{P}\big(f(t_1,\ldots,t_{k-1},t,t_{k+1},\ldots,t_n), f(t_1,\ldots,t_{k-1},t',t_{k+1},\ldots,t_n)\big)\\
&= \Gamma^{\models}_{P}(s,s').
\end{aligned}$$

This argument can be used to show $\Gamma^{\models}_{P}(t,t') \leq \Gamma^{\models}_{P}(s,s')$ even in general case (one can proceed by structural induction over the rank of s).

(38): Holds trivially because of (21).

(39): Take a substitution (x/r) and a mapping $g : X \to T(X)$ defined by $g(x) = r$ and $g(y) = y$ for $y \in X$, $y \neq x$. As in Remark 14, g has a homomorphic extension $h : \mathbf{T}(X) \to \mathbf{T}(X)$, i.e. $h = g^{\sharp}$. Evidently, $h(t) = t(x/r)$ for all terms $t \in T(X)$. Moreover,

$$\big(P(x/r)\big)(t,t') = \bigvee_{\substack{s(x/r)=t\\ s'(x/r)=t'}} P(s,s') = \bigvee_{\substack{h(s)=t\\ h(s')=t'}} P(s,s') = \big(h(P)\big)(t,t').$$

Thus, (22) gives

$$\Gamma^{\models}_{P}(t,t') \leq \Gamma^{\models}_{h(P)}\big(h(t),h(t')\big) = \Gamma^{\models}_{P(x/r)}\big(t(x/r), t'(x/r)\big)$$

which is the required inequality.

(40): Since $\mathcal{S}_{\Gamma^{\models}}$ satisfies (23), we can use the equivalent formulation for finite sets of premises (28) together with (18), (29). Hence,

$$\begin{aligned}
&\Gamma^{\models}_{P}(t,t') \otimes \bigotimes\nolimits_{i=1}^{k} \big(P(t_i,t'_i) \to \Gamma^{\models}_{Q}(t_i,t'_i)\big)^{*} \\
&= \Gamma^{\models}_{P}(t,t') \otimes \bigwedge\nolimits_{i=1}^{k} \big(P(t_i,t'_i) \to \Gamma^{\models}_{Q}(t_i,t'_i)\big)^{*} \\
&= \Gamma^{\models}_{P}(t,t') \otimes \big(\bigwedge\nolimits_{i=1}^{k} \big(P(t_i,t'_i) \to \Gamma^{\models}_{Q}(t_i,t'_i)\big)\big)^{*} \leq \Gamma^{\models}_{Q}(t,t')
\end{aligned}$$

holds for every $P, Q \in \mathcal{P}$, where $\mathrm{Supp}(P) = \big\{\langle t_i, t'_i\rangle \mid i = 1,\ldots,k\big\}$, and for all $t, t' \in T(X)$. □

Remark 16. In the previous proof, we used only condition (29) of Horn truth stressers. Condition (30) has not been used.

* * *

From now on, we focus on the notion of *provability of weighted formulas.* Using similar concepts as in [31], we introduce the notion of a *degree of provability* from a given **L**-set of $\mathcal{P}$-Horn clauses. As we will see later, a suitably defined degree of provability of a $\mathcal{P}$-Horn clause is equal to its membership degree to a deductive closure. This way we establish the completeness theorem.

First, we need to introduce suitable deduction rules and a notion of provability. We proceed in Pavelka-style approach. That is, we infer weighted formulas from weighted formulas. Basic inference steps are accomplished using deduction rules which are partial mappings yielding a weighted formula $\langle\varphi, a\rangle$ (conclusion) from weighted formulas $\langle\varphi_1, a_1\rangle, \ldots, \langle\varphi_n, a_n\rangle$ (premises). Then, the provability degree of φ is defined to be the supremum over all a's such that $\langle\varphi, a\rangle$ is the last member of the proof.

Note in advance that our deduction rules do not conform to the notion of a deduction rule as introduced in [31], see also [22, 29]. According to Pavelka, a deduction rule is a pair $R = \langle R_1, R_2\rangle$, where $R_1 : \mathit{Fml}^n \to \mathit{Fml}$ is a partial mapping on the set of formulas and $R_2 : L^n \to L$ is a mapping on the set of truth degrees. Weighted formula $\langle\varphi, a\rangle$ inferred by R is of the form $\varphi = R_1(\varphi_1,\ldots,\varphi_n)$ and $a = R_2(a_1,\ldots,a_n)$, meaning that one infers validity of φ in degree (at least) $a \in L$ given formulas φ_i valid in degree (at least) a_i ($i = 1,\ldots,n$). Contrary to that, our rules (Ext) and (Mon) compute $a \in L$ in the inferred weighted formula $\langle\varphi, a\rangle$ not only from a_i's but also from truth degrees (represented by constants) which are present in φ_i's. This is, however, only for the sake of convenience. Namely, as we will see in Remark 19, all of our deduction rules are in fact derived rules in a suitably extended Pavelka-style first-order fuzzy logic with ordinary deduction rules of the form $R = \langle R_1, R_2\rangle$ as mentioned above.

Therefore, denoting the set of all $\mathcal{P}$-Horn clauses by Fml, we call every partial mapping

$$R\colon (\mathit{Fml} \times L)^n \to \mathit{Fml} \times L$$

an n-ary ***L*-deduction rule*** *for $\mathcal{P}$-Horn clauses.* A nonempty system $\mathcal{R}$ of deduction rules is called ***L*-deductive system*** *for $\mathcal{P}$-Horn clauses.*

Remark 17. (1) Instead of $R(\langle\varphi_1, a_1\rangle, \ldots, \langle\varphi_n, a_n\rangle) = \langle\varphi, a\rangle$, we use the common notation

$$\frac{\langle\varphi_1, a_1\rangle, \ldots, \langle\varphi_n, a_n\rangle}{\langle\varphi, a\rangle}. \tag{41}$$

On the verbal level, the deduction rule (41) should be read as "From φ_1 in degree a_1, and $\cdots$ and φ_n in degree a_n infer φ in degree a".

(2) An *axiom* can be thought of as nullary deduction rule, i.e. a mapping $A\colon \{\emptyset\} \to \mathit{Fml} \times L$. Hence, an axiom is a weighted formula from $\mathit{Fml} \times L$. In accordance with Remark 1, we can denote an axiom $\big\langle P \Rightarrow (t \approx t'), a\big\rangle$ also by $P \Rightarrow \big\langle t \approx t', a\big\rangle$.

Definition 14. *Let* Γ *be an* $\mathbf{L}$*-set of* $\mathcal{P}$*-Horn clauses and let* $\mathcal{R}$ *be an* $\mathbf{L}^*$*-deductive system for* $\mathcal{P}$*-Horn clauses. An* ***$\mathbf{L}^*$-weighted $\mathcal{R}$-proof*** of a weighted $\mathcal{P}$-Horn clause $\big\langle P \Rightarrow (t \approx t'), a\big\rangle$, $a \in L$, from Γ *is a finite sequence of weighted* $\mathcal{P}$*-Horn clauses* $\langle\varphi_1, a_1\rangle, \ldots, \langle\varphi_l, a_l\rangle$*, where* φ_l *is* $P \Rightarrow (t \approx t')$*,* $a_l = a$*, and for every* $\langle\varphi_i, a_i\rangle$*,* $i = 1, \ldots, l$*, we have either*

$$a_i = \Gamma(\varphi_i)$$

or there is an n*-ary* $\mathbf{L}^*$*-deduction rule* $R \in \mathcal{R}$*, such that*

$$R\big(\big\langle\varphi_{i_1}, a_{i_1}\big\rangle, \ldots, \big\langle\varphi_{i_n}, a_{i_n}\big\rangle\big) = \langle\varphi_i, a_i\rangle,$$

for some $i_1, \ldots, i_n < i$*. The number* l *is called a* ***length*** of the proof.

A weighted $\mathcal{P}$*-Horn clause* $\big\langle P \Rightarrow (t \approx t'), b\big\rangle$ *is said to be* ***$\mathcal{R}$-provable*** from Γ*, if there exists an* $\mathbf{L}^*$*-weighted* $\mathcal{R}$*-proof of* $\big\langle P \Rightarrow (t \approx t'), b\big\rangle$ *from* Γ*. We denote this fact by* $\Gamma \vdash^{\mathcal{R}} \big\langle P \Rightarrow (t \approx t'), b\big\rangle$.

A $\mathcal{P}$*-Horn clause* $P \Rightarrow (t \approx t')$ *is said to be* ***$\mathcal{R}$-provable*** from Γ in degree *(*at least*)* $b \in L$*, if the weighted* $\mathcal{P}$*-Horn clause* $\big\langle P \Rightarrow (t \approx t'), b\big\rangle$ *is* $\mathcal{R}$*-provable from* Γ.

For every $\mathcal{P}$*-Horn clause* $P \Rightarrow (t \approx t')$ *we define a* ***degree*** $\big|P \Rightarrow (t \approx t')\big|_\Gamma^{\mathcal{R}}$ ***of $\mathcal{R}$-provability*** of $P \Rightarrow (t \approx t')$ from Γ *by*

$$\big|P \Rightarrow (t \approx t')\big|_\Gamma^{\mathcal{R}} = \bigvee\big\{a \in L \mid \Gamma \vdash^{\mathcal{R}} \big\langle P \Rightarrow (t \approx t'), a\big\rangle\big\}.$$

Remark 18. In what follows, we use a system of deduction rules (Ref)–(Mon) introduced below. Therefore, if there is no danger of confusion, we omit the "$\mathcal{R}$" from terms "$\mathcal{R}$-proof", "$\big|P \Rightarrow (t \approx t')\big|_\Gamma^{\mathcal{R}}$", etc. and write simply "proof", "$\big|P \Rightarrow (t \approx t')\big|_\Gamma$", etc. In [36], we present another axiomatization which uses more axioms, and less n-ary deduction rules ($n \geq 1$).

First group of rules are the *rules of congruence:*

$$(\mathrm{Ref})\colon \langle P \Rightarrow (t \approx t), 1\rangle,$$
$$(\mathrm{Sym})\colon \frac{\big\langle P \Rightarrow (t \approx t'), a\big\rangle}{\langle P \Rightarrow (t' \approx t), a\rangle},$$
$$(\mathrm{Tra})\colon \frac{\big\langle P \Rightarrow (t \approx t'), a\big\rangle,\ \big\langle P \Rightarrow (t' \approx t''), b\big\rangle}{\langle P \Rightarrow (t \approx t''), a \otimes b\rangle},$$
$$(\mathrm{Rep})\colon \frac{\big\langle P \Rightarrow (t \approx t'), a\big\rangle}{\langle P \Rightarrow (s \approx s'), a\rangle},$$

where $P \in \mathcal{P}, a, b \in L, t, t', t'', s, s' \in T(X)$, and s contains t as a subterm and s' results from s by substitution of one occurrence of t in s by t'.

The second group are the rules of *extensivity, substitution,* and *monotony:*

$$(\mathrm{Ext})\colon \big\langle P \Rightarrow (t \approx t'), P(t, t')\big\rangle,$$
$$(\mathrm{Sub})\colon \frac{\big\langle P \Rightarrow (t \approx t'), a\big\rangle}{\big\langle P(x/r) \Rightarrow \big(t(x/r) \approx t'(x/r)\big), a\big\rangle},$$
$$(\mathrm{Mon})\colon \frac{\big\{\big\langle Q \Rightarrow (t_i \approx t_i'), a_i\big\rangle;\ i = 1, \dots, n\big\},\ \big\langle P \Rightarrow (t \approx t'), b\big\rangle}{\big\langle Q \Rightarrow (t \approx t'), b \otimes \bigotimes_{i=1}^{n} (P(t_i, t_i') \to a_i)^*\big\rangle},$$

where $P, Q \in \mathcal{P}$ such that $\mathrm{Supp}(P) = \big\{\big\langle t_i, t_i'\big\rangle \,|\, i = 1, \dots, n\big\}$, $t, t', r \in T(X)$, $x \in X, a_1, \dots, a_n, a, b \in L$.

Remark 19. (1) On the verbal level, the rule of monotony (Mon) can be read: “if P implies $t \approx t'$ and every premise $t_i \approx t_i'$ from P is implied by Q, then also $t \approx t'$ is implied by Q”. A finer reading of (Mon) is: “Q implies $t \approx t'$ (at least) in degree to which P implies $t \approx t'$ *and* each $t_i \approx t_i'$ is implied by Q at least in degree to which $t_i \approx t_i'$ belongs to P”.

(2) The truth value $b \otimes \bigotimes_{i=1}^{n} (P(t_i, t_i') \to a_i)^*$ as used in (Mon) is computed using truth values of input formulas together with $P(t_i, t_i')$’s. In other words, the *thresholds* of premises represented by degrees $P(t_i, t_i')$ have an influence on the resulting truth value.

Consider the following example. Take a complete residuated lattice $\mathbf{L} = \langle\{0, a, 1\}, \vee, \wedge, \otimes, \to, 0, 1\rangle$, where $0 < a < 1$, and $a \otimes a = 0$, the rest is determined uniquely. Moreover, we equip $\mathbf{L}$ by Horn truth stresser *, for which $a^* = 0^* = 0$, $1^* = 1$. Now, we can use (Mon) to infer

$$\frac{\big\langle r \approx r', 1\big\rangle \Rightarrow \big\langle s \approx s', a\big\rangle,\ \big\langle s \approx s', a\big\rangle \Rightarrow \big\langle t \approx t', 1\big\rangle}{\langle r \approx r', 1\rangle \Rightarrow \langle t \approx t', 1\rangle}.$$

On the other hand, increasing the threshold of $s \approx s'$ from a to 1, we get an inference with empty conclusion

$$\frac{\big\langle r \approx r', 1\big\rangle \Rightarrow \big\langle s \approx s', a\big\rangle,\ \big\langle s \approx s', 1\big\rangle \Rightarrow \big\langle t \approx t', 1\big\rangle}{\langle r \approx r', 1\rangle \Rightarrow \langle t \approx t', 0\rangle}.$$

(3) One way to have deduction rules in Pavelka-style (separate syntactic and semantic parts) is to split the (Mon) rule into separate rules (Mon_P) for every set of premises $P \in \mathcal{P}$. In this case, it would be possible to distinguish the syntactic and semantic part of a rule by two independent mappings as usual. For the above example, we would then use two different deduction rules (Mon_{P_1}), (Mon_{P_2}). However, this would be an artificial way of doing what can be done more naturally. The same applies to (Ext).

(4) We are going to show that (Ref)–(Mon) are derived rules in a natural Pavelka-style first-order fuzzy logic. To that purpose, we assume that some natural (weighted) formulas are provable in the Pavelka-style logic we work with (we mention them in the course of our demonstration). In particular, we assume that we have formulas guaranteeing reflexivity, symmetry, transitivity, and compatibility of $\approx$, and formulas guaranteeing the required properties of logical connectives as axioms. As we will work with truth constants (for every $a \in L$ we consider a truth constant $\overline{a}$), we need to assume the appropriate bookkeeping axioms for the constants [22]. Namely, for $\triangle$ we assume $\vdash \triangle\,\overline{a} \Longleftrightarrow \overline{a^*}$. We use the following deduction rules: *modus ponens* (MP; from $\langle\varphi, a\rangle$ and $\langle\varphi \Rightarrow \psi, b\rangle$ infer $\langle\psi, a \otimes b\rangle$) *logical constant introduction* [29] (from $\langle\varphi, a\rangle$ infer $\langle\overline{a} \Rightarrow \varphi, 1\rangle$), and *truth confirmation* [22, 23] (from $\langle\varphi, 1\rangle$ infer $\langle\triangle\,\varphi, 1\rangle$). For convenience, we write $\vdash \varphi$ instead of $\vdash \langle\varphi, 1\rangle$.

Let $\mathrm{Supp}(P) = \{\langle t_1, t'_1\rangle, \ldots, \langle t_n, t'_n\rangle\}$ and put $P(t_i, t'_i) = p_i$ for every $i = 1, \ldots, n$.

(Ext): Let $P \Rightarrow (t \approx t')$. We show $\vdash \overline{p} \Rightarrow (P \Rightarrow (t \approx t'))$ for $p = P(t, t')$. Observe that for $P(t, t') = 0$, the claim is trivial. Thus, suppose there is some $j \in \{1, \ldots, n\}$ with $t = t_j$ and $t' = t'_j$. Therefore, $p = P(t_j, t'_j) = p_j$. We have

$$\vdash \triangle \textstyle\bigwedge_{i=1}^{n}(\overline{p_i} \Rightarrow (t_i \approx t'_i)) \Rightarrow \bigwedge_{i=1}^{n}(\overline{p_i} \Rightarrow (t_i \approx t'_i)),$$
[instance of $\triangle\,\varphi \Rightarrow \varphi$,]
$$\vdash \textstyle\bigwedge_{i=1}^{n}(\overline{p_i} \Rightarrow (t_i \approx t'_i)) \Rightarrow (\overline{p_j} \Rightarrow (t_j \approx t'_j)),$$
[using $\vdash (\varphi \wedge \psi) \Rightarrow \varphi$,]
$$\vdash \triangle \textstyle\bigwedge_{i=1}^{n}(\overline{p_i} \Rightarrow (t_i \approx t'_i)) \Rightarrow (\overline{p_j} \Rightarrow (t_j \approx t'_j)),$$
[by $\vdash (\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi))$, MP,]
$$\vdash \overline{p_j} \Rightarrow \big(\triangle \textstyle\bigwedge_{i=1}^{n}(\overline{p_i} \Rightarrow (t_i \approx t'_i)) \Rightarrow (t_j \approx t'_j)\big),$$
[by $\vdash (\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow (\psi \Rightarrow (\varphi \Rightarrow \chi))$, MP,]

showing that $\vdash \overline{p} \Rightarrow (P \Rightarrow (t = t'))$.

(Mon): We show that if $\vdash \overline{b} \Rightarrow (P \Rightarrow (t \approx t'))$ and $\vdash \overline{a_i} \Rightarrow (Q \Rightarrow (t_i \approx t'_i))$ for every $i = 1, \ldots, n$ then $\vdash \big(\overline{b} \otimes \bigotimes_{i=1}^{n} \triangle(\overline{p_i} \Rightarrow \overline{a_i})\big) \Rightarrow (Q \Rightarrow (t \approx t'))$. First,

$$\vdash Q \Rightarrow (\overline{a_i} \Rightarrow (t_i \approx t'_i)),$$
[by $\vdash (\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow (\psi \Rightarrow (\varphi \Rightarrow \chi))$, MP, $i = 1, \ldots, n$,]
$$\vdash Q \Rightarrow \textstyle\bigwedge_{i=1}^{n}(\overline{a_i} \Rightarrow (t_i \approx t'_i)),$$
[using $\vdash (\varphi \Rightarrow \psi) \Rightarrow ((\varphi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow (\psi \wedge \chi)))$, MP,]

$\vdash \triangle\big(Q \Rightarrow \bigwedge_{i=1}^{n}(\overline{a_i} \Rightarrow (t_i \approx t_i'))\big)$,
[truth confirmation,]
$\triangle Q \Rightarrow \triangle \bigwedge_{i=1}^{n}(\overline{a_i} \Rightarrow (t_i \approx t_i'))$,
[by $\vdash \triangle(\varphi \Rightarrow \psi) \Rightarrow (\triangle\varphi \Rightarrow \triangle\psi)$, MP,]
$\vdash Q \Rightarrow \triangle Q$,
[instance of $\triangle\varphi \Rightarrow \triangle\triangle\varphi$ (Q is of the form $\triangle\varphi$),]
$\vdash Q \Rightarrow \triangle \bigwedge_{i=1}^{n}(\overline{a_i} \Rightarrow (t_i \approx t_i'))$,
[by $\vdash (\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi))$, MP.]

Now $\vdash (\psi \Rightarrow \chi) \Rightarrow ((\varphi\otimes\psi) \Rightarrow (\varphi\otimes\chi))$ and $\vdash (\triangle\varphi\otimes\triangle\psi) \Rightarrow \triangle(\varphi\otimes\psi)$ give

$$\begin{aligned}
&\vdash \big(\textstyle\bigotimes_{i=1}^{n}\triangle(\overline{p_i} \Rightarrow \overline{a_i})\otimes Q\big) \Rightarrow \big(\bigotimes_{i=1}^{n}\triangle(\overline{p_i} \Rightarrow \overline{a_i})\otimes\triangle\bigwedge_{i=1}^{n}(\overline{a_i} \Rightarrow (t_i \approx t_i'))\big),\\
&\vdash \big(\textstyle\bigotimes_{i=1}^{n}\triangle(\overline{p_i} \Rightarrow \overline{a_i})\otimes\triangle\bigwedge_{i=1}^{n}(\overline{a_i} \Rightarrow (t_i \approx t_i'))\big) \Rightarrow\\
&\qquad \Rightarrow \triangle\big(\textstyle\bigotimes_{i=1}^{n}(\overline{p_i} \Rightarrow \overline{a_i})\otimes\bigwedge_{i=1}^{n}(\overline{a_i} \Rightarrow (t_i \approx t_i'))\big).
\end{aligned}$$

By $\vdash (\varphi\otimes(\psi \wedge \chi)) \Rightarrow ((\varphi\otimes\psi) \wedge (\varphi\otimes\chi))$,
$\vdash (\varphi\otimes\psi) \Rightarrow \varphi$, $\vdash ((\varphi \Rightarrow \psi)\otimes(\psi \Rightarrow \chi)) \Rightarrow (\varphi \Rightarrow \chi)$, and using truth confirmation with isotony of $\otimes$ and $\vdash \triangle(\varphi \Rightarrow \psi) \Rightarrow (\triangle\varphi \Rightarrow \triangle\psi)$:

$$\begin{aligned}
&\vdash \triangle\big(\textstyle\bigotimes_{i=1}^{n}(\overline{p_i} \Rightarrow \overline{a_i})\otimes\bigwedge_{i=1}^{n}(\overline{a_i} \Rightarrow (t_i \approx t_i'))\big) \Rightarrow\\
&\qquad \Rightarrow \triangle\big(\textstyle\bigwedge_{i=1}^{n}((\overline{p_i} \Rightarrow \overline{a_i})\otimes(\overline{a_i} \Rightarrow (t_i \approx t_i')))\big),\\
&\vdash \triangle\big(\textstyle\bigwedge_{i=1}^{n}((\overline{p_i} \Rightarrow \overline{a_i})\otimes(\overline{a_i} \Rightarrow (t_i \approx t_i')))\big) \Rightarrow \triangle\big(\bigwedge_{i=1}^{n}(\overline{p_i} \Rightarrow (t_i \approx t_i'))\big).
\end{aligned}$$

Using transitivity applied on previous formulas, isotony, and MP:

$$\begin{aligned}
&\vdash \big(\textstyle\bigotimes_{i=1}^{n}\triangle(\overline{p_i} \Rightarrow \overline{a_i})\otimes Q\big) \Rightarrow \triangle\big(\bigwedge_{i=1}^{n}(\overline{p_i} \Rightarrow (t_i \approx t_i'))\big),\\
&\vdash \big(\overline{b}\otimes\textstyle\bigotimes_{i=1}^{n}\triangle(\overline{p_i} \Rightarrow \overline{a_i})\otimes Q\big) \Rightarrow \big(\overline{b}\otimes\triangle\big(\bigwedge_{i=1}^{n}(\overline{p_i} \Rightarrow (t_i \approx t_i'))\big)\big).
\end{aligned}$$

Taking into account $\vdash \overline{b} \Rightarrow (P \Rightarrow (t \approx t'))$, we have

$\vdash \big(\overline{b}\otimes\triangle\big(\bigwedge_{i=1}^{n}(\overline{p_i} \Rightarrow (t_i \approx t_i'))\big)\big) \Rightarrow (t \approx t')$,
[by $\vdash (\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi\otimes\psi) \Rightarrow \chi)$, MP,]
$\vdash \big(\overline{b}\otimes\bigotimes_{i=1}^{n}\triangle(\overline{p_i} \Rightarrow \overline{a_i})\otimes Q\big) \Rightarrow (t \approx t')$,
[using transitivity, MP,]
$\vdash \big(\overline{b}\otimes\bigotimes_{i=1}^{n}\triangle(\overline{p_i} \Rightarrow \overline{a_i})\big) \Rightarrow (Q \Rightarrow (t \approx t'))$,
[by $\vdash ((\varphi\otimes\psi) \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \chi))$, MP,]

which is the desired weighted formula. One can proceed similarly for the rest of the deduction rules. Note that the rule of truth confirmation and axioms

$$\begin{aligned}
&\triangle\varphi \Rightarrow \varphi,\\
&\triangle(\varphi \Rightarrow \psi) \Rightarrow (\triangle\varphi \Rightarrow \triangle\psi),\\
&\triangle\varphi \Rightarrow \triangle\triangle\varphi,\\
&(\triangle\varphi\otimes\triangle\psi) \Rightarrow \triangle(\varphi\otimes\psi)
\end{aligned}$$

being used in the previous demonstration naturally correspond to properties of implicational truth stressers, see (6), (7), (8), (16), and (v) of Lemma 1.

The following theorem shows the relationship between deductive closure of Γ and the degree of provability from Γ.

Theorem 12. *Suppose* Γ *is an* $\mathbf{L}$*-set of* $\mathcal{P}$*-Horn clauses, and* * *is a Horn truth stresser. Then*

$$\left|P \Rightarrow (t \approx t')\right|_{\Gamma} = \Gamma^{\vdash}_{P}(t, t')$$

for every $P \in \mathcal{P}$ *and* $t, t' \in T(X)$.

Proof. "$\leq$": It is sufficient to check that for every member $\langle P \Rightarrow (t \approx t'), a\rangle$ of an $\mathbf{L}^*$-weighted proof from Γ, we have $a \leq \Gamma^{\vdash}_{P}(t, t')$. We proceed by induction over the length of an $\mathbf{L}^*$-weighted proof.

Suppose $\langle P \Rightarrow (t \approx t'), a\rangle$ is a member of an $\mathbf{L}^*$-weighted proof from Γ where $a = \Gamma(P \Rightarrow (t \approx t')) = \Gamma_P(t, t')$. Using (33), we have $a = \Gamma_P(t, t') \leq \Gamma^{\vdash}_{P}(t, t')$. Otherwise, $\langle P \Rightarrow (t \approx t'), a\rangle$ was derived using one of (Ref)–(Mon). We check "$\leq$" for all of these rules separately.

(Ref): If $\langle P \Rightarrow (t \approx t), 1\rangle$ was inferred by (Ref), we directly obtain $1 \leq \Gamma^{\vdash}_{P}(t, t) = 1$ due to (34).

(Sym): If $\langle P \Rightarrow (t' \approx t), a\rangle$ was inferred from $\langle P \Rightarrow (t \approx t'), a\rangle$, then induction hypothesis together with (35) give $a \leq \Gamma^{\vdash}_{P}(t, t') \leq \Gamma^{\vdash}_{P}(t', t)$.

(Tra): Suppose $\langle P \Rightarrow (t \approx t''), a \otimes b\rangle$ was inferred from $\langle P \Rightarrow (t \approx t'), a\rangle$ and $\langle P \Rightarrow (t' \approx t''), b\rangle$. By induction hypothesis, $a \leq \Gamma^{\vdash}_{P}(t, t')$, $b \leq \Gamma^{\vdash}_{P}(t', t'')$. Hence, (36) yields $a \otimes b \leq \Gamma^{\vdash}_{P}(t, t') \otimes \Gamma^{\vdash}_{P}(t', t'') \leq \Gamma^{\vdash}_{P}(t, t'')$.

(Rep): If $\langle P \Rightarrow (s \approx s'), a\rangle$ was inferred by (Rep) from $\langle P \Rightarrow (t \approx t'), a\rangle$, induction hypothesis and (37) give $a \leq \Gamma^{\vdash}_{P}(t, t') \leq \Gamma^{\vdash}_{P}(s, s')$.

(Ext): Almost trivial using (38).

(Sub): If $\langle P(x/r) \Rightarrow (t(x/r) \approx t'(x/r)), a\rangle$ was inferred from $\mathcal{P}$-Horn clause $\langle P \Rightarrow (t \approx t'), a\rangle$, induction hypothesis and (39) yield $a \leq \Gamma^{\vdash}_{P}(t, t') \leq \Gamma^{\vdash}_{P(x/r)}\big(t(x/r), t'(x/r)\big)$.

(Mon): Suppose $\langle Q \Rightarrow (t \approx t'), c\rangle$ was inferred from $\langle P \Rightarrow (t \approx t'), b\rangle$ and $\langle Q \Rightarrow (t_i \approx t'_i), a_i\rangle$'s with $\mathrm{Supp}(P) = \{\langle t_i, t'_i\rangle \mid i = 1, \ldots, k\}$. Thus, we have $c = b \otimes \bigotimes_{i=1}^{k} \big(P(t_i, t'_i) \to a_i\big)^*$. Using the induction hypothesis, we have $b \leq \Gamma^{\vdash}_{P}(t, t')$, and $a_i \leq \Gamma^{\vdash}_{Q}(t_i, t'_i)$ for each $i = 1, \ldots, k$. Thus, by (40) we obtain

$$\begin{aligned} c &= b \otimes \bigotimes_{i=1}^{k} \big(P(t_i, t'_i) \to a_i\big)^* \\ &\leq \Gamma^{\vdash}_{P}(t, t') \otimes \bigotimes_{i=1}^{k} \big(P(t_i, t'_i) \to \Gamma^{\vdash}_{Q}(t_i, t'_i)\big)^* \leq \Gamma^{\vdash}_{Q}(t, t'). \end{aligned}$$

"$\geq$": Let $\mathcal{D} = \{D_P \in \mathbf{L}^{T(X)\times T(X)} \mid P \in \mathcal{P}\}$ be a $\mathcal{P}$-indexed system of $\mathbf{L}$-relations, where $D_P(t, t') = \left|P \Rightarrow (t \approx t')\right|_{\Gamma}$ for every $P \in \mathcal{P}$, and $t, t' \in T(X)$. We show that $\mathcal{D}$ satisfies (33)–(40) from which $\mathcal{S}_{\Gamma^{\vdash}} \leq \mathcal{D}$ follows immediately by definition of $\Gamma^{\vdash}$. Observing that $\mathcal{S}_{\Gamma^{\vdash}} \leq \mathcal{D}$ means $\Gamma^{\vdash}_{P}(t, t') \leq \left|P \Rightarrow (t \approx t')\right|_{\Gamma}$, then finishes the proof. Thus, let us check (33)–(40).

(33): Since $\langle P \Rightarrow (t \approx t'), \Gamma_P(t,t')\rangle$ is an $\mathbf{L}^*$-weighted proof of length 1, we have $D_P(t,t') = \left|P \Rightarrow (t \approx t')\right|_\Gamma \geq \Gamma_P(t,t')$, verifying (33).

(34): Evidently, $D_P(t,t) = 1$ since $\langle P \Rightarrow (t \approx t), 1\rangle$ is a proof by (Ref).

(35): Using (Sym), each proof $\delta_{i_1}, \dots, \delta_{i_{n_i}}, \langle P \Rightarrow (t \approx t'), a_i\rangle$ can be extended to a proof

$$\delta_{i_1}, \dots, \delta_{i_{n_i}}, \langle P \Rightarrow (t \approx t'), a_i\rangle, \langle P \Rightarrow (t' \approx t), a_i\rangle.$$

Hence, $D_P(t,t') = \left|P \Rightarrow (t \approx t')\right|_\Gamma \leq \left|P \Rightarrow (t' \approx t)\right|_\Gamma = D_P(t',t)$.

(36): Let $D_P(t,t') = \bigvee_{i \in I} a_i$ and $D_P(t',t'') = \bigvee_{j \in J} b_j$ where for every $i \in I$ and $j \in J$ there are proofs $\delta_{i_1}, \dots, \delta_{i_{n_i}}, \langle P \Rightarrow (t \approx t'), a_i\rangle$ and $\delta_{j_1}, \dots, \delta_{j_{n_j}}, \langle P \Rightarrow (t' \approx t''), b_j\rangle$. Concatenating the proofs and using (Tra) we get a proof

$$\begin{aligned}&\delta_{i_1}, \dots, \delta_{i_{n_i}}, \langle P \Rightarrow (t \approx t'), a_i\rangle,\\ &\delta_{j_1}, \dots, \delta_{j_{n_j}}, \langle P \Rightarrow (t' \approx t''), b_j\rangle, \langle P \Rightarrow (t \approx t''), a_i \otimes b_j\rangle.\end{aligned}$$

Hence, $P \Rightarrow (t \approx t'')$ is provable in degree at least $a_i \otimes b_j$ and so

$$\begin{aligned}D_P(t,t') \otimes D_P(t',t'') &= \textstyle\bigvee_{i \in I} a_i \otimes \bigvee_{j \in J} b_j = \bigvee_{i \in I} \bigvee_{j \in J} (a_i \otimes b_j)\\ &\leq \left|P \Rightarrow (t \approx t'')\right|_\Gamma = D_P(t,t'').\end{aligned}$$

That is, $\mathcal{D}$ satisfies (36).

(37): Similarly, for a term s' resulting from s by substitution of one occurrence of t in s by t', every proof $\delta_{i_1}, \dots, \delta_{i_{n_i}}, \langle P \Rightarrow (t \approx t'), a_i\rangle$ can be extended by (Rep) to a proof

$$\delta_{i_1}, \dots, \delta_{i_{n_i}}, \langle P \Rightarrow (t \approx t'), a_i\rangle, \langle P \Rightarrow (s \approx s'), a_i\rangle.$$

It follows that $D_P(t,t') = \left|P \Rightarrow (t \approx t')\right|_\Gamma \leq \left|P \Rightarrow (s \approx s')\right|_\Gamma = D_P(s,s')$.

(38): Since $\langle P \Rightarrow (t \approx t'), P(t,t')\rangle$ is a proof, $P(t,t') \leq \left|P \Rightarrow (t \approx t')\right|_\Gamma = D_P(t,t')$.

(39): Let $D_P(t,t') = \bigvee_{i \in I} a_i$ where for every $i \in I$ there is a proof $\delta_{i_1}, \dots, \delta_{i_{n_i}}, \langle P \Rightarrow (t \approx t'), a_i\rangle$. Using (Sub), such proofs can be extended to

$$\delta_{i_1}, \dots, \delta_{i_{n_i}}, \langle P \Rightarrow (t \approx t'), a_i\rangle, \langle P(x/r) \Rightarrow \big(t(x/r) \approx t'(x/r)\big), a_i\rangle.$$

Hence, $D_P(t,t') \leq D_{P(x/r)}\big(t(x/r), t'(x/r)\big)$.

(40): This is the second time we are using condition (30). Suppose $P, Q \in \mathcal{P}$ and $\mathrm{Supp}(P) = \{\langle t_k, t'_k\rangle \mid k = 1, \dots, n\}$. For every $t, t' \in T(X)$, let $D_P(t,t') = \bigvee_{i \in I} b_i$, where for every $i \in I$ there is a proof $\delta_{i_1}, \dots, \delta_{i_{n_i}}, \langle P \Rightarrow (t \approx t'), b_i\rangle$. Moreover, let $D_Q(t_k, t'_k) = \bigvee_{j_k \in J_k} a_{k,j_k}$, $k = 1, \dots, n$, where for every $j_k \in J_k$, there is a proof $\delta_{j_k,1}, \ldots, \delta_{j_k,n_{j_k}}, \langle Q \Rightarrow (t_k \approx t'_k), a_{k,j_k}\rangle$. We can take any $i \in I$, $j_1 \in J_1, \ldots, j_n \in J_n$, concatenate the proofs, and apply (Mon) to get a proof

$$
\begin{array}{rll}
 & \delta_{j_1,1}, \ldots, \delta_{j_1,n_{j_1}}, & \\
1: & \langle Q \Rightarrow (t_1 \approx t'_1), a_{1,j_1}\rangle, & \text{proof of } \langle Q \Rightarrow (t_1 \approx t'_1), a_{1,j_1}\rangle \\
 & \vdots \quad \vdots \quad \vdots & \vdots \\
 & \delta_{j_n,1}, \ldots, \delta_{j_n,n_{j_n}}, & \\
n: & \langle Q \Rightarrow (t_n \approx t'_n), a_{n,j_n}\rangle, & \text{proof of } \langle Q \Rightarrow (t_n \approx t'_n), a_{n,j_n}\rangle \\
 & \delta_{i_1}, \ldots, \delta_{i_{n_i}}, & \\
n{+}1: & \langle P \Rightarrow (t \approx t'), b_i\rangle, & \text{proof of } \langle P \Rightarrow (t \approx t'), b_i\rangle \\
n{+}2: & \langle Q \Rightarrow (t \approx t'), b_i \otimes \bigotimes_{k=1}^{n} \big(P(t_k, t'_k) \to a_{k,j_k}\big)^*\rangle. & \text{(Mon) on } 1, \ldots, n+1
\end{array}
$$

Hence, $|Q \Rightarrow (t \approx t')|_\Gamma$ is greater or equal to $b_i \otimes \bigotimes_{k=1}^{n} \big(P(t_k, t'_k) \to a_{k,j_k}\big)^*$. Using (30) we get

$$
\begin{aligned}
D_P(t, t') &\otimes \bigotimes_{k=1}^{n} \big(P(t_k, t'_k) \to D_Q(t_k, t'_k)\big)^* \\
&= \bigvee_{i \in I} b_i \otimes \bigotimes_{k=1}^{n} \big(P(t_k, t'_k) \to \bigvee_{j_k \in J_k} a_{k,j_k}\big)^* \\
&= \bigvee_{i \in I} b_i \otimes \bigotimes_{k=1}^{n} \bigvee_{j_k \in J_k} \big(P(t_k, t'_k) \to a_{k,j_k}\big)^* \\
&= \bigvee_{i \in I} b_i \otimes \bigvee_{j_1 \in J_1, \ldots, j_n \in J_n} \bigotimes_{k=1}^{n} \big(P(t_k, t'_k) \to a_{k,j_k}\big)^* \\
&= \bigvee_{\substack{i \in I \\ j_1 \in J_1, \ldots, j_n \in J_n}} \big(b_i \otimes \bigotimes_{k=1}^{n} \big(P(t_k, t'_k) \to a_{k,j_k}\big)^*\big) \leq D_Q(t, t'),
\end{aligned}
$$

which gives (40). Note that since $\langle Q \Rightarrow (t_k \approx t'_k), \Gamma_Q(t_k, t'_k)\rangle$ is a proof, every J_k is nonempty. This justifies the application of (30). □

Now we have the following consequence.

Theorem 13 (completeness theorem). *Let* $\mathbf{L}^*$ *be a complete residuated lattice equipped with a Horn truth stresser* *. *Let* $\mathcal{P}$ *be a proper family of premises of type* F *in variables* X, *where every* $P \in \mathcal{P}$ *is finite. For every* $\mathbf{L}$*-set* Γ *of* $\mathcal{P}$*-Horn clauses we have*

$$
|P \Rightarrow (t \approx t')|_\Gamma = \Gamma_P^{\vdash}(t, t') = \Gamma_P^{\models}(t, t') = \|P \Rightarrow (t \approx t')\|_\Gamma
$$

for every $P \in \mathcal{P}$ *and* $t, t' \in T(X)$.

Proof. Consequence of Theorem 8, Theorem 11, and Theorem 12. □

For finite $\mathbf{L}$, we can avoid (30) by using an analogous argument as in proof of Theorem 10. For this purpose we introduce an additional deduction rule:

$$
(\text{Sup}): \frac{\langle P \Rightarrow (t \approx t'), a\rangle, \ \langle P \Rightarrow (t \approx t'), b\rangle}{\langle P \Rightarrow (t \approx t'), a \vee b\rangle},
$$

where $P \in \mathcal{P}$, $a, b \in L$, $t, t' \in T(X)$.

Theorem 14 (completeness theorem for finite L). *Let* $\mathbf{L}^*$ *be a finite residuated lattice with an implicational truth stresser* * *satisfying* (29). *Let* $\mathcal{P}$ *be a proper family of premises of type* F *in variables* X, *where every* $P \in \mathcal{P}$ *is finite and let* $\mathcal{R}_\vee$ *denote the* $\mathbf{L}^*$*-deductive system resulting from* $\mathcal{R}$ *by adding* (Sup). *Then for every* $\mathbf{L}$*-set* Γ *of* $\mathcal{P}$*-Horn clauses we have* $\left|P \Rightarrow (t \approx t')\right|_{\Gamma}^{\mathcal{R}_\vee} = \left\|P \Rightarrow (t \approx t')\right\|_{\Gamma}$ *for every* $P \in \mathcal{P}$ *and* $t, t' \in T(X)$.

Proof. Using Theorem 8 and Theorem 11, we only check that

$$\left|P \Rightarrow (t \approx t')\right|_{\Gamma}^{\mathcal{R}_\vee} = \Gamma_P^{\vdash}(t, t').$$

"$\leq$": If $\langle P \Rightarrow (t \approx t'), a \vee b\rangle$ was inferred from weighted $\mathcal{P}$-Horn clauses $\langle P \Rightarrow (t \approx t'), a\rangle$ and $\langle P \Rightarrow (t \approx t'), b\rangle$ using (Sup), then assuming $a \leq \Gamma_P^{\vdash}(t, t')$ and $b \leq \Gamma_P^{\vdash}(t, t')$, we have $a \vee b \leq \Gamma_P^{\vdash}(t, t')$. The rest follows from the "$\leq$"-part of the proof of Theorem 12.

"$\geq$": Put $D_P(t, t') = \left|P \Rightarrow (t \approx t')\right|_{\Gamma}^{\mathcal{R}_\vee}$ for every $P \in \mathcal{P}$, and $t, t' \in T(X)$. We check that $\mathcal{D} = \{D_P \mid P \in \mathcal{P}\}$ satisfies (33)–(40). We focus only on (40) since the rest follows from the proof of Theorem 12. Since $\mathbf{L}$ is finite, we have $D_P(t, t') = \bigvee_{i=1}^{k} \left\{a_i \mid \Gamma \vdash^{\mathcal{R}_\vee} \langle P \Rightarrow (t \approx t'), a_i\rangle\right\}$, i.e. there are proofs $\delta_{i,1}, \ldots, \delta_{i,n_i}, \langle P \Rightarrow (t \approx t'), a_i\rangle$ $(i = 1, \ldots, k)$. We can concatenate these proofs and using (Sup) we get a proof of $\langle P \Rightarrow (t \approx t'), a_1 \vee \cdots \vee a_k\rangle$. As a consequence, $D_P(t, t') = a$ implies $\Gamma \vdash^{\mathcal{R}_\vee} \langle P \Rightarrow (t \approx t'), a\rangle$.

Suppose $P, Q \in \mathcal{P}$ and $\mathrm{Supp}(P) = \left\{\langle t_k, t'_k\rangle \mid k = 1, \ldots, n\right\}$.
Take $t, t' \in T(X)$. Let $D_P(t, t') = b$, i.e. due to the previous observation, there is a proof $\delta_1, \ldots, \delta_n, \langle P \Rightarrow (t \approx t'), b\rangle$. Moreover, let $D_Q(t_k, t'_k) = a_k$ $(k = 1, \ldots, n)$ and denote $\delta_{k,1}, \ldots, \delta_{k,n_k}, \langle Q \Rightarrow (t_k \approx t'_k), a_k\rangle$ the corresponding proofs. Concatenating the proofs and applying (Mon), we get

$$\begin{aligned} &D_P(t, t') \otimes \bigotimes_{k=1}^{n} \left(P(t_k, t'_k) \to D_Q(t_k, t'_k)\right)^* \\ &\quad = b \otimes \bigotimes_{k=1}^{n} \left(P(t_k, t'_k) \to a_k\right)^* \leq \left|Q \Rightarrow (t \approx t')\right|_{\Gamma}^{\mathcal{R}_\vee} = D_Q(t, t'). \end{aligned}$$

□

Example 6. The following are examples of $\mathbf{L}^*$ for which we have a complete fuzzy Horn logic.

(a) Take a finite Heyting algebra $\mathbf{L}$. We can define * by $a^* = a$ for all $a \in L$. Trivially, * is an implicational truth stresser. (29) is satisfied since $\otimes = \wedge$.
(b) For any finite residuated lattice, let * be defined by (9). In this case, (29) holds since $1 \otimes a = 1 \wedge a = a$, and $0 \otimes a = 0 \wedge a = 0$. The usage of globalization as a truth stresser has an important influence on the deduction rule (Mon). If $P(t_i, t'_i) \nleq a_i$ for some $i \in I$, then the resulting formula $Q \Rightarrow (t \approx t')$ is inferred in degree 0 (not interesting). On the other hand, when $P(t_i, t'_i) \leq a_i$ for all $i \in I$, $Q \Rightarrow (t \approx t')$ is inferred in degree b. To sum up, for * being the globalization, (Mon) simplifies to

$$\text{(BMon)}: \frac{\left\{\langle Q \Rightarrow (t_i \approx t'_i), a_i\rangle;\ i = 1, \ldots, n\right\}, \langle P \Rightarrow (t \approx t'), b\rangle}{\langle Q \Rightarrow (t \approx t'), b\rangle},$$

where $P, Q \in \mathcal{P}$, $\mathrm{Supp}(P) = \{\langle t_i, t_i' \rangle \mid i = 1, \ldots, n\}$, and $P(t_i, t_i') \leq a_i$ for all $i = 1, \ldots, n$.

(c) There are other examples of Horn truth stressers. Suppose $\mathbf{L}$ is a finite BL-chain. Prelinearity of $\mathbf{L}$ implies $a \otimes \bigwedge_{i \in I} b_i = \bigwedge_{i \in I}(a \otimes b_i)$. Due to Theorem 1, we can define * on $\mathbf{L}$ by (20), i.e. * sends every element $a \in L$ to the greatest idempotent of $\mathbf{L}$ which is less or equal to a. (29) is a consequence of divisibility.

* * *

The truth degree $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ depends on both * and $\rightarrow$. Using $\rightarrow$ is clear (we deal with implications). We saw that the use of * naturally unifies two possible meanings of $P \Rightarrow (t \approx t')$. Clearly, if * is defined by (9) then $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ does not depend on $\rightarrow$. The effect of $\rightarrow$ is completely displaced by *. Surprisingly, an analogy applies also to general implicational truth stresser satisfying (29).

Since $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ equals $a^* \rightarrow \|t \approx t'\|_{\mathbf{M},v}$ for

$$a = \bigwedge_{s,s' \in T(X)} \left(P(s, s') \rightarrow \|s \approx s'\|_{\mathbf{M},v}\right),$$

we are interested in truth degrees of $a^* \rightarrow b$. For $a^* \leq b$ we have $a^* \rightarrow b = 1$. When $a^* \not\leq b$, $a^* \rightarrow b$ is the greatest element of $\{c \mid a^* \otimes c \leq b\}$. Due to (29), $a^* \rightarrow b$ is the greatest element of $\{c \mid a^* \wedge c \leq b\}$. That is, $a^* \rightarrow b$ is the *relative pseudo-complement* of a^* to b. For instance, when $\mathbf{L}$ is a chain, then $a^* \rightarrow b = b$ for $a^* > b$.

Note that the truth degree $a = \bigwedge_{s,s' \in T(X)} \left(P(s, s') \rightarrow \|s \approx s'\|_{\mathbf{M},v}\right)$ itself is defined using $\rightarrow$. However, the following theorem shows that $a^* = \|P\|_{\mathbf{M},v}$ is not influenced by the definition of $\rightarrow$. As a consequence, it immediately follows that the truth degree $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ is (for given $\mathbf{M}$, v) fully determined by the lattice structure of $\mathbf{L}$ and the implicational truth stresser * satisfying (29).

Theorem 15. *Let* $\mathbf{L}_1 = \langle L, \vee, \wedge, \otimes_1, \rightarrow_1, 0, 1\rangle$, $\mathbf{L}_2 = \langle L, \vee, \wedge, \otimes_2, \rightarrow_2, 0, 1\rangle$ *be complete residuated lattices. Let* * *be an implicational truth stresser satisfying* (29) *for both* $\mathbf{L}_1$ *and* $\mathbf{L}_2$. *Then* $(a \rightarrow_1 b)^* = (a \rightarrow_2 b)^*$ *for all* $a, b \in L$.

Proof. For every $a \in L$, let $H(a) = \{c^* \mid c \in L \text{ and } c^* \leq a\}$. We claim that

$$H(a \rightarrow_1 b) = H(a \rightarrow_2 b) \tag{42}$$

for all $a, b \in L$. Indeed, for every $a, b, c \in L$ we have $c^* \in H(a \rightarrow_1 b)$ iff $c^* \leq a \rightarrow_1 b$ iff $a \otimes_1 c^* \leq b$ by adjointness, iff $a \otimes_2 c^* = a \wedge c^* = a \otimes_1 c^* \leq b$ by (29), iff $c^* \leq a \rightarrow_2 b$ by adjointness, iff $c^* \in H(a \rightarrow_2 b)$. Hence, (42) holds true for all $a, b \in L$. Clearly, $(a \rightarrow_1 b)^* \leq (a \rightarrow_1 b)$ due to (7). Thus, (42) yields $(a \rightarrow_1 b)^* \in H(a \rightarrow_1 b) = H(a \rightarrow_2 b)$. That is, $(a \rightarrow_1 b)^* \leq a \rightarrow_2 b$. Analogously, we have $(a \rightarrow_2 b)^* \leq a \rightarrow_1 b$. Now using monotony of * together with (16) we have $(a \rightarrow_1 b)^* = (a \rightarrow_1 b)^{**} \leq (a \rightarrow_2 b)^*$ and $(a \rightarrow_2 b)^* = (a \rightarrow_2 b)^{**} \leq (a \rightarrow_1 b)^*$. Hence, $(a \rightarrow_1 b)^* = (a \rightarrow_2 b)^*$ for every $a, b \in L$. □

Now an easy inspection of (10) and (11) shows that $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ is not influenced by the definition of $\rightarrow$. This is an immediate consequence of previous observations. A summary follows.

Corollary 2. *Let* $\mathbf{L}_1 = \langle L, \vee, \wedge, \otimes_1, \rightarrow_1, 0, 1\rangle$, $\mathbf{L}_2 = \langle L, \vee, \wedge, \otimes_2, \rightarrow_2, 0, 1\rangle$ *be complete residuated lattices. Let* * *be an implicational truth stresser satisfying* (29) *for both* $\mathbf{L}_1$ *and* $\mathbf{L}_2$. *Let* $\mathbf{M}$ *be an* $\mathbf{L}_i$*-algebra* (*for both* $i = 1, 2$) *and* v *be a valuation on* $\mathbf{M}$. *Then,*

$$\|P \Rightarrow (t \approx t')\|^{\mathbf{L}_1^*}_{\mathbf{M},v} = \|P \Rightarrow (t \approx t')\|^{\mathbf{L}_2^*}_{\mathbf{M},v}$$

for every $\mathcal{P}$*-implication* $P \Rightarrow (t \approx t')$. □

Remark 20. One should not be mislead. Even though the effect of residuum is eliminated as shown in Corollary 2, multiplication and residuum are still relevant. For instance, the notion of *compatibility of functions with* $\approx$ is based on multiplication. Generally, there are $\mathbf{L}_1$-algebras which are not $\mathbf{L}_2$-algebras and vice versa. Hence, the semantic consequence and graded provability are different for $\mathbf{L}_1^*$ and $\mathbf{L}_2^*$. Various definitions of truth stressers on the same structure of truth values yield various semantics of implications.

6. Implications with crisp premises

From now on, we will restrict ourselves to $\mathcal{P}$-Horn clauses with crisp premises, i.e. every $P \in \mathcal{P}$ is crisp. Our motivation is twofold. First, with crisp premises, implications are les fuzzy and correspond better to ordinary ones. Second, restriction to crisp premises might lead to completeness for a wider class of structures of truth values (which is indeed the case, as we will see).

Let $\mathcal{P} = \{P \in \mathbf{L}^{T(X)\times T(X)} \mid P \text{ is finite and crisp}\}$. In such a case, $h(P) = \{\langle h(t), h(t')\rangle \mid \langle t, t'\rangle \in P\}$ for every $P \in \mathcal{P}$ and arbitrary endomorphism h on $\mathbf{T}(X)$. Since every $P \in \mathcal{P}$ is crisp, we have either $P(s, s') = 1$, or $P(s, s') = 0$. Thus, we have

$$\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v} = \Big(\bigwedge\nolimits_{\langle s,s'\rangle \in P} \|s \approx s'\|_{\mathbf{M},v}\Big)^* \rightarrow \|t \approx t'\|_{\mathbf{M},v}.$$

For crisp premises, conditions (21) and (23) simplify to

$$\langle t, t'\rangle \in P \ \text{ implies } \ \Gamma_P(t, t') = 1, \tag{43}$$

$$\Big(\bigwedge\nolimits_{\langle s,s'\rangle \in P} \Gamma_Q(s, s')\Big)^* \leq S(\Gamma_P, \Gamma_Q). \tag{44}$$

Moreover, (44) is equivalent to

$$\Gamma_P(t, t') \otimes \Big(\bigwedge\nolimits_{\langle s,s'\rangle \in P} \Gamma_Q(s, s')\Big)^* \leq \Gamma_Q(t, t').$$

Analogously, (38) and (40) simplify to

$$\langle t, t'\rangle \in P \ \text{ implies } \ \Gamma^{\vdash}_P(t, t') = 1, \tag{45}$$

$$\Gamma^{\vdash}_P(t, t') \otimes \bigotimes\nolimits_{\langle s,s'\rangle \in P} \Gamma^{\vdash}_Q(s, s')^* \leq \Gamma^{\vdash}_Q(t, t'). \tag{46}$$

We are going to show that completeness in case of crisp premises can be established without (30) and without limitation to finite residuated lattices. First, observe that (30) follows from the following conditions:

$$a \to \bigvee_{i\in I} b_i = \bigvee_{i\in I}(a \to b_i) \quad \text{for every nonempty index set } I, \tag{47}$$

$$\left(\bigvee_{i\in I} b_i\right)^* = \bigvee_{i\in I} {b_i}^* \quad \text{for every index set } I. \tag{48}$$

Moreover, since $a \to \bigvee_{i\in I} b_i \geq \bigvee_{i\in I}(a \to b_i)$ and $\left(\bigvee_{i\in I} b_i\right)^* \geq \bigvee_{i\in I} {b_i}^*$ are always true, (47) and (48) are equivalent to $a \to \bigvee_{i\in I} b_i \leq \bigvee_{i\in I}(a \to b_i)$ and $\left(\bigvee_{i\in I} b_i\right)^* \leq \bigvee_{i\in I} {b_i}^*$, respectively.

Remark 21. Condition (48) does not hold for a general implicational truth stresser. For instance, for $\mathbf{L} = \langle\{0, a, b, 1\}, \vee, \wedge, \otimes, \to, 0, 1\rangle$ being a four-element Boolean algebra (i.e., $a, b \in L$ are incomparable), we define an implicational truth stresser * by (9). We have $a \vee b = 1$, i.e. $(a \vee b)^* = 1$. On the other hand $a^* = b^* = 0$, thus $a^* \vee b^* = 0$.

Consider the $\mathbf{L}^*$-deductive system $\mathcal{R}$ as it has been introduced in Section 5. The rules of extensivity and monotony can be equivalently replaced by

$$(\text{CExt}): \left\langle P \Rightarrow (t \approx t'), 1\right\rangle \qquad \text{for every } \left\langle t, t'\right\rangle \in P,$$

$$(\text{CMon}): \frac{\left\{\left\langle Q \Rightarrow (t_i \approx t'_i), a_i\right\rangle;\ i = 1, \dots, n\right\},\ \left\langle P \Rightarrow (t \approx t'), b\right\rangle}{\left\langle Q \Rightarrow (t \approx t'), b \otimes a_1^* \otimes \cdots \otimes a_n^*\right\rangle},$$

where P, Q are crisp sets of premises, $P = \left\{\left\langle t_i, t'_i\right\rangle \mid i = 1, \dots, n\right\}$, $t, t', t_i, t'_i \in T(X)$, and $a_1, \dots, a_n, b \in L$. The key observation is that we do not need (30) in the full scope. Since the truth value of a in (30) is either 1 or 0, a moment's reflection shows that (30) simplifies to (48).

Theorem 16. *Suppose* Γ *is an* $\mathbf{L}$*-set of Horn clauses with crisp premises, and* * *is an implicational truth stresser satisfying* (29), (48). *Then* $\left|P \Rightarrow (t \approx t')\right|_\Gamma = \Gamma_P^{\vdash}(t, t')$ *for every* $P \in \mathcal{P}$ *and* $t, t' \in T(X)$.

Proof. "$\leq$": Follows from the proof of the "$\leq$"-part of Theorem 12.

"$\geq$": We will check only the critical part where (48) is used. The rest follows directly from the proof of the "$\geq$"-part of Theorem 12. We have to check that $\mathcal{D} = \{D_P \mid P \in \mathcal{P}\}$, where $D_P(t, t') = \left|P \Rightarrow (t \approx t')\right|_\Gamma$ satisfies (40) which is in our case simplified to (46).

Let $P = \left\{\left\langle t_k, t'_k\right\rangle \mid k = 1, \dots, n\right\}$. We can assume that $D_P(t, t') = \bigvee_{i\in I} b_i$, where for every $i \in I$ there is a proof $\delta_{i_1}, \dots, \delta_{i_{n_i}}, \left\langle P \Rightarrow (t \approx t'), b_i\right\rangle$. Moreover, let $D_Q(t_k, t'_k) = \bigvee_{j_k \in J_k} a_{k,j_k}$, $k = 1, \dots, n$, where for every $j_k \in J_k$, there is an $\mathbf{L}^*$-weighted proof $\delta_{j_k,1}, \dots, \delta_{j_k,n_{j_k}}, \left\langle Q \Rightarrow (t_k \approx t'_k), a_{k,j_k}\right\rangle$. Analogously as in Theorem 12, we can claim that the provability degree $\left|Q \Rightarrow (t \approx t')\right|_\Gamma$ is greater or equal to $b_i \otimes a_{1,j_1}^* \otimes \cdots \otimes a_{n,j_n}^*$ for all $i \in I$, $j_1 \in J_1, \dots, j_n \in J_n$. Now, using (48) we have

$$
\begin{aligned}
&D_P(t,t') \otimes \bigotimes\nolimits_{k=1}^{n} D_Q(t_k,t'_k)^* = \bigvee\nolimits_{i\in I} b_i \otimes \bigotimes\nolimits_{k=1}^{n} \bigl(\bigvee\nolimits_{j_k\in J_k} a_{k,j_k}\bigr)^* \\
&\quad\le \bigvee\nolimits_{i\in I} b_i \otimes \bigotimes\nolimits_{k=1}^{n} \bigvee\nolimits_{j_k\in J_k} a^*_{k,j_k} = \bigvee\nolimits_{i\in I} b_i \otimes \bigvee\nolimits_{j_1\in J_1,\dots,j_n\in J_n} \bigotimes\nolimits_{k=1}^{n} a^*_{k,j_k} \\
&\quad= \bigvee\nolimits_{\substack{i\in I\\ j_1\in J_1,\dots,j_n\in J_n}} \bigl(b_i \otimes a^*_{1,j_1} \otimes \cdots \otimes a^*_{n,j_n}\bigr) \le D_Q(t,t').
\end{aligned}
$$

which proves the simplified inequality (46). Hence, $\mathcal{D}$ satisfies (40). □

Summarizing previous observations, we get the following.

Corollary 3. *Let* $\mathbf{L}^*$ *be a complete residuated lattice equipped with an implicational truth stresser* * *satisfying* (29), (48). *For every* $\mathbf{L}$*-set* Γ *of Horn clauses with crisp premises we have*

$$
\bigl|P \Rightarrow (t \approx t')\bigr|_\Gamma = \Gamma_P^{\vdash}(t,t') = \Gamma_P^{\models}(t,t') = \bigl\|P \Rightarrow (t \approx t')\bigr\|_\Gamma
$$

for every $P \in \mathcal{P}$ *and* $t, t' \in T(X)$. □

As the next theorem shows, (30) can be replaced by (48) even in Theorem 9. Since Theorem 9 and Theorem 12 were the only ones where we used (30) in the proof, we see that in case of crisp premises, (48) equivalently replaces (30).

Theorem 17. *Let* $\mathbf{L}^*$ *be a complete residuated lattice with an implicational truth stresser* * *satisfying* (29) *and* (48). *Let* Γ *be an* $\mathbf{L}^*$*-implicational* $\mathcal{P}_{\mathrm{Fin}}$*-theory, where* $\mathcal{P}$ *is a proper family of crisp premises which is closed under finite restrictions. Then putting*

$$
\Sigma_P = \bigcup\nolimits_{P'\in \mathrm{Fin}(P)} \Gamma_{P'}, \qquad \textit{for } P \in \mathcal{P},
$$

Σ *is an* $\mathbf{L}^*$*-implicational* $\mathcal{P}$*-theory. Moreover,* Σ *is the least* $\mathbf{L}^*$*-implicational* $\mathcal{P}$*-theory with* $\Gamma = \Sigma_{\mathrm{Fin}}$.

Proof. The proof is analogous to that of Theorem 9 and therefore omitted. Let us note, that the crucial observation is that

$$
\begin{aligned}
&\bigotimes\nolimits_{i=1}^{k} \bigvee\nolimits_{Q'\in \mathrm{Fin}(Q)} \bigl(P'(t_i,t'_i) \to \Gamma_{Q'}(t_i,t'_i)\bigr)^* = \bigotimes\nolimits_{i=1}^{k} \bigvee\nolimits_{Q'\in \mathrm{Fin}(Q)} \Gamma_{Q'}(t_i,t'_i)^* \\
&\quad= \bigotimes\nolimits_{i=1}^{k} \bigl(\bigvee\nolimits_{Q'\in \mathrm{Fin}(Q)} \Gamma_{Q'}(t_i,t'_i)\bigr)^* \\
&\quad= \bigotimes\nolimits_{i=1}^{k} \bigl(P'(t_i,t'_i) \to \bigvee\nolimits_{Q'\in \mathrm{Fin}(Q)} \Gamma_{Q'}(t_i,t'_i)\bigr)^*,
\end{aligned}
$$

where $P, Q \in \mathcal{P}$, $P' \in \mathrm{Fin}(P)$, $\mathrm{Supp}\bigl(P'\bigr) = \bigl\{\langle t_i, t'_i\rangle \mid i = 1,\dots,k\bigr\}$, and $t, t' \in T(X)$ are arbitrary. The rest follows from the proof of Theorem 9. □

* * *

Now we will show some examples of implicational truth stressers satisfying (29) and (48). This will demonstrate that crispness of premises allows for a larger class of truth value structures.

Example 7. (a) Let $\mathbf{L}$ be any complete Heyting algebra (i.e. a complete residuated lattice, where $\otimes = \wedge$, the residuum is then a relative pseudo-complement). The implicational truth stresser * defined by $a^* = a$, for all $a \in L$, satisfies (29) and (48) trivially. In other words, for every complete Heyting algebra, we have a syntactico-semantically complete calculus for reasoning about Horn clauses with crisp premises.

(b) Let $\mathbf{L}$ be an arbitrary complete residuated lattice equipped with an implicational truth stresser * defined by (9). Evidently, * satisfies (29) trivially, because $0 \otimes a = 0 \wedge a$, and $1 \otimes a = 1 \wedge a$ for all $a \in L$. Furthermore, * satisfies (48) iff 1 (the greatest element of L) is $\vee$-irreducible, i.e. iff for every family $\{b_i < 1 \mid i \in I\}$ we have $\bigvee_{i \in I} b_i < 1$. Indeed, suppose 1 is $\vee$-irreducible. It suffices to check that $\left(\bigvee_{i \in I} b_i\right)^* = 1$ implies $\bigvee_{i \in I} b_i^* = 1$. But $\left(\bigvee_{i \in I} b_i\right)^* = 1$ yields $\bigvee_{i \in I} b_i = 1$, i.e. there is an index i_0 such that $b_{i_0} = 1$ since 1 is assumed to be $\vee$-irreducible. Hence, $\bigvee_{i \in I} b_i^* = 1$. Conversely, if 1 is not $\vee$-irreducible, then there exists a family $\{b_i < 1 \mid i \in I\}$ such that $\bigvee_{i \in I} b_i = 1$. As a consequence, we have $1 = \left(\bigvee_{i \in I} b_i\right)^* \not\leq \bigvee_{i \in I} b_i^* = \bigvee_{i \in I} 0 = 0$.

Unlike the case of fuzzy equational logic [5], there are still structures of truth values for which we do not have completeness even for the globalization taken as the truth stresser. We are going to show that each complete residuated lattice can be slightly modified in such a way that for the modified $\mathbf{L}'$, the globalization satisfies (29) and (48), i.e. conditions which guarantee completeness in case of crisp premises. The modified $\mathbf{L}'$ results from $\mathbf{L}$ by adding new top element. Clearly, this element is $\vee$-irreducible (supremally irreducible). There are two aspects here. First, an epistemic one: The top element of a structure of truth values represents full truth, while the other elements represent partial truths. Therefore, $\vee$-irreducibility of 1 then means that full truth cannot be approximated by (better and better) partial truths, which might seem appropriate. Second, a technical one (however, related to the first one): The degree of provability of a formula φ equals 1 (in Pavelka's style) iff there is a proof $\ldots, \langle\varphi, 1\rangle$, i.e. a proof with degree 1.

In what follows, we use the well-known construction of ordinal sum (see [26–28] for $L = [0, 1]$ and [21] for BL-algebras). Namely, we add a new top element to $\mathbf{L}$ by ordinally adding the two-element Boolean algebra $\mathbf{2}$ to $\mathbf{L}$, i.e. we get $\mathbf{L}' = \mathbf{L} \oplus \mathbf{2}$. For the sake of completeness, we present a definition of ordinal sum of finitely many residuated lattices.

Definition 15. *Suppose $\langle I, \leq\rangle$ is a finite chain with the least element* 0 *and the greatest element* 1. *For $i, j \in I$ put $i \prec j$ iff j covers i (i.e. $i \leq j$ and $i \leq k \leq j$ implies $i = k$ or $j = k$). Let $\{\mathbf{L}_i \mid i \in I\}$ be a family of complete residuated lattices $\mathbf{L}_i = \langle L_i, \vee_i, \wedge_i, \otimes_i, \rightarrow_i, 0_i, 1_i\rangle$. Furthermore, let us assume that for every $i, j \in I$, $i \prec j$, we have $1_i = 0_j$ and $L_i \cap L_j = \{0_j\}$, and for every $i, j \in I$ such that $i \neq j$, $i \not\prec j$, $j \not\prec i$ we have $L_i \cap L_j = \emptyset$.*

*An **ordinal sum** $\bigoplus_{i \in I} \mathbf{L}_i$* of the family $\{\mathbf{L}_i \mid i \in I\}$ *is a complete residuated lattice $\mathbf{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$, where $L = \bigcup_{i \in I} L_i$, $0 = 0_0$, $1 = 1_1$ and $a \leq b$ iff either $a, b \in L_i$, $a \leq_i b$, or $a \in L_i$, $b \in L_j$ and $i < j$. The lattice operations*

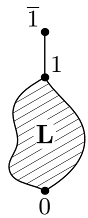

Fig. 1. Complete residuated lattice with new top element

$\vee, \wedge$ *are naturally induced by* $\leq$*, and* $\otimes, \rightarrow$ *are defined by*

$$a \otimes b = \begin{cases} a \otimes_i b & \text{if } a, b \in L_i, \\ a \wedge b & \text{otherwise,} \end{cases}$$

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{if } a \in L_j - L_i,\ b \in L_i - L_j,\ i < j, \\ a \rightarrow_i b & \text{if } a \not\leq b,\ a, b \in L_i. \end{cases}$$

Remark 22. An ordinal sum of $\{\mathbf{L}_i \mid i \in I\}$ is indeed a complete residuated lattice [21] (proof in [21] uses BL-algebras but it applies to all residuated lattices).

In particular, for an ordinal sum $\mathbf{L}' = \mathbf{L} \oplus \mathbf{2}$, i.e. $I = \{0, 1\}$, where $\mathbf{L}_0 = \mathbf{L}$, $\mathbf{L}_1 = \mathbf{2} = \langle\{1, \overline{1}\}, \vee, \wedge, \otimes, \rightarrow, 1, \overline{1}\rangle$, and 1 is the greatest element of $\mathbf{L}$, we have

$$a \otimes^{\mathbf{L}'} b = \begin{cases} a & \text{for } b = \overline{1}, \\ b & \text{for } a = \overline{1}, \\ a \otimes^{\mathbf{L}} b & \text{otherwise.} \end{cases} \qquad a \rightarrow^{\mathbf{L}'} b = \begin{cases} \overline{1} & \text{for } a \leq^{\mathbf{L}'} b, \\ b & \text{for } a = \overline{1}, \\ a \rightarrow^{\mathbf{L}} b & \text{otherwise.} \end{cases}$$

Constructing $\mathbf{L} \oplus \mathbf{2}$ corresponds to adding a new top element to $\mathbf{L}$, see Fig. 1. That is, we have

Corollary 4. *Globalization* (9) *defined on* $\mathbf{L} \oplus \mathbf{2}$ *is an implicational truth stresser satisfying* (29) *and* (48). □

This gives a syntactico-semantically complete calculus for reasoning about Horn clauses with crisp premises. In this case, if $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v} = 1$, then it follows that either for some $\langle s, s'\rangle \in P$ we have $\|s \approx s'\|_{\mathbf{M},v} \neq 1$, or $\|t \approx t'\|_{\mathbf{M},v} = 1$. Thus, the notion of a validity in degree 1 is very similar to that of bivalent case. Nevertheless, we are still capable to prove partially true statements from a given $\mathbf{L}$-set of Horn clauses with crisp premises.

Complete residuated lattices with implicational truth stressers satisfying (29) and (48) can be ordinally added yielding new structures with nontrivial implicational truth stressers satisfying (29) and (48). This is justified by the next lemma.

Lemma 5. *Let* $\langle I, \leq\rangle$ *be a finite chain with the least element* 0 *and the greatest element* 1*. Let* $\{\mathbf{L}_i^{*_i} \mid i \in I\}$ *be a family of complete residuated lattices* $\mathbf{L}_i = \langle L_i, \vee_i, \wedge_i, \otimes_i, \rightarrow_i, 0_i, 1_i\rangle$ *with implicational truth stressers* ^{*_i} *satisfying* (29) *and* (48)*. Suppose* $\{\mathbf{L}_i \mid i \in I\}$ *can be ordinally added. Then the operation* * *on* $\bigcup_{i \in I} L_i$ *defined by* $a^* = a^{*_i}$ *for* $a \in L_i$ *is an implicational truth stresser on* $\bigoplus_{i \in I} \mathbf{L}_i$ *satisfying* (29) *and* (48).

Proof. Clearly, conditions (6), (7), (16), and (17) follow directly from the definition. Denote the operations on $\bigoplus_{i\in I}\mathbf{L}_i$ by $\wedge, \vee, \otimes, \rightarrow, 0, 1$.

(8): We consider three separate cases. If $a \leq b$ then clearly $a^* \leq b^*$, i.e. $(a \rightarrow b)^* = 1 = a^* \rightarrow b^*$. If $a \rightarrow b = b$, then $(a \rightarrow b)^* = b^* \leq a^* \rightarrow b^*$. Finally, when $a \not\leq b$ and $a \rightarrow b = a \rightarrow_i b$, i.e. $a, b \in L_i$, then we have $(a \rightarrow b)^* = (a \rightarrow_i b)^* = (a \rightarrow_i b)^{*_i} \leq a^{*_i} \rightarrow_i b^{*_i} \leq a^* \rightarrow b^*$.

(18): Take $\{a_j \mid j \in J\}$.
Let $k = \min\{i \in I \mid \text{there is } j \in J \text{ such that } a_j \in L_i\}$. For $J' = \{j \mid a_j \in L_k\}$, we have $\bigwedge_{j\in J} a_j = \bigwedge_{j\in J'} a_j$. Thus, $\bigwedge_{j\in J} a_j^* = \bigwedge_{j\in J'} a_j^{*_k} = \left(\bigwedge_{j\in J'} a_j\right)^{*_k} = \left(\bigwedge_{j\in J} a_j\right)^*$.

(29): If $a, b^* \in L_i$, the claim is trivial. If $b^* \in L_i$ and $a \notin L_i$, we have $a \otimes b^* = a \wedge b^*$ by definition.

(48): For $\{b_j \mid j \in J\}$, take $l = \max\{i \in I \mid \text{there is } j \in J \text{ such that } b_j \in L_i\}$, put $J' = \{j \mid b_j \in L_l\}$, and observe that $\left(\bigvee_{j\in J} b_j\right)^* = \left(\bigvee_{j\in J'} b_j\right)^{*_l} = \bigvee_{j\in J'} b_j^{*_l} = \bigvee_{j\in J} b_j^*$. □

Remark 23. (1) The construction of Lemma 5 applied to $\mathbf{L}_i \oplus \mathbf{2}$ from Corollary 4 is illustrated in Fig. 2.

(2) The idea of having crisp premises can be naturally generalized. Suppose $\mathbf{L}^*$ is a complete residuated lattice with implicational truth stresser * satisfying (29). Take a finite chain $K \subseteq L$ such that $\{0, 1\} \subseteq K$ and consider $\mathcal{P} \subseteq \mathbf{L}^{T(X)\times T(X)}$ such that $P \in \mathcal{P}$ iff $\{P(s, s') \mid s, s' \in T(X)\} \subseteq K$ and P is finite. $\mathcal{P}$ is a proper family of finite premises (for $K = \{0, 1\}$, we obtain exactly the family of all finite crisp premises). An inspection of the proof of Theorem 12 shows that it is sufficient to consider (30) for any b_i's ($I \neq \emptyset$) and all $a \in K$ since K consists of all truth values used in premises of $\mathcal{P}$-Horn clauses under consideration. For instance, $\bigoplus_{i\in I}(\mathbf{L}_i \oplus \mathbf{2})$ (see Fig. 2) satisfies (30) for any $a \in K$, where $K = \{0_0\} \cup \{\overline{1}_i \mid i \in I\}$ (easy to check using Corollary 4 and Lemma 5).

7. Fuzzy equational logic and equation implications

We are going to show that the fuzzy equational logic [5] is a special case of our approach. Putting $\mathcal{P} = \{\emptyset\}$, $\mathcal{P}$-implications are exactly formulas of the form $\emptyset \Rightarrow (t \approx t')$, i.e. $\mathcal{P}$-implications with empty premises. It is easily seen that

$$\left\|\emptyset \Rightarrow (t \approx t')\right\|_{\mathbf{M},v} = \|\emptyset\|_{\mathbf{M},v} \rightarrow \left\|t \approx t'\right\|_{\mathbf{M},v} = \left\|t \approx t'\right\|_{\mathbf{M},v}.$$

Hence, an identity $t \approx t'$ can be thought of as the implication $\emptyset \Rightarrow (t \approx t')$. Clearly, the truth degree $\left\|\emptyset \Rightarrow (t \approx t')\right\|_{\mathbf{M},v}$ does not depend on *.

In [5], the algebraic counterpart of a semantically closed $\mathbf{L}$-set Σ of identities is a fully invariant closure $\theta_{\mathrm{FI}}(\Sigma)$ of the corresponding $\mathbf{L}$-relation on $T(X)$. Now, in our approach, we can consider $\mathcal{S}_\Sigma$, which is a one-element system $\mathcal{S}_\Sigma = \{\Sigma_\emptyset\}$, where $\Sigma_\emptyset = \Sigma$. Hence, due to Corollary 1 and Theorem 8, the semantic closure of $\mathcal{S}_\Sigma$ is a one-element system $\mathcal{S}_{\Sigma^\models} = \{\Sigma_\emptyset^\models\}$, where $\Sigma_\emptyset^\models$ is the least congruence on $\mathbf{T}(X)$ which contains Σ and $\mathcal{S}_{\Sigma^\models}$ satisfies (21)–(23). (21), i.e. $\emptyset \subseteq \Sigma_\emptyset^\models$, is a

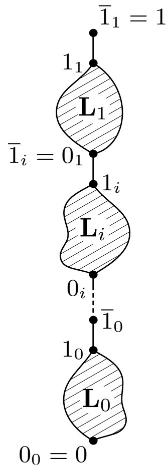

Fig. 2. Ordinal sum of $\{\mathbf{L}_i \oplus \mathbf{2} \mid i \in I\}$

trivial property. (23) holds trivially as well since $S(\Sigma^{\models}_{\emptyset}, \Sigma^{\models}_{\emptyset}) = 1$. Thus, the only one nontrivial property of $\mathcal{S}_{\Sigma^{\models}}$ is (22), i.e. $\Sigma^{\models}_{\emptyset}(t, t') \leq \Sigma^{\models}_{\emptyset}(h(t), h(t'))$. But this is nothing but the full-invariance of $\Sigma^{\models}_{\emptyset}$. That is, $\Sigma^{\models}_{\emptyset}$ is the least fully invariant congruence on $\mathbf{T}(X)$ which contains Σ. Hence, up to a slightly different formalism, the representation of semantic consequence from [5] and Section 3 coincides. As a consequence,

$$\left\|t \approx t'\right\|_{\Sigma} = \left\|t \approx t'\right\|_{\mathbf{T}(X)/\Sigma^{\models}_{\emptyset}}.$$

As to the notion of provability, [5] uses the following *equational deduction rules:*

$$(\text{ERef})\colon \langle t \approx t, 1\rangle\,, (\text{ESym})\colon \frac{\langle t \approx t', a\rangle}{\langle t' \approx t, a\rangle}, \quad (\text{ETra})\colon \frac{\langle t \approx t', a\rangle, \langle t' \approx t'', b\rangle}{\langle t \approx t'', a \otimes b\rangle},$$

$$(\text{ERep})\colon \frac{\langle t \approx t', a\rangle}{\langle s \approx s', a\rangle}, (\text{ESub})\colon \frac{\langle t \approx t', a\rangle}{\langle t(x/r) \approx t'(x/r), a\rangle},$$

where $t, t', t'', s \in T(X)$, $a, b \in L$ are arbitrary, $x \in X$, term s has an occurrence of t as a subterm and s' is a term resulting from s by substitution of one occurrence of t by t'.

Evidently, (ERef)–(ESub) are instances of (Ref)–(Sub) for $P = \emptyset$. Furthermore, for $\mathcal{P} = \{\emptyset\}$, both rules (Ext) and (Mon) can be omitted. Namely, (Ext) yields $\langle t \approx t', 0\rangle$ and (Mon) derives $\langle t \approx t', b\rangle$ from $\langle t \approx t', b\rangle$. It is thus easily seen that the degree of provability of $t \approx t'$ from an $\mathbf{L}$-set of identities as defined in [5] coincides with the degree of provability of $\emptyset \Rightarrow (t \approx t')$ from the $\mathbf{L}$-set Σ' of $\mathcal{P}$-implications where $\Sigma'(\emptyset \Rightarrow (t, t')) = \Sigma(t, t')$.

Similarly as for semantic closure, $\mathcal{P} = \{\emptyset\}$ leads to a simplified notion of a deductive closure $\Sigma^{\vdash}$ which in fact coincides with that of [5] (we omit details).

Now it is essential to observe that the proof of Theorem 12 does not involve (30) any longer. Hence, Theorem 12 yields that $\left|t \approx t'\right|_{\Sigma} = \Sigma^{\vdash}_{\emptyset}(t, t')$ for every $\mathbf{L}$-set Σ of $\{\emptyset\}$-implications (i.e. identities). The latter relationship is true for every complete residuated lattice $\mathbf{L}$ since for every $\mathbf{L}$ we can define an implicational truth stresser satisfying (29), e.g. by (9). Finally, Theorem 11 yields

$$\left\|t \approx t'\right\|_{\Sigma} = \left\|t \approx t'\right\|_{\mathbf{T}(X)/\Sigma^{\models}_{\emptyset}} = \Sigma^{\vdash}_{\emptyset}(t, t') = \left|t \approx t'\right|_{\Sigma}.$$

which is just the same *completeness theorem* as presented in [5].

* * *

In his paper [32], Selman has proven a completeness theorem for the so-called *equation implications.* An equation implication is simply a formula either of the form $(s \approx s') \Rightarrow (t \approx t')$ or $t \approx t'$, where s, s', t, t' are arbitrary terms. In our terminology, these formulas are $\mathcal{P}$-implications determined by a proper family of premises $\mathcal{P} = \{P \mid |\mathrm{Supp}(P)| \leq 1\}$, and $\mathbf{L} = \mathbf{2}$. Equation implications are very simple formulas. From this point of view, we may ask how does the class of complete residuated lattices narrow down if we inspect the completeness considering (weighted) equation implications instead of (weighted) identities.

We can introduce a slightly more general concept. For every $n \in \mathbb{N}$ let $\mathcal{P}_n$ denote a proper family of premises defined by

$$\mathcal{P}_n = \left\{P \in \mathbf{L}^{T(X) \times T(X)} \mid |\mathrm{Supp}(P)| \leq n\right\}.$$

Clearly, $\mathcal{P}_1$ is a proper family of (weighted) equation implications. Note that $\mathcal{P}_n$ is a proper family for every $n \in \mathbb{N}$ since the support of an endomorphic image $h(P)$ of $P \in \mathcal{P}$ can have at most n elements.

But unlike the equational case or the case with crisp premises, an inspection on theory in previous sections shows that $\mathcal{P}_n$ simplify neither the properties of * sufficient to express a semantic consequence nor the deduction rules. In certain sense, the general rule (Mon) simplifies so that it is not necessary to formalize it by infinitely many rules, but only by $n + 1$ rules. For instance, in case of equational implications, it is sufficient to consider

$$\frac{\left\langle t \approx t', b\right\rangle}{\langle r \approx r', a'\rangle \Rightarrow \langle t \approx t', b\rangle}$$

and

$$\frac{\left\langle r \approx r', a'\right\rangle \Rightarrow \left\langle s \approx s', a\right\rangle, \left\langle s \approx s', b'\right\rangle \Rightarrow \left\langle t \approx t', b\right\rangle}{\langle r \approx r', a'\rangle \Rightarrow \langle t \approx t', b \otimes (b' \to a)^{*}\rangle}$$

instead of the general (Mon). However, we have to keep (30) to be able to prove completeness.

Acknowledgements. Supported by grant no. B1137301 of the Grant Agency of the Academy of Sciences of Czech Republic and by instititutional support, research plan MSM 6198959214.

References

1. Baaz, M.: Infinite-valued Gödel logics with 0-1 projections and relativizations. GÖDEL '96 – Logical Foundations of Mathematics, Computer Sciences and Physics, Lecture Notes in Logic vol. 6, Springer-Verlag 1996, pp. 23–33
2. Banaschewski. B., Herrlich, H.: Subcategories defined by implications. Houston J. Math. **2**, 149–171 (1976)
3. Bělohlávek, R.: Fuzzy closure operators. J. Math. Anal. Appl. **262**, 473–489 (2001)
4. Bělohlávek, R.: Fuzzy closure operators II. Soft Computing **7** (1), 53–64 (2002)
5. Bělohlávek, R.: Fuzzy equational logic. Arch. Math. Log. **41**, 83–90 (2002)
6. Bělohlávek, R.: Fuzzy Relational Systems: Foundations and Principles. Kluwer Academic/Plenum Publishers, New York, 2002
7. Bělohlávek, R.: Birkhoff variety theorem and fuzzy logic. Arch. Math. Log. **42**, 781–790 (2003)
8. Bělohlávek, R., Funioková, T., Vychodil, V.: Fuzzy closure operators with truth stressers (submitted)
9. Bělohlávek, R., Vychodil, V.: Algebras with fuzzy equalities (submitted)
10. Bělohlávek, R., Vychodil, V.: Fuzzy Horn logic II: implicationally defined classes (submitted)
11. Birkhoff, G.: On the structure of abstract algebras. Proc. Camb. Philos. Soc. **31**, 433–454 (1935)
12. Bloom, S.L., Wright, J.B.: Finitary quasi varieties. J. Pure and Appl. Algebra **25**, 121–154 (1982)
13. Burris, S., Sankappanavar, H. P.: A Course in Universal Algebra. Springer-Verlag, New York, 1981
14. Chang, C.C., Keisler, H.J.: Continuous Model Theory. Princeton University Press, Princeton, NJ, 1966
15. Chang, C.C., Keisler, H.J.: Model Theory. North–Holland, Amsterdam, 1973
16. Feynman, R.P., Leighton, R.B., Sands, M.: The Feynman Lectures on Physics. Addison Wesley, Reading, MA, 1963
17. Fujiwara, T.: On the construction of the least universal Horn class containing a given class. Osaka J. Math. **8**, 425–436 (1971)
18. Goguen, J.A.: L-fuzzy sets. J. Math. Anal. Appl. **18**, 145–174 (1967)
19. Goguen, J.A.: The logic of inexact concepts. Synthese **18**, 325–373 (1968–9)
20. Gottwald, S.: A Treatise on Many-Valued Logics. Research Studies Press, Baldock, Hertfordshire, England, 2001
21. Hájek, P.: Basic fuzzy logic and BL-algebras. Soft Computing **2**, 124–128 (1998)
22. Hájek, P.: Metamathematics of Fuzzy Logic. Kluwer, Dordrecht, 1998
23. Hájek, P.: On very true. Fuzzy Sets and Systems **124**, 329–333 (2001)
24. Hájek, P.: Monadic fuzzy predicate logic. Preprint
25. Höhle, U.: On the fundamentals of fuzzy set theory. J. Math. Anal. Appl. **201**, 786–826 (1996)
26. Klement, E.P., Mesiar, R., Pap, E.: Triangular Norms. Kluwer, Dordrecht, 2000
27. Ling, C.H.: Representation of associative functions. Publ. Math. Debrecen **12**, 189–212 (1965)
28. Mostert, P.S., Shields, A.L.: On the structure of semigroups on a compact manifold with boundary. Ann. Math. **65**, 117–143 (1957)
29. Novák, V., Perfilieva, I., Močkoř, J.: Mathematical Principles of Fuzzy Logic. Kluwer, Boston, 1999
30. Palasińska, K., Pigozzi, G.: Gentzen-style axiomatization in equational logic. Algebra Universalis **34**, 128–143 (1995)

31. Pavelka, J.: On fuzzy logic I, II, III. Z. Math. Logik Grundlagen Math. **25**, 45–52, 119–134, 447–464 (1979)
32. Selman, A.: Completeness of calculii for axiomatically defined classes of algebras. Algebra Universalis **2**, 20–32 (1972)
33. Takeuti, G., Titani, S.: Globalization of intuitionistic set theory. Annals of Pure and Applied Logic **33**, 195–211 (1987)
34. Vojtáš, P.: Fuzzy logic programming. Fuzzy Sets and Systems **124** (3), 361–370 (2001)
35. Vychodil, V.: Direct limits and reduced products of algebras with fuzzy equalities (submitted)
36. Vychodil, V.: Cut and weakening in fuzzy Horn logic (submitted)
37. Vychodil, V.: Extended fuzzy equational logic (submitted)
38. Wechler, W.: Universal Algebra for Computer Scientists. Springer-Verlag, Berlin Heidelberg, 1992