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Fuzzy Horn logic II

implicationally defined classes

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Abstract. The paper studies closure properties of classes of fuzzy structures defined by fuzzy implicational theories, i.e. theories whose formulas are implications between fuzzy identities. We present generalizations of results from the bivalent case. Namely, we characterize model classes of general implicational theories, finitary implicational theories, and Horn theories by means of closedness under suitable algebraic constructions.

1. Introduction and preliminaries

The present paper is a follow up to [10] where we introduced fuzzy Horn logic and presented its general completeness theorem plus some completeness theorems for some important special cases. The main aim of this paper is to study implicationally defined model classes. In particular, we characterize the model classes by means of closedness under suitable algebraic constructions. Our results generalize analogous results for the ordinary case which are well-known in universal algebra. Note that a characterization of equationally defined classes generalizing the well-known Birkhoff variety theorem was presented in [8]. As equationally defined classes are special cases of implicationally defined classes (equations correspond to implications with empty premises), the present paper shifts the results of [8] to a more general setting.

In the rest of this section we briefly recall the necessary notions. For further details, we refer mainly to [10] (implications between identities), [9, 24] (constructions related to algebras with fuzzy equalities), [7] (general first-order fuzzy structures), [15, 17, 20] (fuzzy logic).

We use complete residuated lattices as the structures of truth degrees. A (*complete*) *residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ such that (i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, (ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid,

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(iii) $\langle\otimes, \to\rangle$ is an *adjoint pair,* i.e. $a \otimes b \leq c$ iff $a \leq b \to c$ is valid for each $a, b, c \in L$ (so called *adjointness property*).

An **L**-*set* A (or fuzzy set with truth degrees in **L**) in a universe set U is a mapping $A : U \to L$, $A(u) \in L$ being interpreted as the truth value of "element u belongs to A". Let $\mathbf{L}^U$ denote the collection of all **L**-sets in universe U. A mapping $\emptyset_U : U \to L$ with $\emptyset_U(u) = 0$ ($u \in U$) is called an empty **L**-set in U. For every **L**-set $A : U \to L$, we define an ordinary set Supp(A) by $\mathrm{Supp}(A) = \{u \in U \mid A(u) > 0\}$. Supp($A$) is called the *support set of* A. **L**-set A is called *finite* if Supp(A) is finite. For every **L**-set $A : U \to L$ and $a \in L$, we define an ordinary set ${}^aA$ by ${}^aA = \{u \in U \mid A(u) \geq a\}$. ${}^aA$ is called an *a-cut of* A. **L**-sets can be represented by L-indexed systems of a-cuts [7]. For **L**-sets A and B, $S(A, B)$ defined by $S(A, B) = \bigwedge_{u \in U}\big(A(u) \to B(u)\big)$ is called a degree of subsethood of A in B. We write $A \subseteq B$ (A is a *subset* of B) iff $S(A, B) = 1$, i.e. iff for each $u \in U$, $A(u) \leq B(u)$. Basic operations with **L**-sets are defined componentwise using operations of **L**. A *binary* **L**-*relation* R (or binary fuzzy relation with truth degrees in **L**) on U is an **L**-set in the universe set $U \times U$, i.e. it is a mapping $R : U \times U \to L$. An **L**-*equivalence* (or similarity) E on U is a binary **L**-relation on U satisfying $E(u, u) = 1$ (reflexivity), $E(u, v) = E(v, u)$ (symmetry), $E(u, v) \otimes E(v, w) \leq E(u, w)$ (transitivity) for all $u, v, w \in U$. An **L**-equivalence on U for which $E(u, v) = 1$ implies $u = v$ is called an **L**-*equality*. Function $f : U^n \to U$ is said to be *compatible* with binary **L**-relation R on U if $R(u_1, v_1) \otimes \cdots \otimes R(u_n, v_m) \leq R\big(f(u_1, \ldots, u_n), f(v_1, \ldots, v_n)\big)$ for all $u_1, v_1 \ldots u_n, v_n \in U$. An **L**-set A is called *crisp* if $A(u) \in \{0, 1\}$ for each $u \in U$. As usual, we sometimes identify crisp **L**-sets with the corresponding ordinary sets.

A *type* is a collection F of function symbols, each with its arity. Given a complete residuated lattice **L**, the *language* of **L**-Horn logic consists of (at least denumerable) set X of variables, a type F, a binary predicate symbol $\approx$ standing for (fuzzy) equality, a set $\{\overline{a};\ a \in L\}$ of symbols of truth values (however, for the sake of convenience and since there is no danger of misunderstanding, we identify $\overline{a}$ with a), and symbols of logical connectives $\Rightarrow$ (implication), $\mathbin{\&}$ (conjunction) and $\bigwedge$ (generalized conjunction). The set $T(X)$ of all terms over F and X is defined as usual. Terms are denoted by $p, q, \ldots, t$, possibly with indices. The set of all variables occurring in t is denoted by var(t).

An *algebra with* **L**-*equality* (shortly an **L**-*algebra*) of type F is a triplet $\mathbf{M} = \big\langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\big\rangle$, where $\big\langle M, F^{\mathbf{M}}\big\rangle$ is an (ordinary) algebra of type F and $\approx^{\mathbf{M}}$ is an **L**-equality on M such that each $f^{\mathbf{M}} \in F^{\mathbf{M}}$ is compatible with $\approx^{\mathbf{M}}$.

A *subuniverse* of **M** is any subset $N \subseteq M$ which is closed under all operations of **M**. **L**-algebra $\mathbf{N} = \big\langle N, \approx^{\mathbf{N}}, F^{\mathbf{N}}\big\rangle$ is called a *subalgebra* of **M** (denoted $\mathbf{N} \in \mathrm{Sub}(\mathbf{M})$), if $\big\langle N, F^{\mathbf{N}}\big\rangle$ is a subalgebra of $\big\langle M, F^{\mathbf{M}}\big\rangle$ (as an ordinary algebra) and $\approx^{\mathbf{N}}$ is a restriction of $\approx^{\mathbf{M}}$ to N. The notion of an **L**-algebra generated by a set of elements is defined as in ordinary case. We denote $\big[N'\big]_{\mathbf{M}}$ the least subuniverse of **M** containing N'. If $\big[N'\big]_{\mathbf{M}} \neq \emptyset$, the corresponding **L**-algebra **N** with universe $\big[N'\big]_{\mathbf{M}}$ is called an **L**-algebra generated by N'. If $|N'| < \kappa$ for an infinite cardinal κ then **N** is called κ-*generated.* ω-generated **L**-algebra is called *finitely generated.*

Let $\mathbf{M}, \mathbf{N}$ be $\mathbf{L}$-algebras of the same type. A mapping $h: M \to N$ satisfying $\mathrm{a} \approx^{\mathbf{M}} \mathrm{b} \leq h(\mathrm{a}) \approx^{\mathbf{N}} h(\mathrm{b})$ is called an *≈-morphism.* An ≈-morphism $h: M \to N$ is called a *morphism* (*of* $\mathbf{L}$*-algebras*) if h is a morphism between ordinary algebras $\langle \mathbf{M}, F^{\mathbf{M}} \rangle$ and $\langle \mathbf{N}, F^{\mathbf{N}} \rangle$. A morphism $h: \mathbf{M} \to \mathbf{M}$ is called an endomorphism. A morphism $h: \mathbf{M} \to \mathbf{N}$ satisfying $\mathrm{a} \approx^{\mathbf{M}} \mathrm{b} = h(\mathrm{a}) \approx^{\mathbf{N}} h(\mathrm{b})$ $(\mathrm{a}, \mathrm{b} \in M)$ is called an *embedding.* Surjective embedding is called an *isomorphism.* For an isomorphism $h: \mathbf{M} \to \mathbf{N}$, $\mathbf{M}$ and $\mathbf{N}$ are called *isomorphic* ($\mathbf{M} \cong \mathbf{N}$). If $h: \mathbf{M} \to \mathbf{N}$ is surjective (*epimorphism*), then $\mathbf{N}$ is called an *image* of $\mathbf{M}$. For morphisms $h: \mathbf{M} \to \mathbf{M}'$ and $g: \mathbf{M}' \to \mathbf{M}''$ we consider a composed morphism $(h \circ g): \mathbf{M} \to \mathbf{M}''$ as a composed mapping.

An $\mathbf{L}$-relation θ on M such that (i) θ is an $\mathbf{L}$-equivalence on M, (ii) $\approx^{\mathbf{M}} \subseteq \theta$, (iii) all functions $f^{\mathbf{M}} \in F^{\mathbf{M}}$ are compatible with θ, is called a *congruence on* $\mathbf{M}$. Congruences on an $\mathbf{L}$-algebra $\mathbf{M}$ form a complete lattice [9] denoted by $\mathrm{Con}_{\mathbf{L}}(\mathbf{M})$. $\theta(R) \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M})$ denotes the congruence generated by $R: M \times M \to L$. For a congruence θ on an $\mathbf{L}$-algebra $\mathbf{M}$, an $\mathbf{L}$-algebra $\mathbf{M}/\theta = \langle M/\theta, \approx^{\mathbf{M}/\theta}, F^{\mathbf{M}/\theta} \rangle$, where (i) $\langle M/\theta, F^{\mathbf{M}/\theta} \rangle$ is an ordinary factor algebra of $\langle \mathbf{M}, F^{\mathbf{M}} \rangle$ modulo $\{\langle \mathrm{a}, \mathrm{b} \rangle \mid \theta(\mathrm{a}, \mathrm{b}) = 1\}$, (ii) $[\mathrm{a}]_\theta \approx^{\mathbf{M}/\theta} [\mathrm{b}]_\theta = \theta(\mathrm{a}, \mathrm{b})$ for all $\mathrm{a}, \mathrm{b} \in M$, is called a *factor* $\mathbf{L}$*-algebra of* $\mathbf{M}$ *modulo* θ. An epimorphism $h_\theta: M \to M/\theta$, where $h_\theta(\mathrm{a}) = [\mathrm{a}]_\theta$ $(\mathrm{a} \in M)$, is called a *natural morphism.* For a morphism $h: \mathbf{M} \to \mathbf{N}$ let $\theta_h \in \mathbf{L}$ be a congruence defined by $\theta_h(\mathrm{a}_1, \mathrm{a}_2) = h(\mathrm{a}_1) \approx^{N} h(\mathrm{a}_2)$ $(\mathrm{a}, \mathrm{b} \in M)$. θ_h is called a *kernel* of h. The above-introduced concepts obey the usual theorems on morphisms (see [7] for first-order fuzzy structures and [9] for $\mathbf{L}$-algebras in particular).

A *direct product* $\bigtimes_{i \in I} \mathbf{M}_i = \langle \bigtimes_{i \in I} M_i, \approx^{\bigtimes_{i \in I} \mathbf{M}_i}, F^{\bigtimes_{i \in I} \mathbf{M}_i} \rangle$ of $\mathbf{L}$-algebras $\mathbf{M}_i$ $(i \in I)$ is an $\mathbf{L}$-algebra such that $\langle \bigtimes_{i \in I} M_i, F^{\bigtimes_{i \in I} \mathbf{M}_i} \rangle$ is a direct product of ordinary algebras $\langle M_i, F^{\mathbf{M}_i} \rangle$ and $\approx^{\bigtimes_{i \in I} \mathbf{M}_i}$ is defined by $\mathrm{a} \approx^{\bigtimes_{i \in I} \mathbf{M}_i} \mathrm{b} = \bigwedge_{i \in I} \mathrm{a}(i) \approx^{\mathbf{M}_i} \mathrm{b}(i)$. For $I = \emptyset$, $\bigtimes_{i \in I} \mathbf{M}_i$ is a trivial (one-element) $\mathbf{L}$-algebra. For every $j \in I$ an epimorphism $\pi_j : \bigtimes_{i \in I} \mathbf{M}_i \to \mathbf{M}_j$, where $\pi_j(\mathrm{a}) = \mathrm{a}(j)$ $(\mathrm{a} \in M)$ is called a j-th *projection* of $\bigtimes_{i \in I} \mathbf{M}_i$. A *subdirect product* of $\mathbf{L}$-algebras $\mathbf{M}_i$ $(i \in I)$ is a sublagebra of the direct product of $\mathbf{M}_i$'s such that each π_j is a surjective mapping.

A partially ordered index set $\langle I, \leq \rangle$ is called *directed*, if $I \neq \emptyset$ and for every $i, j \in I$ there is $k \in I$ such that $i, j \leq k$. A family $\{\mathbf{M}_i \mid i \in I\}$ of $\mathbf{L}$-algebras, where $\langle I, \leq \rangle$ is a directed index set and $\mathbf{M}_i \in \mathrm{Sub}(\mathbf{M}_j)$ for $i \leq j$ is called a *directed family* of $\mathbf{L}$-algebras. A *direct union* $\bigcup_{i \in I} \mathbf{M}_i = \langle \bigcup_{i \in I} M_i, \approx^{\bigcup_{i \in I} \mathbf{M}_i}, F^{\bigcup_{i \in I} \mathbf{M}_i} \rangle$ of a directed family $\{\mathbf{M}_i \mid i \in I\}$ of $\mathbf{L}$-algebras is an $\mathbf{L}$-algebra, where $\langle \bigcup_{i \in I} M_i, F^{\bigcup_{i \in I} \mathbf{M}_i} \rangle$ is a direct union of ordinary algebras $\langle M_i, F^{\mathbf{M}_i} \rangle$ (that is, for $\mathrm{a}, \ldots, \mathrm{b} \in \bigcup_{i \in I} M_i$, $f^{\bigcup_{i \in I} \mathbf{M}_i}(\mathrm{a}, \ldots, \mathrm{b})$ is defined to be $f^{\mathbf{M}_i}(\mathrm{a}, \ldots, \mathrm{b})$ for i such that $\mathrm{a}, \ldots, \mathrm{b} \in M_i$) and for $\mathrm{a}, \mathrm{b} \in \bigcup_{i \in I} M_i$ with $\mathrm{a} \in M_i$ and $\mathrm{b} \in M_k$ we put $\mathrm{a} \approx^{\bigcup_{i \in I} \mathbf{M}_i} \mathrm{b} = \mathrm{a} \approx^{\mathbf{M}_k} \mathrm{b}$ for $k \geq i, j$. Every $\mathbf{L}$-algebra is isomorphic to a direct union of its finitely generated subalgebras [9].

A *weak direct family* of $\mathbf{L}$-algebras of type F consists of: (i) a directed index set $\langle I, \leq \rangle$; (ii) a family $\{\mathbf{M}_i \mid i \in I\}$ of pairwise disjoint $\mathbf{L}$-algebras of type F; (iii) a family $\{h_{ij}: \mathbf{M}_i \to \mathbf{M}_j \mid i \leq j\}$ of morphisms, where $h_{ii} = \mathrm{id}_{\mathbf{M}_i}$ $(i \in I)$ and $h_{ik} = h_{ij} \circ h_{jk}$ $(i \leq j \leq k)$. A weak direct family is called a *direct family* if for every $\mathrm{a} \in M_i$, $\mathrm{b} \in M_j$ there exists $k \in I$, $i, j \leq k$ such that for each $l \in I$, $k \leq l$ we have $h_{ik}(a) \approx^{\mathbf{M}_k} h_{jk}(b) = h_{il}(a) \approx^{\mathbf{M}_l} h_{jl}(b)$ (this condition is automatically satisfied if $\mathbf{L}$ is a finite residuated lattice). For every weak direct family of $\mathbf{L}$-algebras $\{\mathbf{M}_i \mid i \in I\}$ let θ_∞ denote an $\mathbf{L}$-equivalence on $\bigcup_{i\in I} M_i$ defined by $\theta_\infty(a, b) = \bigvee_{k\geq i,j} h_{ik}(a) \approx^{\mathbf{M}_k} h_{jk}(b)$ ($a \in M_i, b \in M_j$). Furthermore, an $\mathbf{L}$-algebra $\lim \mathbf{M}_i = \left\langle\left(\bigcup_{i\in I} M_i\right)/\theta_\infty, \approx^{\lim \mathbf{M}_i}, F^{\lim \mathbf{M}_i}\right\rangle$, where (i) $\left(\bigcup_{i\in I} M_i\right)/\theta_\infty$ is a factorization of $\bigcup_{i\in I} M_i$ by ${}^1\theta_\infty$, i.e. $\left(\bigcup_{i\in I} M_i\right)/\theta_\infty = \left\{[a]_{\theta_\infty} \mid a \in \bigcup_{i\in I} M_i\right\}$, $[a]_{\theta_\infty} = \left\{a' \mid \theta_\infty(a, a') = 1\right\}$; (ii) $f^{\lim \mathbf{M}_i}\left([a_1]_{\theta_\infty}, \ldots, [a_n]_{\theta_\infty}\right) = \left[f^{\mathbf{M}_k}(h_{i_1k}(a_1), \ldots, h_{i_nk}(a_n))\right]_{\theta_\infty}$ for every n-ary $f \in F$ and arbitrary $[a_1]_{\theta_\infty}, \ldots, [a_n]_{\theta_\infty} \in \left(\bigcup_{i\in I} M_i\right)/\theta_\infty$ such that $a_1 \in M_{i_1}, \ldots, a_n \in M_{i_n}$, and $k \in I$, $k \geq i_1, \ldots, i_n$; (iii) $[a]_{\theta_\infty} \approx^{\lim \mathbf{M}_i} [b]_{\theta_\infty} = \theta_\infty(a, b)$ for all $[a]_{\theta_\infty}, [b]_{\theta_\infty} \in \left(\bigcup_{i\in I} M_i\right)/\theta_\infty$; is called a *direct limit of a (weak) direct family* $\{\mathbf{M}_i \mid i \in I\}$. A family $\{h_i : \mathbf{M}_i \to \lim \mathbf{M}_i \mid i \in I\}$ of morphisms $h_i : \mathbf{M}_i \to \lim \mathbf{M}_i$, where $h_i(a) = [a]_{\theta_\infty}$ ($i \in I$, $a \in M_i$) is called a *limit cone* of $\{\mathbf{M}_i \mid i \in I\}$.

Let $\{\mathbf{M}_i \mid i \in I\}$ be a family of $\mathbf{L}$-algebras of the same type and let F be a filter over I. Then for every $a, b \in \bigtimes_{i\in I} M_i$ and $X \in F$ we define a truth degree $[\![a \approx b]\!]_X \in L$ by $[\![a \approx b]\!]_X = \bigwedge_{i\in X} a(i) \approx^{\mathbf{M}_i} b(i)$. We define a congruence $\theta_F \in \mathrm{Con}_{\mathbf{L}}(\bigtimes_{i\in I} \mathbf{M}_i)$ by $\theta_F(a, b) = \bigvee_{X\in F} [\![a \approx b]\!]_X$. $\left(\bigtimes_{i\in I} \mathbf{M}_i\right)/\theta_F$ denoted by $\bigtimes_F \mathbf{M}_i$ is called a *reduced product* of $\{\mathbf{M}_i \mid i \in I\}$ modulo F. Reduced products of $\mathbf{L}$-algebras correspond in a natural way to direct limits [24]. A filter F over I is *safe* w.r.t. $\{\mathbf{M}_i \mid i \in I\}$ if for every $a, b \in \bigtimes_{i\in I} M_i$ there is $X \in F$ such that $\theta_F(a, b) = [\![a \approx b]\!]_X$. If $\mathcal{K}$ is a class of $\mathbf{L}$-algebras of the same type and F is safe w.r.t. every family of $\mathbf{L}$-algebras taken from $\mathcal{K}$ then F is said to be $\mathcal{K}$-*safe*. If F is $\mathcal{K}$-safe for arbitrary class $\mathcal{K}$ of $\mathbf{L}$-algebras then F is called *safe*. If F is safe with respect to a family $\{\mathbf{M}_i \mid i \in I\}$ then $\bigtimes_F \mathbf{M}_i$ is called a *safe reduced product.*

$\mathbf{T}(X) = \left\langle T(X), \approx^{\mathbf{T}(X)}, F^{\mathbf{T}(X)}\right\rangle$ denotes the *term* $\mathbf{L}$-*algebra of type* F (*in* X), i.e. $\langle T(X), F^{\mathbf{T}(X)}\rangle$ is an ordinary term algebra and $\approx^{\mathbf{T}(X)}$ is crisp $\mathbf{L}$-equality. Every $\approx$-morphism $h : X \to M$ (M being a universe of $\mathbf{M}$) has a uniquely determined homomorphic extension $h^\sharp : \mathbf{T}(X) \to \mathbf{M}$. If $\mathbf{M} \cong \mathbf{T}(X)/\theta(R)$, where X and R are finite, then $\mathbf{M}$ is called a *finitely presented* $\mathbf{L}$-algebra. Every $\mathbf{L}$-algebra is isomorphic to a direct limit of a direct family of finitely presented $\mathbf{L}$-algebras [24]. For every morphism $h : \mathbf{M} \to \lim \mathbf{M}_i$ from a finitely presented $\mathbf{L}$-algebra $\mathbf{M}$ to a direct limit $\lim \mathbf{M}_i$ of a direct family of $\mathbf{L}$-algebras there is $k \in I$ and a morphism $g : \mathbf{M} \to \mathbf{M}_k$ such that $h = g \circ h_k$. The previous claim does not apply for direct limits of general weak direct families of $\mathbf{L}$-algebras [24].

Classes of $\mathbf{L}$-algebras are denoted by $\mathcal{K}, \mathcal{K}', \ldots$ A class $\mathcal{K}$ of $\mathbf{L}$-algebras is an *abstract class,* if it is closed under isomorphic images. In the sequel we work mostly with abstract classes of $\mathbf{L}$-algebras. Class operators corresponding to the above presented constructions are denoted by H (homomorphic images), I (isomorphic images), S (subalgebras), P (direct products), U (direct unions), L (direct limits of direct families), P_R (safe reduced products). That is, $\mathrm{H}(\mathcal{K})$ denotes the class of all homomorphic images of $\mathbf{L}$-algebras from $\mathcal{K}$ etc.

The paper is organized as follows. Section 2 introduces implicationally defined classes and shows basic closure properties. Section 3 characterizes classes defined by general implications and classes defined by finitary implications, Section 4 characterizes classes defined by Horn implications.

2. Implicationally defined classes and closure properties

The notion of an implication of identities in fuzzy setting was introduced in [10]; we recall the basic concepts.

For $P \in \mathbf{L}^{T(X)\times T(X)}$, and endomorphism $h : \mathbf{T}(X) \to \mathbf{T}(X)$ we define an *endomorphic image* $h(P) \in \mathbf{L}^{T(X)\times T(X)}$ by

$$h(P)(t,t') = \bigvee \left\{P(s,s') \mid h(s) = t, h(s') = t'\right\}$$

for every $t, t' \in T(X)$. Let $\emptyset \neq \mathcal{P} \subseteq \mathbf{L}^{T(X)\times T(X)}$. For $P \in \mathbf{L}^{T(X)\times T(X)}$, put $\mathrm{var}(P) = \bigcup \left\{\mathrm{var}(t) \cup \mathrm{var}(t') \mid P(t,t') > 0\right\}$, i.e. $\mathrm{var}(P)$ is a set of variables occurring in identities which belong to P in some nonzero degree.

A family $\mathcal{P} \subseteq \mathbf{L}^{T(X)\times T(X)}$ is called a *proper family of premises* of type F (in variables X) if for every $P \in \mathcal{P}$ and every endomorphism h on $\mathbf{T}(X)$ we have $h(P) \in \mathcal{P}$. Every $P \in \mathcal{P}$ is then called an $\mathbf{L}$-*set of premises.* A $\mathcal{P}$-*implication* is an expression of the form

$$\textstyle\bigwedge_{P(s,s')>0} \left\langle s \approx s', P(s,s')\right\rangle \Rightarrow (t \approx t').$$

where $P \in \mathcal{P}$ and $t, t' \in T(X)$. For a $\mathcal{P}$-implication φ and a truth value $a \in L$, the couple $\langle \varphi, a\rangle$ is called a *weighted* $\mathcal{P}$-*implication*. $\mathcal{P}$-implications (weighted $\mathcal{P}$-implications) will be denoted by $P \Rightarrow (t \approx t')$ ($\left\langle P \Rightarrow t \approx t'), a\right\rangle$ or $P \Rightarrow \left\langle t \approx t', a\right\rangle$). $\mathbf{L}$-sets of $\mathcal{P}$-implications are denoted usually by $\Sigma, \Gamma, \ldots$. Note that $\mathbf{L}$-sets of $\mathcal{P}$-implications naturally correspond to (ordinary) sets of weighted $\mathcal{P}$-implications and vice versa.

Any finite $\mathbf{L}$-relation $P' \in \mathbf{L}^{T(X)\times T(X)}$, where $P'(t,t') > 0$ implies $P'(t,t') = P(t,t')$ for every terms $t, t' \in T(X)$, is called a *finite restriction* of $P \in \mathcal{P}$. In the sequel, a *set of all finite restrictions* of $P \in \mathcal{P}$ will be denoted by $\mathrm{Fin}(P)$.

A general semantics of $\mathcal{P}$-implications can be approached in a unified way using an additional unary operation "$*$" on $\mathbf{L}$. A unary operation $^* : L \to L$ satisfying

$$1^* = 1, \tag{1}$$
$$a^* \leq a, \tag{2}$$
$$(a \to b)^* \leq a^* \to b^*, \tag{3}$$

for every $a, b \in L$, is called a *truth stresser* for $\mathbf{L}$ [17, 18]. Let $\mathbf{L}^*$ denote $\mathbf{L}$ equipped with a truth stresser *. A truth stresser * satisfying

$$a^{**} = a^*, \tag{4}$$
$$a^* \otimes a^* = a^*, \tag{5}$$
$$\textstyle\bigwedge_{i\in I} a_i^* = \left(\bigwedge_{i\in I} a_i\right)^*, \tag{6}$$

for every $a \in L$, $a_i \in L$ for all $i \in I$, is called an *implicational truth stresser.* Unless stated otherwise, we assume that * is an implicational truth stresser.

For examples of truth stressers in our context, see [10]. An important example of an implicational truth stresser is a so-called globalization [23]:

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

If $\mathbf{L}$ is a chain, the globalization coincides with Baaz's operation [2, 17]. For an $\mathbf{L}$-algebra $\mathbf{M}$ and a valuation $v\colon X \to M$, we define a *truth degree* $\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ of $P \Rightarrow (t \approx t')$ in $\mathbf{M}$ under v with respect to * by

$$\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v} = \|P\|_{\mathbf{M},v} \to \|t \approx t'\|_{\mathbf{M},v},$$

where

$$\|P\|_{\mathbf{M},v} = \left(\textstyle\bigwedge_{s,s' \in T(X)} \left(P(s,s') \to \|s \approx s'\|_{\mathbf{M},v}\right)\right)^*.$$

For technical reasons, for every weighted $\mathcal{P}$-implication $\langle P \Rightarrow (t \approx t'), a\rangle$, we define a degree $\|P \Rightarrow (t \approx t'), a\|_{\mathbf{M},v}$ by $a \to \|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$. The truth stresser * plays a role of a thresholding function. For instance, for * defined by (7) we have

$$\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v} = \begin{cases} \|t \approx t'\|_{\mathbf{M},v} & \text{if } P(s,s') \le \|s \approx s'\|_{\mathbf{M},v} \\ & \text{for all } s, s' \in T(X), \\ 1 & \text{otherwise.} \end{cases}$$

As usual, for an $\mathbf{L}$-algebra $\mathbf{M}$ and a class $\mathcal{K}$ of $\mathbf{L}$-algebras we define the truth degrees of $P \Rightarrow (t \approx t')$ in $\mathbf{M}$ and $\mathcal{K}$ by

$$\|P \Rightarrow (t \approx t')\|_{\mathbf{M}} = \textstyle\bigwedge_{v:X \to M} \|P \Rightarrow (t \approx t')\|_{\mathbf{M},v},$$
$$\|P \Rightarrow (t \approx t')\|_{\mathcal{K}} = \textstyle\bigwedge_{\mathbf{M} \in \mathcal{K}} \|P \Rightarrow (t \approx t')\|_{\mathbf{M}}.$$

$\mathrm{Mod}(\Sigma)$ denotes the class of all models of Σ, i.e.

$$\mathrm{Mod}(\Sigma) = \left\{\mathbf{M} \mid \Sigma(\varphi) \le \|\varphi\|_{\mathbf{M}} \text{ for every } \mathcal{P}\text{-implication } \varphi\right\}.$$

Let $\mathcal{P}$ be a proper family of premises. Put $\mathcal{P}_{\mathrm{Fin}} = \{P \in \mathcal{P} \mid P \text{ is finite}\}$, where P finite means that $\mathrm{Supp}(P)$ is a finite set. We will use the term *$\mathcal{P}$-Horn clauses* instead of $\mathcal{P}_{\mathrm{Fin}}$-implications. Analogously, we can define a restriction $\mathcal{P}_\omega$ of $\mathcal{P}$ on premises with finitely many variables:

$$\mathcal{P}_\omega = \{P \in \mathcal{P} \mid \mathrm{var}(P) \text{ is finite}\}.$$

$\mathcal{P}_\omega$ is a proper family of premises, see [10]. In what follows, we will use the term *$\mathcal{P}$-finitary implications* instead of $\mathcal{P}_\omega$-implications. More generally, for an infinite cardinal κ we put

$$\mathcal{P}_\kappa = \{P \in \mathcal{P} \mid |\mathrm{var}(P)| < \kappa\}.$$

Clearly, $\mathcal{P}_\kappa$ is proper since $|\mathrm{var}(h(P))| \le |\mathrm{var}(h(\mathrm{var}(P)))| < \kappa$ for every $P \in \mathcal{P}_\kappa$ (each $\mathrm{var}(h(x))$ is finite).

Definition 1. *A class $\mathcal{K}$ of* $\mathbf{L}$*-algebras is called an* ***$\mathcal{P}$-implicational class*** determined by * *if there is an* $\mathbf{L}$*-set* Σ *of* $\mathcal{P}$*-implications such that* $\mathcal{K} = \mathrm{Mod}(\Sigma)$. *A* $\mathcal{P}$*-implicational class* $\mathcal{K}$ *is called a* ***$\mathcal{P}$-finitary implicational class*** (***$\mathcal{P}$-Horn class***) *if* Σ *can be taken so that it is an* $\mathbf{L}$*-set of* $\mathcal{P}$*-finitary implications* ($\mathcal{P}$*-Horn clauses*). *For a class* $\mathcal{K}$ *of* $\mathbf{L}$*-algebras of a type* F *and* $\mathcal{Q} \subseteq \mathcal{P}$ *we define an* $\mathbf{L}$*-set* $\mathrm{Impl}_{\mathcal{Q}}(\mathcal{K})$ *of* $\mathcal{P}$*-implications by*

$$\big(\mathrm{Impl}_{\mathcal{Q}}(\mathcal{K})\big)\big(P \Rightarrow (t \approx t')\big) = \begin{cases} \|P \Rightarrow (t \approx t')\|_{\mathcal{K}} & \text{if } P \in \mathcal{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

We write $\mathrm{Impl}(\mathcal{K})$, $\mathrm{Impl}_\kappa(\mathcal{K})$, *and* $\mathrm{Horn}(\mathcal{K})$ *instead of* $\mathrm{Impl}_{\mathcal{P}}(\mathcal{K})$, $\mathrm{Impl}_{\mathcal{P}_\kappa}(\mathcal{K})$, *and* $\mathrm{Impl}_{\mathcal{P}_{\mathrm{Fin}}}(\mathcal{K})$, *respectively.*

Remark 1. Since $\mathcal{P}_{\mathrm{Fin}} \subseteq \mathcal{P}_\omega \subseteq \mathcal{P}$, every $\mathcal{P}$-Horn class is a $\mathcal{P}$-finitary class and every $\mathcal{P}$-finitary class is a $\mathcal{P}$-implicational class.

We are now going to show some closure properties of implicational classes.

Lemma 1. *Let* $\Rightarrow Pt \approx t'$ *be a* $\mathcal{P}$*-implication. Then,*

(i) $\|P \Rightarrow (t \approx t')\|_{\mathbf{M}} \leq \|P \Rightarrow (t \approx t')\|_{\mathbf{N}}$ *for every* $\mathbf{N} \in \mathrm{Sub}(\mathbf{M})$,

(ii) $\|P \Rightarrow (t \approx t')\|_{\{\mathbf{M}_i \mid i \in I\}} \leq \|P \Rightarrow (t \approx t')\|_{\bigtimes_{i\in I}\mathbf{M}_i}$ *for all* $\mathbf{M}_i$*'s* $(i \in I)$,

(iii) $\|P \Rightarrow (t \approx t')\|_{\{\mathbf{M}_i \mid i \in I\}} \leq \|P \Rightarrow (t \approx t')\|_{\mathbf{M}}$
for $\mathbf{M}$ *being a subdirect product of* $\mathbf{M}_i$*'s* $(i \in I)$.

Proof. (i): Follows from the fact that $\approx^{\mathbf{N}}$ is a restriction of $\approx^{\mathbf{M}}$ on N. Note that (i) holds true for general truth stressers (conditions (4)–(6) are not required).

(ii): Denoting the i-th projection by π_i, we have
$\|r \approx r'\|_{\bigtimes_{i\in I}\mathbf{M}_i, v} = \bigwedge_{i\in I} \|r \approx r'\|_{\mathbf{M}_i, v\circ\pi_i}$. Using (6),

$$\begin{aligned}
\textstyle\bigwedge_{i\in I} \|P\|_{\mathbf{M}_i, v\circ\pi_i} &= \textstyle\bigwedge_{i\in I} \big(\bigwedge_{s,s'\in T(X)}\big(P(s,s') \to \|s \approx s'\|_{\mathbf{M}_i, v\circ\pi_i}\big)\big)^* \\
&= \textstyle\big(\bigwedge_{i\in I} \bigwedge_{s,s'\in T(X)}\big(P(s,s') \to \|s \approx s'\|_{\mathbf{M}_i, v\circ\pi_i}\big)\big)^* \\
&= \textstyle\big(\bigwedge_{s,s'\in T(X)} \bigwedge_{i\in I}\big(P(s,s') \to \|s \approx s'\|_{\mathbf{M}_i, v\circ\pi_i}\big)\big)^* \\
&= \textstyle\big(\bigwedge_{s,s'\in T(X)}\big(P(s,s') \to \bigwedge_{i\in I} \|s \approx s'\|_{\mathbf{M}_i, v\circ\pi_i}\big)\big)^* \\
&= \textstyle\big(\bigwedge_{s,s'\in T(X)}\big(P(s,s') \to \|s \approx s'\|_{\bigtimes_{i\in I}\mathbf{M}_i, v}\big)\big)^* = \|P\|_{\bigtimes_{i\in I}\mathbf{M}_i, v}\,.
\end{aligned}$$

We further obtain

$$\begin{aligned}
&\textstyle\|P \Rightarrow (t \approx t')\|_{\{\mathbf{M}_i \mid i\in I\}} \bigwedge_{i\in I} \bigwedge_{v_i : X\to M_i} \|P \Rightarrow (t \approx t')\|_{\mathbf{M}_i, v_i} \\
&\quad= \textstyle\bigwedge_{v:X\to\bigtimes_{i\in I}M_i} \bigwedge_{i\in I} \|P \Rightarrow (t \approx t')\|_{\mathbf{M}_i, v\circ\pi_i} \\
&\quad= \textstyle\bigwedge_{v:X\to\bigtimes_{i\in I}M_i} \bigwedge_{i\in I}\big(\|P\|_{\mathbf{M}_i, v\circ\pi_i} \to \|t \approx t'\|_{\mathbf{M}_i, v\circ\pi_i}\big) \leq \\
&\quad\leq \textstyle\bigwedge_{v:X\to\bigtimes_{i\in I}M_i}\big(\bigwedge_{i\in I} \|P\|_{\mathbf{M}_i, v\circ\pi_i} \to \bigwedge_{i\in I} \|t \approx t'\|_{\mathbf{M}_i, v\circ\pi_i}\big) \\
&\quad= \textstyle\bigwedge_{v:X\to\bigtimes_{i\in I}M_i}\big(\|P\|_{\bigtimes_{i\in I}\mathbf{M}_i, v} \to \|t \approx t'\|_{\bigtimes_{i\in I}\mathbf{M}_i, v}\big) \\
&\quad= \textstyle\bigwedge_{v:X\to\bigtimes_{i\in I}M_i} \|P \Rightarrow (t \approx t')\|_{\bigtimes_{i\in I}\mathbf{M}_i, v} = \|P \Rightarrow (t \approx t')\|_{\bigtimes_{i\in I}\mathbf{M}_i}\,.
\end{aligned}$$

(iii): Consequence of (i) and (ii). □

Lemma 2. *For an* $\mathbf{L}$*-set* Σ *of* $\mathcal{P}$*-implications,* $\mathrm{Mod}(\Sigma)$ *is an abstract class of* $\mathbf{L}$*-algebras closed under the formations of subalgebras and direct products.*

Proof. It is almost immediate that if $\mathbf{M} \in \mathrm{I}(\mathrm{Mod}(\Sigma))$ then $\mathbf{M} \in \mathrm{Mod}(\Sigma)$, i.e. $\mathrm{Mod}(\Sigma)$ is an abstract class of $\mathbf{L}$-algebras.

The facts that $\mathrm{S}(\mathrm{Mod}(\Sigma)) \subseteq \mathrm{Mod}(\Sigma)$ and $\mathrm{P}(\mathrm{Mod}(\Sigma)) \subseteq \mathrm{Mod}(\Sigma)$ follow by Lemma 1. For instance, if $\mathbf{N} \in \mathrm{S}(\mathrm{Mod}(\Sigma))$, i.e. $\mathbf{N} \in \mathrm{Sub}(\mathbf{M})$ for some $\mathbf{L}$-algebra $\mathbf{M} \in \mathrm{Mod}(\Sigma)$ then $\Sigma(P \Rightarrow (t \approx t')) \leq \|P \Rightarrow (t \approx t')\|_{\mathbf{M}}$ since $\mathbf{M} \in \mathrm{Mod}(\Sigma)$ and $\|P \Rightarrow (t \approx t')\|_{\mathbf{M}} \leq \|P \Rightarrow (t \approx t')\|_{\mathbf{N}}$ by Lemma 1, whence $\Sigma(P \Rightarrow (t \approx t')) \leq \|P \Rightarrow (t \approx t')\|_{\mathbf{N}}$ proving $\mathbf{N} \in \mathrm{Mod}(\Sigma)$. □

Analogously as in the ordinary case, one may easily show that $\mathcal{P}$-implicational classes are not closed under homomorphic images (e.g. using implications expressing cancellation rule).

Lemma 3. *Let* $\bigcup_{i \in I} \mathbf{M}_i$ *be a direct union of a directed family* $\{\mathbf{M}_i \mid i \in I\}$ *of* $\mathbf{L}$*-algebras. Then*

$$\|P \Rightarrow (t \approx t')\|_{\{\mathbf{M}_i \mid i \in I\}} \leq \|P \Rightarrow (t \approx t')\|_{\bigcup_{i \in I} \mathbf{M}_i}$$

for every $\mathcal{P}$*-finitary implication* $P \Rightarrow (t \approx t')$.

Proof. Since $P \Rightarrow (t \approx t')$ is a $\mathcal{P}$-finitary implication, $Y = \mathrm{var}(P) \cup \mathrm{var}(t) \cup \mathrm{var}(t')$ is a finite set. Thus, for every valuation v of X in $\bigcup_{i \in I} M_i$, we can consider a finite set $M' = \{v(x) \mid x \in Y\}$. Moreover, M' is a finite subset of $\bigcup_{i \in I} M_i$, i.e. there is some index $k \in I$ such that $M' \subseteq M_k$. Hence the restriction v_Y of v on Y can be thought of as a valuation in $\mathbf{M}_k \in \mathrm{Sub}(\bigcup_{i \in I} \mathbf{M}_i)$. Thus, $\|r \approx r'\|_{\bigcup_{i \in I} \mathbf{M}_i, v_Y} = \|r \approx r'\|_{\mathbf{M}_k, v_Y}$ for every $\langle r, r' \rangle \in \{\langle s, s' \rangle \mid P(s, s') > 0\} \cup \{\langle t, t' \rangle\}$. Furthermore,

$$\begin{aligned}\|P \Rightarrow (t \approx t')\|_{\{\mathbf{M}_i \mid i \in I\}} &\leq \|P \Rightarrow (t \approx t')\|_{\mathbf{M}_k, v_Y} \\ &= \|P \Rightarrow (t \approx t')\|_{\bigcup_{i \in I} \mathbf{M}_i, v_Y} = \|P \Rightarrow (t \approx t')\|_{\bigcup_{i \in I} \mathbf{M}_i, v}.\end{aligned}$$

As a consequence, $\|P \Rightarrow (t \approx t')\|_{\{\mathbf{M}_i \mid i \in I\}} \leq \|P \Rightarrow (t \approx t')\|_{\bigcup_{i \in I} \mathbf{M}_i}$. □

Note that Lemma 3 is true for general truth stressers.

Lemma 4. *For an* $\mathbf{L}$*-set* Σ *of* $\mathcal{P}$*-finitary implications,* $\mathrm{Mod}(\Sigma)$ *is an abstract class of* $\mathbf{L}$*-algebras closed under the formations of subalgebras, direct products, and direct unions.*

Proof. Closedness under direct unions follows from Lemma 3. The rest follows from Lemma 2. □

Lemma 5. *Let* $\lim \mathbf{M}_i$ *be a direct limit of a direct family* $\{\mathbf{M}_i \mid i \in I\}$ *of* $\mathbf{L}$*-algebras. Then*

$$\|P \Rightarrow (t \approx t')\|_{\{\mathbf{M}_i \mid i \in I\}} \leq \|P \Rightarrow (t \approx t')\|_{\lim \mathbf{M}_i}$$

for every $\mathcal{P}$*-Horn clause* $P \Rightarrow (t \approx t')$.

Proof. Let us have a valuation $v\colon X \to \left(\bigcup_{i\in I} M_i\right)/\theta_\infty$ and let $P \Rightarrow (t \approx t')$ be a $\mathcal{P}$-Horn clause, where $\mathrm{Supp}(P) = \left\{\langle t_m, t'_m\rangle \mid m = 1, \dots, n\right\}$. Take $Y = \mathrm{var}(P) \cup \mathrm{var}(t) \cup \mathrm{var}(t')$ and consider a restriction v_Y of v on Y and the homomorphic extension $v_Y^\sharp : \mathbf{T}(Y) \to \lim \mathbf{M}_i$. Since Y is finite, $\mathbf{T}(Y)$ is finitely presented. Due to the image factorization (see [24]), there is an index $k \in I$ and a morphism $g\colon \mathbf{T}(Y) \to \mathbf{M}_k$ such that $v_Y^\sharp = g \circ h_k$. Let g_Y denote a restriction of g on Y. We have

$$\|r\|_{\lim \mathbf{M}_i, v} = \|r\|_{\lim \mathbf{M}_i, v_Y} = v_Y^\sharp(r) = h_k(g(r)) = h_k\left(\|r\|_{\mathbf{M}_k, g_Y}\right)$$

for any $r \in T(Y)$. Moreover for $r, r' \in T(Y)$ it follows that

$$\begin{aligned}
\left\|r \approx r'\right\|_{\lim \mathbf{M}_i, v} &= \|r\|_{\lim \mathbf{M}_i, v} \approx^{\lim \mathbf{M}_i} \left\|r'\right\|_{\lim \mathbf{M}_i, v} \\
&= h_k\left(\|r\|_{\mathbf{M}_k, g_Y}\right) \approx^{\lim \mathbf{M}_i} h_k\left(\left\|r'\right\|_{\mathbf{M}_k, g_Y}\right) \\
&= \left[\|r\|_{\mathbf{M}_k, g_Y}\right]_{\theta_\infty} \approx^{\lim \mathbf{M}_i} \left[\left\|r'\right\|_{\mathbf{M}_k, g_Y}\right]_{\theta_\infty} \\
&= \theta_\infty\left(\|r\|_{\mathbf{M}_k, g_Y}, \left\|r'\right\|_{\mathbf{M}_k, g_Y}\right).
\end{aligned}$$

For every $r, r' \in T(Y)$ there is an index $l \in I$, $k \le l$ such that

$$\begin{aligned}
\left\|r \approx r'\right\|_{\lim \mathbf{M}_i, v} &= \theta_\infty\left(\|r\|_{\mathbf{M}_k, g_Y}, \left\|r'\right\|_{\mathbf{M}_k, g_Y}\right) \\
&= h_{kl}\left(\|r\|_{\mathbf{M}_k, g_Y}\right) \approx^{\mathbf{M}_l} h_{kl}\left(\left\|r'\right\|_{\mathbf{M}_k, g_Y}\right) \\
&= \|r\|_{\mathbf{M}_l, g_Y \circ h_{kl}} \approx^{\mathbf{M}_l} \left\|r'\right\|_{\mathbf{M}_l, g_Y \circ h_{kl}} = \left\|r \approx r'\right\|_{\mathbf{M}_l, g_Y \circ h_{kl}}.
\end{aligned}$$

Therefore there are indices $j_0, j_1, \dots, j_n \ge k$ such that

$$\begin{aligned}
\left\|t_m \approx t'_m\right\|_{\lim \mathbf{M}_i, v} &= \left\|t_m \approx t'_m\right\|_{\mathbf{M}_{j_m}, g_Y \circ h_{kj_m}} \quad \text{for each } m = 1, \dots, n, \\
\left\|t \approx t'\right\|_{\lim \mathbf{M}_i, v} &= \left\|t \approx t'\right\|_{\mathbf{M}_{j_0}, g_Y \circ h_{kj_0}}.
\end{aligned}$$

Moreover, I is directed, i.e. there is an index $j \in I$ with $j_0, j_1, \dots, j_n \le j$. Using properties of direct families it follows that

$$\begin{aligned}
\left\|t_m \approx t'_m\right\|_{\lim \mathbf{M}_i, v} &= \left\|t_m \approx t'_m\right\|_{\mathbf{M}_j, (g_Y \circ h_{kj_m}) \circ h_{j_m j}} \quad \text{for each } m = 1, \dots, n, \\
\left\|t \approx t'\right\|_{\lim \mathbf{M}_i, v} &= \left\|t \approx t'\right\|_{\mathbf{M}_j, (g_Y \circ h_{kj_0}) \circ h_{j_0 j}}.
\end{aligned}$$

Now observe that for each $m = 0, \dots, n$ we have

$$(g_Y \circ h_{kj_m}) \circ h_{j_m j} = g_Y \circ (h_{kj_m} \circ h_{j_m j}) = g_Y \circ h_{kj}.$$

Denoting $g_Y \circ h_{kj}$ by w, we get

$$\left\|P \Rightarrow (t \approx t')\right\|_{\{\mathbf{M}_i \mid i \in I\}} \le \left\|P \Rightarrow (t \approx t')\right\|_{\mathbf{M}_j, w} = \left\|P \Rightarrow (t \approx t')\right\|_{\lim \mathbf{M}_i, v}.$$

Hence, $\left\|P \Rightarrow (t \approx t')\right\|_{\{\mathbf{M}_i \mid i \in I\}} \le \left\|P \Rightarrow (t \approx t')\right\|_{\lim \mathbf{M}_i}$. □

As one can see from the proof, Lemma 5 holds true for general truth stressers.

Lemma 6. *For an* $\mathbf{L}$*-set* Σ *of* $\mathcal{P}$*-Horn clauses,* $\mathrm{Mod}(\Sigma)$ *is an abstract class of* $\mathbf{L}$*-algebras closed under the formations of subalgebras, direct products, direct unions, and direct limits.*

Proof. Closedness under direct limits follows from Lemma 5. The rest follows from Lemma 4. □

* * *

In ordinary case, validity of an implication can be expressed using the notion of injectivity. Namely, an algebra $\mathbf{M}$ is injective w.r.t. an implication $P \Rightarrow (t \approx t')$ iff $P \Rightarrow (t \approx t')$ is valid in $\mathbf{M}$. This criterion is known as the Banaschewski-Herrlich criterion. Such a criterion can be used to prove that a Horn class is closed under L. In what follows, we present an analogy to Banaschewski-Herrlich criterion.

Definition 2. *Let* $\mathbf{L}^*$ *be a complete residuated lattice with a truth stresser* **. For a weighted* $\mathcal{P}$*-implication* $\langle P \Rightarrow (t \approx t'), a\rangle$ *we define*

$$Q(s,s') = \begin{cases} P(t,t') \vee a & \text{if } s = t, \text{ and } s' = t', \\ P(s,s') & \text{otherwise.} \end{cases}$$

Let $h_{PQ}\colon \mathbf{T}(X)/\theta(P) \to \mathbf{T}(X)/\theta(Q)$ *be a morphism defined by*

$$h_{PQ}\bigl([t]_{\theta(P)}\bigr) = [t]_{\theta(Q)} \tag{8}$$

for all $t \in T(X)$*. An* $\mathbf{L}$*-algebra* $\mathbf{M}$ *is said to be* ***injective*** w.r.t. $\langle P \Rightarrow (t \approx t'), a\rangle$ *iff for every morphism* $h\colon \mathbf{T}(X)/\theta(P) \to \mathbf{M}$*, there exists a morphism* $g\colon \mathbf{T}(X)/\theta(Q) \to \mathbf{M}$ *such that* $h = h_{PQ} \circ g$*.*

Remark 2. One can check that h_{PQ} defined by (8) is a well-defined morphism.

Theorem 1. *Let* $\mathbf{L}^*$ *be a complete residuated lattice with a truth stresser* * *and let* $\langle P \Rightarrow (t \approx t'), a\rangle$ *be a weighted* $\mathcal{P}$*-implication.*

(i) *If* $a \leq \|P \Rightarrow (t \approx t')\|_{\mathbf{M}}$*, then* $\mathbf{M}$ *is injective w.r.t.* $\langle P \Rightarrow (t \approx t'), a\rangle$*.*
(ii) *For* * *defined by* (7) *if* $\mathbf{M}$ *is injective w.r.t.* $\langle P \Rightarrow (t \approx t'), a\rangle$*, then* $a \leq \|P \Rightarrow (t \approx t')\|_{\mathbf{M}}$*.*

Proof. (i): By assumption, $a \leq \|P \Rightarrow (t \approx t')\|_{\mathbf{M},v}$ for every $v\colon X \to M$.

Consider a morphism $h\colon \mathbf{T}(X)/\theta(P) \to \mathbf{M}$ and a valuation $v\colon X \to M$, where $v(x) = h\bigl([x]_{\theta(P)}\bigr)$, $x \in X$. Hence, for a homomorphic extension $v^\sharp$ of v we have $v^\sharp = h_{\theta(P)} \circ h$. Furthermore, it follows that $\|s \approx s'\|_{\mathbf{M},v} = v^\sharp(s) \approx^{\mathbf{M}} v^\sharp(s') = \theta_{v^\sharp}(s,s') = \theta_{h_{\theta(P)}\circ h}(s,s')$ for all terms $s, s' \in T(X)$. Therefore,

$$\begin{aligned} P(s,s') &\leq \theta(P)(s,s') = [s]_{\theta(P)} \approx^{\mathbf{T}(X)/\theta(P)} \bigl[s'\bigr]_{\theta(P)} \\ &\leq h\bigl([s]_{\theta(P)}\bigr) \approx^{\mathbf{M}} h\bigl(\bigl[s'\bigr]_{\theta(P)}\bigr) = \theta_{h_{\theta(P)}\circ h}(s,s') = \theta_{v^\sharp}(s,s') \end{aligned}$$

for all $s, s' \in T(X)$. By (1),

$$\begin{aligned} a &\leq \left\| P \Rightarrow (t \approx t') \right\|_{\mathbf{M},v} \\ &= \left(\bigwedge_{s,s' \in T(X)} \left(P(s,s') \to \left\| s \approx s' \right\|_{\mathbf{M},v}\right)\right)^* \to \left\| t \approx t' \right\|_{\mathbf{M},v} \\ &= \left(\bigwedge_{s,s' \in T(X)} \left(P(s,s') \to \theta_{v^\sharp}(s,s')\right)\right)^* \to \theta_{v^\sharp}(t,t') \\ &= 1^* \to \theta_{v^\sharp}(t,t') = 1 \to \theta_{v^\sharp}(t,t') = \theta_{v^\sharp}(t,t'). \end{aligned}$$

We thus have $P(t,t') \leq \theta_{v^\sharp}(t,t')$ and $a \leq \theta_{v^\sharp}(t,t')$, i.e. $Q(t,t') = P(t,t') \vee a \leq \theta_{v^\sharp}(t,t')$. Since $\theta(Q) \in \mathrm{Con}_{\mathbf{L}}(\mathbf{T}(X))$ is generated by Q, it readily follows that $\theta(Q) \subseteq \theta_{v^\sharp}$. Now, by standard argument, there is a morphism $g\colon \mathbf{T}(X)/\theta(Q) \to \mathbf{M}$ such that $v^\sharp = h_{\theta(Q)} \circ g$. Hence, $h_{\theta(P)} \circ h = h_{\theta(Q)} \circ g$, that is $h_{\theta(P)} \circ h = (h_{\theta(P)} \circ h_{PQ}) \circ g$. Surjectivity of $h_{\theta(P)}$ implies $h = h_{PQ} \circ g$. Hence, $\mathbf{M}$ is injective w.r.t. $\langle P \Rightarrow (t \approx t'), a\rangle$.

(ii): Let * be defined by (7) and let $\mathbf{M}$ be injective w.r.t. $\langle P \Rightarrow (t \approx t'), a\rangle$. We have to show that $a \leq \left\| P \Rightarrow (t \approx t') \right\|_{\mathbf{M},v}$ for every valuation v of X in $\mathbf{M}$. Take a valuation $v : X \to M$. If there are terms $s, s' \in T(X)$ such that $P(s,s') \nleq \left\| s \approx s' \right\|_{\mathbf{M},v}$, then $\|P\|_{\mathbf{M},v} = 0$ and thus $a \leq \left\| P \Rightarrow (t \approx t') \right\|_{\mathbf{M},v} = 1$.

Hence, we focus on the nontrivial case. Let $P(s,s') \leq \left\| s \approx s' \right\|_{\mathbf{M},v}$ for all $s, s' \in T(X)$. From (7) it follows that $\|P\|_{\mathbf{M},v} = 1$. Hence, we need to check $a \leq \left\| t \approx t' \right\|_{\mathbf{M},v}$. For a homomorphic extension $v^\sharp$ of v we have $P \subseteq \theta_{v^\sharp}$, that is $\theta(P) \subseteq \theta_{v^\sharp}$. Furthermore, it follows that there is a morphism $g'\colon \mathbf{T}(X)/\theta(P) \to \mathbf{M}$ such that $v^\sharp = h_{\theta(P)} \circ g'$. Since $\mathbf{M}$ is injective w.r.t. $\langle P \Rightarrow (t \approx t'), a\rangle$, there is a morphism $g : \mathbf{T}(X)/\theta(Q) \to \mathbf{M}$ with $g' = h_{PQ} \circ g$. Thus, $v^\sharp = h_{\theta(P)} \circ g' = h_{\theta(P)} \circ h_{PQ} \circ g = h_{\theta(Q)} \circ g$. As a consequence,

$$\begin{aligned} a \leq \theta(Q)(t,t') = [t]_{\theta(Q)} \approx^{\mathbf{T}(X)/\theta(Q)} \left[t'\right]_{\theta(Q)} &\leq g\left([t]_{\theta(Q)}\right) \approx^{\mathbf{M}} g\left([t]_{\theta(Q)}\right) \\ &= v^\sharp(t) \approx^{\mathbf{M}} v^\sharp(t') = \|t\|_{\mathbf{M},v} \approx^{\mathbf{M}} \left\| t' \right\|_{\mathbf{M},v} = \left\| t \approx t' \right\|_{\mathbf{M},v}. \end{aligned}$$

By (1), we obtain $a \leq \left\| t \approx t' \right\|_{\mathbf{M},v} = 1^* \to \left\| t \approx t' \right\|_{\mathbf{M},v} = \left\| P \Rightarrow (t \approx t') \right\|_{\mathbf{M},v}$ showing $a \leq \left\| P \Rightarrow (t \approx t') \right\|_{\mathbf{M}}$. □

Remark 3. (1) For * being the globalization, Theorem 1 gives an "if and only if" criterion for a $\mathcal{P}$-implication to be true in $\mathbf{M}$ in degree at least a.

(2) Since identities can be thought of as $\mathcal{P}$-implications for $\mathcal{P} = \{\emptyset\}$, the truth stresser does not play any role and we can apply Theorem 1 without mentioning *.

(3) Theorem 1 can be used to show that for globalization,

$$\left\| P \Rightarrow (t \approx t') \right\|_{\{\mathbf{M}_i \mid i \in I\}} \leq \left\| P \Rightarrow (t \approx t') \right\|_{\lim \mathbf{M}_i}$$

is true for every direct family $\{\mathbf{M}_i \mid i \in I\}$ of $\mathbf{L}$-algebras (this is already covered by Theorem 5): Put $a = \left\| P \Rightarrow (t \approx t') \right\|_{\{\mathbf{M}_i \mid i \in I\}}$, i.e. $a \leq \left\| P \Rightarrow (t \approx t') \right\|_{\mathbf{M}_i}$ for all $i \in I$. That is, every $\mathbf{M}_i$ is injective w.r.t. $\langle P \Rightarrow (t \approx t'), a\rangle$. It remains to show that $\lim \mathbf{M}_i$ is injective w.r.t. $\langle P \Rightarrow (t \approx t'), a\rangle$ as well. Consider a morphism $h\colon \mathbf{T}(X)/\theta(P) \to \lim \mathbf{M}_i$. Since $P \Rightarrow (t \approx t')$ is a $\mathcal{P}$-Horn clause, $\mathbf{T}(X)/\theta(P)$, where $X = \mathrm{var}(P) \cup \mathrm{var}(t) \cup \mathrm{var}(t')$ is a finitely presented $\mathbf{L}$-algebra. Due to image factorization (see [24]), for some index $k \in I$ the mapping h factorizes through some component of $\lim \mathbf{M}_i$, i.e. $h = h' \circ h_k$, where $h' : \mathbf{T}(X)/\theta(P) \to \mathbf{M}_k$ is a morphism. By assumption, $\mathbf{M}_k$ is injective w.r.t. $\langle P \Rightarrow (t \approx t'), a\rangle$, thus there is a morphism $g : \mathbf{T}(X)/\theta(Q) \to \mathbf{M}_k$ such that $h' = h_{PQ} \circ g$. As a consequence, $h = h' \circ h_k = h_{PQ} \circ (g \circ h_k)$, i.e. $g \circ h_k$ is the desired morphism. Thus, $\lim \mathbf{M}_i$ is injective with respect to $\langle P \Rightarrow (t \approx t'), a\rangle$.

3. Sur-reflections and sur-reflective classes

In this section, we characterize $\mathcal{P}$-implicational classes as abstract classes of $\mathbf{L}$-algebras closed under subalgebras and direct products. We start by sur-reflections. Note that in ordinary implicational classes, sur-reflections play a role analogous to that of free algebras in the theory of varieties.

Definition 3. *Let $\mathcal{K}$ be an abstract class of $\mathbf{L}$-algebras of type F, $\mathbf{M}$ be an $\mathbf{L}$-algebra of type F. A morphism $r : \mathbf{M} \to \mathbf{R}$, where $\mathbf{R} \in \mathcal{K}$, is called a* ***reflection*** of $\mathbf{M}$ in $\mathcal{K}$, *if for every morphism $h : \mathbf{M} \to \mathbf{N}$, $\mathbf{N} \in \mathcal{K}$, there exists a uniquely determined morphism $h' : \mathbf{R} \to \mathbf{N}$ such that $h = r \circ h'$. Moreover, if r is an epimorphism (surjective morphism), then r is called a* ***sur-reflection*** of $\mathbf{M}$ in $\mathcal{K}$.

An abstract class $\mathcal{K}$ of $\mathbf{L}$-algebras of type F is called ***sur-reflective*** *if every $\mathbf{L}$-algebra $\mathbf{M}$ of type F admits a sur-reflection $r : \mathbf{M} \to \mathbf{R}$ in $\mathcal{K}$.*

Example 1. Let $\mathcal{K}$ be a an abstract class of $\mathbf{L}$-algebras. If $\mathbf{F}_{\mathcal{K}}(\overline{X}) \in \mathcal{K}$ [8, 9] then the natural mapping $h_{\theta_{\mathcal{K}}}(X) : \mathbf{T}(X) \to \mathbf{F}_{\mathcal{K}}(\overline{X})$ is a sur-reflection of the term $\mathbf{L}$-algebra $\mathbf{T}(X)$ in $\mathcal{K}$. Indeed, $h_{\theta_{\mathcal{K}}}(X)$ is an epimorphism. Moreover, for every morphism $h : \mathbf{T}(X) \to \mathbf{N}$, where $\mathbf{N} \in \mathcal{K}$, we can consider a mapping $g : \overline{X} \to N$ defined by $g(\overline{x}) = h(x)$ for all $\overline{x} \in \overline{X}$. Since $\mathbf{N} \in \mathcal{K}$, we have $\mathbf{T}(X)/\theta_h \in \mathrm{IS}(\mathcal{K})$, thus $\theta_{\mathcal{K}}(X) \subseteq \theta_h$. As a consequence,

$$\overline{x} \approx^{\mathbf{F}_{\mathcal{K}}(\overline{X})} \overline{y} = \theta_{\mathcal{K}}(X)(x, y) \leq \theta_h(x, y) = h(x) \approx^{\mathbf{N}} h(y) = g(\overline{x}) \approx^{\mathbf{N}} g(\overline{y}).$$

That is, g is an $\approx$-morphism. As a consequence, g has a uniquely determined homomorphic extension $g^\sharp : \mathbf{F}_{\mathcal{K}}(\overline{X}) \to \mathbf{N}$ (see [9]). Hence, $h(t) = g^\sharp\big([t]_{\theta_{\mathcal{K}}(X)}\big) = (h_{\theta_{\mathcal{K}}(X)} \circ g^\sharp)(t)$ holds for all $t \in T(X)$, i.e. $h = h_{\theta_{\mathcal{K}}(X)} \circ g^\sharp$. To sum up, $h_{\theta_{\mathcal{K}}(X)} : \mathbf{T}(X) \to \mathbf{F}_{\mathcal{K}}(\overline{X})$ is a sur-reflection of $\mathbf{T}(X)$ in $\mathcal{K}$.

Theorem 2. *Let $\mathcal{K}$ be an abstract class of $\mathbf{L}$-algebras and let $r_1 : \mathbf{M} \to \mathbf{R}_1$, $r_2 : \mathbf{M} \to \mathbf{R}_2$ be sur-reflections of $\mathbf{M}$ in $\mathcal{K}$. Then $\mathbf{R}_1 \cong \mathbf{R}_2$.*

Proof. Since $\mathbf{R}_1, \mathbf{R}_2 \in \mathcal{K}$ by the definition of sur-reflections, there are uniquely determined morphisms $r_1' : \mathbf{R}_2 \to \mathbf{R}_1$, $r_2' : \mathbf{R}_1 \to \mathbf{R}_2$ such that $r_1 = r_2 \circ r_1'$, and $r_2 = r_1 \circ r_2'$. Thus, $r_1 = (r_1 \circ r_2') \circ r_1'$, and $r_2 = (r_2 \circ r_1') \circ r_2'$. As a consequence, $r_2' \circ r_1' = \mathrm{id}_{\mathbf{R}_1}$, $r_1' \circ r_2' = \mathrm{id}_{\mathbf{R}_2}$. Hence, $\mathbf{R}_1 \cong \mathbf{R}_2$. □

Remark 4. According to Theorem 2, a sur-reflection of $\mathbf{M}$ in $\mathcal{K}$ is determined up to an isomorphism. This observation enables us to denote a sur-reflection of $\mathbf{M}$ in $\mathcal{K}$ by $r_M : \mathbf{M} \to \mathbf{R}_{\mathcal{K}}(\mathbf{M})$. Moreover, when considering sur-reflections, we sometimes omit the surjective mapping r_M and use the term “sur-reflection” for $\mathbf{R}_{\mathcal{K}}(\mathbf{M})$ instead. In such a case we assume that r_M is the corresponding mapping.

A sur-reflection of $\mathbf{M}$ in an abstract class $\mathcal{K}$ can be thought of as the greatest image of $\mathbf{M}$ in $\mathcal{K}$. The notion of a greatest image can be defined as follows. An $\mathbf{L}$-algebra $\mathbf{M}' \in \mathcal{K}$ is said to be *the greatest image of* $\mathbf{M}$ *in* $\mathcal{K}$ if there is an epimorphism $h : \mathbf{M} \to \mathbf{M}'$ and for every epimorphism $g : \mathbf{M} \to \mathbf{N}$, $\mathbf{N} \in \mathcal{K}$ we have $\theta_h \subseteq \theta_g$. Obviously, $\mathbf{M}' \cong \mathbf{M}/\theta_h$, and $\mathbf{M}/\theta_h$ is a "greater factor $\mathbf{L}$-algebra" than $\mathbf{M}/\theta_g \cong \mathbf{N}$. That is, the definition of the greatest image corresponds well to the intuition. The following theorem characterizes the relationship between sur-reflections and greatest images in more detail.

Theorem 3. *Suppose* $\mathcal{K}$ *is an abstract class of* $\mathbf{L}$*-algebras,* $\mathrm{S}(\mathcal{K}) \subseteq \mathcal{K}$*. Then an epimorphism* $r : \mathbf{M} \to \mathbf{R}$ *is a sur-reflection of* $\mathbf{M}$ *in* $\mathcal{K}$ *iff* $\mathbf{R}$ *is the greatest image of* $\mathbf{M}$ *in* $\mathcal{K}$*.*

Proof. "$\Rightarrow$": Let $r : \mathbf{M} \to \mathbf{R}$ be a sur-reflection of $\mathbf{M}$ in $\mathcal{K}$. That is, for every epimorphism $g : \mathbf{M} \to \mathbf{N}$ there is a morphism $g' : \mathbf{R} \to \mathbf{N}$ such that $g = r \circ g'$. Thus, $\theta_r \subseteq \theta_{r\circ g'} = \theta_g$, i.e. $\mathbf{R}$ is the greatest image of $\mathbf{M}$ in $\mathcal{K}$.

"$\Leftarrow$": Let $\mathbf{R}$ be the greatest image of $\mathbf{M}$ in $\mathcal{K}$. Take arbitrary $\mathbf{N} \in \mathcal{K}$ and a morphism $h : \mathbf{M} \to \mathbf{N}$. Note that h is not supposed to be surjective. On the other hand, the first isomorphism theorem [9] yields $h = h' \circ g$, where $h' : \mathbf{M} \to \mathbf{M}/\theta_h$ is an epimorphism, and $g : \mathbf{M}/\theta_h \to \mathbf{N}$ is an embedding. Since $\mathcal{K}$ is closed under subalgebras, we have $\mathbf{M}/\theta_h \in \mathrm{IS}(\mathcal{K}) = \mathcal{K}$. Moreover, $\mathbf{R}$ is supposed to be the greatest image, i.e. $\theta_r \subseteq \theta_{h'}$ for some epimorphism $r : \mathbf{M} \to \mathbf{R}$. Since $\mathbf{R} \cong \mathbf{M}/\theta_r$, it follows that there is a uniquely determined morphism $g' : \mathbf{R} \to \mathbf{M}/\theta_h$ such that $h' = r \circ g'$. Therefore, $h = h' \circ g = (r \circ g') \circ g = r \circ (g' \circ g)$. Altogether, $r : \mathbf{M} \to \mathbf{R}$ is a sur-reflection of $\mathbf{M}$ in $\mathcal{K}$. □

Theorem 4. *An abstract class of* $\mathbf{L}$*-algebras is sur-reflective iff it is closed under the formations of subalgebras and direct products.*

Proof. "$\Rightarrow$": Let $\mathcal{K}$ be a sur-reflective class of $\mathbf{L}$-algebras. We check closedness under S and P. Take $\mathbf{M} \in \mathcal{K}$, and $\mathbf{N} \in \mathrm{Sub}(\mathbf{M})$. We show $\mathbf{N} \in \mathcal{K}$ by checking that $\mathbf{N}$ is isomorphic to its sur-reflection $\mathbf{R}_{\mathcal{K}}(\mathbf{N})$ with $r_N : \mathbf{N} \to \mathbf{R}_{\mathcal{K}}(\mathbf{N})$. Consider an embedding $h : \mathbf{N} \to \mathbf{M}$. Then $h = r_N \circ h'$ for some morphism $h' : \mathbf{R}_{\mathcal{K}}(\mathbf{N}) \to \mathbf{M}$. Since h is an embedding, we have

$$\begin{aligned} \mathrm{a} \approx^{\mathbf{N}} \mathrm{b} &\leq r_N(\mathrm{a}) \approx^{\mathbf{R}_{\mathcal{K}}(\mathbf{N})} r_N(\mathrm{b}) \\ &\leq h'(r_N(\mathrm{a})) \approx^{\mathbf{M}} h'(r_N(\mathrm{b})) = h(\mathrm{a}) \approx^{\mathbf{M}} h(\mathrm{b}) = \mathrm{a} \approx^{\mathbf{N}} \mathrm{b} \end{aligned}$$

for every $\mathrm{a}, \mathrm{b} \in N$. Thus, r_N is an embedding. Since r_N is a sur-reflection it is also an epimorphism. Hence, $\mathbf{N} \cong \mathbf{R}_{\mathcal{K}}(\mathbf{N}) \in \mathcal{K}$ proving $\mathbf{N} \in \mathcal{K}$, i.e. $\mathrm{S}(\mathcal{K}) \subseteq \mathcal{K}$.

Take a family $\{\mathbf{M}_i \mid i \in I\} \subseteq \mathcal{K}$. We will show $\bigtimes_{i\in I}\mathbf{M}_i \in \mathcal{K}$ by proving $\bigtimes_{i\in I}\mathbf{M}_i \cong \mathbf{R}_{\mathcal{K}}(\bigtimes_{i\in I}\mathbf{M}_i)$, where $r : \bigtimes_{i\in I}\mathbf{M}_i \to \mathbf{R}_{\mathcal{K}}(\bigtimes_{i\in I}\mathbf{M}_i)$ is a sur-reflection of $\bigtimes_{i\in I}\mathbf{M}_i$ in $\mathcal{K}$. Every projection $\pi_j : \bigtimes_{i\in I}\mathbf{M}_i \to \mathbf{M}_j$ is an epimorphism. Hence, for every $j \in I$ there exists a morphism $p_j : \mathbf{R}_{\mathcal{K}}(\bigtimes_{i\in I}\mathbf{M}_i) \to \mathbf{M}_j$ such that $\pi_j = r \circ p_j$. By standard argument, there is a morphism $h : \mathbf{R}_{\mathcal{K}}(\bigtimes_{i\in I}\mathbf{M}_i) \to \bigtimes_{i\in I}\mathbf{M}_i$ such that $h \circ \pi_j = p_j$ for every $j \in I$. Thus, $r \circ h \circ \pi_j = r \circ p_j = \pi_j$ and so $r \circ h = \mathrm{id}_{\bigtimes_{i\in I}\mathbf{M}_i}$. Now, we have

$$\mathfrak{a} \approx^{\bigtimes_{i\in I}\mathbf{M}_i} \mathfrak{b} \le r(\mathfrak{a}) \approx^{\mathbf{R}_{\mathcal{K}}(\bigtimes_{i\in I}\mathbf{M}_i)} r(\mathfrak{b}) \le h(r(\mathfrak{a})) \approx^{\bigtimes_{i\in I}\mathbf{M}_i} h(r(\mathfrak{b})) = \mathfrak{a} \approx^{\bigtimes_{i\in I}\mathbf{M}_i} \mathfrak{b}$$

for all $\mathfrak{a}, \mathfrak{b} \in \bigtimes_{i\in I} M_i$. Hence, $r : \bigtimes_{i\in I}\mathbf{M}_i \to \mathbf{R}_{\mathcal{K}}(\bigtimes_{i\in I}\mathbf{M}_i)$ is an isomorphism, i.e. $\bigtimes_{i\in I}\mathbf{M}_i \in \mathcal{K}$.

"⇐": Suppose $\mathcal{K}$ is an abstract class of $\mathbf{L}$-algebras of type F and let $\mathrm{S}(\mathcal{K}) \subseteq \mathcal{K}$ and $\mathrm{P}(\mathcal{K}) \subseteq \mathcal{K}$. We show that every $\mathbf{L}$-algebra $\mathbf{M}$ of type F has a sur-reflection in $\mathcal{K}$. Let

$$H_{\mathcal{K}}(\mathbf{M}) = \{\theta \in \mathrm{Con}_{\mathbf{L}}(\mathbf{M}) \mid \mathbf{M}/\theta \in \mathcal{K}\}.$$

$H_{\mathcal{K}}(\mathbf{M})$ is nonempty since $\mathcal{K}$ is closed under P. Putting,

$$\mathbf{P}_{\mathcal{K}}(\mathbf{M}) = \bigtimes_{\theta\in H_{\mathcal{K}}(\mathbf{M})} \mathbf{M}/\theta,$$

$\mathrm{P}(\mathcal{K}) \subseteq \mathcal{K}$ implies $\mathbf{P}_{\mathcal{K}}(\mathbf{M}) \in \mathcal{K}$. A family $\{h_\theta : \mathbf{M} \to \mathbf{M}/\theta \mid \theta \in H_{\mathcal{K}}(\mathbf{M})\}$ of natural morphisms induces a uniquely determined morphism $p : \mathbf{M} \to \mathbf{P}_{\mathcal{K}}(\mathbf{M})$ with $p \circ \pi_\theta = h_\theta$. Finally, we have $p = r \circ s$ for an epimorphism $r : \mathbf{M} \to \mathbf{R}$ and an embedding $s : \mathbf{R} \to \mathbf{P}_{\mathcal{K}}(\mathbf{M})$. That is, $\mathbf{R} \in \mathrm{IS}(\mathcal{K}) \subseteq \mathcal{K}$.

We claim that $r : \mathbf{M} \to \mathbf{R}$ is a sur-reflection of $\mathbf{M}$ in $\mathcal{K}$. Take a morphism $h : \mathbf{M} \to \mathbf{N}$, $\mathbf{N} \in \mathcal{K}$. Using the first isomorphism theorem [9] we have $h = h_{\theta_h} \circ g$, where $h_{\theta_h} : \mathbf{M} \to \mathbf{M}/\theta_h$ is a natural morphism, and $g : \mathbf{M}/\theta_h \to \mathbf{N}$ is an embedding, see Fig. 1. Thus, $\mathbf{M}/\theta_h \in \mathcal{K}$, i.e. $\theta_h \in H_{\mathcal{K}}(\mathbf{M})$. We have,

$$h = h_{\theta_h} \circ g = p \circ \pi_{\theta_h} \circ g = r \circ s \circ \pi_{\theta_h} \circ g.$$

Hence, for $h' : \mathbf{R} \to \mathbf{N}$ being $s \circ \pi_{\theta_h} \circ g$ we have $h = r \circ h'$. Since r is surjective, the uniqueness of h' is immediate. Altogether, $r : \mathbf{M} \to \mathbf{R}$ is a sur-reflection of $\mathbf{M}$ in $\mathcal{K}$, i.e. $\mathcal{K}$ is sur-reflective. □

Corollary 1. *Suppose* $\mathbf{L}^*$ *is a complete residuated lattice equipped with an implicational truth stresser* *. *Then every* $\mathcal{P}$*-implicational class of* $\mathbf{L}$*-algebras is sur-reflective.*

Proof. Consequence of Lemma 2 and Theorem 4. □

For an $\mathbf{L}$-algebra $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ we can consider a set of variables X with $|X| = |M|$. For the sake of convenience, we can assume $X = M$. Then

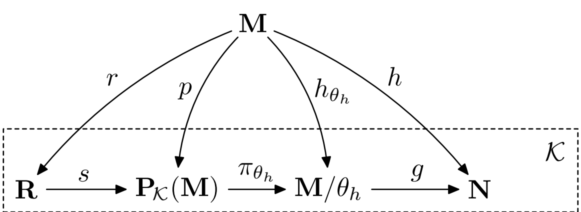

Fig. 1. Construction of a sur-reflection of $\mathbf{M}$ in $\mathcal{K}$

$\mathbf{T}(M)$ is a term $\mathbf{L}$-algebra of type F. The terms of type F over M are denoted by $a, b, f(a_1, \ldots, a_n)$, and so on while the elements of $\mathbf{M}$ are denoted by $\mathfrak{a}, \mathfrak{b}, f^{\mathbf{M}}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)$.

Evidently, for the identical mapping $\mathrm{id}_M : M \to M$ and the corresponding homomorphic extension $\mathrm{id}_M^\sharp : \mathbf{T}(M) \to \mathbf{M}$ we have

$$\mathrm{id}_M^\sharp(f(a_1, \ldots, a_n)) = f^{\mathbf{M}}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n).$$

We will need the following $\mathbf{L}$-sets of $\mathcal{P}$-implications.

Definition 4. *Suppose $\mathcal{K}$ is a sur-reflective class of $\mathbf{L}$-algebras of type F. For every $\mathbf{L}$-algebra $\mathbf{M}$ of type F let $\mathcal{P}_M = \mathbf{L}^{T(M)\times T(M)}$. Moreover, we define an $\mathbf{L}$-set $P_M \in \mathcal{P}_M$ of premises by*

$$P_M(s, s') = \mathrm{id}_M^\sharp(s) \approx^{\mathbf{M}} \mathrm{id}_M^\sharp(s')$$

*for all $s, s' \in T(M)$. A **$\mathcal{P}_M$-theory** of $\mathbf{M}$ over $\mathcal{K}$ is an $\mathbf{L}$-set $\Sigma_{\mathbf{M}}^{\mathcal{K}}$ of $\mathcal{P}_M$-implications defined by*

$$\Sigma_{\mathbf{M}}^{\mathcal{K}}\big(P \Rightarrow (t \approx t')\big) = \begin{cases} r_M\big(\mathrm{id}_M^\sharp(t)\big) \approx^{\mathbf{R}_{\mathcal{K}}(\mathbf{M})} r_M\big(\mathrm{id}_M^\sharp(t')\big) & \text{if } P = P_M, \\ 0 & \text{otherwise,} \end{cases}$$

where $r_M : \mathbf{M} \to \mathbf{R}_{\mathcal{K}}(\mathbf{M})$ is a sur-reflection of $\mathbf{M}$ in $\mathcal{K}$.

Lemma 7. *Suppose $\mathbf{L}^*$ is a complete residuated lattice with a truth stresser * defined by* (7). *Let $\mathcal{K}$ be a sur-reflective class of $\mathbf{L}$-algebras of type F. Then $\mathcal{K} = \bigcap_{\mathbf{M}} \mathrm{Mod}(\Sigma_{\mathbf{M}}^{\mathcal{K}})$ where $\mathbf{M}$ ranges over all $\mathbf{L}$-algebras of type F.*

Proof. "$\mathcal{K} \subseteq \bigcap_{\mathbf{M}} \mathrm{Mod}(\Sigma_{\mathbf{M}}^{\mathcal{K}})$": It suffices to show $\mathcal{K} \subseteq \mathrm{Mod}(\Sigma_{\mathbf{M}}^{\mathcal{K}})$ for every $\mathbf{L}$-algebra $\mathbf{M}$ of type F, i.e. to check that $\Sigma_{\mathbf{M}}^{\mathcal{K}}(P \Rightarrow (t \approx t')) \leq \big\|P \Rightarrow (t \approx t')\big\|_{\mathbf{N}}$ holds for every $\mathcal{P}_M$-implication $P \Rightarrow (t \approx t')$ and every $\mathbf{N} \in \mathcal{K}$.

So, let us have $\mathbf{N} \in \mathcal{K}$, and let $v : M \to N$ be a valuation of M in $\mathbf{N}$ with its homomorphic extension $v^\sharp : \mathbf{T}(M) \to \mathbf{N}$. Thus, we have $v^\sharp = g \circ g'$, where $g : \mathbf{T}(M) \to \mathbf{T}(M)/\theta_{v^\sharp}$ is a natural morphism and $g' : \mathbf{T}(M)/\theta_{v^\sharp} \to \mathbf{N}$ is an embedding.

Let $P \Rightarrow (t \approx t')$ be an $\mathcal{P}_M$-implication such that $P = P_M$. If $P(s, s') \nleq v^\sharp(s) \approx^{\mathbf{N}} v^\sharp(s')$ for some $s, s' \in T(M)$, then obviously $\big\|P \Rightarrow (t \approx t')\big\|_{\mathbf{N},v} = 1$. Thus, let $P(s, s') \leq v^\sharp(s) \approx^{\mathbf{N}} v^\sharp(s')$ for all $s, s' \in T(M)$. Consider a mapping $h : M \to T(M)/\theta_{v^\sharp}$ defined by $h(\mathfrak{a}) = [a]_{\theta_{v^\sharp}}$ for every $\mathfrak{a} \in M$. Since

$$\mathrm{id}_M^\sharp(s) \approx^{\mathbf{M}} \mathrm{id}_M^\sharp(s') = P(s, s') \leq v^\sharp(s) \approx^{\mathbf{N}} v^\sharp(s'),$$

for all $s, s' \in T(M)$, it follows that

$$\begin{aligned} \mathfrak{a} \approx^{\mathbf{M}} \mathfrak{b} &= \mathrm{id}_M^\sharp(a) \approx^{\mathbf{M}} \mathrm{id}_M^\sharp(b) \leq v^\sharp(a) \approx^{\mathbf{N}} v^\sharp(b) \\ &= \theta_{v^\sharp}(a, b) = [a]_{\theta_{v^\sharp}} \approx^{\mathbf{T}(M)/\theta_{v^\sharp}} [b]_{\theta_{v^\sharp}} \end{aligned}$$

for all $a, b \in M$. That is, h is an $\approx$-morphism. Moreover, for any n-ary $f^{\mathbf{M}} \in F^{\mathbf{M}}$ and arbitrary elements $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \in M$ let $f^{\mathbf{M}}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n) = \mathfrak{b}$. Then, $[f(a_1, \ldots, a_n)]_{\theta_{v^\sharp}} = [b]_{\theta_{v^\sharp}}$ and it follows that

$$\begin{aligned} h\big(f^{\mathbf{M}}(\mathfrak{a}_1, \ldots, \mathfrak{a}_n)\big) &= h(\mathfrak{b}) = [b]_{\theta_{v^\sharp}} = [f(a_1, \ldots, a_n)]_{\theta_{v^\sharp}} \\ &= f^{\mathbf{T}(M)/\theta_{v^\sharp}}\big([a_1]_{\theta_{v^\sharp}}, \ldots, [a_n]_{\theta_{v^\sharp}}\big) = f^{\mathbf{T}(M)/\theta_{v^\sharp}}\big(h(\mathfrak{a}_1), \ldots, h(\mathfrak{a}_n)\big). \end{aligned}$$

Altogether, $h\colon \mathbf{M} \to \mathbf{T}(M)/\theta_{v^\sharp}$ is a morphism. Clearly, $g = \mathrm{id}^\sharp_M \circ h$.

Since $\mathcal{K}$ is sur-reflective, we have $h \circ g' = r_M \circ h'$, where $h'\colon \mathbf{R}_{\mathcal{K}}(\mathbf{M}) \to \mathbf{N}$, see Fig. 2. Thus, $v^\sharp = g \circ g' = \mathrm{id}^\sharp_M \circ h \circ g' = \mathrm{id}^\sharp_M \circ r_M \circ h'$, which implies

$$\begin{aligned} \Sigma^{\mathcal{K}}_{\mathbf{M}}(P \Rightarrow (t \approx t')) &= r_M\big(\mathrm{id}^\sharp_M(t)\big) \approx^{\mathbf{R}_{\mathcal{K}}(\mathbf{M})} r_M\big(\mathrm{id}^\sharp_M(t')\big) \\ &\leq h'\big(r_M\big(\mathrm{id}^\sharp_M(t)\big)\big) \approx^{\mathbf{N}} h'\big(r_M\big(\mathrm{id}^\sharp_M(t')\big)\big) \\ &= v^\sharp(t) \approx^{\mathbf{N}} v^\sharp(t') = \big\|t \approx t'\big\|_{\mathbf{N},v} = \big\|P \Rightarrow (t \approx t')\big\|_{\mathbf{N},v}. \end{aligned}$$

Therefore, $\Sigma^{\mathcal{K}}_{\mathbf{M}}(P \Rightarrow (t \approx t')) \leq \big\|P \Rightarrow (t \approx t')\big\|_{\mathbf{N}}$, i.e. $\mathbf{N} \in \mathrm{Mod}(\Sigma^{\mathcal{K}}_{\mathbf{M}})$.

"$\mathcal{K} \supseteq \bigcap_{\mathbf{M}} \mathrm{Mod}(\Sigma^{\mathcal{K}}_{\mathbf{M}})$": Let $\mathbf{N} \in \bigcap_{\mathbf{M}} \mathrm{Mod}(\Sigma^{\mathcal{K}}_{\mathbf{M}})$. It suffices to show that the sur-reflection $r_N : \mathbf{N} \to \mathbf{R}_{\mathcal{K}}(\mathbf{N})$ is an embedding, since then $\mathbf{N} \cong \mathbf{R}_{\mathcal{K}}(\mathbf{N})$, i.e. $\mathbf{N} \in \mathcal{K}$. Evidently, $\mathbf{N} \in \bigcap_{\mathbf{M}} \mathrm{Mod}(\Sigma^{\mathcal{K}}_{\mathbf{M}})$ implies $\mathbf{N} \in \mathrm{Mod}(\Sigma^{\mathcal{K}}_{\mathbf{N}})$. Hence, we can consider a valuation $\mathrm{id}_N : N \to N$ and its homomorphic extension $\mathrm{id}^\sharp_N : \mathbf{T}(N) \to \mathbf{N}$. Taking into account $\mathbf{N} \in \mathrm{Mod}(\Sigma^{\mathcal{K}}_{\mathbf{N}})$, it follows that

$$\begin{aligned} r_N\big(\mathrm{id}^\sharp_N(t)\big) \approx^{\mathbf{R}_{\mathcal{K}}(\mathbf{N})} r_N\big(\mathrm{id}^\sharp_N(t')\big) &= \Sigma^{\mathcal{K}}_{\mathbf{N}}(P_N \Rightarrow (t \approx t')) \\ &\leq \big\|P_N \Rightarrow (t \approx t')\big\|_{\mathbf{N},\mathrm{id}_N} = \big\|t \approx t'\big\|_{\mathbf{N},\mathrm{id}_N} = \mathrm{id}^\sharp_N(t) \approx^{\mathbf{N}} \mathrm{id}^\sharp_N(t'). \end{aligned}$$

Thus, $r_N : \mathbf{N} \to \mathbf{R}_{\mathcal{K}}(\mathbf{N})$ is an embedding, i.e. $\mathbf{N} \in \mathcal{K}$. □

Now we face the following problem. Given a sur-reflective class $\mathcal{K}$, we have shown that there is a class of $\mathbf{L}$-sets $\Sigma^{\mathcal{K}}_{\mathbf{M}}$ of $\mathcal{P}_M$-implications such that $\mathcal{K} = \bigcap_{\mathbf{M}} \mathrm{Mod}(\Sigma^{\mathcal{K}}_{\mathbf{M}})$. For every $\mathbf{M}$ we use a separate proper family of premises $\mathcal{P}_M$. In addition to that, we deal with a proper class of $\mathbf{L}$-sets since $\mathbf{M}$ ranges over a proper class of all $\mathbf{L}$-algebras of type F. Thus, Lemma 7 itself does not yield that $\mathcal{K}$ is a $\mathcal{P}$-implicational class.

In ordinary case, this problem has been solved by J. Adámek, see [1]. In what follows, we adopt Adámek's approach to get the desired result. The key point of

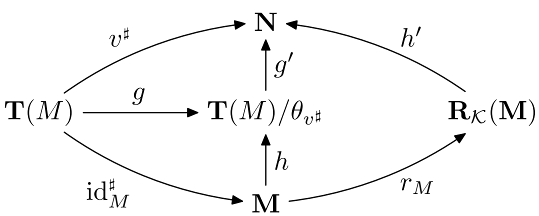

Fig. 2. Scheme for the proof of Lemma 7

[1] is that one can show that every sur-reflective class (in [1] called a quasivariety) is definable by a set of implications using the so-called Vopěnka's Principle. Moreover, it has been shown that assuming the negation of Vopěnka's Principle, there is always a sur-reflective class which cannot be defined by implications. For our purpose, we use a principle concerning classes of $\mathbf{L}$-algebras.

Principle 1. *Given any proper class $\mathcal{K}$ of $\mathbf{L}$-algebras of the same type, there are distinct $\mathbf{L}$-algebras $\mathbf{M}, \mathbf{N} \in \mathcal{K}$ such that $\mathbf{M}$ can be embedded into $\mathbf{N}$.*

Theorem 5. *Vopěnka's Principle implies Principle 1.*

Proof. Let $\mathcal{K}$ be a proper class of $\mathbf{L}$-algebras of type F. We use $\mathcal{K}$ to construct a proper class $\mathcal{K}_c$ of ordinary first-order structures corresponding to $\mathbf{L}$-algebras from $\mathcal{K}$ and then apply Vopěnka's Principle for $\mathcal{K}$ to show that there are distinct $\mathbf{L}$-algebras $\mathbf{M}, \mathbf{N} \in \mathcal{K}$ and an embedding $h \colon \mathbf{M} \to \mathbf{N}$.

Let $R = \{\approx_a \mid a \in L\}$ be a set of binary relation symbols. For each $\mathbf{M} \in \mathcal{K}$, we can consider a first-order structure $\mathbf{M}_c = \left\langle M_c, R^{\mathbf{M}_c}, F^{\mathbf{M}_c}\right\rangle$ of type $\langle R, F, \sigma\rangle$, where $M_c = M$, $F^{\mathbf{M}_c} = F^{\mathbf{M}}$ (i.e. the functional parts of $\mathbf{M}$ and $\mathbf{M}_c$ coincide), and each $\approx_a^{\mathbf{M}_c}$ is the a-cut of $\approx^{\mathbf{M}}$, i.e. $u \approx_a^{\mathbf{M}_c} v$ iff $u \approx^{\mathbf{M}} v \geq a$. Thus, we have $u \approx^{\mathbf{M}} v = \bigvee\left\{a \mid u \approx_a^{\mathbf{M}_c} v\right\}$ $(u, v \in M)$. Clearly, $\mathcal{K}_c$ is a proper class of first-order structures. Hence, by Vopěnka's Principle, there are $\mathbf{M}_c, \mathbf{N}_c \in \mathcal{K}_c$ such that $\mathbf{M}_c$ can be (isomorphically) embedded into $\mathbf{N}_c$. That is, there is a mapping $h \colon M_c \to N_c$ such that $h\left(f^{\mathbf{M}_c}(u_1, \ldots, u_n)\right) = f^{\mathbf{N}_c}\left(h(u_1), \ldots, h(u_n)\right)$ $(f \in F$, $u_1, \ldots, u_n \in M_c)$ and $u \approx_a^{\mathbf{M}_c} v$ iff $h(u) \approx_a^{\mathbf{N}_c} h(v)$ $(u, v \in M_c$, $a \in L)$. As a consequence,

$$u \approx^{\mathbf{M}} v = \bigvee\left\{a \mid u \approx_a^{\mathbf{M}_c} v\right\} = \bigvee\left\{a \mid h(u) \approx_a^{\mathbf{N}_c} h(v)\right\} = h(u) \approx^{\mathbf{N}} h(v),$$

showing that h is an embedding of $\mathbf{L}$-algebras $\mathbf{M}, \mathbf{N}$. □

We need to generalize the notion of a directed family of $\mathbf{L}$-algebras [1, 9, 25].

Definition 5. *Let κ be an infinite cardinal. A family $\{\mathbf{M}_i \mid i \in I\} \neq \emptyset$ of $\mathbf{L}$-algebras of type F is called a* ***κ-directed family****, if for every $J \subseteq I$, $|J| < \kappa$ there exists an index $i \in I$ such that $\mathbf{M}_j \in \mathrm{Sub}(\mathbf{M}_i)$ for all $j \in J$.*

Lemma 8. *Let κ be an infinite cardinal. Then*

(i) *every κ-directed family is a directed family;*
(ii) *every directed family is an ω-directed family.*

Proof. (i): Easy, just putting $i \leq j$ iff $\mathbf{M}_i \in \mathrm{Sub}(\mathbf{M}_j)$, $\langle I, \leq\rangle$ is a directed set and $\{\mathbf{M}_i \mid i \in I\}$ is a directed family.

(ii): Obvious. □

Lemma 8 (i) justifies the following definition.

Definition 6. *Given a κ-directed family $\{\mathbf{M}_i \mid i \in I\}$, the direct union of $\{\mathbf{M}_i \mid i \in I\}$, denoted by $\bigcup_{i\in I}^{\kappa} \mathbf{M}_i = \left\langle \bigcup_{i\in I} M_i, \approx^{\bigcup_{i\in I}^{\kappa} \mathbf{M}_i}, F^{\bigcup_{i\in I}^{\kappa} \mathbf{M}_i}\right\rangle$, is said to be a* ***$\kappa$-direct union*** of a κ-directed family $\{\mathbf{M}_i \mid i \in I\}$. *An abstract class $\mathcal{K}$ of $\mathbf{L}$-algebras is said to be* ***closed under κ-direct unions****, if for every κ-directed family $\{\mathbf{M}_i \in \mathcal{K} \mid i \in I\}$ we have $\bigcup_{i\in I}^{\kappa} \mathbf{M}_i \in \mathcal{K}$.*

The following is easy to see.

Lemma 9. *Suppose $\mathcal{K}$ is an abstract class of* $\mathbf{L}$*-algebras which is closed under κ-direct unions. Then $\mathcal{K}$ is closed under κ'-direct unions for all $\kappa' > \kappa$. Therefore, if* $\mathrm{U}(\mathcal{K}) \subseteq \mathcal{K}$, *then $\mathcal{K}$ is closed under κ-direct unions for arbitrary κ.* □

Lemma 10. *Let κ be any infinite cardinal. Every* $\mathbf{L}$*-algebra is isomorphic to a κ-direct union of a κ-directed family of κ-generated* $\mathbf{L}$*-algebras.*

Proof. As in [9] we get that $\{[M']_{\mathbf{M}} \mid M' \in I_{\mathbf{M}}\}$, where $I_{\mathbf{M}} = \{M' \subseteq M \mid |M'| < \kappa\}$, is a κ-directed family of subalgebras of $\mathbf{M}$. Since every finitely generated subalgebra of $\mathbf{M}$ is in $\{[M']_{\mathbf{M}} \mid M' \in I_{\mathbf{M}}\}$, we can use the same arguments as in [9] to obtain that $\mathbf{M}$ is isomorphic to the κ-direct union of $\{[M']_{\mathbf{M}} \mid M' \in I_{\mathbf{M}}\}$. □

Lemma 11. *Let* $\mathbf{L}^*$ *be a complete residuated lattice with a truth stresser* * *defined by* (7)*. Suppose $\mathcal{K}$ is an abstract class of* $\mathbf{L}$*-algebras, κ is an infinite cardinal,* $\mathcal{P} = \mathbf{L}^{T(X)\times T(X)}$, $|X| = \kappa$*. Then* $\mathcal{K} = \mathrm{Mod}(\Sigma)$ *for some* $\mathbf{L}$*-set* Σ *of* $\mathcal{P}_\kappa$*-implications iff $\mathcal{K}$ is a sur-reflective class which is closed under κ-direct unions.*

Proof. "⇒:" Let $\mathcal{K} = \mathrm{Mod}(\Sigma)$ for some $\mathbf{L}$-set Σ of $\mathcal{P}_\kappa$-implications. From Lemma 2 and Theorem 4 it follows that $\mathcal{K}$ is sur-reflective class. Thus, it suffices to show that $\mathcal{K}$ is closed under κ-direct unions.

Let $\{\mathbf{M}_i \in \mathcal{K} \mid i \in I\}$ be a κ-directed family of $\mathbf{L}$-algebras. We have to show that $\Sigma(P \Rightarrow (t \approx t')) \leq \|P \Rightarrow (t \approx t')\|_{\bigcup_{i\in I}^{\kappa} \mathbf{M}_i}$ for every $\mathcal{P}_\kappa$-implication $P \Rightarrow (t \approx t')$. Take a valuation $v : X \to \bigcup_{i\in I} M_i$ and its homomorphic extension $v^\sharp : \mathbf{T}(X) \to \bigcup_{i\in I}^{\kappa} \mathbf{M}_i$. Put $Y = \mathrm{var}(P) \cup \mathrm{var}(t) \cup \mathrm{var}(t')$. For every $x \in Y$ we can choose an index $i_x \in I$ such that $v(x) \in M_{i_x}$. Let us have an index set $J = \{i_x \mid v(x) \in M_{i_x} \text{ and } x \in Y\}$. Since $|Y| < \kappa$, it follows that $|J| < \kappa$, i.e. there is $i \in I$ such that $v(x) \in M_i$ for every $x \in Y$. Consequently, $\|P \Rightarrow (t \approx t')\|_{\bigcup_{i\in I}^{\kappa} \mathbf{M}_i, v} = \|P \Rightarrow (t \approx t')\|_{\mathbf{M}_i, v}$. Hence, $\bigcup_{i\in I}^{\kappa} \mathbf{M}_i \in \mathrm{Mod}(\Sigma) = \mathcal{K}$.

"⇐": Let $\mathcal{K}$ be a sur-reflective class which is closed under κ-direct unions. Put $\Sigma = \mathrm{Impl}_\kappa(\mathcal{K})$. We claim that $\mathcal{K} = \mathrm{Mod}(\mathrm{Impl}_\kappa(\mathcal{K}))$. Trivially, $\mathcal{K} \subseteq \mathrm{Mod}(\mathrm{Impl}_\kappa(\mathcal{K}))$. Thus, it remains to check the converse inequality. Doing so, it is sufficient to show that every κ-generated $\mathbf{L}$-algebra from $\mathrm{Mod}(\mathrm{Impl}_\kappa(\mathcal{K}))$ belongs to $\mathcal{K}$. Indeed, due to Lemma 10, every $\mathbf{M} \in \mathrm{Mod}(\mathrm{Impl}_\kappa(\mathcal{K}))$ is isomorphic to a κ-direct union of $\{[M']_{\mathbf{M}} \mid M' \subseteq M,\ |M'| < \kappa\} \subseteq \mathrm{Mod}(\mathrm{Impl}_\kappa(\mathcal{K}))$ and $\mathcal{K}$ is assumed to be closed under κ-direct unions.

So, let us have a κ-generated $\mathbf{M} \in \mathrm{Mod}(\mathrm{Impl}_\kappa(\mathcal{K}))$. Since $\mathcal{K}$ is sur-reflective, $\mathbf{M}$ has a sur-reflection $r_M : \mathbf{M} \to \mathbf{R}_{\mathcal{K}}(\mathbf{M})$ in $\mathcal{K}$. We will show that r_M is an embedding. By contradiction, suppose there are $\mathrm{b}, \mathrm{b}' \in M$ such that $\mathrm{b} \approx^{\mathbf{M}} \mathrm{b}' \not\geq r_M(\mathrm{b}) \approx^{\mathbf{R}_{\mathcal{K}}(\mathbf{M})} r_M(\mathrm{b}')$.

Let M', $|M'| < \kappa$, denote the set of generators of $\mathbf{M}$. For a subset of variables $Y \subseteq X$, $|Y| = |M'|$ we can consider a surjective valuation $v\colon Y \to M'$ and its surjective homomorphic extension $v^\sharp\colon \mathbf{T}(Y) \to \mathbf{M}$. Define an $\mathbf{L}$-set $P \in \mathbf{L}^{T(X)\times T(X)}$ by

$$P(s, s') = \begin{cases} v^\sharp(s) \approx^{\mathbf{M}} v^\sharp(s') & \text{for } s, s' \in T(Y), \\ 0 & \text{otherwise.} \end{cases}$$

Since $|Y| < \kappa$, it follows that $P \in \mathcal{P}_\kappa$. The surjectivity of $v^\sharp$ yields that there are terms $t, t' \in T(Y)$, where $v^\sharp(t) = \mathrm{b}$, $v^\sharp(t') = \mathrm{b}'$. Hence,

$$\|P \Rightarrow (t \approx t')\|_{\mathbf{M},v} = \|t \approx t'\|_{\mathbf{M},v} = v^\sharp(t) \approx^{\mathbf{M}} v^\sharp(t') \\ = \mathrm{b} \approx^{\mathbf{M}} \mathrm{b}' \not\geq r_M(\mathrm{b}) \approx^{\mathbf{R}_{\mathcal{K}}(\mathbf{M})} r_M(\mathrm{b}').$$

Since $\mathbf{M} \in \mathrm{Mod}(\mathrm{Impl}_\kappa(\mathcal{K}))$, we have

$$\big(\mathrm{Impl}_\kappa(\mathcal{K})\big)(P \Rightarrow (t \approx t')) \not\geq r_M(\mathrm{b}) \approx^{\mathbf{R}_{\mathcal{K}}(\mathbf{M})} r_M(\mathrm{b}').$$

Thus, there is an $\mathbf{L}$-algebra $\mathbf{N} \in \mathcal{K}$ and a valuation $w: Y \to N$, where $P(s, s') \leq \|s \approx s'\|_{\mathbf{N},w}$ holds for all terms $s, s' \in T(Y)$, and $\|t \approx t'\|_{\mathbf{N},w} \not\geq r_M(\mathrm{b}) \approx^{\mathbf{R}_{\mathcal{K}}(\mathbf{M})} r_M(\mathrm{b}')$.

On the other hand, we clearly have $\theta_{v^\sharp} \subseteq \theta_{w^\sharp}$. Thus, there is a morphism $g: \mathbf{M} \to \mathbf{N}$ such that $w^\sharp = v^\sharp \circ g$, see Fig. 3. Since $\mathbf{N} \in \mathcal{K}$, there is a morphism $g' : \mathbf{R}_{\mathcal{K}}(\mathbf{M}) \to \mathbf{N}$, where $g = r_M \circ g'$. As a consequence, $w^\sharp = v^\sharp \circ r_M \circ g'$. Moreover,

$$\|t \approx t'\|_{\mathbf{N},w} = w^\sharp(t) \approx^{\mathbf{N}} w^\sharp(t') = g'(r_M(v^\sharp(t))) \approx^{\mathbf{N}} g'(r_M(v^\sharp(t'))) \\ = g'(r_M(\mathrm{b})) \approx^{\mathbf{N}} g'(r_M(\mathrm{b}')) \geq r_M(\mathrm{b}) \approx^{\mathbf{R}_{\mathcal{K}}(\mathbf{M})} r_M(\mathrm{b}')$$

which is a contradiction. Altogether, $\mathcal{K} = \mathrm{Mod}(\Sigma)$ for $\Sigma = \mathrm{Impl}_\kappa(\mathcal{K})$. □

Remark 5. Note that the "⇐" part of the original proof of Lemma 11 for ordinary algebras (see [1]) differs from that one presented above. In [1], the author presents a direct construction of an algebra in $\mathcal{K}$, while we proceed by contradiction and use properties of sur-reflections. From the viewpoint of theory of $\mathbf{L}$-algebras, the original construction pertains only to trivial $\approx^{\mathbf{M}}$'s and is thus not applicable for general $\mathbf{L}$-algebras.

Analogously as for $\mathbf{L}$-sets of $\mathcal{P}$-implications and accordingly to the relationship between $\mathbf{L}$-set of $\mathcal{P}$-implications and sets of weighted $\mathcal{P}$-implications we denote the class of all models of a class Σ of weighted $\mathcal{P}$-implications by $\mathrm{Mod}(\Sigma)$, i.e.

$$\mathrm{Mod}(\Sigma) = \left\{\mathbf{M} \,|\, a \leq \|P \Rightarrow (t \approx t')\|_{\mathbf{M}} \text{ for all } \langle P \Rightarrow (t \approx t'), a\rangle \in \Sigma\right\}.$$

Lemma 12. *Let* $\mathbf{L}^*$ *be a complete residuated lattice with globalization,* $\mathcal{K}$ *be a sur-reflective class of* $\mathbf{L}$*-algebras of type* F*. Then there is a class* Σ *of weighted implications with weighted premises such that* $\mathcal{K} = \bigcap_{\mathbf{M}} \mathrm{Mod}(\Sigma^{\mathcal{K}}_{\mathbf{M}}) = \mathrm{Mod}(\Sigma)$*, where* $\mathbf{M}$ *ranges over all* $\mathbf{L}$*-algebras of type* F*.*

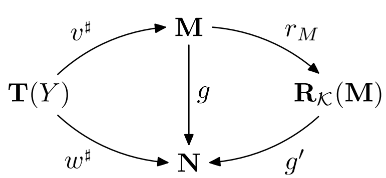

Fig. 3. Scheme for the proof of Lemma 11

Proof. Let Σ be a class, where $\langle P \Rightarrow (t \approx t'), a\rangle \in \Sigma$ iff $\Sigma^{\mathcal{K}}_{\mathbf{M}}(P \Rightarrow (t \approx t')) = a$ for some $\mathbf{L}$-algebra $\mathbf{M}$ of type F. One can easily verify that $\bigcap_{\mathbf{M}} \mathrm{Mod}(\Sigma^{\mathcal{K}}_{\mathbf{M}}) = \mathrm{Mod}(\Sigma)$. The rest follows from Lemma 7. □

Lemma 13. *Let* $\mathbf{L}^*$ *be a complete residuated lattice with a truth stresser* * *defined by* (7). *Assuming Principle 1, for every sur-reflective class* $\mathcal{K}$ *of* $\mathbf{L}$*-algebras there exists an* $\mathbf{L}$*-set* Σ *of* $\mathcal{P}$*-implications such that* $\mathcal{K} = \mathrm{Mod}(\Sigma)$.

Proof. We will show by contradiction that if $\mathcal{K}$ were not definable by any $\mathbf{L}$-set of $\mathcal{P}$-implications then Principle 1 would be violated. Thus, assume $\mathcal{K}$ to be a sur-reflective class such that $\mathcal{K} \neq \mathrm{Mod}(\Sigma)$ for every $\mathbf{L}$-set Σ of $\mathcal{P}$-implications, where $\mathcal{P}$ is an arbitrary proper family of premises. Lemma 12 yields that $\mathcal{K}$ is definable by a class Σ of weighted implications with weighted premises.

For every infinite cardinal κ we can consider a proper family of premises $\mathcal{P}_\kappa$ such that $\mathcal{P} = \mathbf{L}^{T(X)\times T(X)}$, $|X| = \kappa$. Clearly, for every $\mathcal{P}_\kappa$ there is only a set of weighted $\mathcal{P}_\kappa$-implications in Σ. Moreover, from Lemma 11 it follows that $\mathcal{K}$ cannot be closed under κ-direct unions. That is, for every infinite cardinal κ there is an $\mathbf{L}$-algebra $\mathbf{M}_\kappa \notin \mathcal{K}$ such that $\mathbf{M}_\kappa$ is a κ-direct union of some κ-directed family $\{\mathbf{M}_{\kappa,i} \in \mathcal{K} \mid i \in I_\kappa\}$. Let us define an ordinal sequence of $\mathbf{L}$-algebras formed of such $\mathbf{M}_\kappa$'s. Put $\mathbf{N}_0 = \mathbf{M}_\omega$. For every ordinal α let $\mathbf{N}_\alpha$ be $\mathbf{M}_\kappa$ such that $\kappa > |\bigcup_{\beta<\alpha} N_\beta|$.

Observe that $|M_\kappa| < \kappa$ implies $\mathbf{M}_{\kappa,i} \notin \mathcal{K}$ for some index $i \in I_\kappa$, which contradicts $\mathbf{M}_{\kappa,i} \in \mathcal{K}$. Hence, for every $\mathbf{M}_\kappa$, we have $|M_\kappa| \geq \kappa$. It immediately follows that $|N_\alpha| > |N_\beta|$ for every $\beta < \alpha$. Thus, for $\beta < \alpha$ there cannot be an injective mapping sending elements of N_α to N_β. Since every embedding is injective, $\mathbf{N}_\alpha$ cannot be embedded into $\mathbf{N}_\beta$. On the other hand, $\mathbf{N}_\beta$ cannot be embedded into $\mathbf{N}_\alpha$ either. Indeed, suppose $h : \mathbf{N}_\beta \to \mathbf{N}_\alpha$ is an embedding. Since $\mathbf{N}_\alpha$ is a κ-direct union and $|N_\beta| < \kappa$ by definition, for every $\mathrm{a} \in N_\beta$ there is an index $i_{\mathrm{a}} \in I_\kappa$ such that $h(\mathrm{a}) \in M_{\kappa,i_{\mathrm{a}}}$. Moreover, $|\{i_{\mathrm{a}} \mid \mathrm{a} \in N_\beta\}| < \kappa$, i.e., there is some $k \in I_\kappa$ such that $h(N_\beta) \subseteq M_{\kappa,k}$. Thus, $\mathbf{N}_\beta \notin \mathcal{K}$ is a subalgebra of $\mathbf{M}_{\kappa,k} \in \mathcal{K}$, which is a contradiction.

To sum up, the class of all $\mathbf{N}_\alpha$'s is proper and there is no $\mathbf{N}_\alpha$ which can be embedded into another $\mathbf{N}_\beta$ (for $\alpha \neq \beta$), i.e. Principle 1 is violated. □

The following theorem summarizes the equivalent characterizations of $\mathcal{P}$-implicational classes being defined using globalization.

Theorem 6. *Assume Principle 1. Suppose* $\mathbf{L}^*$ *is a complete residuated lattice with a truth stresser* * *defined by* (7). *Then for any abstract class* $\mathcal{K}$ *of* $\mathbf{L}$*-algebras the following are equivalent:*

(i) $\mathcal{K}$ *is a* $\mathcal{P}$*-implicational class,*
(ii) $\mathcal{K}$ *is closed under* S *and* P,
(iii) $\mathcal{K} = \mathrm{SP}(\mathcal{K})$,
(iv) $\mathcal{K} = \mathrm{SP}(\mathcal{K}')$ *for some abstract class* $\mathcal{K}'$ *of* $\mathbf{L}$*-algebras,*
(v) $\mathcal{K}$ *is a sur-reflective class,*
(vi) $\mathcal{K} = \mathrm{Mod}(\Sigma)$ *for some class* Σ *of weighted implications,*
(vii) $\mathcal{K} = \mathrm{Mod}(\mathrm{Impl}(\mathcal{K}))$,
(viii) $\mathcal{K} = \mathrm{Mod}(\mathrm{Impl}(\mathcal{K}'))$ *for some abstract class* $\mathcal{K}'$ *of* $\mathbf{L}$*-algebras.*

Proof. "(i) ⇒ (ii)": Consequence of Lemma 2.

"(ii) ⇒ (iii)": Clearly, $\mathrm{SP}(\mathcal{K}) = \mathrm{S}(\mathcal{K}) = \mathcal{K}$.

"(iii) ⇒ (iv)": Trivial.

"(iv) ⇒ (v)": Evidently, $\mathrm{S}(\mathcal{K}) = \mathrm{SSP}(\mathcal{K}') = \mathrm{SP}(\mathcal{K}') = \mathcal{K}$. Analogously, $\mathrm{P}(\mathcal{K}) = \mathrm{PSP}(\mathcal{K}') \subseteq \mathrm{SPP}(\mathcal{K}') = \mathrm{SP}(\mathcal{K}') = \mathcal{K}$. Now, apply Theorem 4.

"(v) ⇒ (vi)": Apply Lemma 12.

"(vi) ⇒ (vii)": Clearly, $\mathrm{Mod}(\Sigma)$ is sur-reflective since it is an abstract class closed under S, P (routine to check as in Theorem 2). From Theorem 13 it follows that $\mathcal{K} = \mathrm{Mod}(\Sigma')$ for some $\mathbf{L}$-set Σ' of $\mathcal{P}_\kappa$-implications. Therefore, we get $\mathcal{K} = \mathrm{Mod}(\Sigma') = \mathrm{Mod}(\mathrm{Impl}(\mathrm{Mod}(\Sigma'))) = \mathrm{Mod}(\mathrm{Impl}(\mathcal{K}))$.

"(vii) ⇒ (viii)": Trivial.

"(viii) ⇒ (i)": By definition. □

A class $\mathcal{K}$ of $\mathbf{L}$-algebras is called *semivariety* if it is an abstract class closed under formations of subalgebras, direct products, and direct unions. Using analogous arguments as in ordinary case, one can show that USP is a closure operator on abstract classes of $\mathbf{L}$-algebras and $\mathrm{USP}(\mathcal{K})$ is the semivariety generated by $\mathcal{K}$. Now the following characterization is a consequence of Lemma 4, Theorem 4, Lemma 9, and Lemma 11.

Theorem 7. *Suppose* $\mathbf{L}^*$ *is a complete residuated lattice with a truth stresser* * *defined by* (7) *and let* $\mathcal{K}$ *be an abstract class of* $\mathbf{L}$*-algebras. Then* $\mathcal{K}$ *is a* $\mathcal{P}$*-finitary implicational class for some proper family of premises* $\mathcal{P}$ *iff* $\mathcal{K}$ *is a semivariety.* □

From now on, we shall assume Principle 1 when necessary.

4. Quasivarieties

Finally, we study $\mathcal{P}$-Horn classes. Quasivarieties of algebras have been widely studied. There are various approaches to quasivarieties. Our investigation is based mainly on the approach described in [12], but we comment on some other approaches as well.

Definition 7. *A class* $\mathcal{K}$ *of* $\mathbf{L}$*-algebras is called a* ***quasivariety*** *if it is an abstract class closed under formations of subalgebras, direct products, and direct limits of direct families of* $\mathbf{L}$*-algebras from* $\mathcal{K}$.

Remark 6. By Theorem 4, quasivarieties are sur-reflective classes closed under direct limits of direct families.

Since every quasivariety $\mathcal{K}$ is a sur-reflective class, we can consider a sur-reflection of $\mathbf{M}$ in $\mathcal{K}$. We will use mainly sur-reflections of finitely presented $\mathbf{L}$-algebras. Thus, we will adopt the following convention. For a finitely presented $\mathbf{L}$-algebra $\mathbf{T}(X)/\theta(R)$ let $\mathbf{R}_{\mathcal{K}}(X, R)$ denote the sur-reflection of $\mathbf{T}(X)/\theta(R)$ in $\mathcal{K}$, the corresponding epimorphism will be usually denoted simply by r.

In ordinary case, sur-reflections of finitely presented algebras play an important role in the theory of quasivarieties since every quasivariety can be reconstructed by direct limits of such sur-reflections. In what follows we focus on this fact in fuzzy setting.

Definition 8. *Let $\mathcal{K}$ be an abstract class of* $\mathbf{L}$*-algebras of type F. An* $\mathbf{L}$*-algebra* $\mathbf{M}$ *of type F is said to* satisfy the ***QF condition*** w.r.t. $\mathcal{K}$*, if every morphism $h : \mathbf{T}(X)/\theta(R) \to \mathbf{M}$, where $\mathbf{T}(X)/\theta(R)$ is a finitely presented* $\mathbf{L}$*-algebra factorizes through $\mathbf{R}_{\mathrm{SP}(\mathcal{K})}(X, R)$, i.e. $h = r \circ h'$ for a sur-reflection $r : \mathbf{T}(X)/\theta(R) \to \mathbf{R}_{\mathrm{SP}(\mathcal{K})}(X, R)$ of $\mathbf{T}(X)/\theta(R)$ in* $\mathrm{SP}(\mathcal{K})$ *and a morphism h' sending elements of $\mathbf{R}_{\mathrm{SP}(\mathcal{K})}(X, R)$ to* $\mathbf{M}$.

Lemma 14. *Suppose* $\mathbf{L}$ *is a finite residuated lattice. Let $\mathcal{K}$ be an abstract class of* $\mathbf{L}$*-algebras of type F and let* $\mathbf{M}$ *be an* $\mathbf{L}$*-algebra satisfying the QF condition w.r.t. $\mathcal{K}$. Then $\mathbf{M} \cong \mathbf{R}$, where $\mathbf{R}$ is a direct limit of a direct family which consists of sur-reflections $\mathbf{R}_{\mathrm{SP}(\mathcal{K})}(X, R)$ of certain finitely presented* $\mathbf{L}$*-algebras.*

Proof. By adoption of the argument from ordinary case (see [25]) to fuzzy setting, we give only a sketch.

Let $\mathbf{M}$ satisfy QF w.r.t. $\mathcal{K}$. $\mathbf{M}$ is isomorphic to a direct limit $\lim \mathbf{T}(Y_i)/\theta(S_i)$ of a direct family $\left\{h_{ij} : \mathbf{T}(Y_i)/\theta(S_i) \to \mathbf{T}(Y_j)/\theta(S_j) \mid i \leq j\right\}$ of finitely presented $\mathbf{L}$-algebras, see [24].

One can show that for sur-reflections $r_i : \mathbf{T}(Y_i)/\theta(S_i) \to \mathbf{R}_{\mathrm{SP}(\mathcal{K})}(Y_i, S_i)$ $(i \in I)$ there are morphisms $g_{ij} : \mathbf{R}_{\mathrm{SP}(\mathcal{K})}(Y_i, S_i) \to \mathbf{R}_{\mathrm{SP}(\mathcal{K})}(Y_j, S_j)$ such that $\left\{\mathbf{R}_{\mathrm{SP}(\mathcal{K})}(Y_i, S_i) \mid i \in I\right\}$ together with such g_{ij}'s is a weak direct family of $\mathbf{L}$-algebras. Moreover, finiteness of $\mathbf{L}$ implies that this family is in fact a direct family. Now it suffices to check that $\mathbf{M} \cong \lim \mathbf{R}_{\mathrm{SP}(\mathcal{K})}(Y_i, S_i)$ which can be shown using the QF condition and direct limit property, see [24]. □

Note the restriction on finiteness of $\mathbf{L}$ in Lemma 14. It is used to guarantee that a certain weak direct family is a direct family (see the proof of Lemma 14). Since every $\mathbf{M}$ of a quasivariety $\mathcal{K}$ satisfies the QF condition w.r.t. $\mathcal{K}$, we have the following corollary.

Corollary 2. *Suppose* $\mathbf{L}$ *is a finite residuated lattice. Let $\mathcal{K}$ be a quasivariety of* $\mathbf{L}$*-algebras, and let $\mathbf{M} \in \mathcal{K}$. Then $\mathbf{M} \cong \mathbf{R}$, where $\mathbf{R}$ is a direct limit of a direct family of certain sur-reflections $\mathbf{R}_{\mathcal{K}}(X, R)$ of finitely presented algebras. Hence, given a class $\mathcal{K}' = \{\mathbf{R}_{\mathcal{K}}(X, R) \mid X, R \text{ are finite}\}$ it follows that $\mathcal{K} = \mathrm{IL}(\mathcal{K}')$.* □

Remark 7. Analogously as in ordinary case, using QF condition and properties of direct limits, for finite $\mathbf{L}$ it can be shown that LSP is a closure operator on abstract classes of $\mathbf{L}$-algebras and $\mathrm{LSP}(\mathcal{K})$ is the quasivariety generated by $\mathcal{K}$, generalizing the result of T. Fujiwara, see [12].

Finally, we characterize quasivarieties as $\mathcal{P}$-Horn classes.

Lemma 15. *Suppose* $\mathbf{L}^*$ *is a finite residuated lattice with a truth stresser * defined by* (7)*. If $\mathcal{K}$ is a quasivariety then $\mathcal{K} = \mathrm{Mod}(\mathrm{Horn}(\mathcal{K}))$.*

Proof. Let $\mathcal{P} = \mathbf{L}^{T(Y) \times T(Y)}$, where Y is a denumerable set of variables. $\mathcal{K} \subseteq \mathrm{Mod}(\mathrm{Horn}(\mathcal{K}))$ is obvious. It remains to show the converse inclusion.

For the sake of brevity, put $\mathcal{K}' = \mathrm{Mod}(\mathrm{Horn}(\mathcal{K}))$. From Lemma 6 it follows that $\mathcal{K}'$ is a quasivariety. Thus, every $\mathbf{M} \in \mathcal{K}'$ is a isomorphic to direct limit of some sur-reflections $\mathbf{R}_{\mathcal{K}'}(X, R)$ of finitely presented $\mathbf{L}$-algebras, see Corollary 2. Hence, it suffices to show that every sur-reflection $\mathbf{R}_{\mathcal{K}'}(X, R)$ of a finitely presented $\mathbf{L}$-algebra $\mathbf{T}(X)/\theta(R)$ belongs to $\mathcal{K}$. Then clearly, since $\mathcal{K}$ is an abstract class closed under L, we obtain $\mathbf{M} \in \mathcal{K}$.

Let $\mathbf{T}(X)/\theta(R)$, where X, R are finite (without loss of generality, we can assume $X \subseteq Y$), and let $r : \mathbf{T}(X)/\theta(R) \to \mathbf{R}_{\mathcal{K}}(X, R)$, $r' : \mathbf{T}(X)/\theta(R) \to \mathbf{R}_{\mathcal{K}'}(X, R)$ be the sur-reflections of $\mathbf{T}(X)/\theta(R)$ in $\mathcal{K}, \mathcal{K}'$. Since $\mathcal{K} \subseteq \mathcal{K}'$ it follows that $\mathbf{R}_{\mathcal{K}}(X, R) \in \mathcal{K}'$. As a consequence, the sur-reflection $r : \mathbf{T}(X)/\theta(R) \to \mathbf{R}_{\mathcal{K}}(X, R)$ factorizes through $\mathbf{R}_{\mathcal{K}'}(X, R)$, i.e. there is a morphism $g' : \mathbf{R}_{\mathcal{K}'}(X, R) \to \mathbf{R}_{\mathcal{K}}(X, R)$ such that $r = r' \circ g'$. We finish the proof by demonstrating that g' is an embedding, since then $\mathbf{R}_{\mathcal{K}'}(X, R) \in \mathrm{S}(\mathcal{K}) = \mathcal{K}$.

We proceed by contradiction. So, let there be elements b, b' such that

$$\mathrm{b} \approx^{\mathbf{R}_{\mathcal{K}'}(X,R)} \mathrm{b}' \not\geq g'(\mathrm{b}) \approx^{\mathbf{R}_{\mathcal{K}}(X,R)} g'(\mathrm{b}').$$

Moreover, X, R are finite, i.e. we can consider a finite $\mathbf{L}$-set of premises $P \in \mathbf{L}^{T(X) \times T(X)}$ such that

$$P(s, s') = \begin{cases} \theta(R)(s, s') & \text{for } R(s, s') > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define a valuation v of X in $\mathbf{R}_{\mathcal{K}'}(X, R)$ by $v(x) = r'\big([x]_{\theta(R)}\big)$ for every $x \in X$. Hence, for the homomorphic extension $v^\sharp : \mathbf{T}(X) \to \mathbf{R}_{\mathcal{K}'}(X, R)$ we have $v^\sharp(s) = r'\big([s]_{\theta(R)}\big)$ for all terms $s \in T(X)$. Clearly,

$$\begin{aligned} P(s, s') &\leq \theta(R)(s, s') = [s]_{\theta(R)} \approx^{\mathbf{T}(X)/\theta(R)} \big[s'\big]_{\theta(R)} \\ &\leq r'\big([s]_{\theta(R)}\big) \approx^{\mathbf{R}_{\mathcal{K}'}(X,R)} r'\big(\big[s'\big]_{\theta(R)}\big) \\ &= v^\sharp(s) \approx^{\mathbf{R}_{\mathcal{K}'}(X,R)} v^\sharp(s') = \big\| s \approx s' \big\|_{\mathbf{R}_{\mathcal{K}'}(X,R),v} \end{aligned}$$

holds for all $s, s' \in T(X)$. Since r' is surjective, there are terms $t, t' \in T(X)$ such that $r'\big([t]_{\theta(R)}\big) = \mathrm{b}$, and $r'\big(\big[t'\big]_{\theta(R)}\big) = \mathrm{b}'$. Thus,

$$\begin{aligned} \big\| P \Rightarrow (t \approx t') \big\|_{\mathbf{R}_{\mathcal{K}'}(X,R),v} &= \big\| t \approx t' \big\|_{\mathbf{R}_{\mathcal{K}'}(X,R),v} = v^\sharp(t) \approx^{\mathbf{R}_{\mathcal{K}'}(X,R)} v^\sharp(t') \\ &= r'\big([t]_{\theta(R)}\big) \approx^{\mathbf{R}_{\mathcal{K}'}(X,R)} r'\big(\big[t'\big]_{\theta(R)}\big) \\ &= \mathrm{b} \approx^{\mathbf{R}_{\mathcal{K}'}(X,R)} \mathrm{b}' \not\geq g'(\mathrm{b}) \approx^{\mathbf{R}_{\mathcal{K}}(X,R)} g'(\mathrm{b}'). \end{aligned}$$

Therefore, $\big\| P \Rightarrow (t \approx t') \big\|_{\mathbf{R}_{\mathcal{K}'}(X,R)} \not\geq g'(\mathrm{b}) \approx^{\mathbf{R}_{\mathcal{K}}(X,R)} g'(\mathrm{b}')$. Moreover, we have $\mathbf{R}_{\mathcal{K}'}(X, R) \in \mathcal{K}' = \mathrm{Mod}(\mathrm{Horn}(\mathcal{K}))$. Thus,

$$\big(\mathrm{Horn}(\mathcal{K})\big)(P \Rightarrow (t \approx t')) \not\geq g'(\mathrm{b}) \approx^{\mathbf{R}_{\mathcal{K}}(X,R)} g'(\mathrm{b}').$$

Hence, there is $\mathbf{N} \in \mathcal{K}$, and $w : X \to N$ such that $P(s, s') \leq \big\| s \approx s' \big\|_{\mathbf{N},w}$ for all $s, s' \in T(X)$ but $\big\| t \approx t' \big\|_{\mathbf{N},w} \not\geq g'(\mathrm{b}) \approx^{\mathbf{R}_{\mathcal{K}}(X,R)} g'(\mathrm{b}')$. Observe that for a homomorphic extension $w^\sharp : \mathbf{T}(X) \to \mathbf{N}$ we have $R \subseteq P \subseteq \theta_{w^\sharp}$, and thus $\theta(R) \subseteq \theta_{w^\sharp}$. As a consequence, there is a morphism $h : \mathbf{T}(X)/\theta(R) \to \mathbf{N}$ with

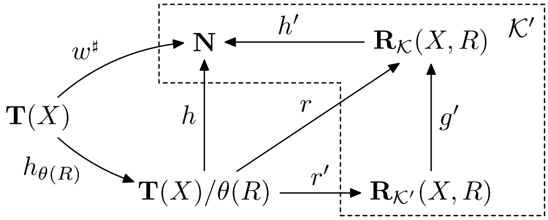

Fig. 4. Scheme for the proof of Theorem 15

$w^\sharp = h_{\theta(R)} \circ h$. Since $\mathbf{N} \in \mathcal{K}$, h factorizes through $\mathbf{R}_{\mathcal{K}}(X, R)$, i.e. there is a morphism $h' : \mathbf{R}_{\mathcal{K}}(X, R) \to \mathbf{N}$ such that $h = r \circ h'$, see Fig. 4. Now it readily follows that $w^\sharp = h_{\theta(R)} \circ h = h_{\theta(R)} \circ r \circ h' = h_{\theta(R)} \circ r' \circ g' \circ h'$. Thus, we have

$$\begin{aligned}
\|t \approx t'\|_{\mathbf{N},w} &= w^\sharp(t) \approx^{\mathbf{N}} w^\sharp(t') \\
&= (h_{\theta(R)} \circ r' \circ g' \circ h')(t) \approx^{\mathbf{N}} (h_{\theta(R)} \circ r' \circ g' \circ h')(t') \\
&= (r' \circ g' \circ h')\big([t]_{\theta(R)}\big) \approx^{\mathbf{N}} (r' \circ g' \circ h')\big([t']_{\theta(R)}\big) \\
&\geq (r' \circ g')\big([t]_{\theta(R)}\big) \approx^{\mathbf{R}_{\mathcal{K}}(X,R)} (r' \circ g')\big([t']_{\theta(R)}\big) \\
&= g'(\mathrm{b}) \approx^{\mathbf{R}_{\mathcal{K}}(X,R)} g'(\mathrm{b}')
\end{aligned}$$

which contradicts $\|t \approx t'\|_{\mathbf{N},w} \not\geq g'(\mathrm{b}) \approx^{\mathbf{R}_{\mathcal{K}}(X,R)} g'(\mathrm{b}')$. Hence, g' is an embedding, $\mathbf{R}_{\mathcal{K}'}(X, R) \in \mathcal{K}$. □

Remark 8. Note that the proof of the bivalent version of Lemma 15 as presented in [25] is not correct. In [25], it is claimed that if $\mathbf{M}$ is isomorphic to a direct limit $\lim \mathbf{T}(Y_i)/\theta(S_i)$ of finitely presented algebras, and if $\mathbf{M} \in \mathrm{Mod}(\mathrm{Horn}(\mathcal{K}))$, then every $\mathbf{T}(Y_i)/\theta(S_i) \in \mathrm{Mod}(\mathrm{Horn}(\mathcal{K}))$. This is not true as it is demonstrated by the following counterexample.

Let us have a term algebra $\mathbf{T}(X)$, where X is finite. Clearly, $\mathbf{T}(X)$ is finitely presented. Take $\mathbf{M} \in \mathcal{K}$, where $\mathcal{K}$ is a variety such that $\mathbf{T}(X) \notin \mathcal{K}$. Moreover, algebra $\mathbf{M}$ is isomorphic to a direct limit $\lim \mathbf{T}(Y_i)/\theta(S_i)$ of a direct family $\{\mathbf{T}(Y_i)/\theta(S_i) \mid i \in I\}$ of finitely presented algebras. We can assume $\mathbf{T}(X) \in \{\mathbf{T}(Y_i)/\theta(S_i) \mid i \in I\}$ (if $\mathbf{T}(X) \notin \{\mathbf{T}(Y_i)/\theta(S_i) \mid i \in I\}$, $\mathbf{T}(X)$ can be added to $\{\mathbf{T}(Y_i)/\theta(S_i) \mid i \in I\}$ using morphisms $g_i : \mathbf{T}(X) \to \mathbf{T}(Y_i)/\theta(S_i)$ defined by $g_i(t) = [t]_{\theta(S_i)}$ for every $i \in I$ with $X \subseteq Y_i$—this can be made without loss of generality, see [24]). Using the argument from [25], one can conclude $\{\mathbf{T}(Y_i)/\theta(S_i) \mid i \in I\} \subseteq \mathrm{Mod}(\mathrm{Horn}(\mathcal{K})) = \mathcal{K}$. That is, the term algebra $\mathbf{T}(X)$ would be a member of $\mathcal{K}$—a contradiction.

Hence, for finite $\mathbf{L}^*$ with globalization we have the following characterization.

Theorem 8. *Suppose* $\mathbf{L}^*$ *is a finite residuated lattice equipped with* * *defined by* (7) *and let* $\mathcal{K}$ *be an abstract class of* $\mathbf{L}$*-algebras. Then* $\mathcal{K}$ *is a* $\mathcal{P}$*-Horn class for some proper family of premises* $\mathcal{P}$ *iff* $\mathcal{K}$ *is a quasivariety.* □

$\approx^{\mathbf{M}_i}$	$\mathfrak{a}_i$	$\mathfrak{a}'_i$	$\mathfrak{b}_i$	$\mathfrak{b}'_i$
$\mathfrak{a}_i$	1	i	0	0
$\mathfrak{a}'_i$	i	1	0	0
$\mathfrak{b}_i$	0	0	1	0
$\mathfrak{b}'_i$	0	0	0	1

Fig. 5. **L**-equality from Example 2

Remark 9. It is well known that in ordinary case, the collections of all varieties, quasivarieties, semivarieties, and sur-reflective classes are pairwise distinct. This applies to the fuzzy case as well. Namely, suppose $\mathbf{L}^*$ is a complete residuated lattice equipped with * defined by (7). Let Σ be an **L**-set of $\mathcal{P}$-implications given by

$$\Sigma = \{\langle x \approx y, a\rangle \Rightarrow \langle x \approx y, 1\rangle \,|\, a \in L,\ a \neq 0\}\,.$$

Evidently, $\mathrm{Mod}(\Sigma)$ consists of all **L**-algebras (of the given type) with crisp **L**-equalities. Thus, $\mathrm{Mod}(\Sigma)$ is a quasivariety which is not a variety since $\mathrm{Mod}(\Sigma)$ is not closed under homomorphic images. To see that quasivarieties and semivarieties of **L**-algebras are distinct, take an ordinary semivariety $\mathcal{K}$ which is not a quasivariety (such $\mathcal{K}$ exists). Consider a class $\mathcal{K}'$ of **L**-algebras such that $\mathbf{M}' \in \mathcal{K}'$ results from $\mathbf{M} \in \mathcal{K}$ by equipping $\mathbf{M}$ with the crisp **L**-equality. Then $\mathcal{K}'$ is a semivariety of **L**-algebras which is not a quasivariety (observe that $\mathcal{K}'$ is closed under any of I, S, P, U, L iff $\mathcal{K}$ is closed under the corresponding crisp operator). In a similar way one can get a sur-reflective class of **L**-algebras which is not a semivariety.

Let us comment on the restriction of finiteness of **L** present in our characterization of quasivarieties. This restriction does not pertain sur-reflective classes and semivarieties. In case of quasivarieties, finiteness of **L** was used to ensure that a weak direct family of **L**-algebras is a direct family. One can ask whether it is possible to work with unrestricted weak direct families and arbitrary complete residuated lattices instead. The following counterexample gives a negative answer by showing that $\mathcal{P}$-Horn classes are not closed under direct limits of arbitrary weak direct families.

Example 2. Take $\mathbf{L}^*$, where **L** is a structure of truth values on the unit interval $[0, 1]$, and * is defined by (7). Let $F = \{f_1, f_2, g_1, g_2\}$ be a type of **L**-algebras such that f_1, f_2, g_1, g_2 are nullary function symbols (constants). Consider a proper family of premises $\mathcal{P} = \mathbf{L}^{T(\emptyset)\times T(\emptyset)}$ and $\mathcal{P}$-Horn clause

$$\langle f_1 \approx f_2, 1\rangle \Rightarrow g_1 \approx g_2.$$

Moreover, let Σ be an **L**-set of $\mathcal{P}$-Horn clauses such that

$$\mathrm{Supp}(\Sigma) = \{\langle f_1 \approx f_2, 1\rangle \Rightarrow g_1 \approx g_2\}\,.$$

Thus, $\mathrm{Mod}(\Sigma)$ is a $\mathcal{P}$-Horn class of **L**-algebras.

Now we introduce a weak direct family of **L**-algebras. Let $\langle I, \leq\rangle$ with $I = [0, 1)$ be a directed index set and let $\mathbf{M}_i = \left\langle M_i, \approx^{\mathbf{M}_i}, f_1^{\mathbf{M}_i}, f_2^{\mathbf{M}_i}, g_1^{\mathbf{M}_i}, g_2^{\mathbf{M}_i}\right\rangle$, $i \in I$ be

$\mathbf{L}$-algebras of type F such that $M_i = \{\mathrm{a}_i, \mathrm{a}'_i, \mathrm{b}_i, \mathrm{b}'_i\}$, $f_1^{\mathbf{M}_i} = \mathrm{a}_i$, $f_2^{\mathbf{M}_i} = \mathrm{a}'_i$, $g_1^{\mathbf{M}_i} = \mathrm{b}_i$, $g_2^{\mathbf{M}_i} = \mathrm{b}'_i$, and $\approx^{\mathbf{M}_i}$ is an $\mathbf{L}$-equality defined in Fig. 5. In addition to that, we define a family $\{h_{ij} : \mathbf{M}_i \to \mathbf{M}_j \mid i \leq j\}$ of morphisms by $h_{ij}(\mathrm{a}_i) = \mathrm{a}_j$, $h_{ij}(\mathrm{a}'_i) = \mathrm{a}'_j$, $h_{ij}(\mathrm{b}_i) = \mathrm{b}_j$, $h_{ij}(\mathrm{b}'_i) = \mathrm{b}'_j$. It is immediate that this defines a weak direct family of $\mathbf{L}$-algebras which is not a direct family.

Clearly, $\|f_1 \approx f_2\|_{\mathbf{M}_i} = i < 1$ for every $i \in I$, thus we have

$$\|\langle f_1 \approx f_2, 1\rangle \Rightarrow g_1 \approx g_2\|_{\mathbf{M}_i} = 1.$$

Hence, $\mathbf{M}_i \in \mathrm{Mod}(\Sigma)$ for all $i \in I$. On the other hand, for the direct limit $\lim \mathbf{M}_i$ it follows that $\theta_\infty(\mathrm{a}_i, \mathrm{a}'_i) = 1$, and $\theta_\infty(\mathrm{b}_i, \mathrm{b}'_i) = 0$. That is,

$$f_1^{\lim \mathbf{M}_i} = [\mathrm{a}_i]_{\theta_\infty} = \left[\mathrm{a}'_i\right]_{\theta_\infty} = f_2^{\lim \mathbf{M}_i}, \;\; g_1^{\lim \mathbf{M}_i} = [\mathrm{b}_i]_{\theta_\infty} \neq \left[\mathrm{b}'_i\right]_{\theta_\infty} = g_2^{\lim \mathbf{M}_i},$$

i.e. $f_1^{\lim \mathbf{M}_i} \approx^{\lim \mathbf{M}_i} f_2^{\lim \mathbf{M}_i} = 1$ while $g_1^{\lim \mathbf{M}_i} \approx^{\lim \mathbf{M}_i} g_2^{\lim \mathbf{M}_i} = 0$. Thus,

$$\|\langle f_1 \approx f_2, 1\rangle \Rightarrow g_1 \approx g_2\|_{\lim \mathbf{M}_i} = \|g_1 \approx g_2\|_{\lim \mathbf{M}_i} = 0.$$

As a consequence, $\lim \mathbf{M}_i \notin \mathrm{Mod}(\Sigma)$ since $\Sigma(\langle f_1 \approx f_2, 1\rangle \Rightarrow g_1 \approx g_2) > 0$. In other words, a $\mathcal{P}$-Horn class $\mathrm{Mod}(\Sigma)$ is not closed under direct limits of arbitrary weak direct families.

* * *

Now we present a characterization of $\mathcal{P}$-Horn classes as abstract classes of $\mathbf{L}$-algebras closed under formations of subalgebras and reduced products. It is worth to add that as in Lemma 15, we need to suppose a finite structure of truth values. We start by closedness of $\mathcal{P}$-implicational classes under safe reduced products.

Lemma 16. *Let* $\mathbf{L}^*$ *be a complete residuated lattice with an implicational truth stresser* *. *Suppose an* $\mathbf{L}$-*set* Σ *of* $\mathcal{P}$-*Horn clauses is given. Then* $\mathrm{Mod}(\Sigma)$ *is closed under safe reduced products.*

Proof. Since every safe reduced product $\bigtimes_F \mathbf{M}_i$ ($\mathbf{M}_i \in \mathcal{K}$) is isomorphic to a direct limit $\lim \mathbf{M}_Z$ of certain direct family $\{\mathbf{M}_Z \in \mathrm{Mod}(\Sigma) \mid Z \in F\}$ (see [24]), the closedness of $\mathrm{Mod}(\Sigma)$ under safe reduced products is a consequence of Lemma 5. □

Before delving into the converse problem, note that $\mathcal{P}$-Horn classes are not closed under arbitrary reduced products as shown by the following counterexample.

Example 3. Take $\mathbf{L}^*$ such that $L = [0, 1]$, and * is defined by (7). Consider the same type of $\mathbf{L}$-algebras and Σ as in Example 2. Let $\mathbb{N}$ be an index set and for every $i \in \mathbb{N}$ let $\mathbf{M}_i = \langle M_i, \approx^{\mathbf{M}_i}, f_1^{\mathbf{M}_i}, f_2^{\mathbf{M}_i}, g_1^{\mathbf{M}_i}, g_2^{\mathbf{M}_i}\rangle$ be defined the same way as in Example 2 except that for $\mathrm{a}_i, \mathrm{a}'_i$ we put $\mathrm{a}_i \approx^{\mathbf{M}_i} \mathrm{a}'_i = \mathrm{a}'_i \approx^{\mathbf{M}_i} \mathrm{a}_i = 1 - \frac{1}{i}$. Clearly, $\mathbf{M}_i \in \mathrm{Mod}(\Sigma)$ for every $i \in \mathbb{N}$. Let F be a Fréchet filter over $\mathbb{N}$. Put $\mathbf{M} = \bigtimes_{i \in \mathbb{N}} \mathbf{M}_i$. For every $X \in F$ we have

$$\begin{aligned}[\![f_1^{\mathbf{M}} \approx f_2^{\mathbf{M}}]\!]_X &= \textstyle\bigwedge_{i \in X} f_1^{\mathbf{M}}(i) \approx^{\mathbf{M}_i} f_2^{\mathbf{M}}(i) \\ &= \textstyle\bigwedge_{i \in X} f_1^{\mathbf{M}_i} \approx^{\mathbf{M}_i} f_2^{\mathbf{M}_i} = \bigwedge_{i \in X} \mathrm{a}_i \approx^{\mathbf{M}_i} \mathrm{a}'_i < 1.\end{aligned}$$

Thus, F is not safe w.r.t. $\{\mathbf{M}_i \mid i \in \mathbb{N}\}$ since

$$\theta_F(f_1^{\mathbf{M}}, f_2^{\mathbf{M}}) = \bigvee_{X\in F} [\![f_1^{\mathbf{M}} \approx f_2^{\mathbf{M}}]\!]_X = \lim_{n\to\infty}\left(1 - \tfrac{1}{n}\right) = 1.$$

As a consequence,

$$\begin{aligned}\|\langle f_1 \approx f_2, 1\rangle \Rightarrow g_1 \approx g_2\|_{\bigtimes_F \mathbf{M}_i} &= \|g_1 \approx g_2\|_{\bigtimes_F \mathbf{M}_i} = \theta_F(g_1^{\mathbf{M}}, g_2^{\mathbf{M}}) \\ &= \bigvee_{X\in F} [\![g_1^{\mathbf{M}} \approx g_2^{\mathbf{M}}]\!]_X = \bigvee_{X\in F} \bigwedge_{i\in X} \mathfrak{b}_i \approx^{\mathbf{M}_i} \mathfrak{b}'_i = 0,\end{aligned}$$

showing $\bigtimes_F \mathbf{M}_i \notin \mathrm{Mod}(\Sigma)$.

Lemma 17. *Suppose* $\mathbf{L}^*$ *is a residuated lattice with a truth stresser* * *defined by* (7). *Let* $\mathcal{K}$ *be an abstract class of* $\mathbf{L}$*-algebras which is closed under subalgebras and safe reduced products. If every filter* F *is* $\mathcal{K}$*-safe, then* $\mathcal{K} = \mathrm{Mod}(\Sigma)$ *for an* $\mathbf{L}$*-set* Σ *of* $\mathcal{P}$*-Horn clauses, where* $\mathcal{P} = \mathbf{L}^{T(X)\times T(X)}$.

Proof. Observe that $\mathcal{K}$ is closed under direct products since $F = \{I\}$ is safe with respect to an arbitrary family $\{\mathbf{M}_i \mid i \in I\}$, and $\bigtimes_F \mathbf{M}_i \cong \bigtimes_{i\in I} \mathbf{M}_i$. As a consequence, $\mathcal{K}$ is a sur-reflective class. Thus, $\mathcal{K} = \mathrm{Mod}(\mathrm{Impl}(\mathcal{K}))$ due to Theorem 6. We claim that $\mathcal{K} = \mathrm{Mod}(\Sigma)$, where $\Sigma = \mathrm{Horn}(\mathcal{K})$. Trivially, $\mathcal{K} \subseteq \mathrm{Mod}(\mathrm{Horn}(\mathcal{K}))$, i.e. we have to check the converse inclusion. We will proceed by contradiction.

Let $\mathbf{M} \in \mathrm{Mod}(\mathrm{Horn}(\mathcal{K}))$ and $\mathbf{M} \notin \mathrm{Mod}(\mathrm{Impl}(\mathcal{K}))$. That is,

$$\left\|P \Rightarrow (t \approx t')\right\|_{\mathbf{M}} \not\geq \big(\mathrm{Impl}(\mathcal{K})\big)(P \Rightarrow (t \approx t')) \tag{9}$$

for some $\mathcal{P}$-implication $P \Rightarrow (t \approx t')$. $P \Rightarrow (t \approx t')$ induces a family

$$\left\{P' \Rightarrow (t \approx t') \mid P' \in \mathrm{Fin}(P)\right\}$$

of $\mathcal{P}$-Horn clauses, where $\mathrm{Fin}(P)$ denotes the set of all finite restrictions of P.

Take any $P' \in \mathrm{Fin}(P)$. Clearly, $\left\|P' \Rightarrow (t \approx t')\right\|_{\mathbf{M}} \leq \left\|P \Rightarrow (t \approx t')\right\|_{\mathbf{M}}$. Thus, from (9) it follows that $\left\|P' \Rightarrow (t \approx t')\right\|_{\mathbf{M}} \not\geq \big(\mathrm{Impl}(\mathcal{K})\big)(P \Rightarrow (t \approx t'))$. Since $\mathbf{M} \in \mathrm{Mod}(\mathrm{Horn}(\mathcal{K}))$, we have

$$\big(\mathrm{Horn}(\mathcal{K})\big)(P' \Rightarrow (t \approx t')) \not\geq \big(\mathrm{Impl}(\mathcal{K})\big)(P \Rightarrow (t \approx t')).$$

That is, for every $P' \in \mathrm{Fin}(P)$ there is an $\mathbf{L}$-algebra $\mathbf{N}_{P'} \in \mathcal{K}$ and a valuation $v_{P'}$ of X in $\mathbf{N}_{P'}$ such that $P'(s, s') \leq \left\|s \approx s'\right\|_{\mathbf{N}_{P'}, v_{P'}}$ for all terms $s, s' \in T(X)$ and $\left\|t \approx t'\right\|_{\mathbf{N}_{P'}, v_{P'}} \not\geq \big(\mathrm{Impl}(\mathcal{K})\big)(P \Rightarrow (t \approx t'))$. In the following, we construct certain safe reduced product of a family $\left\{\mathbf{N}_{P'} \mid P' \in \mathrm{Fin}(P)\right\}$ to obtain a contradiction. Let us introduce a proper filter over $\mathrm{Fin}(P)$. First, we can consider a family $P_{s,s'} = \left\{P' \in \mathrm{Fin}(P) \mid P'(s, s') = P(s, s')\right\}$ for every $s, s' \in T(X)$. Evidently, $\emptyset \neq P_{s,s'} \subseteq \mathrm{Fin}(P)$ for all $s, s' \in T(X)$. Put $J = \left\{P_{s,s'} \mid s, s' \in T(X)\right\}$. Obviously, for every $s_1, s'_1, \ldots, s_n, s'_n \in T(X)$ there is a finite restriction $P' \in \mathrm{Fin}(P)$ such that $P'(s_i, s'_i) = P(s_i, s'_i)$ for all $i = 1, \ldots, n$. As a consequence, $\bigcap\left\{P_{s_i,s'_i} \mid i = 1, \ldots, n\right\} \neq \emptyset$ showing that J has the finite intersection property. This enables us to define a proper filter F over $\mathrm{Fin}(P)$ to be a filter generated by J.

A family $\{v_{P'}\colon X \to N_{P'} \mid P' \in \mathrm{Fin}(P)\}$ of valuations induces a valuation $v\colon X \to \bigtimes_{P' \in \mathrm{Fin}(P)} N_{P'}$ such that $v(x)(P') = v_{P'}(x)$ for all $x \in X$, $P' \in \mathrm{Fin}(P)$. By standard argument, there is a valuation w of X in $\bigtimes_F \mathbf{N}_{P'}$ such that $w(x) = [v(x)]_{\theta_F}$ for every $x \in X$. Thus, for $s, s' \in T(X)$ we have

$$\begin{aligned}\|s \approx s'\|_{\bigtimes_F \mathbf{N}_{P'}, w} &= w^\sharp(s) \approx^{\bigtimes_F \mathbf{N}_{P'}} w^\sharp(s') = \left[v^\sharp(s)\right]_{\theta_F} \approx^{\bigtimes_F \mathbf{N}_{P'}} \left[v^\sharp(s')\right]_{\theta_F} \\ &= \theta_F\left(v^\sharp(s), v^\sharp(s')\right) = \bigvee_{Z \in F} [\![v^\sharp(s) \approx v^\sharp(s')]\!]_Z \\ &= \bigvee_{Z \in F} \bigwedge_{P' \in Z} v^\sharp(s)(P') \approx^{\mathbf{N}_{P'}} v^\sharp(s')(P') \\ &= \bigvee_{Z \in F} \bigwedge_{P' \in Z} v^\sharp_{P'}(s) \approx^{\mathbf{N}_{P'}} v^\sharp_{P'}(s').\end{aligned}$$

Recall that $P'(s, s') \le \|s \approx s'\|_{\mathbf{N}_{P'}, v_{P'}} = v^\sharp_{P'}(s) \approx^{\mathbf{N}_{P'}} v^\sharp_{P'}(s')$ holds for every $P' \in \mathrm{Fin}(P)$ and all $s, s' \in T(X)$. Moreover, $P_{s,s'} \in F$. It readily follows that

$$\begin{aligned}\|s \approx s'\|_{\bigtimes_F \mathbf{N}_{P'}, w} &\ge [\![v^\sharp(s) \approx v^\sharp(s')]\!]_{P_{s,s'}} = \bigwedge_{P' \in P_{s,s'}} v^\sharp_{P'}(s) \approx^{\mathbf{N}_{P'}} v^\sharp_{P'}(s') \\ &\ge \bigwedge_{P' \in P_{s,s'}} P'(s, s') = \bigwedge_{P' \in P_{s,s'}} P(s, s') = P(s, s'),\end{aligned}$$

i.e. $P(s, s') \le \|s \approx s'\|_{\bigtimes_F \mathbf{N}_{P'}, w}$ for all terms $s, s' \in T(X)$. Moreover, since F is $\mathcal{K}$-safe by the assumption, we have

$$\|t \approx t'\|_{\bigtimes_F \mathbf{N}_{P'}, w} = [\![v^\sharp(t) \approx v^\sharp(t')]\!]_{Z_0} = \bigwedge_{P' \in Z_0} v^\sharp_{P'}(t) \approx^{\mathbf{N}_{P'}} v^\sharp_{P'}(t')$$

for some $Z_0 \in F$. In addition to that,

$$v^\sharp_{P'}(t) \approx^{\mathbf{N}_{P'}} v^\sharp_{P'}(t') = \|t \approx t'\|_{\mathbf{N}_{P'}, v_{P'}} \not\ge \left(\mathrm{Impl}(\mathcal{K})\right)(P \Rightarrow (t \approx t'))$$

for all $P' \in Z_0$. Putting previous facts together, we have

$$\begin{aligned}\|P \Rightarrow (t \approx t')\|_{\bigtimes_F \mathbf{N}_{P'}, w} &= \|t \approx t'\|_{\bigtimes_F \mathbf{N}_{P'}, w} \\ &= \bigwedge_{P' \in Z_0} \|t \approx t'\|_{\mathbf{N}_{P'}, v_{P'}} \not\ge \left(\mathrm{Impl}(\mathcal{K})\right)(P \Rightarrow (t \approx t')).\end{aligned} \tag{10}$$

Since every $\mathbf{N}_{P'}$ belongs to $\mathcal{K}$ which is supposed to be closed under safe reduced products, $\bigtimes_F \mathbf{N}_{P'}$ belongs to $\mathcal{K}$. As a consequence, $\|P \Rightarrow (t \approx t')\|_{\bigtimes_F \mathbf{N}_{P'}, w} \ge \left(\mathrm{Impl}(\mathcal{K})\right)(P \Rightarrow (t \approx t'))$ which contradicts (10). □

An interesting point to stress is that Lemma 17 requires safeness of a filter even without invoking the connection to direct limits. If **L** is finite, then every filter is safe with respect to any family of **L**-algebras of the same type. As a corollary, we have the following theorem.

Theorem 9. *Suppose* $\mathbf{L}^*$ *is a finite residuated lattice with a truth stresser* * *defined by* (7). *Then* $\mathcal{K}$ *is an abstract class of* **L**-*algebras closed under subalgebras and reduced products iff* $\mathcal{K}$ *is a* $\mathcal{P}$-*Horn class.* □

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