On Proofs and Rule of Multiplication in Fuzzy Attribute Logic*

Radim Belohlavek[1,2] and Vilem Vychodil[2]

[1] Dept. Systems Science and Industrial Engineering, Binghamton University—SUNY
Binghamton, NY 13902, U. S. A., rbelohla@binghamton.edu

[2] Dept. Computer Science, Palacky University, Olomouc
Tomkova 40, CZ-779 00 Olomouc, Czech Republic, vilem.vychodil@upol.cz

Abstract. The paper develops fuzzy attribute logic, i.e. a logic for reasoning about formulas of the form $A \Rightarrow B$ where A and B are fuzzy sets of attributes. A formula $A \Rightarrow B$ represents a dependency which is true in a data table with fuzzy attributes iff each object having all attributes from A has also all attributes from B, membership degrees in A and B playing a role of thresholds. We study axiomatic systems of fuzzy attribute logic which result by adding a single deduction rule, called a rule of multiplication, to an ordinary system of deduction rules complete w.r.t. bivalent semantics, i.e. to well-known Armstrong axioms. In this paper, we concentrate on the rule of multiplication and its role in fuzzy attribute logic. We show some advantageous properties of the rule of multiplication. In addition, we show that these properties enable us to reduce selected problems concerning proofs in fuzzy attribute logic to the corresponding problems in the ordinary case. As an example, we discuss the problem of normalization of proofs and present, in the setting of fuzzy attribute logic, a counterpart to a well-known theorem from database theory saying that each proof can be transformed to a so-called RAP-sequence.

1 Introduction

If-then rules in their various variants are perhaps the most common way to express our knowledge. Usually, if-then rules are extracted from data to bring up a new knowledge about the data or are formulated by a user/expert to represent a constraint on the data. If-then rules of the form $A \Rightarrow B$, where A and B are collections of attributes, have been used in data mining and in databases. In data mining, rules $A \Rightarrow B$ are called association rules or attribute implications, and have the following basic meaning: If an object has all attributes from A then it has all attributes from B. This gives a rise to the first semantics. The rules $A \Rightarrow B$ are interpreted in tables with crisp attributes, i.e., with rows corresponding to objects, columns corresponding to "yes-or-no" attributes, and

* Supported by grant No. 1ET101370417 of GA AV ČR, by grant No. 201/05/0079 of the Czech Science Foundation, and by institutional support, research plan MSM 6198959214.

table entries containing 1 or 0 indicating whether an object does or does not have an attribute. The goal then is to extract "all interesting rules" from data. In databases, rules $A \Rightarrow B$ are called functional dependencies, see [17] for a comprehensive overview, and have the following meaning: If any two objects (items, rows) of a database table agree in their values on all attributes from A then they agree on all attributes from B. This gives a rise to the second semantics in which rules $A \Rightarrow B$ are interpreted in database tables.

The two ways of interpreting rules $A \Rightarrow B$ are closely connected. Namely, semantic entailment coincides for both of them. This entailment can also be captured syntactically. This goes back to Armstrong [1] who introduced a set of inference rules, a modified version of which became known as Armstrong axioms. Armstrong axioms are very well known to be complete w.r.t. database semantics and, due to the above-mentioned connection, also to the other semantics. This means that a rule $A \Rightarrow B$ is provable (using Armstrong axioms) from a set T of rules if and only if $A \Rightarrow B$ semantically follows from T.

In a series of papers, see e.g. [3–5, 7–10] and also an overview paper [6], we started to develop the above-described issues from the point of view of fuzzy logic (note that the first attempt is [18]). We introduced fuzzy attribute implications, i.e., our counterparts to the ordinary if-then rules $A \Rightarrow B$ described above. Among other issues, we studied two types of semantics, the first one given by tables with fuzzy attributes and the second one given by ranked tables over domains with similarities. We proved that these types of semantics have the same semantic entailment. In [4], we introduced a logical calculus, called fuzzy attribute logic, for reasoning with fuzzy attribute implications. We proved its syntactico-semantical completeness, both in the ordinary style ("provable = semantically entailed") and in the graded style ("degree of provability = degree of semantical entailment"). In [5], we presented further results on fuzzy attribute logic. For our present purpose, the most important result of [5] is an invention of a single deduction rule, called a rule of multiplication, which has the following property: Adding the rule of multiplication to an ordinary system of Armstrong axioms (Armstrong axioms are, in fact, deduction rules) yields a syntactico-semantically complete system for reasoning with fuzzy attribute implications. The main aim of the present paper is to focus in more detail on the rule of multiplication and its role in fuzzy attribute logic. As emphasized in [5], the rule of multiplication allows one to consider fuzzy attribute logic as consisting of a system of "ordinary Armstrong rules" plus a single "fuzzy rule" (which is the rule of multiplication). We will show that in addition to this role which is more or less an aesthetic one, the rule of multiplication has also its practical role. Among other things, its properties allow us to almost automatically transfer results known from the ordinary case. As an example, we focus on the problem of normalization of proofs and present, in the setting of fuzzy attribute logic, a counterpart to a well-known theorem from database theory saying that each proof can be transformed to a RAP-sequence.

The paper is organized as follows. Section 2 surveys preliminaries from fuzzy sets and fuzzy logic. Fuzzy attribute logic and its completeness is presented in Section 3. Section 4 deals with the rule of multiplication and related issues. Section 5 presents conclusions.

2 Preliminaries

This section surveys preliminaries from fuzzy sets and fuzzy logic. Further details can be found, e.g., in [2, 14, 16]. As a structure of truth degrees, i.e., a set of truth degrees equipped with truth functions of logical connectives, we use complete residuated lattices with a truth-stressing hedge (shortly, a hedge) [14, 15], i.e., algebras $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e., $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$ (adjointness property); hedge * satisfies $1^* = 1$, $a^* \leq a$, $(a \rightarrow b)^* \leq a^* \rightarrow b^*$, $a^{**} = a^*$ for each $a, b \in L$. Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true", see [14, 15].

A favorite choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ or a subchain of $[0, 1]$, equipped with well-known pairs of $\otimes$ (t-norms or restrictions of t-norms) and the corresponding $\rightarrow$ (residuum to $\otimes$).

Two boundary cases of (truth-stressing) hedges are (i) identity, i.e., $a^* = a$ ($a \in L$); (ii) globalization [19]: $1^* = 1$ and $a^* = 0$ for $a < 1$. A special case of a complete residuated lattice with hedge is the two-element Boolean algebra $\langle\{0, 1\}, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of classical logic. That is, the operations $\wedge, \vee, \otimes, \rightarrow$ of $\mathbf{2}$ are the truth functions (interpretations) of the corresponding logical connectives of the classical logic and $0^* = 0$, $1^* = 1$. Note that if we prove an assertion for general $\mathbf{L}$, then, in particular, we obtain a "crisp version" of this assertion for $\mathbf{L}$ being $\mathbf{2}$.

Having $\mathbf{L}$, we define usual notions of an $\mathbf{L}$-set (fuzzy set), $\mathbf{L}$-relation (fuzzy relation), etc. $\mathbf{L}^U$ denotes the collection of all $\mathbf{L}$-sets in U. Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A, B) = \textstyle\bigwedge_{u \in U}\bigl(A(u) \rightarrow B(u)\bigr), \tag{1}$$

which generalizes the classical subsethood relation $\subseteq$. $S(A, B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A, B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$.

3 Fuzzy Attribute Logic and Its Completeness

We now introduce basic concepts of fuzzy attribute logic (FAL). Suppose Y is a finite set of attributes. A (*fuzzy*) *attribute implication* (*over attributes* Y), shortly FAI, is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ (A and B are fuzzy sets of attributes). Fuzzy attribute implications are the basic formulas of FAL.

The next step is the semantics of FAL given by interpreting FAIs in data tables with fuzzy attributes. The intended meaning of $A \Rightarrow B$ is: "if it is (very)

Table 1. Data table with fuzzy attributes

I	y_1	y_2	y_3	y_4	y_5	y_6
x_1	1.0	1.0	0.0	1.0	1.0	0.2
x_2	1.0	0.4	0.3	0.8	0.5	1.0
x_3	0.2	0.9	0.7	0.5	1.0	0.6
x_4	1.0	1.0	0.8	1.0	1.0	0.5

$X = \{x_1, \ldots, x_4\}$

$Y = \{y_1, \ldots, y_6\}$

true that an object has all attributes from A, then it has also all attributes from B" with the logical connectives being given by $\mathbf{L}$. A *data table with fuzzy attributes* can be seen as a triplet $\langle X, Y, I \rangle$ where X is a set of objects, Y is a finite set of attributes (the same as above in the definition of a fuzzy attribute implication), and $I \in \mathbf{L}^{X \times Y}$ is a binary $\mathbf{L}$-relation between X and Y assigning to each object $x \in X$ and each attribute $y \in Y$ a degree $I(x, y)$ to which x has y. $\langle X, Y, I \rangle$ can be thought of as a table with rows and columns corresponding to objects $x \in X$ and attributes $y \in Y$, respectively, and table entries containing degrees $I(x, y)$, see Tab. 1. A row of a table $\langle X, Y, I \rangle$ corresponding to an object $x \in X$ can be seen as a fuzzy set I_x of attributes to which an attribute $y \in Y$ belongs to a degree $I_x(y) = I(x, y)$. Forgetting now for a while about the data table, any fuzzy set $M \in \mathbf{L}^Y$ can be seen as a fuzzy set of attributes of some object with $M(y)$ being a degree to which the object has attribute y. For a fuzzy set $M \in \mathbf{L}^Y$ of attributes, we define a *degree* $||A \Rightarrow B||_M \in L$ *to which* $A \Rightarrow B$ *is true in* M by

$$||A \Rightarrow B||_M = S(A, M)^* \to S(B, M). \tag{2}$$

It is easily seen that if M is a fuzzy set of attributes of some object x then $||A \Rightarrow B||_M$ is the degree to which "if it is (very) true that x has all attributes from A then x has all attributes from B". For a system $\mathcal{M}$ of $\mathbf{L}$-sets in Y, define a degree $||A \Rightarrow B||_{\mathcal{M}}$ to which $A \Rightarrow B$ is true in (each M from) $\mathcal{M}$ by

$$||A \Rightarrow B||_{\mathcal{M}} = \textstyle\bigwedge_{M \in \mathcal{M}} ||A \Rightarrow B||_M. \tag{3}$$

Finally, given a data table $\langle X, Y, I \rangle$ and putting $\mathcal{M} = \{I_x \mid x \in X\}$, $||A \Rightarrow B||_{\mathcal{M}}$ is a degree to which it is true that $A \Rightarrow B$ is true in each row of table $\langle X, Y, I \rangle$, i.e., a degree to which "for each object $x \in X$: if it is (very) true that x has all attributes from A, then x has all attributes from B". This degree is denoted by $||A \Rightarrow B||_{\langle X,Y,I \rangle}$ and is called a degree to which $A \Rightarrow B$ is true in data table $\langle X, Y, I \rangle$.

Remark 1. (1) For a fuzzy attribute implication $A \Rightarrow B$, both A and B are fuzzy sets of attributes. Particularly, both A and B can be crisp (i.e., $A(y) \in \{0, 1\}$ and $B(y) \in \{0, 1\}$ for each $y \in Y$). Ordinary attribute implications (association rules, functional dependencies) are thus a special case of fuzzy attribute implications.

(2) For a fuzzy attribute implication $A \Rightarrow B$, degrees $A(y) \in L$ and $B(y) \in L$ can be seen as thresholds. This is best seen when * is globalization, i.e., $1^* = 1$

and $a^* = 0$ for $a < 1$. Since for $a, b \in L$ we have $a \leq b$ iff $a \rightarrow b = 1$, we have

$$(a \rightarrow b)^* = \begin{cases} 1 & \text{iff } a \leq b, \\ 0 & \text{iff } a \not\leq b. \end{cases}$$

Therefore, $||A \Rightarrow B||_{\langle X,Y,I \rangle} = 1$ means that a proposition "for each object $x \in X$: if for each attribute $y \in Y$, x has y in degree greater than or equal to (a threshold) $A(y)$, then for each $y \in Y$, x has y in degree at least $B(y)$" has a truth degree 1 (is fully true). In general, $||A \Rightarrow B||_{\langle X,Y,I \rangle}$ is a truth degree of the latter proposition. As a particular example, if $A(y) = a$ for $y \in Y_A \subseteq Y$ (and $A(y) = 0$ for $y \notin Y_A$) $B(y) = b$ for $y \in Y_B \subseteq Y$ (and $B(y) = 0$ for $y \notin Y_B$), the proposition says "for each object $x \in X$: if x has all attributes from Y_A in degree at least a, then x has all attributes from Y_B in degree at least b", etc. That is, having A and B fuzzy sets allows for a rich expressibility of relationships between attributes which is why we want A and B to be fuzzy sets in general.

We are now coming to the concept of semantic entailment. For simplicity, we consider ordinary sets of FAIs as theories in FAL, i.e., a theory in FAL is a set T of FAIs. Note that more generally, one can consider fuzzy sets of FAIs which is more in the spirit of fuzzy logic, see [5, 10]. For a theory T, the set $\mathrm{Mod}(T)$ of all *models* of T is defined by

$$\mathrm{Mod}(T) = \{M \in \mathbf{L}^Y \mid \text{for each } A \Rightarrow B \in T\colon ||A \Rightarrow B||_M = 1\}.$$

Therefore, $M \in \mathrm{Mod}(T)$ means that each $A \Rightarrow B$ from T is fully true in M. Then, a *degree* $||A \Rightarrow B||_T \in L$ to which $A \Rightarrow B$ *semantically follows* from a set T of attribute implications is defined by

$$||A \Rightarrow B||_T = \textstyle\bigwedge_{M \in \mathrm{Mod}(T)} ||A \Rightarrow B||_M.$$

That is, $||A \Rightarrow B||_T$ can be seen as a truth degree of "$A \Rightarrow B$ is true in each model of T".

Consider now the following system of *deduction rules*:

(Ax) infer $A \cup B \Rightarrow A$,

(Cut) from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow D$,

(Mul) from $A \Rightarrow B$ infer $c^* \otimes A \Rightarrow c^* \otimes B$,

for each $A, B, C, D \in \mathbf{L}^Y$, and $c \in L$. Rules (Ax)–(Mul) are to be understood as usual deduction rules: having FAIs which are of the form of FAIs in the input part (the part preceding "infer") of a rule, a rule allows us to infer (in one step) the corresponding fuzzy attribute implication in the output part (the part following "infer") of a rule. (Ax) is a nullary rule (axiom) which says that each $A \cup B \Rightarrow A$ ($A, B \in \mathbf{L}^Y$), i.e., each $C \Rightarrow D$ with $D \subseteq C$, can be inferred in one step.

As usual, a FAI $A \Rightarrow B$ is called *provable* from a set T of fuzzy attribute implications using a set $\mathcal{R}$ of deduction rules, written $T \vdash_{\mathcal{R}} A \Rightarrow B$, if there is a sequence $\varphi_1, \ldots, \varphi_n$ of fuzzy attribute implications such that φ_n is $A \Rightarrow B$ and for each φ_i we either have $\varphi_i \in T$ or φ_i is inferred (in one step) from some of the preceding formulas (i.e., $\varphi_1, \ldots, \varphi_{i-1}$) using some deduction rule from $\mathcal{R}$. If $\mathcal{R}$ consists of (Ax)–(Mul), we say just "provable ..." instead of "provable ... using $\mathcal{R}$" and write just $T \vdash A \Rightarrow B$ instead of $T \vdash_{\mathcal{R}} A \Rightarrow B$. The following theorem was proved in [5].

Theorem 1 (completeness). *Let* $\mathbf{L}$ *and* Y *be finite. Let* T *be a set of fuzzy attribute implications. Then*

$$T \vdash A \Rightarrow B \quad \textit{iff} \quad ||A \Rightarrow B||_T = 1.$$

□

Therefore, $A \Rightarrow B$ is provable from T iff $A \Rightarrow B$ semantically follows from T in degree 1.

4 Properties and Role of the Rule of Multiplication

Rule (Mul) is called a *rule of multiplication.* System (Ax)–(Mul) improves a previously known complete system presented in [4]. Namely, the system in [4] consists of the following deduction rules:

(Ax') infer $A \Rightarrow S(B, A) \otimes B$,

(Wea') from $A \Rightarrow B$ infer $A \cup C \Rightarrow B$,

(Cut') from $A \Rightarrow e \otimes B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow e^* \otimes D$

for each $A, B, C, D \in \mathbf{L}^Y$, and $e \in L$.

Comparing (Ax)–(Mul) to (Ax')–(Cut'), we can see the following distinction. (Ax)–(Mul) results by directly taking two ordinary Armstrong rules, namely (Ax) and (Cut), and by adding a new "fuzzy rule", namely (Mul). In more detail, rules "infer $A \cup B \Rightarrow A$" and "from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow D$" with A, B, C, D being ordinary sets of attributes are well-known ordinary Armstrong rules, see e.g. [17]. Therefore, (Ax) and (Cut) are just these rules with sets replaced by fuzzy sets. Rule (Mul) is a new rule. In the ordinary setting, (Mul) is trivial since it reads "from $A \Rightarrow B$ infer $A \Rightarrow B$" when $c = 1$ and "from $A \Rightarrow B$ infer $\emptyset \Rightarrow \emptyset$" when $c = 0$ (this is easily seen because the only hedge * in the ordinary setting is the identity mapping). On the other hand, (Ax')–(Cut') result by modifying ordinary rules. For instance, (Cut') results from the above ordinary rule "from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ infer $A \cup C \Rightarrow D$" (with A, B, C, D being ordinary sets) by replacing sets by fuzzy sets and by inserting multiplication by truth degrees e and e^*.

The first apparent advantage of system (Ax)–(Mul) is thus aesthetic. System (Ax)–(Mul) can be seen as having two parts, the "ordinary one" consisting of ordinary rules (Ax) and (Cut), and the "fuzzy one" consisting of (Mul). Intuitively, it therefore keeps separated the "ordinary" and the "fuzzy part" with the rule of multiplication "taking care of fuzziness". We are now going to show that there are practical advantages of keeping the ordinary and fuzzy rules separated as well.

The first example is an easily observable fact mentioned already in [5] that if * is globalization, (Mul) can be omitted. Namely, for $c = 1$, (Mul) becomes "from $A \Rightarrow B$ infer $A \Rightarrow B$" which does not yield anything new. For $c < 1$, (Mul) becomes "from $A \Rightarrow B$ infer $\emptyset \Rightarrow \emptyset$" but $\emptyset \Rightarrow \emptyset$ can be inferred using (Ax) and so (Mul) can be omitted. In the rest of our paper, we are concerned with properties of the rule of multiplication related to the structure of proofs in FAL.

Let us say that (Mul) *commutes* (*backwards*) with a rule (R) of the form "from $A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n$ infer $A \Rightarrow B$" if any FAI $C \Rightarrow D$ which results by first inferring $A \Rightarrow B$ from $A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n$, using (R) and then inferring $C \Rightarrow D$ from $A \Rightarrow B$ using (Mul) can be obtained by first inferring $C_1 \Rightarrow D_1$, $\ldots$, $C_n \Rightarrow D_n$, from $A_1 \Rightarrow B_1$, $\ldots$, $A_n \Rightarrow B_n$, using (Mul), respectively, and then inferring $C \Rightarrow D$ from $C_1 \Rightarrow D_1$, $\ldots$, $C_n \Rightarrow D_n$, using (R).

A practical meaning of commutativity of (Mul) with (R) is that in proofs, one can change the order of rules (Mul) and (R) from "first (R), then (Mul)" to "first (Mul), then (R)".

Lemma 1 (commutativity of (Mul)). (Mul) *commutes with both* (Ax) *and* (Cut).

Proof. Commutativity with (Ax): Let $c^* \otimes (A \cup B) \Rightarrow c^* \otimes A$ result by first inferring $A \cup B \Rightarrow A$ by (Ax) and then inferring $c^* \otimes (A \cup B) \Rightarrow c^* \otimes A$ from $A \cup B \Rightarrow A$ by (Mul). In this case, $c^* \otimes (A \cup B) \Rightarrow c^* \otimes A$ can be obtained by applying (Mul) 0-times (i.e., not applying at all) and then inferring $c^* \otimes (A \cup B) \Rightarrow c^* \otimes A$ by (Ax) since $c^* \otimes (A \cup B) = (c^* \otimes A) \cup (c^* \otimes B)$ and thus, $c^* \otimes (A \cup B) \Rightarrow c^* \otimes A$ is of the form $C \cup D \Rightarrow C$.

Commutativity with (Cut): Let $c^* \otimes (A \cup C) \Rightarrow c^* \otimes D$ result by first inferring $A \cup C \Rightarrow D$ from $A \Rightarrow B$ and $B \cup C \Rightarrow D$, and then inferring $c^* \otimes (A \cup C) \Rightarrow c^* \otimes D$ from $A \cup C \Rightarrow D$ by (Mul). Then, one can first infer $c^* \otimes A \Rightarrow c^* \otimes B$ and $c^* \otimes (B \cup C) \Rightarrow c^* \otimes D$ from $A \Rightarrow B$ and $B \cup C \Rightarrow D$ by (Mul), respectively. Since $c^* \otimes (B \cup C) = (c^* \otimes B) \cup (c^* \otimes C)$, and $c^* \otimes (A \cup C) = (c^* \otimes A) \cup (c^* \otimes C)$, one can infer $c^* \otimes (A \cup C) \Rightarrow c^* \otimes D$ from $c^* \otimes A \Rightarrow c^* \otimes B$ and $c^* \otimes (B \cup C) \Rightarrow c^* \otimes D$ using (Cut). □

Another property of (Mul) is the following one.

Lemma 2 (idempotency of (Mul)). *Two* (*or more*) *consecutive inferences by* (Mul) *can be replaced by a single inference by* (Mul).

Proof. Suppose we start with $A \Rightarrow B$, apply (Mul) to infer $c^* \otimes A \Rightarrow c^* \otimes B$ and then apply (Mul) again to infer $d^* \otimes c^* \otimes A \Rightarrow d^* \otimes c^* \otimes B$. Then, the assertion follows by observing that $d^* \otimes c^* = (d^* \otimes c^*)^*$. Indeed, $d^* \otimes c^* \otimes A \Rightarrow d^* \otimes c^* \otimes B$ is then of the form $a^* \otimes A \Rightarrow a^* \otimes B$, with $a = d^* \otimes c^*$. It remains to prove $d^* \otimes c^* = (d^* \otimes c^*)^*$. "$\geq$" follows directly from subdiagonality of *, i.e., from $b^* \leq b$. "$\leq$" is equivalent to $d^* \leq c^* \to (d^* \otimes c^*)^*$ which is true. To see this, observe that $d^* \leq c^* \to (d^* \otimes c^*)$ from which we get

$$d^* = d^{**} \leq (c^* \to (d^* \otimes c^*))^* \leq c^{**} \to (d^* \otimes c^*)^* = c^* \to (d^* \otimes c^*)^*.$$

□

The above-observed commutativity of (Mul) has the following consequence.

Theorem 2 (normal form of proofs in FAL). *If $A \Rightarrow B$ is provable from T using* (Ax)–(Mul)*, then there exists a proof $A_1 \Rightarrow B_1, \ldots, A_n \Rightarrow B_n$ of $A \Rightarrow B$ from T using* (Ax)–(Mul) *and integers $1 \leq k \leq l \leq n$ such that*

1. *for $i = 1, \ldots, k$, we have $A_i \Rightarrow B_i \in T$,*
2. *for $i = k+1, \ldots, l$, $A_i \Rightarrow B_i$ results by application of* (Mul) *to some $A_j \Rightarrow B_j$ with $1 \leq j \leq k$,*
3. *for $i = l+1, \ldots, n$, $A_i \Rightarrow B_i$ results by application of* (Ax) *or* (Cut) *to some $A_j \Rightarrow B_j$'s with $1 \leq j < i$.*

Proof. The proof follows by induction from Lemma 1 and Lemma 2, we omit details. □

According to Theorem 2, each proof from T in FAL can be transformed to a proof which starts by formulas from T, continues by applications of (Mul) to these formulas, and then by applications of (Ax) and (Cut) to preceding formulas. Therefore, the part of the proof which uses "ordinary rules" (Ax) and (Cut) is separated from the part which uses "fuzzy rule" (Mul).

Another way of formulating Theorem 2 is the following. Denote

$$T^* = \{c^* \otimes A \Rightarrow c^* \otimes B \mid A \Rightarrow B \in T, c \in L\}.$$

Then we have

Theorem 3 (provable from T = provable from T^* using ordinary rules). *$A \Rightarrow B$ is provable from T using* (Ax)–(Mul) *iff $A \Rightarrow B$ is provable from T^* using ordinary rules* (Ax) *and* (Cut).

Proof. Easy consequence of Theorem 2 and definitions. □

As an application of the presented results, we now present an analogy of a well-known theorem from relational databases saying that each proof of a functional dependence can be transformed into a RAP-sequence. Due to lack of space, we omit discussion on ramifications of this theorem and refer to [17]. Consider the following deduction rules:

(Ref) infer $A \Rightarrow A$,
(Acc) from $A \Rightarrow B \cup C$ and $C \Rightarrow D \cup E$ infer $A \Rightarrow B \cup C \cup D$,
(Pro) from $A \Rightarrow B \cup C$ infer $A \Rightarrow B$,

for each $A, B, C, D, E \in \mathbf{L}^Y$. The rules result from the well-known ordinary rules of reflexivity, accumulation, and projectivity by replacing sets with fuzzy sets, see [17]. An *MRAP-sequence* for $A \Rightarrow B$ from T is a proof of $A \Rightarrow B$ from T using (Mul), (Ref), (Acc), (Pro), such that

1. the proof starts with FAIs from T,
2. continues with FAIs which result by application of (Mul) to formulas from 1.,

3. continues with $A \Rightarrow A$,
4. continues with formulas which result by application of (Acc) to formulas from 1., 2., and 3.,
5. ends with application of (Pro) which results in $A \Rightarrow B$, the last member of the proof.

Theorem 4 (MRAP-sequence theorem). *If $A \Rightarrow B$ follows from T in degree 1, then there exists an MRAP-sequence for $A \Rightarrow B$ from T.*

Proof. Sketch: If $A \Rightarrow B$ follows from T in degree 1, then there exists a proof of $A \Rightarrow B$ from T by Theorem 1. According to Theorem 2, this proof can be transformed into a normal form described in Theorem 2. The transformed proof starts with a sequence $A_1 \Rightarrow B_1, \ldots, A_l \Rightarrow B_l$ which satisfies conditions 1. and 2. of an MRAP-sequence (formulas from T and formulas resulting by application of (Mul)) and continues with a sequence $A_{l+1} \Rightarrow B_{l+1}, \ldots$, in which only (Ax) and (Cut) are used. It can be shown that the latter sequence can be transformed into a sequence which uses only rules (Ref), (Acc), and (Pro). Since these rules are just "ordinary rules" with ordinary sets replaced by fuzzy sets, one can repeat the ordinary proof, verbatim, see e.g. [17, Theorem 4.2]), showing that sequence $A_{l+1} \Rightarrow B_{l+1}, \ldots$, can be transformed into a sequence starting with $A \Rightarrow A$, continuing by applications of (Acc), and ending with an application of (Pro) which yields $A \Rightarrow B$. Altogether, this gives an MRAP-sequence for $A \Rightarrow B$. ☐

5 Concluding Remarks

We showed properties of the rule of multiplication in FAL related to the structure of proofs. Proofs in FAL are both of theoretical and practical interest (for instance, proofs are used in algorithms testing redundancy of a set of FAIs, see e.g. [9]). As an application of the properties, we presented theorems concerning normal forms of proofs in FAL. The main benefit of our approach is that separating a system for FAL into an "ordinary part" and a "fuzzy part", which is represented by the rule of multiplication, enables us to use automatically results known from the ordinary case (from the theory of relational databases, formal concept analysis, etc.). Due to the existing relationship to database interpretation of FAIs described above, our results apply to reasoning about functional dependencies in fuzzy setting as well, see e.g. [7] and [11] for comparison of some approaches.

Other issues which we did not present due to the limited scope and issues for further research include

- further properties of the rule of multiplication,
- rule of multiplication in Pavelka-style FAL, see [10],
- further study of the possibility to automatically convey ordinary results to fuzzy setting,
- this includes a more rigorous treatment of issues like the relationship between ordinary deduction rules and their counterparts resulting by replacing sets by fuzzy sets (this was described more or less intuitively in our paper due to lack of space),

– as a long-term goal, a further study of data dependencies in a fuzzy setting.

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