Arch. Math. Logic (2015) 54:785–802
DOI 10.1007/s00153-015-0440-0

Mathematical Logic

A logic of graded attributes

Radim Belohlavek[1] · **Vilem Vychodil**[1]

Received: 26 February 2014 / Accepted: 19 June 2015 / Published online: 10 July 2015

Abstract We present a logic for reasoning about attribute dependencies in data involving degrees such as a degree to which an object is red or a degree to which two objects are similar. The dependencies are of the form $A \Rightarrow B$ and can be interpreted in two ways: first, in data tables with entries representing degrees to which objects (rows) have attributes (columns); second, in database tables where each domain is equipped with a similarity relation. We assume that the degrees form a scale equipped with operations representing many-valued logical connectives. If 0 and 1 are the only degrees, the algebra of degrees becomes the two-element Boolean algebra and the two interpretations become well-known dependencies in Boolean data and functional dependencies of relational databases. In a setting with general scales, we obtain a new kind of dependencies with naturally arising degrees of validity, degrees of entailment, and related logical concepts. The deduction rules of the proposed logic are inspired by Armstrong rules and make it possible to infer dependencies to degrees—the degrees of provability. We provide a soundness and completeness theorem for such a setting asserting that degrees of entailment coincide with degrees of provability, prove the independence of deduction rules, and present further observations.

Keywords Attribute implication · Fuzzy logic · Graded-style completeness · Pavelka-style logic

Mathematics Subject Classification 03B50 · 03B52 · 68T37

✉ Radim Belohlavek
radim.belohlavek@acm.org

Vilem Vychodil
vychodil@acm.org

1 Department of Computer Science, Palacký University, 17. listopadu 12,
771 46 Olomouc, Czech Republic

1 Introduction

Rules of the form $A \Rightarrow B$ where A and B are collections of attributes, such as

$$\{\text{car, registered in US, registered after 1994}\} \Rightarrow \{\text{meets Clean Air Act}\},$$

are abundant in human reasoning. Such rules play a substantial role in data processing. In data analysis, such rules are known as attribute implications [6,9,12]. They are interpreted in tables with Boolean attributes describing which objects have which attributes and have the following meaning: each object which has all attributes in A has also all attributes in B. Association rules [29] generalize this semantics in that they allow for exceptions from this rule. In relational databases, such rules are known as functional dependencies. They are interpreted in relations (tables with arbitrarily-valued attributes) and have the following meaning: every two tuples with the same values on attributes in A have the same values on attributes in B.

The two interpretations mentioned above share a bivalent character of the truth conditions: an object either has an attribute or not in the first case; two attribute values either match (are equal) or not. It has been argued on various accounts, see e.g. [1,10,28], that there is a need to extend the currently available methods to account for indeterminacy, imprecision, and approximation, which relate both to data and human understanding and reasoning about the data. Two examples relevant to our coniderations are graded (or, fuzzy) attributes, such as *red* or *obese*, and similarity relations which enable similarity queries and reasoning involving similarity in general. Both of these cases are conveniently modeled by degrees, namely a degree to which a given attribute applies to a particular object and a degee to which two attribute values are similar. The approach based on degrees, sometimes referred to as a graded approach, constitutes the core idea of fuzzy logic [27], whose formal facet has recently been considerably developed, see [7,17] for an overview.

In our previous work, see e.g. [3,5], we developed an approach to the rules $A \Rightarrow B$ suitable for describing dependencies in data with grades. In particular, we developed two semantics. One based on data tables with graded attributes and the other based on an extension of Codd's model of data in which domains are equipped with similarity relations. We developed a logic for reasoning with such dependencies and proved its completeness theorem which says that $A \Rightarrow B$ follows from a set T of dependencies if and only if $A \Rightarrow B$ is provable from T. Note that similar attempts appeared e.g. in [24,26] (see also [4] for a comparative overview).

These attempts, however, do not truly capture an important facet of such dependencies. Namely, in presence of grades, the key logic notions involved, such as validity or entailment, naturally come in degrees. That is to say, instead of "$A \Rightarrow B$ is valid in data (or not)" and "T entails $A \Rightarrow B$ (or not)", we naturally come to a degree to which $A \Rightarrow B$ is valid in data and a degree to which T entails $A \Rightarrow B$. In this perspective, the ordinary validity and entailment are particular cases of the more broadly conceived notions of degrees of validity and entailment, and correspond to the boundary cases, i.e. validity and entailment to degree 1.

In this paper, we develop a logic for attribute dependencies which captures validity and entailment to degrees and enables inference from partially true dependencies. Our logic is a particular case of Pavelka's abstract fuzzy logic, which is surveyed in Sect. 2, and uses deduction rules inspired by the well-known Armstrong rules [2,20]. We prove that the logic is sound and complete, i.e. that the degree to which a dependence follows from a theory T of partially true dependencies equals the degree of its provability from T, and provide further observations on the proposed logic.

2 Preliminaries

2.1 Residuated lattices

We assume that the truth degrees form a scale L that is equipped with truth functions of logical connectives. In particular, we assume that L conforms to the structure of a complete residuated lattice with a hedge, i.e. an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$; for each $a, b, c \in L$; hedge * satisfies (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, (iv) $a^{**} = a^*$, for all $a, b \in L$. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Accordingly, if $||\varphi||$ and $||\psi||$ denote the truth degrees of formulas φ and ψ, then $||\varphi|| \otimes ||\psi||$ and $||\varphi|| \rightarrow ||\psi||$ are the truth degrees of "φ and ψ" and "φ implies ψ", respectively. *, called an (idempotent truth-stressing) hedge, is a (truth function of) logical connective "very true", see [18,19]. That is, $||\varphi||^*$ is the truth degree of "φ is very true".

The conditions assumed for $\mathbf{L}$ are justified by natural requirements on properties of logical connectives and requirements on *modus ponens* in presence of degrees. Residuated structures of truth degrees are the main structures used in modern fuzzy logic [7,17]. The reader may find more information, including the properties of residuated lattices, in [11,15,16,18].

Examples of $\mathbf{L}$ include those where $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm and $\rightarrow$ being the residuum of $\otimes$. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz ($a \otimes b = \max(a+b-1, 0)$, $a \rightarrow b = \min(1-a+b, 1)$), Gödel: ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = b$ else), Goguen (product): ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$, $a \rightarrow b = \frac{b}{a}$ else). Examples of finite $\mathbf{L}$ include equidistant subchains of $[0, 1]$ equipped with the restrictions of Łukasiewicz or Gödel operations, or any other discrete t-norm [21]. Importantly, if $L = \{0, 1\}$ then $\mathbf{L}$ essentially becomes the two-element Boolean algebra of classical logic. Two boundary cases of hedges are (i) identity: $a^* = a$ $(a \in L)$; (ii) globalization [25]: $a^* = 1$ if $a = 1$, $a^* = 0$ else. In what follows, $\mathbf{L}$ always denotes a complete residuated lattice with hedge.

For a given $\mathbf{L}$, we define usual notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A : U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A"; A is also denoted by $A = \{{}^{a}\!/u, \ldots\}$ where $a = A(u)$. Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that

$(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. For $a \in L$ and $A \in \mathbf{L}^U$, we define $\mathbf{L}$-sets $a \otimes A$ (a-multiple of A) and $a \to A$ (a-shift of A) by

$$(a \otimes A)(u) = a \otimes A(u) \quad \text{and} \quad (a \to A)(u) = a \to A(u), \tag{1}$$

for each $u \in U$. Given $A, B \in \mathbf{L}^U$, we define the degree $S(A, B)$ of inclusion of A in B by

$$S(A, B) = \bigwedge\nolimits_{u \in U} \bigl(A(u) \to B(u)\bigr), \tag{2}$$

which generalizes the classical subsethood relation $\subseteq$. Described verbally, $S(A, B)$ represents the degree to which every element from A belongs to B. It is easily shown that $S(A, B) = 1$ if and only if $A(u) \leq B(u)$ for each $u \in U$, in which case we say that A is (fully) included in B and denote this fact by $A \subseteq B$.

2.2 Pavelka's abstract fuzzy logic

A suitable logical framework which captures degrees of entailment and other notions and which is nowadays known as *abstract fuzzy logic* has been proposed in part I of Pavelka's seminal paper [23]. We first present this framework (see also [14,18]) and then comment on related work.

An abstract fuzzy logic is a tuple $\langle \mathcal{F}, \mathbf{L}, \mathcal{S}, \mathcal{R} \rangle$[1] consisting of

- an (abstract) set $\mathcal{F}$ of *formulas*;
- a complete lattice $\mathbf{L} = \langle L, \leq, \cdots \rangle$ (possibly with additional operations);
- an $\mathbf{L}$-*semantics* $\mathcal{S}$, which is an arbitrary set $\mathcal{S} \subseteq \mathbf{L}^{\mathcal{F}}$ of $\mathbf{L}$-sets of formulas;
- a set $\mathcal{R}$ of *deduction rules*, as explained below.

The formulas in $\mathcal{F}$ may be built up inductively from atomic ones as usual but need not have any inner structure. The elements in L are called truth degrees and $\mathbf{L}$ plays the role of a structure (algebra) of truth degrees; they include the boundary 0 and 1; in general $a \in L$ may but need not be numbers. The elements $E \in \mathcal{S}$ play the role of truth evaluations (semantic structures in which formulas assume truth degrees); for $\varphi \in \mathcal{F}$ and $E \in \mathcal{S}$, we denote $E(\varphi) \in L$ also by $||\varphi||_E$ and call it the truth degree of φ in E. Each (n-ary) deduction rule $R \in \mathcal{R}$ is a pair $R = \langle R_{\text{syn}}, R_{\text{sem}} \rangle$ consisting of a partial function $R_{\text{syn}} : \mathcal{F}^n \to \mathcal{F}$ (syntactic part) and a function $R_{\text{sem}} : L^n \to L$ (semantic part) and is usually visualized as

$$\frac{\langle \varphi_1, a_1 \rangle, \ldots, \langle \varphi_n, a_n \rangle}{\langle \varphi, a \rangle}, \tag{3}$$

where $\varphi = R_{\text{syn}}(\varphi_1, \ldots, \varphi_n)$ and $a = R_{\text{sem}}(a_1, \ldots, a_n)$. The intended meaning is that from the validity of φ_i to degree (at least) a_i, $i = 1, \ldots, n$, we may infer that φ is valid to degree (at least) a. An example, which in fact motivated this conception, is

[1] The literature contains variations of this view; e.g. a fuzzy set of axioms is sometimes added as another member of the tuple.

Goguen's generalization of ordinary *modus ponens* [15]: $\frac{\langle\varphi, a\rangle, \langle\varphi\Rightarrow\psi, c\rangle}{\langle\psi, a\otimes c\rangle}$, where $\otimes$ is a truth function of a many-valued conjunction. We shall utilize the fact that without any impact on the results in question, R_{sem} may be conceived as depending also on φ_is, i.e. $a = R_{\text{sem}}(\langle\varphi_1, a_1\rangle, \ldots, \langle\varphi_n, a_n\rangle)$, cf. Remark 4.

A *theory* T is an arbitrary fuzzy set of formulas, i.e. $T \in \mathbf{L}^{\mathcal{F}}$; a *model* of a theory T is any fuzzy set $E \in \mathcal{S}$ for which $T(\varphi) \leq E(\varphi)$ for any $\varphi \in \mathcal{F}$ (degree to which φ is true in E is greater than or equal to that prescribed by T). Denoting the models of T by $\text{Mod}(T)$, the *degree* $||\varphi||_T$ *to which* φ *semantically follows from* T is defined by

$$||\varphi||_T = \bigwedge\nolimits_{E\in\text{Mod}(T)} ||\varphi||_E,$$

i.e. the infimum of degrees to which φ is true in models of T.

An $\mathbf{L}$-*weighted proof* from T is a finite sequence $\langle\varphi_1, a_1\rangle$, ..., $\langle\varphi_n, a_n\rangle$ of $\mathbf{L}$-weighted formulas, i.e. pairs $\langle\varphi_i, a_i\rangle$ of $\varphi_i \in \mathcal{F}$ and $a_i \in L$, such that for each i, $a_i = T(\varphi_i)$ (assumption) or $\langle\varphi_i, a_i\rangle$ is obtained from some $\langle\varphi_j, a_j\rangle$s, $j < i$, by some rule $R \in \mathcal{R}$, i.e. $\varphi_i = R_{\text{syn}}(\ldots, \varphi_j, \ldots)$ and $a_i = R_{\text{sem}}(\ldots, a_j, \ldots)$. A *degree* $|\varphi|_T$ *of provability* of a formula $\varphi \in \mathcal{F}$ from a theory $T \in \mathbf{L}^{\mathcal{F}}$ is defined as

$$|\varphi|_T = \bigvee \{a_n \mid \langle\varphi_1, a_1\rangle, \ldots, \langle\varphi_n, a_n\rangle \text{ is a proof from } T \text{ with } \varphi_n = \varphi\}.$$

An abstract fuzzy logic $\langle\mathcal{F}, \mathbf{L}, \mathcal{S}, \mathcal{R}\rangle$ is (*Pavelka-style*) *complete* if for each theory $T \in \mathbf{L}^{\mathcal{F}}$ and formula $\varphi \in \mathcal{F}$ we have $|\varphi|_T = ||\varphi||_T$, i.e. degrees of semantic entailment equal degrees of provability.

Pavelka's abstract fuzzy logics encompass both truth-functional and non-truth-functional calculi. The first example of the former is represented by Pavelka's propositional logics developed in part II and III of [23], whose predicate extension with Łukasiewicz operations has later been extensively studied by Novák, see e.g. [22]. Abstract fuzzy logics have been thoroughly examined by Gerla—[14] contains many particular logics including non-truth-functional logics such as probabilistic logics [13]. Interestingly, Hájek showed, see e.g. [18], that Pavelka's truth-functional Łukasiewicz logic may be "simulated" within the ordinary Łukasiewicz logic expanded by truth constants for rationals (in that the notion of degree of provability may be defined in it) and simplified considerably the resulting system, which stimulated a stream of papers on fuzzy logics with truth constants in language, see e.g. [8] for an overview. Note, however, that these logics are conceptually different from Pavelka's abstract fuzzy logic and that general abstract fuzzy logics cannot be "simulated" in the sense mentioned above.

3 Logic of graded attributes

3.1 Graded attribute implications and basic semantic notions

We now present our logic for reasoning about particular dependencies regarding graded attributes. The logic is defined over a given nonempty set Y of attributes and we present it as a particular abstract fuzzy logic $\langle\mathcal{F}, \mathbf{L}, \mathcal{S}, \mathcal{R}\rangle$ (Sect. 2.2). First, the structure $\mathbf{L}$ is an arbitrary complete residuated lattice (Sect. 2.1). Second, the formulas are the so-called *graded attribute implications* over Y [3,5] which may be seen as expressions of the form

$$A \Rightarrow B$$

where A and B are $\mathbf{L}$-sets of attributes in Y, thus of the form

$$\{^{a_1}/y_1, \ldots, {}^{a_p}/y_p\} \Rightarrow \{^{b_1}/z_1, \ldots, {}^{b_q}/z_q\},$$

such as

$$\{^{0.5}/\text{unhealthy food}, {}^{0.9}/\text{little activity}\} \Rightarrow \{^{0.7}/\text{high blood pressure}\}.$$

Formally, such formulas are pairs $\langle A, B\rangle$ of $\mathbf{L}$-sets, i.e. the set of formulas is $\mathcal{F} = \{\langle A, B\rangle \mid A, B \in \mathbf{L}^Y\}$, but for convenience and following common usage in database theory we denote them $A \Rightarrow B$.

The $\mathbf{L}$-semantics $\mathcal{S}$ is derived from the intended interpretation of graded attribute implications. Namely, the implications are interpreted in tables representing to what degrees (taken from $\mathbf{L}$) the graded attributes (columns) apply to objects (rows), such as

I	y_1	y_2	y_3
x_1	1.0	0.9	0.8
x_2	1.0	0.7	0.8
x_3	0.9	0.5	0.8

. (4)

An implication in such tables is “tested” in objects. As objects are represented by $\mathbf{L}$-sets of attributes, such as the $\mathbf{L}$-set $M = \{^1/y_1, {}^{0.9}/y_2, {}^{0.8}/y_3\}$ representing x_1, the basic semantic components involved are $\mathbf{L}$-sets in Y—these are the *structures* of our logic in which te formulas are interpreted. The *degree* $||A \Rightarrow B||_M \in L$ *to which* $A \Rightarrow B$ *is true in* $M \in \mathbf{L}^Y$ is defined by

$$||A \Rightarrow B||_M = S(A, M)^* \to S(B, M), \tag{5}$$

where $S(\cdots)$ denotes the degree of inclusion defined in (2), $*$ is the hedge and $\to$ the residuum of $\mathbf{L}$ (the rationale is explained in Remark 1). Each $M \in \mathbf{L}^Y$ thus induces a mapping $E_M : \mathcal{F} \to L$ defined by $E_M(A \Rightarrow B) = ||A \Rightarrow B||_M$ and the $\mathbf{L}$-semantics $\mathcal{S}$ consists of all such mappings (evaluations), i.e. $\mathcal{S} = \{E_M \mid M \in \mathbf{L}^Y\}$. For convenience and again to follow common practice, we identify E_M with M. That is to say, the semantics structures in which the formulas $A \Rightarrow B$ are evaluated are $\mathbf{L}$-sets M of attributes in Y (representing table rows).

Remark 1 (a) Let $M \in \mathbf{L}^Y$ represent an object x. According to the principles of fuzzy logic, $||A \Rightarrow B||_M$ is the truth degree of the proposition “if (it is very true that) x has all attributes in A, then x has all attributes in B”. Hence, (5) generalizes the notion of validity of ordinary attribute implications. In particular, one easily checks that if $L = \{0, 1\}$, we get the notion of ordinary attribute implication and (5) becomes the ordinary condition of validity. The hedge $*$ may be thought of as a parameter. In particular, setting $*$ to globalization and identity (Sect. 2.1) yields two natural truth conditions, as explained in (b).

(b) Let us look at the truth condition (5) from an intuitive point of view. Consider the implication

$$A \Rightarrow B = \left\{{}^{1}\!/y_1, {}^{0.5}\!/y_3\right\} \Rightarrow \left\{{}^{0.8}\!/y_2\right\}$$

and the object x_1 from (4), which is represented by $M_1 = \{{}^{1}\!/y_1, {}^{0.9}\!/y_2, {}^{0.8}\!/y_3\}$. Since x_1 has all the attributes to the degrees prescribed by A as well as B, i.e. $A(y_1) \leq M_1(y_1)$, $A(y_3) \leq M_1(y_3)$, and $B(y_2) \leq M_1(y_2)$, the implicatiion should be ruled true on intuitive grounds. Indeed, in this case, $S(A, M_1) = 1$ and $S(B, M_1) = 1$, and thus $||A \Rightarrow B||_{M_1} = S(A, M_1)^* \rightarrow S(B, M_1) = 1 \rightarrow 1 = 1$ for any possible choice of $\rightarrow$ and $*$.

Next, consider x_2 and its corresponding M_2. Now, x_2 has all the attributes to the degrees prescribed by A, i.e. $S(A, M_2) = 1$, but does not have y_2 to the degree required by B because $B(y_2) = 0.8 > 0.7 = M_2(y_2)$. Nevertheless, the prescribed degree 0.8 is almost attained by x_2 and one would consider $A \Rightarrow B$ almost true on intuitive grounds. In particular, if $\rightarrow$ is the Łukasiewicz implication, we have $0.8 \rightarrow 0.7 = 0.9$, whence $||A \Rightarrow B||_{M_2} = 1^* \rightarrow 0.9 = 1 \rightarrow 0.9 = 0.9$ (for every $*$), and we obtain similar results with Goguen and Gödel connectives.

The object x_3 does not even have the attributes to the degrees prescribed by A, and hence $S(A, M_3) < 1$. Arguably, one has two reasonable options now. First, one may consider $A \Rightarrow B$ satisfied for free because x_3 does not satisfy A. This option corresponds to the choice of globalization for $*$ because then, $||A \Rightarrow B||_{M_3} = S(A, M_3)^* \rightarrow S(B, M_3) = 0 \rightarrow S(B, M_3) = 1$ for arbitrary $\rightarrow$. Second, one may want to take into account the degree $S(A, M_3) < 1$ to which x_3 has all the attributes in A in evaluating the validity of $A \Rightarrow B$. This option corresponds to the choice of identity of $*$ because then, for $\rightarrow$ being the Łukasiewicz implication, $||A \Rightarrow B||_{M_3} = S(A, M_3) \rightarrow S(B, M_3) = 0.9 \rightarrow 0.7 = 0.8$. This example explains the role of $*$ in (5): We keep the semantics general as to account for both of the options mentioned above.

(c) Consider now the meaning of $A \Rightarrow B$ being fully true for an object x represented by M, i.e. $||A \Rightarrow B||_M = 1$. If $*$ is the identity, then since $a \rightarrow b = 1$ iff $a \leq b$, being fully true means $S(A, M) \leq S(B, M)$, i.e. the degree to which x has all attributes in A is less than or equal to the degree to which x has all attributes in B, representing one possible meaning of $A \Rightarrow B$ being fully true. If $*$ is the globalization, being fully true means that if x has y to degree $\geq A(y)$ for each $y \in Y$, then x also has y to degree $\geq B(y)$ for each $y \in Y$, representing another possible meaning of $A \Rightarrow B$ being fully true, in which the degrees $A(y)$ and $B(y)$ are naturally understood as validity thresholds.

The abstract fuzzy logic now yields the following notions (cf. Sect. 2.2). A *theory* is an $\mathbf{L}$-set T of graded attribute implications, the degree $T(A \Rightarrow B)$ being intuitively understood as a degree to which we can assume validity of $A \Rightarrow B$ when making inferences from T; the set $\mathrm{Mod}(T)$ of all *models* of T is defined by

$$\text{Mod}(T) = \{M \in \mathbf{L}^Y \mid T(A \Rightarrow B) \leq ||A \Rightarrow B||_M \text{ for each } A \Rightarrow B\};$$

the *degree* $||A \Rightarrow B||_T \in L$ *to which* $A \Rightarrow B$ *semantically follows from a theory* T is defined by

$$||A \Rightarrow B||_T = \bigwedge_{M \in \text{Mod}(T)} ||A \Rightarrow B||_M.$$

Remark 2 Note that for $L = \{0, 1\}$ (ordinary case in which 0 and 1 are the only truth degrees), the above notions yield their well-known ordinary counterparts.

Remark 3 The choice of Pavelka's abstract fuzzy logic for treating the present attribute dependencies is motivated by the simplicity and generality of this framework. Importantly, the notions and, as we shall see in the next section, also the arguments within this framework are conceptually similar to those of the ordinary, bivalent theory of attribute dependencies.

It should, nevertheless, be mentioned that we could alternatively formalize the dependencies within the propositional or predicate logics developed by Pavelka [23, partII,III] and Novák [22] which are particular abstract fuzzy logics and are mentioned at the end of Sect. 2.2. Alternatively, we could formalize them within the logics with truth constants mentioned in Sect. 2.2, which are based on a bivalent notion of entailment and in which degrees of provability may additionally be defined. In these calculi, considered over an arbitrary but fixed complete residuated lattice with a hedge, graded attribute implications could naturally be represented by particular propositional formulas, such as $\Delta((\overline{a} \Rightarrow y_1) \wedge (\overline{b} \Rightarrow y_2)) \Rightarrow (\overline{c} \Rightarrow y_3)$, where $\Rightarrow$ is the symbol of implication in the particular logic, representing the attribute implication $\{{}^{a}\!/y_1, {}^{b}\!/y_2\} \Rightarrow \{{}^{c}\!/y_3\}$. This would thus amount to studying certain fragments of these calculi. It follows from our results below that such fragments would obey a Pavelka-style completeness even though, as is well-known, Pavelka-style completeness fails in general for such calculi (as was first observed in [23]). A study of this and other fragments thus represents a possibly interesting direction of study.

3.2 Deduction rules, proofs, and degrees of provability

The deduction rules are pairs $R = \langle R_{\text{syn}}, R_{\text{sem}} \rangle$ of mappings, as described in Sect. 2.2, which we depict using schemes such as (3). In particular, our logic uses the following deduction rules:

$$\text{(Ax)}\ \frac{}{\langle A \cup B \Rightarrow A,\ 1\rangle},$$

$$\text{(Cut)}\ \frac{\langle A \Rightarrow B,\ a\rangle, \langle B \cup C \Rightarrow D,\ b\rangle}{\langle A \cup C \Rightarrow D,\ a^* \otimes b\rangle},$$

$$\text{(Sh)}\ \frac{\langle A \Rightarrow B,\ a\rangle}{\langle A \Rightarrow C,\ S(C, a \otimes B)\rangle},$$

where $A, B, C, D \in \mathbf{L}^Y$, $a, b \in L$, and $S(\cdots)$ denotes the degree of inclusion defined by (2). The rules are inspired by Armstrong axioms, see [20].

Remark 4 Note that in order for our rules to conform to the form desribed above, i.e. $R_{\mathrm{syn}}(\varphi_1, \ldots, \varphi_n)$ and $R_{\mathrm{sem}}(\langle \varphi_1, a_1 \rangle, \ldots, \langle \varphi_n, a_n \rangle)$, we may consider (Ax) as a shorthand for a collection of rules $(\mathrm{Ax}_{A,B})$ for $A, B \in \mathbf{L}^Y$, (Cut) as a collection of (Cut_C) for $C \in \mathbf{L}^Y$, (Sh) as a collection of (Sh_C) for $C \in \mathbf{L}^Y$, $c \in L$. Similar remarks apply also to the derivable rules which are presented below.

Moreover, instead of (Sh), we might consider a collection of rules $(\mathrm{Sh}_{B,C})$ for $B, C \in \mathbf{L}^Y$. The semantic part of each such rule is a mapping $a \mapsto S(C, a \otimes B)$, i.e. a mapping of L to L. The semantic parts of all the rules are then of the form $R_{\mathrm{sem}}(a_1, \ldots, a_n)$ and all the rules are then rules in the sense of the original Pavelka's approach [23].

Notice that if $L = \{0, 1\}$, (Ax) and (Cut) can be identified with the ordinary rule of axiom and the rule of cut, because then, the effective instances of the rules read "infer $\langle A \cup B \Rightarrow A,\ 1 \rangle$" and "from $\langle A \Rightarrow B,\ 1 \rangle$ and $\langle B \cup C \Rightarrow D,\ 1 \rangle$ infer $\langle A \cup C \Rightarrow D,\ 1 \rangle$".

With these rules, we now obtain from the general notions of Pavelka's abstract fuzzy logic (Sect. 2.2) the notion of an (**L**-weighted) *proof* of an **L**-weighted formula $\langle A \Rightarrow B, a \rangle$ from a theory T; and that of *degree* $|A \Rightarrow B|_T$ *of provability* of $A \Rightarrow B$ from T i.e.

$$|A \Rightarrow B|_T = \bigvee \{a \mid \ \ldots, \langle A \Rightarrow B,\ a \rangle \text{ is a proof from } T\}.$$

As usual, we call a rule of the form (3) derivable from a set $\mathcal{R}$ of rules if there exists a weighted proof $\langle \varphi_1, a_1 \rangle, \ldots, \langle \varphi_n, a_n \rangle, \ldots, \langle \varphi, b \rangle$ with $a \leq b$ using rules from $\mathcal{R}$, for every $\langle \varphi_i, a_i \rangle$ for which R is defined.

Lemma 1 *Rules* (Ax), (Cut)*, and* (Sh) *are independent, i.e. none of the rules is derivable from the remanining two.*

Proof (Ax): Clearly, (Ax) is not derivable from (Cut) and (Sh).

(Cut): Observe first that for $L = \{0, 1\}$, (Ax) and (Cut) may be identified with the ordinary rules of axiom and cut, and (Sh) becomes the rule (Pro) of projectivity saying "from $A \Rightarrow B$ infer $A \Rightarrow C$, provided $C \subseteq B$". Therefore, if (Cut) were derivable from (Ax) and (Sh), this would imply that in the ordinary case, (Cut) is derivable from (Ax) and (Pro) which is not the case, since (Ax) and (Pro) only yield rules $A \Rightarrow B$ with $A \supseteq B$.

(Sh): If T is crisp, i.e. $T(A \Rightarrow B) = 0$ or $T(A \Rightarrow B) = 1$ for every $A \Rightarrow B$, then the only $\langle C \Rightarrow D,\ a \rangle$ which may be inferred from T using (Ax) and (Cut) clearly have $a = 1$. On the other hand, (Sh) may infer $\langle C \Rightarrow D,\ a \rangle$ with $a < 1$, finishing the proof.

The following lemma provides further, derivable rules which shall be used below.

Lemma 2 *The following rules are derivable from* (Ax)–(Sh)*:*

$$\text{(Ref)}\ \frac{}{\langle A \Rightarrow A,\ 1 \rangle},\quad \text{(Wea)}\ \frac{\langle A \Rightarrow B,\ a \rangle}{\langle A \cup C \Rightarrow B,\ a \rangle},\quad \text{(Pro)}\ \frac{\langle A \Rightarrow B \cup C,\ a \rangle}{\langle A \Rightarrow B,\ a \rangle},$$

$$\text{(Add)}\ \frac{\langle A \Rightarrow B,\ a \rangle, \langle A \Rightarrow C,\ b \rangle}{\langle A \Rightarrow B \cup C,\ a \wedge b \rangle},\quad \text{(Tra)}\ \frac{\langle A \Rightarrow B,\ a \rangle, \langle B \Rightarrow C,\ b \rangle}{\langle A \Rightarrow C,\ a^* \otimes b \rangle},$$

$$\text{(Mul)}\ \frac{\langle A \Rightarrow B,\ a\rangle}{\langle c^* \otimes A \Rightarrow c^* \otimes B,\ a\rangle},\quad \text{(Sh}\uparrow\text{)}\ \frac{\langle A \Rightarrow B,\ a\rangle}{\langle A \Rightarrow a \otimes B,\ 1\rangle},\quad \text{(Sh}\downarrow\text{)}\ \frac{\langle A \Rightarrow B,\ 1\rangle}{\langle A \Rightarrow a \rightarrow B,\ a\rangle},$$

for each $A, B, C \in \mathbf{L}^Y$, *and* $a, b, c \in L$.

Proof (Ref): This rule results from (Ax) when putting $B = \emptyset$.
(Wea): Apply (Cut) to $\langle A \cup C \Rightarrow A,\ 1\rangle$ (infer this by (Ax)) and $\langle A \cup A \Rightarrow B,\ a\rangle = \langle A \Rightarrow B,\ a\rangle$.
(Pro): Apply (Sh) and observe that $a \leq S(B, a \otimes (B \cup C))$.
(Tra): Use (Cut).
(Sh↑): Use (Sh) for $C = a \otimes B$ and observe that $1 = S(a \otimes B, a \otimes B)$.
(Sh↓): Use (Sh) for $C = a \rightarrow B$ and observe that $a \leq S(a \rightarrow B, 1 \otimes B)$.
(Mul): Apply (Sh↑) to $\langle A \Rightarrow B, a\rangle$ to obtain

$$\langle A \Rightarrow a \otimes B, 1\rangle. \tag{6}$$

Use (Ax) to infer $\langle c^* \otimes A \Rightarrow c^* \otimes A, 1\rangle$ from which (Sh) yields

$$\langle c^* \otimes A \Rightarrow A, S(A, c^* \otimes A)\rangle. \tag{7}$$

Now (Cut) applied to (7) and (6) yields $\langle c^* \otimes A \Rightarrow a \otimes B, S(A, c^* \otimes A)^*\rangle$, from which we obtain

$$\langle c^* \otimes A \Rightarrow c^* \otimes a \otimes B, S(c^* \otimes a \otimes B, S(A, c^* \otimes A)^* \otimes a \otimes B)\rangle. \tag{8}$$

using (Sh). Now observe that $S(c^* \otimes a \otimes B, S(A, c^* \otimes A)^* \otimes a \otimes B) = 1$, i.e. $c^* \otimes a \otimes B \subseteq S(A, c^* \otimes A)^* \otimes a \otimes B$. Indeed, due to adjointness, $c^* \leq S(A, c^* \otimes A)$, whence isotony and idempotency of $*$ yields $c^* \leq S(A, c^* \otimes A)^*$ from which $c^* \otimes a \otimes B \subseteq S(A, c^* \otimes A)^* \otimes a \otimes B$ readily follows. Therefore, the inferred weighted formula (8) is

$$\langle c^* \otimes A \Rightarrow c^* \otimes a \otimes B, 1\rangle,$$

from which (Sh↓) yields

$$\langle c^* \otimes A \Rightarrow a \rightarrow (c^* \otimes a \otimes B), a\rangle.$$

Since $a \rightarrow (c^* \otimes a \otimes B) \supseteq c^* \otimes B$, we may apply (Pro) to obtain

$$\langle c^* \otimes A \Rightarrow c^* \otimes B), 1\rangle,$$

verifying that (Mul) is derivable.
(Add): Use (Sh↑) to infer

$$\langle A \Rightarrow a \otimes B,\ 1\rangle$$

from $\langle A \Rightarrow B, a\rangle$; use (Wea) to infer $\langle A \cup a \otimes B \Rightarrow C, b\rangle$ from $\langle A \Rightarrow C, b\rangle$ and then use (Sh↑) to infer $\langle A \cup a \otimes B \Rightarrow b \otimes C, 1\rangle$; then apply (Cut) to the last weighted formula and to $\langle b \otimes C \cup a \otimes B \Rightarrow b \otimes C \cup a \otimes B, 1\rangle$ to infer

$$\langle a \otimes B \cup A \Rightarrow b \otimes C \cup a \otimes B, 1\rangle;$$

then use (Cut) to $\langle A \Rightarrow a \otimes B, 1\rangle$ and $\langle a \otimes B \cup A \Rightarrow b \otimes C \cup a \otimes B, 1\rangle$ and infer

$$\langle A \Rightarrow b \otimes C \cup a \otimes B, 1\rangle.$$

Use (Sh↓) to the last weighted formula to infer

$$\langle A \Rightarrow (a \wedge b) \rightarrow (b \otimes C \cup a \otimes B), a \wedge b\rangle$$

Finally, since $B \cup C \subseteq (a \wedge b) \rightarrow b \otimes C \cup a \otimes B$, use (Pro) to the last displayed formula to infer $\langle A \Rightarrow B \cup C, a \wedge b\rangle$.

Remark 5 Note that in the system consisting of (Ax) (Cut), and (Sh), one can replace (Sh) by more elementary rules, namely by (Sh↑), (Sh↓), and

$$\text{(E)}\ \frac{\langle A \Rightarrow a \rightarrow (a \otimes B), a\rangle}{\langle A \Rightarrow B, a\rangle},$$

for $A, B \in \mathbf{L}^Y$ and $a \in L$. Indeed, since (E) is a particular instance of (Pro), Lemma 2 and its proof imply that (Sh↑), (Sh↓), and (E) are derivable from (Sh).

Conversely, (Sh) is derivable from (Sh↑), (Sh↓), and (E) as follows. Apply (Sh↑) to $\langle A \Rightarrow B, a\rangle$ to obtain $\langle A \Rightarrow a \otimes B, 1\rangle$ and then (Sh↓) to obtain

$$\langle A \Rightarrow S(C, a \otimes B) \rightarrow (a \otimes B), S(C, a \otimes B)\rangle.$$

Apply (Sh↑) again to obtain

$$\langle A \Rightarrow S(C, a \otimes B) \otimes (S(C, a \otimes B) \rightarrow (a \otimes B)), 1\rangle.$$

Since $S(C, a \otimes B) \otimes (S(C, a \otimes B) \rightarrow (a \otimes B)) \supseteq S(C, a \otimes B) \otimes C$ and since the instance of (Pro) with $a = 1$ is readily derivable from (Cut), applying this instance to the last weighted formula, we obtain

$$\langle A \Rightarrow S(C, a \otimes B) \otimes C, 1\rangle,$$

from which we get by (Sh↑)

$$\langle A \Rightarrow S(C, a \otimes B) \rightarrow (S(C, a \otimes B) \otimes C), S(C, a \otimes B)\rangle.$$

Finally, (E) yields

$$\langle A \Rightarrow C, S(C, a \otimes B)\rangle.$$

3.3 Soundness and completeness

A rule R is called *sound* if for each $\varphi_1, \ldots, \varphi_n$ (for which R_{syn} is defined) and for each $a_1, \ldots a_n \in L$ we have

$$\mathrm{Mod}(\{{}^{a_1}\!/\varphi_1, \ldots, {}^{a_n}\!/\varphi_n\}) \subseteq \mathrm{Mod}(\{{}^{R_{\mathrm{sem}}(\langle\varphi_1, a_1\rangle, \ldots)}\!/R_{\mathrm{syn}}(\varphi_1, \ldots)\}),$$

i.e., for arbitrary $M \in \mathbf{L}^Y$: if each $\varphi_1, \ldots$ is true in M at least to degree $a_1, \ldots$, then $R_{\mathrm{syn}}(\varphi_1, \ldots)$ is true in M at least to degree $R_{\mathrm{sem}}(\langle\varphi_1,\ a_1\rangle, \ldots)$.

Lemma 3 *Each of the rules* (Ax)–(Sh) *is sound. Therefore,* (Ref)–(Sh↓) *are sound as well.*

Proof (Ax): Soundness means that $||A \cup B \Rightarrow A||_M = 1$ for every M, which is obvious because $S(A \cup B, M)^* \leq S(A \cup B, M) \leq S(A, M)$ and $a \leq b$ iff $a \rightarrow b = 1$.

(Cut): We need to show that if $a \leq ||A \Rightarrow B||_M$ and $b \leq ||B \cup C \Rightarrow D||_M$ then $a^* \otimes b \leq ||A \cup C \Rightarrow D||_M$. $a \leq ||A \Rightarrow B||_M = S(A, M)^* \rightarrow S(B, M)$ is equivalent to

$$a \otimes S(A, M)^* \leq S(B, M). \tag{9}$$

From $(\alpha \vee \beta) \rightarrow \gamma = (\alpha \rightarrow \gamma) \wedge (\beta \rightarrow \gamma)$ it follows that $S(B \cup C, M) = S(B, M) \cap S(C, M)$. Therefore, $b \leq ||B \cup C \Rightarrow D||_M$ is equivalent to

$$\begin{aligned} b \otimes S(B \cup C, M)^* = b \otimes (S(B, M) \wedge S(C, M))^* &\leq S(D, M) \\ &= \bigwedge_{y \in Y} (D(y) \rightarrow M(y)), \end{aligned}$$

which holds true iff for each $y \in Y$,

$$b \otimes (S(B, M) \wedge S(C, M))^* \otimes D(y) \leq M(y). \tag{10}$$

Now, $a^* \otimes b \leq ||A \cup C \Rightarrow D||_M$ is equivalent to $a^* \otimes b \otimes S(A \cup C, M)^* \rightarrow S(D, M)$ which holds iff for each $y \in Y$ we have $a^* \otimes b \otimes S(A \cup C, M)^* \otimes D(y) \leq M(y)$. That is, since again, $S(A \cup C, M) = S(A, M) \wedge S(C, M)$, we need to check

$$b \otimes a^* \otimes (S(A, M) \wedge S(C, M))^* \otimes D(y) \leq M(y). \tag{11}$$

Let us first verify

$$a^* \otimes (S(A, M) \wedge S(C, M))^* \leq (a \otimes S(A, M)^* \wedge S(C, M))^*. \tag{12}$$

For one, since $\alpha^* = \alpha^{**}$, $(\alpha \wedge \beta)^* \leq \alpha^* \wedge \beta^*$, and $\alpha^* \leq \alpha$, we have

$$\begin{aligned} a^* \otimes (S(A, M) \wedge S(C, M))^* &= a^* \otimes (S(A, M) \wedge S(C, M))^{**} \\ &\leq a^* \otimes (S(A, M)^* \wedge S(C, M)^*)^* \leq a^* \otimes (S(A, M)^* \wedge S(C, M))^*. \end{aligned}$$

Therefore, to verify (12), it is enough to verify

$$a^* \otimes (S(A, M)^* \wedge S(C, M))^* \leq (a \otimes S(A, M)^* \wedge S(C, M))^*.$$

Due to adjointness, this inequality is equivalent to

$$(S(A, M)^* \wedge S(C, M))^* \leq a^* \to (a \otimes S(A, M)^* \wedge S(C, M))^*$$

which holds true because it is derived as follows. Due to $\alpha \leq \beta \to (\beta \otimes \alpha)$, $(\alpha \to \beta)^* \leq \alpha^* \to \beta^*$, and $\alpha \otimes (\beta \wedge \gamma) \leq (\alpha \otimes \beta) \wedge \gamma$, we have

$$\begin{aligned}&(S(A, M)^* \wedge S(C, M))^*\\&\quad \leq (a \to (a \otimes (S(A, M)^* \wedge S(C, M))))^*\\&\quad \leq a^* \to (a \otimes (S(A, M)^* \wedge S(C, M)))^*\\&\quad \leq a^* \to (a \otimes S(A, M)^* \wedge S(C, M))^*.\end{aligned}$$

We established (12). Now, from (12), (9), and (10) we get

$$\begin{aligned}&b \otimes a^* \otimes (S(A, M) \wedge S(C, M))^* \otimes D(y)\\&\quad \leq b \otimes (a \otimes S(A, M)^* \wedge S(C, M))^* \otimes D(y)\\&\quad \leq b \otimes (S(B, M) \wedge S(C, M))^* \otimes D(y) \leq M(y),\end{aligned}$$

proving (11).

(Sh): We need to check that if $a \leq ||A \Rightarrow B||_M$, i.e. $a \otimes S(A, M)^* \leq S(B, M)$, then $S(C, a \otimes B) \leq ||A \Rightarrow C||_M$, i.e. $S(C, a \otimes B) \otimes S(A, M)^* \leq S(C, M)$. The last inequality holds true iff for each $y \in Y$ we have $C(y) \otimes S(C, a \otimes B) \otimes S(A, M)^* \leq M(y)$. Since the assumption clearly implies $S(A, M)^* \leq a \to S(B, M)$, we get

$$\begin{aligned}&C(y) \otimes S(C, a \otimes B) \otimes S(A, M)^*\\&\quad \leq C(y) \otimes S(C, a \otimes B) \otimes (a \to (B(y) \to M(y)))\\&\quad \leq C(y) \otimes (C(y) \to a \otimes B(y)) \otimes (a \to (B(y) \to M(y)))\\&\quad \leq a \otimes B(y) \otimes (a \to (B(y) \to M(y))) \leq B(y) \otimes (B(y) \to M(y)) \leq M(y),\end{aligned}$$

by repeating application of $\alpha \otimes (\alpha \to \beta) \leq \beta$, verifying soundness of (Sh).

Soundness of (Ref)–(Sh$\downarrow$) is now a direct consequence of Lemma 2.

Before turning to the soundness and completeness theorem, we recall the notions of a syntactic and semantic closure of a theory, which we essentially borrow from [23]. Let us first note that in the ordinary Pavelka-style framework, R_{sem}-parts of rules are required to preserve suprema, which property has important consequences we mention below. Our rules do not have this property, i.e. in general they do not satisfy $R_{\text{sem}}(\ldots, \langle\varphi, \bigvee_{J_i} a_{j_i}\rangle, \ldots) = \bigvee_{J_i} R_{\text{sem}}(\ldots, \langle\varphi, a_{j_i}\rangle, \ldots)$ for any $J_i \subseteq L$. As one easily checks, however, they satisfy a weaker condition, namely isotony, i.e. if

$a_1 \leq b_1$, ..., then $R_{\rm sem}(\langle\varphi, a_1\rangle, \dots) \leq R_{\rm sem}(\langle\varphi, b_1\rangle, \dots)$, which turns out sufficient to establish soundness and completeness in our setting.

We call a theory T *syntactically closed* w.r.t. $\mathcal{R}$ if for each rule $R \in \mathcal{R}$ and all implications $\dots, A_i \Rightarrow B_i, \dots$ for which $R_{\rm syn}$ is defined we have

$$R_{\rm sem}(\dots, \langle A_i \Rightarrow B_i,\ T(A_i \Rightarrow B_i)\rangle, \dots) \leq T(R_{\rm syn}(\dots, A_i \Rightarrow B_i, \dots)).$$

This means that "nothing new" may be derived from T by rules in $\mathcal{R}$ in that if $A \Rightarrow B$ may be derived from T to degree a, then $a \leq T(A \Rightarrow B)$. This is clear because then, for some rule $R \in \mathcal{R}$, we have $A \Rightarrow B = R_{\rm syn}(\dots, A_i \Rightarrow B_i, \dots)$ and $a = R_{\rm sem}(\dots, \langle A_i \Rightarrow B_i,\ T(A_i \Rightarrow B_i)\rangle, \dots)$. As in the original Pavelka's framework, it is easily observed that due to the isotony of $R_{\rm sem}$-parts of deduction rules, syntactically closed theories form a closure system, hence we may denote by $\text{syn}(T)$ the least syntactically closed theory containing T. We call a theory T *semantically closed* if for each $A \Rightarrow B$ we have

$$||A \Rightarrow B||_T \leq T(A \Rightarrow B),$$

i.e., T already contains all semantic consequences of T. Since semantically closed theories form a closure system [23], we may again denote by $\text{sem}(T)$ the least semactically closed theory containing T.

The following lemma is essential for the proof of the soundness and completeness theorem for the system of rules (Ax)–(Sh). Note that the restriction to finite $\mathbf{L}$ which we utilize to establish Lemma 4 (a), along with our restriction to finite Y, comes in fact from our adherence to finitary deduction rules in a similar manner as the restriction to finite Y in the proof of completeness of classical Armstrong rules [20] (this is seen in the proof of Theorem 1).

Lemma 4 *For any theory T and $A \Rightarrow B$ we have*

(a) $(\text{syn}(T))(A \Rightarrow B) = |A \Rightarrow B|_T$ *for any finite* $\mathbf{L}$*;*
(b) $(\text{sem}(T))(A \Rightarrow B) = ||A \Rightarrow B||_T$.

Proof (a): The part $(\text{syn}(T))(A \Rightarrow B) \geq |A \Rightarrow B|_T$ is proven the same way as in the ordinary Pavelka's framework [23], since isotony—which is needed in the proof—is satisfied by (Ax)–(Sh). The proof of part $(\text{syn}(T))(A \Rightarrow B) \leq |A \Rightarrow B|_T$, however, makes use of preservation of suprema in the ordinary Pavelka's framework [23]. An inspection of the argument reveals that the step in which preservation of suprema is needed has, the form

$$R_{\rm sem}(\dots, \langle\varphi, \textstyle\bigvee_{a\in P} a\rangle, \dots) = \textstyle\bigvee_{a\in P} R_{\rm sem}(\dots, \langle\varphi, a\rangle, \dots),$$

where P is the set of all a's such that there exists a weighted proof $\dots, \langle\varphi, a\rangle$ from T. Clearly, such equality is satisfied if P has a largest element, which is our case because $\mathbf{L}$ is finite and because

$$\text{(Sup)}\ \frac{\langle A \Rightarrow B,\ a\rangle, \langle A \Rightarrow B,\ b\rangle}{\langle A \Rightarrow B,\ a \vee b\rangle}$$

is a derived rule. To verify the latter claim, observe that applying (Sh↑) to $\langle A \Rightarrow B,\ a\rangle$ and $\langle A \Rightarrow B,\ b\rangle$ we obtain

$$\langle A \Rightarrow a \otimes B,\ 1\rangle \text{ and } \langle A \Rightarrow b \otimes B,\ 1\rangle,$$

respectively, from which we get

$$\langle A \Rightarrow a \otimes B \cup b \otimes B,\ 1\rangle$$

by (Add). Due to $a \otimes B \cup b \otimes B = (a \vee b) \otimes B$, the last weighted formula is equal to $\langle A \Rightarrow (a \vee b) \otimes B,\ 1\rangle$ from which we obtain

$$\langle A \Rightarrow (a \vee b) \rightarrow ((a \vee b) \otimes B),\ a \vee b\rangle$$

by (Sh↓). Since $B \subseteq (a \vee b) \rightarrow ((a \vee b) \otimes B)$, an application of (Pro) finally yields

$$\langle A \Rightarrow (a \vee b) \rightarrow ((a \vee b) \otimes B),\ a \vee b\rangle,$$

veryfying that (Sup) is a derived rule.

(b): Since the semantic notions of our logic are a particular case of the semantic notions of the ordinary Pavelka's framework, the result is a particular case of the corresponding result from [23].

The following theorem presents the main result of this paper. Note that provability refers to rules (Ax)–(Sh), i.e. $|A \Rightarrow B|_T$ denotes a degree of provability using rules (Ax)–(Sh).

Theorem 1 (Soundness and completeness) *Let L be finite. For each theory T of graded attribute implications and graded attribute implication $A \Rightarrow B$ we have*

$$||A \Rightarrow B||_T = |A \Rightarrow B|_T.$$

Proof "$\geq$": As in Pavelka's framework, each semantically closed T is also syntactically closed w.r.t. any set of sound rules. Since each of (Ax)–(Sh) is sound (Lemma 3) and since $\mathrm{sem}(T)$ is semantically closed, $\mathrm{sem}(T)$ is also syntactically closed and so $\mathrm{syn}(\mathrm{sem}(T)) = \mathrm{sem}(T)$. Therefore, using Lemma 4 (a), the fact that $T \subseteq \mathrm{sem}(T)$ and hence also $\mathrm{syn}(T) \subseteq \mathrm{syn}(\mathrm{sem}(T))$, we get $|A \Rightarrow B|_T = (\mathrm{syn}(T))(A \Rightarrow B) \subseteq (\mathrm{syn}(\mathrm{sem}(T)))(A \Rightarrow B) = (\mathrm{sem}(T))(A \Rightarrow B) = ||A \Rightarrow B||_T$.

"$\leq$": We aim at establishing that each syntactically closed theory is semantically closed, because then $\mathrm{sem}(\mathrm{syn}(T)) = \mathrm{syn}(T)$ and, analogously to the "$\geq$"-part, Lemma 4 (b), and $T \subseteq \mathrm{syn}(T)$ imply $||A \Rightarrow B||_T = (\mathrm{sem}(T))(A \Rightarrow B) \subseteq (\mathrm{sem}(\mathrm{syn}(T)))(A \Rightarrow B) = (\mathrm{syn}(T))(A \Rightarrow B) = |A \Rightarrow B|_T$.

Let thus T be syntactically closed. To establish that T is semantically closed, we need to show $\mathrm{sem}(T) \subseteq T$. Due to Lemma 4, $(\mathrm{sem}(T))(A \Rightarrow B) = ||A \Rightarrow B||_T$. By way of contradiction, assume that there exists $A \Rightarrow B$ for which

$$||A \Rightarrow B||_T \nleq T(A \Rightarrow B). \tag{13}$$

It is now sufficient to show that there exists a model $A^+ \in \mathrm{Mod}(T)$ such that $||A \Rightarrow B||_{A^+} \leq T(A \Rightarrow B)$ since then,

$$||A \Rightarrow B||_T = \bigwedge_{M\in\mathrm{Mod}(T)} ||A \Rightarrow B||_M \leq ||A \Rightarrow B||_{A^+} \leq T(A \Rightarrow B)$$

is a contradiction to (13).

To this end, put $A^+ = \bigcup_{B\in\mathbf{L}^Y} T(A \Rightarrow B) \otimes B$, where $T(A \Rightarrow B) \otimes B$ is the multiplication of B defined by (1). Observe first that $T(A \Rightarrow A^+) = 1$. Indeed, since T is syntactically closed, (Sh↑) applied to $\langle A \Rightarrow B, T(A \Rightarrow B)\rangle$ and syntactical closedness of T yield $1 = T(A \Rightarrow T(A \Rightarrow B) \otimes B)$ for each $B \in \mathbf{L}^Y$. Since both Y and L are finite, a repeated application of (Add) and syntactical closedness of T yields $T(A \Rightarrow A^+) = 1$.

We now verify: (a) $||A \Rightarrow B||_{A^+} \leq T(A \Rightarrow B)$ and (b) A^+ is a model of T.
(a): Observe first that $A \subseteq A^+$ and thus $S(A, A^+)^* = 1$. Indeed, syntactical closedness of T applied to (Ref) yields $T(A \Rightarrow A) = 1$ and thus $A = T(A \Rightarrow A) \otimes A \subseteq A^+$. Now,

$$||A \Rightarrow B||_{A^+} = S(A, A^+)^* \to S(B, A^+) = 1 \to S(B, A^+) = S(B, A^+).$$

It now suffices to show $S(B, A^+) \leq T(A \Rightarrow B)$. Since $S(B, A^+) \otimes B \subseteq A^+$, using (Ax) we infer the weighted formula

$$\langle S(B, A^+) \otimes B \cup A^+ \Rightarrow S(B, A^+) \otimes B,\ 1\rangle = \langle A^+ \Rightarrow S(B, A^+) \otimes B,\ 1\rangle.$$

Now, we apply (Sh↓) with $a = S(B, A^+)$ and obtain

$$\langle A^+ \Rightarrow S(B, A^+) \to (S(B, A^+) \otimes B),\ S(B, A^+)\rangle.$$

Since $B \subseteq S(B, A^+) \to (S(B, A^+) \otimes B$, (Pro) now yields

$$\langle A^+ \Rightarrow B,\ S(B, A^+)\rangle.$$

Now, (Tra) applied to $\langle A \Rightarrow A^+, T(A \Rightarrow A^+)\rangle = \langle A \Rightarrow A^+, 1\rangle$ and the previous weighted formula yields the weighted formula

$$\langle A \Rightarrow B,\ 1 \otimes S(B, A^+)\rangle = \langle A \Rightarrow B,\ S(B, A^+)\rangle.$$

Syntactical closedness of T now yields $S(B, A^+) \leq T(A \Rightarrow B)$, proving (a).
(b): We need to show $T(C \Rightarrow D) \leq ||C \Rightarrow D||_{A^+}$ for any $C \Rightarrow D$. This amounts to checking $T(C \Rightarrow D) \leq S(C, A^+)^* \to S(D, A^+)$ which is equivalent to

$$S(C, A^+)^* \otimes T(C \Rightarrow D) \otimes D \subseteq A^+. \tag{14}$$

To verify (14), it suffices to show that

$$T(A \Rightarrow S(C, A^+)^* \otimes T(C \Rightarrow D) \otimes D) = 1. \tag{15}$$

Indeed, denoting $E = S(C, A^+)^* \otimes T(C \Rightarrow D) \otimes D$, (15) implies

$$A^+ = \bigcup_{B \in \mathbf{L}^Y} T(A \Rightarrow B) \otimes B \supseteq T(A \Rightarrow E) \otimes E = E,$$

establishing (14). Therefore, it remains to show (15). We have $T(A \Rightarrow A^+) = 1$. Because $A^+ \supseteq S(C, A^+)^* \otimes C$, (Ax) yields the weighted formula

$$\langle A^+ \Rightarrow S(C, A^+)^* \otimes C,\ 1\rangle. \tag{16}$$

(Tra) applied to $\langle A \Rightarrow A^+,\ T(A \Rightarrow A^+)\rangle = \langle A \Rightarrow A^+,\ 1\rangle$ and (16) yields

$$\langle A \Rightarrow S(C, A^+)^* \otimes C,\ 1\rangle. \tag{17}$$

Now, (Mul) applied to $\langle C \Rightarrow D,\ T(C \Rightarrow D)\rangle$ and $a = S(C, A^+)$ yields

$$\langle S(C, A^+)^* \otimes C \Rightarrow S(C, A^+)^* \otimes D,\ T(C \Rightarrow D)\rangle. \tag{18}$$

The last step is to apply (Tra) to (16) and (18) which gives $\langle A \Rightarrow S(C, A^+)^* \otimes D,\ T(C \Rightarrow D)\rangle$. Next, (Sh↑) yields $\langle A \Rightarrow S(C, A^+)^* \otimes T(C \Rightarrow D) \otimes D,\ 1\rangle$. The fact that T is syntactically closed now implies (15), finishing the proof.

Remark 6 Note that, as is seen from the proof and the fact that $|A \Rightarrow B|_T \leq (\mathrm{syn}(T))(A \Rightarrow B)$ holds for every isotone deduction rules, $|A \Rightarrow B|_T \leq ||A \Rightarrow B||_T$ holds for any, possibly infinite L. Note also that the completeness theorem may be obtained for arbitrary L when using infinitary deduction rules but we refrain from elaborating on this issue.

Acknowledgments This work has been partly supported by grants No. P202/10/0262 (both authors till 2012) and No. 15-17899S (Belohlavek) of the Czech Science Foundation.

References

1. Abiteboul, S. et al.: The Lowell database research self-assessment. Commun. ACM **48**(5), 111–118 (2005)
2. Armstrong, W.W.: Dependency structures in data base relationships. IFIP Congress, Geneva, Switzerland, pp. 580–583 (1974)
3. Belohlavek, R., Vychodil, V.: Attribute implications in a fuzzy setting. Lect. Notes Artif. Intell. **3874**, 45–60 (2006)
4. Belohlavek, R., Vychodil, V.: Codd's relational model from the point of view of fuzzy logic. J. Logic Comput. **21**(5), 851–862 (2011)
5. Belohlavek, R., Vychodil, V.: Attribute dependencies for data with grades (submitted, arXiv:1402.2071 [cs.LO]).
6. Carpineto, C., Romano, G.: Concept Data Analysis: Theory and Applications. Wiley, New Jersey (2004)
7. Cintula, P., Hájek, P., Noguera, C.: Handbook of Mathematical Fuzzy Logic, vol. I, II, College Publ., London 2011 (vols. 37 and 38 of series Studies in Logic)
8. Esteva, F., Godo, L., Marchioni, E.: Fuzzy logics with enriched language. In [7], vol. II, pp. 627–711
9. Fagin, R.: Functional dependencies in a relational database and propositional logic. IBM J. Res. Dev. **21**(6), 543–544 (1977)
10. Fagin, R.: Combining fuzzy information: an overview. SIGMOD Rec **31**(2), 109–118 (2002)

11. Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics. Elsevier, New York (2007)
12. Ganter, B., Wille, R.: Formal Concept Analysis. Mathematical Foundations. Springer, Berlin (1999)
13. Gerla, G.: Inferences in probability logic. Artif. Intell. **70**(1–2), 33–52 (1994)
14. Gerla, G.: Fuzzy Logic. Mathematical Tools for Approximate Reasoning. Kluwer, Dordrecht (2001)
15. Goguen, J.A.: The logic of inexact concepts. Synthese **18**, 325–373 (1968–1969)
16. Gottwald, S.: A Treatise on Many-Valued Logic. Studies in Logic and Computation, vol. 9, Research Studies Press: Baldock, Hertfordshire, England (2001)
17. Gottwald, S.: Mathematical fuzzy logics. Bull. Symb. Logic **14**, 210–239 (2008)
18. Hájek, P.: Metamathematics of Fuzzy Logic. Kluwer, Dordrecht (1998)
19. Hájek, P.: On very true. Fuzzy Sets Syst. **124**, 329–333 (2001)
20. Maier, D.: The Theory of Relational Databases. Computer Science Press, Rockville (1983)
21. Mayor, G., Torrens, J.: Triangular norms on discrete settings. In: Klement, E.P., Mesiar, R. Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms, pp. 189–230. Elsevier, New York (2005)
22. Novák, V., Perfilieva, I., Močkoř, J.: Mathematical Principles of Fuzzy Logic. Kluwer, Dodrecht (1999)
23. Pavelka, J.: On fuzzy logic I, II, III. Z. Math. Logik Grundlagen Math. **25**, 45–52, 119–134, 447–464 (1979)
24. Raju, K.V.S.V.N., Majumdar, A.K.: Fuzzy functional dependencies and lossless join decomposition of fuzzy relational database systems. ACM Trans. Database Syst. **13**(2), 129–166 (1988)
25. Takeuti, G., Titani, S.: Globalization of intuitionistic set theory. Ann. Pure Appl. Logic **33**, 195–211 (1987)
26. Tyagi, B.K., Sharfuddin, A., Dutta, R.N., Tayal, D.K.: A complete axiomatization of fuzzy functional dependencies using fuzzy function. Fuzzy Sets Syst. **151**(2), 363–379 (2005)
27. Zadeh, L.A.: Fuzzy Logic. IEEE Comput. **21**(4), 83–93 (1988)
28. Zadeh, L.A.: Fuzzy logic, neural networks, and soft computing. Commun. ACM **37**(3), 77–84 (1994)
29. Zhang, C., Zhang, S.: Association Rule Mining. Models and Algorithms. Springer, Berlin (2002)