Order (2010) 27:147–161
DOI 10.1007/s11083-010-9143-7

Residuated Lattices of Size ≤ 12

Radim Belohlavek · Vilem Vychodil

Received: 30 October 2008 / Accepted: 19 January 2010 / Published online: 5 February 2010

Abstract We present the numbers of all non-isomorphic residuated lattices with up to 12 elements and a link to a database of these lattices. In addition, we explore various characteristics of these lattices such as the width, length, and various properties considered in the literature and provide the corresponding statistics. We also present algorithms for computing finite residuated lattices including a fast heuristic test of non-isomorphism.

Keywords Finite residuated lattices · Finite lattices · Algorithms

1 Introduction and Preliminaries

Ordered sets and lattices play a crucial role in several areas, e.g. in data visualization and analysis, uncertainty modeling, many-valued and fuzzy logics, graph theory, etc. Residuated lattices, in particular, were pioneered in the 1930s by Dilworth and Ward [9, 33]. In the late 1960s, residuated lattices were introduced into many-valued logics and, in particular, into fuzzy logics as structures of truth values (truth degrees) [14, 15]. Residuated lattices and various special residuated lattices are now used as the main structures of truth values in fuzzy logic and fuzzy set theory, see

R. Belohlavek · V. Vychodil
Department of Computer Science, Palacky University, Olomouc 17. listopadu 12, 771 46 Olomouc, Czech Republic

R. Belohlavek · V. Vychodil (✉)
SUNY Binghamton, PO Box 6000, Binghamton, NY 13902-6000, USA
e-mail: vychodil@binghamton.edu

R. Belohlavek
e-mail: rbelohla@binghamton.edu

e.g. [4, 11, 16, 18, 21, 27, 31], and are subject to algebraic investigation, see e.g. [5, 13, 23]. In addition to general motivations for enumerating finite structures, the motivation for enumerating finite residuated lattices derives from their role as scales of truth degrees. Namely, in many application areas, see e.g. [27], an expert defines a fuzzy set by assigning truth degrees (elements of a residuated lattice) to the elements of a particular universe. Now, according to Miller's 7±2 phenomenon, well known from psychology [30], humans are able to work consistently with a scale of degrees containing up to 7±2 elements. With more than 7±2 elements, the assignments become inconsistent. From this perspective, by computing all residuated lattices with up to 12 elements, we cover all the residuated lattices which are practically useful in such scenario.

A previous work on related problems includes [22] where the author provides the numbers and descriptions of non-isomorphic residuated lattices with up to six elements. Bartušek and Navara [2], De Baets [7] and De Baets and Mesiar [8] are also studies related to our paper. Namely, the authors compute the numbers of all t-norms [26] on finite chains but do not pay attention to general nonlinear residuated lattices. With respect to the previous work, we improve the size up to which we compute all the residuated lattices, from 6 (see [22]) to 12. Moreover, we systematically explore various properties of the residuated lattices which have not been provided in the previously published papers.

We use standard notions and notation and refer to [3, 17] (ordered sets and lattices) and [4, 13, 16, 18, 21, 23] (residuated lattices) for details. In particular, for a partially ordered set $\mathbf{L} = \langle L, \leq \rangle$, we denote by $\mathcal{L}(A)$ and $\mathcal{U}(A)$ the lower and upper cones of $A \subseteq L$, and by $\bigwedge A$ and $\bigvee A$ the infimum and supremum of $A \subseteq L$. Recall that a *residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ where $\langle L, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice, $\langle L, \otimes, 1 \rangle$ is a commutative monoid, and $\otimes$ and $\rightarrow$ satisfy $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ for each $a, b, c \in L$ (adjointness property). Binary operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) serve as truth functions of connectives "fuzzy conjunction" and "fuzzy implication" [4, 11, 15, 16, 18, 21]. Various subclasses of residuated lattices have been investigated in many-valued and fuzzy logics, e.g. MTL-algebras [11], BL-algebras [18] and its three important subclasses, namely MV-algebras, Gödel algebras, and Π-algebras.

We proceed as follows. First, as described in Section 2, we generate non-isomorphic finite lattices. Second, we use these lattices and generate non-isomorphic residuated lattices, as described in Section 3. Section 4 presents a summary regarding selected properties of the computed structures. The corresponding tables are presented in the Appendix. A more detailed version of this paper, containing particularly the pseudocodes of all the algorithms and proofs of their correctness is available at http://lattice.inf.upol.cz/order/reslat12.pdf.

2 Generating Non-isomorphic Finite Lattices

As described above, we first need to compute non-isomorphic finite lattices. The problem of counting and listing all non-isomorphic partial orders and, in particular, lattices has been studied in several papers in the past, see e.g. [10, 19, 20, 24, 25, 28, 29], see also Chapter XI in [12]. In [20], the numbers of all finite lattices with up to 18 elements are presented along with the algorithm for listing the lattices. For our purposes, we could have used the algorithm from [20]. However, we briefly describe another algorithm, which we used for computing finite lattices. The algorithm is simple and is based on a new heuristic test of isomorphism. The test is based on a general idea and we use it also in Section 3 to generate non-isomorphic residuated lattices.

We represent a lattice $\mathbf{L} = \langle L, \leq \rangle$ with $|L| = n$ by the adjacency matrix of $\leq$. Moreover, since $\leq$ can be extended to a linear order, we can safely assume that the adjacency matrices are upper triangular (the linear order corresponds to the ordering of matrix rows and columns), cf. Fig. 1. The essential information is contained in the inner area of the upper triangle (grey area in Fig. 1) which we encode by a binary vector (vector 001110, i.e. a concatenation of vectors 001, 11, and 0 encoding the rows of the grey area in Fig. 1). Such a representation makes it possible to efficiently compute transitive closures of general relations represented by triangular matrices, and to compute infima and suprema of lattice orders, cf. [1].

To recognize non-isomorphic lattices, we employ the following heuristic test. To every element a of a lattice $\mathbf{L} = \langle L, \leq \rangle$, we assign a vector $v(a) = \langle v_1(a), v_2(a), v_3(a), v_4(a) \rangle$ defined as follows. $v_1(a) = |\mathcal{L}(\{a\})|$ and $v_2(a) = |\mathcal{U}(\{a\})|$ (numbers of elements less/greater than or equal to a); $v_3(a)$ and $v_4(a)$ are the numbers of two-element subsets $\{c, d\}$ of L for which $0, 1 \notin \{c, d\}$ and whose infimum/supremum yields a. Roughly speaking, $v_1(a), \ldots, v_4(a)$ represent a "position" of a in the lattice. We now lexicographically order and concatenate the vectors $v(a)$ of all $a \in L$ and obtain a vector of non-negative integers which we call the *charateristic vector* of $\mathbf{L} = \langle L, \leq \rangle$.

Example 1 The values of v_is for the lattice from Fig. 1 are shown in the following table:

$\mathbf{L}$	0	a	b	c	d	1
v_1	1	2	2	3	4	6
v_2	6	3	4	2	2	1
v_3	2	1	3	0	0	0
v_4	0	0	0	1	3	2

That is, $v(0) = \langle 1, 6, 2, 0 \rangle$, etc. With respect to the lexicographic order $\leqslant_{\text{lex}}$,

$$v(0) \leqslant_{\text{lex}} v(a) \leqslant_{\text{lex}} v(b) \leqslant_{\text{lex}} v(c) \leqslant_{\text{lex}} v(d) \leqslant_{\text{lex}} v(1).$$

$\leq$	0	a	b	c	d	1
0	×	×	×	×	×	×
a		×			×	×
b			×	×	×	×
c				×		×
d					×	×
1						×

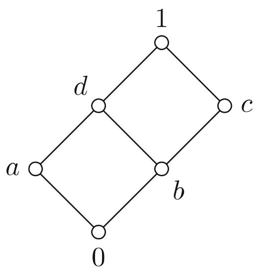

Fig. 1 Upper triangular adjacency matrix (*left*) of a finite lattice (*right*)

Therefore, the characteristic vector of the lattice is

$$\langle 1, 6, 2, 0, 2, 3, 1, 0, 2, 4, 3, 0, 3, 2, 0, 1, 4, 2, 0, 3, 6, 1, 0, 2\rangle.$$

Determining the characteristic vector of a given n-element lattice can be solved with an asymptotic complexity of $O(n^3)$. Indeed, traversing trough the binary vector representing the adjacency matrix is done in $O(n^2)$ steps, each such a step requires a computation of infima and suprema, which can be done in $O(n)$ steps. Thus, we need $O(n^3)$ steps to find the values of all $v(a)$s. Finally, an efficient sorting algorithm like heap-sort can be used to sort $v(a)$s according to $\leqslant_{\text{lex}}$ in $O(n \log n)$ steps which does not increase the asymptotic complexity. Thus, the overall time complexity of determining the characteristic vector is $O(n^3)$.

A direct procedure to test whether two n-element lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ are isomorphic requires, in the worst case, to test $n!$ bijective maps between two n-element lattices and to check the isomorphism condition for them. As we will see, characteristic vectors allow us to disqualify quickly most pairs of non-isomorphic lattices, i.e. to reach quickly the conclusion that two given lattices are not isomorphic. The heuristic test of non-isomorphism of two lattices consists in computing their characteristic vectors and checking whether the vectors are equal. Clearly, if the vectors are not equal, $\mathbf{L}_1$ and $\mathbf{L}_2$ are not isomorphic. If the vectors are equal, one cannot conclude that $\mathbf{L}_1$ and $\mathbf{L}_2$ are isomorphic because two non-isomorphic lattices may have the same characteristic vectors. Therefore, the *heuristic test may fail*. Hence, if the vectors are equal, we proceed by testing all isomorphism candidates to check by brute force whether there is at least one which is indeed an isomorphism. An *isomorphism candidate* between $\mathbf{L}_1$ and $\mathbf{L}_2$ is a bijection $h\colon L_1 \to L_2$ which satisfies $v^{\mathbf{L}_1}(a) = v^{\mathbf{L}_2}\big(h(a)\big)$ for every $a \in L_1$, i.e. the vector assigned to a in $\mathbf{L}_1$ equals the vector assigned to a in $\mathbf{L}_2$. Clearly, the bijections which are not isomorphism candidates cannot be isomorphisms and, therefore, need not be tested. We will see in Remark 1 that testing only isomorphism candidates significantly reduces the number of bijections to test.

Is a failure of the heuristic test rare? We have investigated this problem for lattices with up to 12 elements. Suppose c is a characteristic vector of a finite lattice. By an *order of c,* denoted $||c||$, we mean the number of pairwise non-isomorphic lattices whose characteristic vector is c. If $||c|| = 1$, there is just one finite lattice (up to isomorphism) with c in which case the heuristic test does not fail. Table 1 shows the numbers of characteristic vectors of given orders. The columns of the table correspond to sizes of lattices, the rows correspond to orders of characteristic vectors, and the table entries show how many characteristic vectors (of orders given by rows and sizes given by columns) there are. We can see that for $n \leq 7$, the heuristic test does not fail and that in the other cases, the heuristic test performs reasonably well.

We employ the heuristic test in an algorithm which incrementally generates finite lattices. The following observation allows us to use an n-element lattice $\mathbf{L}'$ to generate an $(n+1)$-element lattice $\mathbf{L}$ by adding a column and a row representing a new coatom to an upper triangular matrix representing the n-element lattice.

Theorem 1 *Let* $\mathbf{L} = \langle L, \leq\rangle$ *be a finite lattice with* $|L| > 1$*,* $c \in L$ *be a coatom in* $\mathbf{L}$*. Then* $L' = L - \{c\}$ *equipped with* $\leq'$ *which is a restriction of* $\leq$ *on* L' *is a lattice which is a* $\wedge$*-sublattice of* $\mathbf{L}$*.*

Table 1 Numbers of characteristic vectors of given orders

	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	2	5	15	53	220	1,049	5,682	34,502	232,070
2	0	0	0	0	0	0	0	1	13	125	1,159	10,963
3	0	0	0	0	0	0	0	0	1	18	212	2,035
4	0	0	0	0	0	0	0	0	0	2	28	388
5	0	0	0	0	0	0	0	0	0	0	6	102
6	0	0	0	0	0	0	0	0	0	0	4	65
7	0	0	0	0	0	0	0	0	0	0	0	16
8	0	0	0	0	0	0	0	0	0	0	0	6
10	0	0	0	0	0	0	0	0	0	0	0	1
11	0	0	0	0	0	0	0	0	0	0	0	1
13	0	0	0	0	0	0	0	0	0	0	0	2
16	0	0	0	0	0	0	0	0	0	0	0	1

We iteratively add all possible rows and columns to obtain all lattices which can be obtained from $\mathbf{L}'$. For every new row and column, we need to check whether the new matrix represents a lattice order. If not, we change entries in the new column and row and use transitive closure to compute another candidate lattice order. When a new lattice is obtained, we use the heuristic test of isomorphism to check whether an isomorphic lattice has already been generated.

Remark 1 Interestingly, the average number of isomorphism candidates that are used during each isomorphism test is low. The following table shows the numbers of isomorphism tests performed when computing all lattices of a given size and the average number of isomorphism candidates per one test.

Size of L	1	2	3	4	5	6	7	8	9	10	11	12
Tests of isomorphism	0	0	0	0	3	22	148	1,055	8,661	80,921	859,881	10,277,785
Generated candidates	0	0	0	0	3	22	155	1,158	10,054	97,113	1,058,787	12,765,905
Ratio (candidates/test)	–	–	–	–	1.00	1.00	1.05	1.10	1.16	1.20	1.23	1.24

Hence, roughly speaking, if two lattices are isomorphic, the map proving the isomorphism is usually the first candidate considered. The isomorphism test using characteristic vectors and isomorphism candidates is thus efficient for lattices with $|L| \leq 12$.

3 Generating Finite Residuated Lattices

In this section we describe a way to generate residuated lattices of a given size. We describe an algorithm which, for a given finite lattice $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$, generates all pairs $\langle \otimes, \rightarrow \rangle$ of adjoint operations on $\mathbf{L}$. The algorithm from Section 2 and this algorithm provide us with an algorithm for generating residuated lattices up to a given size. Let $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a finite lattice with $L = \{0 = a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1} = 1\}$.

We use the following assertion which follows from well-known properties of residuated lattices [16, 21] taking into account the finiteness of L.

Theorem 2 *Let $\mathbf{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a finite lattice, $\langle L, \otimes, 1\rangle$ be a commutative monoid such that $\otimes$ is monotone w.r.t. $\leq$. Then the following are equivalent:*

(1) *there exists (unique) $\rightarrow$ satisfying adjointness w.r.t. $\otimes$;*
(2) *for each $a, b, c \in L$: $a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c)$;*
(3) *$\rightarrow$ given by $a \rightarrow b = \bigvee\{c \in L \mid a \otimes c \leq b\}$ satisfies adjointness w.r.t. $\otimes$.*

Due to Theorem 2, in order to generate all adjoint pairs $\langle\otimes, \rightarrow\rangle$ on $\mathbf{L}$, it suffices to generate all monotone, commutative, and associative operations $\otimes$ for which (1) (greatest element of $\mathbf{L}$) is a neutral element and which satisfy condition (2) of Theorem 2. Then, we can use (3) to compute the corresponding residuum $\rightarrow$.

We generate the multiplications $\otimes$ by filling the table of $\otimes$ (entry given by row i and column j stores the value $a_i \otimes a_j$) using a recursive backtracking procedure and make use of the below-mentioned properties of $\otimes$. For this purpose, we assume that $a_i \leq a_j$ implies $i \leq j$ for all $a_i, a_j \in L$ (indexing of lattice elements extends the lattice order). Due to commutativity of $\otimes$, $a \otimes 0 = 0$, and $a \otimes 1 = a$, we only need to fill in the inner part of the upper triangle of the table of $\otimes$. The following theorem provides us with bounds on the values in the inner part (the first part is well known, the second follows from the monotony of $\otimes$).

Theorem 3 *Let $\mathbf{L}$ be a residuated lattice. Then, for each $a, b \in L$,*

(1) *$a \otimes b \leq a \wedge b$;*
(2) *$\bigvee\{c \otimes d \mid c, d \in L$ such that $c \leq a$ and $d \leq b\} \leq a \otimes b$.*

As a result, when filling the entry at row i and column j, i.e. a candidate value $a_i \otimes a_j$, we can restrict to the values from

$$\mathrm{Bounds}(i, j) = \left[b, a_i \wedge a_j\right]$$

with

$$b = \bigvee\{a_{\min(k,l)} \otimes a_{\max(k,l)} \mid (k = i \text{ and } a_l \prec a_j) \text{ or } (a_k \prec a_i \text{ and } l = j)\}$$

where $a_m \prec a_n$ denotes that a_m is covered by a_n, i.e. $a_m \leq a_n$ and $a_m \leq c \leq a_n$ implies $a_m = c$ or $a_n = c$. We consider every element from Bounds(i, j) a candidate for $a_i \otimes a_j$. To check that a candidate value may indeed be used, we check for each $a, b, c \in L$ the following conditions:

$$a \otimes (b \otimes c) = (a \otimes b) \otimes c, \tag{1}$$

$$a \otimes (b \vee c) = (a \otimes b) \vee (a \otimes c), \tag{2}$$

$$a \leq b \ \text{ implies } \ a \otimes c \leq b \otimes c, \tag{3}$$

provided that the expressions in (1)–(3) are defined. The table entries are traversed in the following order: $a_1 \otimes a_1$, $a_1 \otimes a_2, \ldots, a_1 \otimes a_{n-3}$, $a_1 \otimes a_{n-2}$, $a_2 \otimes a_2$, $a_2 \otimes a_3, \ldots, a_{n-2} \otimes a_{n-2}$, to ensure that the bounds for Bounds(i, j) are available when needed. It is easy to see that such a procedure is sound and generates all multiplications $\otimes$ which appear in adjoint couples $\langle \otimes, \rightarrow \rangle$ on a given finite lattice $\mathbf{L}$.

In order to generate all non-isomorphic residuated lattices with the lattice part $\mathbf{L} = \langle L, \wedge, \vee, 0, 1 \rangle$, we exclude the isomorphic copies, which may arise when computing the multiplications $\otimes$ as described above, by selecting one representative. The representative is selected using the following lexicographic order. For two multiplications $\otimes_1$ and $\otimes_2$, we put $\otimes_1 <_\ell \otimes_2$ iff there exist $a_i, a_j \in L$ such that the following two conditions are both satisfied:

(1) $k < l$ for $a_k = a_i \otimes_1 a_j$ and $a_l = a_i \otimes_2 a_j$,
(2) $a_k \otimes_1 a_l = a_k \otimes_2 a_l$ for all $a_k, a_l \in L$ such that $k < i$ or ($k = i$ and $l < j$).

Obviously, $<_\ell$ defines a strict total order on all possible multiplications (binary operations, in general) on $\mathbf{L} = \langle L, \wedge, \vee, 0, 1 \rangle$. Denote by $\leqslant_\ell$ the reflexive closure of $<_\ell$.

Consider now two distinct adjoint pairs $\langle \otimes_1, \rightarrow_1 \rangle$ and $\langle \otimes_2, \rightarrow_2 \rangle$ computed by the above backtracking algorithm and the corresponding residuated lattices $\mathbf{L}_1 = \langle L, \wedge, \vee, \otimes_1, \rightarrow_1, 0, 1 \rangle$ and $\mathbf{L}_2 = \langle L, \wedge, \vee, \otimes_2, \rightarrow_2, 0, 1 \rangle$. It is easily seen that $\mathbf{L}_1$ and $\mathbf{L}_2$ are isomorphic iff there is a lattice automorphism $h\colon L \to L$ such that $a \otimes_2 b = h(h^{-1}(a) \otimes_1 h^{-1}(b))$. Thus, we proceed as follows. After $\otimes$ is generated, we compute the set of all automorphic images $\{\otimes_i \mid i \in I\}$ of $\otimes$ and store $\otimes$ iff $\otimes$ is lexicographically least among all $\{\otimes_i \mid i \in I\}$, i.e., iff $\otimes \leqslant_\ell \otimes_i$ for all $i \in I$. Each automorphic image $\otimes_i$ of $\otimes$ is defined by $a \otimes_i b = h(h^{-1}(a) \otimes h^{-1}(b))$ where h is a lattice automorphism of $\mathbf{L} = \langle L, \wedge, \vee, 0, 1 \rangle$. Therefore, in order to apply the procedure, we have to generate all automorphisms of a given finite lattice $\mathbf{L}$. This can be done in a straightforward manner using characteristic vectors and automorphism candidates introduced in Section 2.

4 Selected Properties of Generated Structures

In this section we present basic characteristics of finite residuated lattices generated by our algorithms. We used the algorithms to generate all non-isomorphic residuated lattices with up to 12 elements. Prior to that, we generated all non-isomorphic lattices up to 12 elements. The tables summarizing the observations from this section can be found in the Appendix. A database of generated lattices is available at: http://lattice.inf.upol.cz/order/.

Numbers of Finite (Residuated) Lattices Table 2 (see Appendix) contains a basic summary. The table columns correspond to sizes of lattices (numbers of their elements). The first row contains the numbers of non-isomorphic lattices. These numbers agree with observations concerning the numbers of lattices from [20]. The second row contains the numbers of non-isomorphic residuated lattices. The third row contains the numbers of non-isomorphic linearly ordered residuated lattices (i.e., lattices with every pair of elements comparable). We can see from the table that small residuated lattices tend to be linear: for $|L| = 5$, 22 residuated lattices out of 26 are linear. With growing sizes of $|L|$, the portion of linear residuated lattices decreases: for $|L| = 11$, one fifth of all the residuated lattices are linear; for $|L| = 12$ only one seventh of all the residuated lattices are linear. Another observation concerns the relationship between (numbers of) residuated lattices and (numbers of) their distinct lattice reducts. Recall that if $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ is a residuated lattice, its reduct $\langle L, \wedge, \vee, 0, 1\rangle$ is a lattice. Thus, we may ask how many n-element lattices are reducts of n-element residuated lattices. This is shown in the last row of Table 2 which contains the numbers of pairwise distinct non-isomorphic lattice reducts of all non-isomorphic residuated lattices. For instance, the values in column corresponding to $|L| = 12$ mean: there are 262776 non-isomorphic lattices but only 38165 of them can be equipped with $\otimes$ and $\rightarrow$ to form a residuated lattice. Notice that even if the number of residuated lattices rapidly grows with growing $|L|$, the number of their lattice reducts compared to the number of all lattices (of that size) decreases. This means that with growing $|L|$, the average number of residuated lattices with the same lattice part increases. For instance, for $|L| = 8$ the average number of residuated lattices sharing the same lattice part is approximately 77 while for $|L| = 12$ it is 803.

Heights and Widths of Finite (Residuated) Lattices The values in Table 2 may suggest that most residuated lattices can be found on n-element chains. This is so for smaller residuated lattices but it is no longer true for larger lattices. Namely, consider the heights and widths of the lattices. A *height*/*width* of a lattice is the length of the longest maximal chain/antichain contained in that lattice. For instance, for $|L| = 12$, we can depict the numbers of lattices according to their width and height as in Table 3 (see Appendix). The rows and columns in Table 3 represent heights and widths of lattices, respectively. The table entries represent the numbers of non-isomorphic lattices with the dimensions given by the corresponding rows and columns. In a similar way, we can depict the numbers of distinct residuated lattices according to their width and height as in Table 4. Table 4 shows that the lattice parts of most residuated lattices are "tall and thin" and that in case of $|L| = 12$, the most frequent residuated lattices are those with width 2 (second column of Table 4). Let us mention that the distribution of all lattices and all lattice reducts according to their dimensions is quite different from that of residuated lattices. Indeed, the distributions of lattices in Tables 3 and 5 are similar but quite different from that in Table 4. Analogous observations can be made for all generated finite residuated lattices and their lattice reducts with $|L| < 12$.

Numbers of Finite (Residuated) Lattices Satisfying Additional Conditions Table 6 provides a summary of the numbers of non-isomorphic lattices which are modular, distributive, have complements, are Boolean, have relative complements, pseudo-complements, and relative pseudo-complements [17]. Note that the lines for all, modular, and distributive lattices are known [32], and that it is also know that the numbers in line 3 and line 8 coincide [6]. In addition to that, we consider the following properties of residuated lattices (see [4, 11, 17, 18]):

(MOD) $a \leq c$ implies $a \vee (b \wedge c) = (a \vee b) \wedge c$ (*modularity*)
(DIS) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ (*distributivity*)

(MTL)	$(a \rightarrow b) \vee (b \rightarrow a) = 1$	(*prelinearity*)
(Π1)	$(c \rightarrow 0) \rightarrow 0 \leq ((a \otimes c) \rightarrow (b \otimes c)) \rightarrow (a \rightarrow b)$	(Π1-*property*)
(Π2)	$a \wedge (a \rightarrow 0) = 0$	(Π2-*property*)
(STR)	$(a \otimes b) \rightarrow 0 = (a \rightarrow 0) \vee (b \rightarrow 0)$	(*strictness*)
(WNM)	$((a \otimes b) \rightarrow 0) \vee ((a \wedge b) \rightarrow (a \otimes b)) = 1$	(*weak nilpotent minimum*)
(DIV)	$a \wedge b = a \otimes (a \rightarrow b)$	(*divisibility*)
(INV)	$a = (a \rightarrow 0) \rightarrow 0$	(*involution*)
(IDM)	$a = a \otimes a$	(*idempotency*)

These properties are of interest, e.g., when lattices are considered as structures of truth values in many-valued logics and fuzzy logics [4, 18]. Table 7 contains the numbers of residuated lattices satisfying these conditions. Table 8 summarizes the numbers of algebras (particular residuated lattices) which are defined by a combination of the above-mentioned properties. The tables show that BL-algebras are very rare among residuated lattices up to 12 elements. The situation for MTL-algebras is better but still, only 15 % of all 12-element residuated lattices are MTL-algebras. An observation which may be surprising is that (Π1) is far more frequent a property than prelinearity (for $|L| \leq 12$).

Relationship Between Properties of Finite (Residuated) Lattices Table 6 shows the numbers of lattices having each property but does not show, e.g., how many modular lattices are pseudo-complemented; similarly for Tables 7 and 8. To reveal dependencies among properties of lattices and residuated lattices, we constructed Tables 9, 10, and 11. These tables show all combinations of lattice and residuated lattice properties which appear in the generated databases. In case of Table 9, the columns denote the same properties of lattices considered in Table 6. The left-most column contains numbers of non-isomorphic lattices with given combinations of properties. Each row of the tables represents one combination of properties (properties which are present are marked by "×"). From Table 9 we can see that some combinations of properties are rare. In addition to that, some combinations of properties do not appear in "small" lattices (up to certain number of elements). For instance, the least lattice which is only relatively complemented and (in consequence) complemented has 9 elements and it is depicted in Fig. 2 (left). The least lattice which does not satisfy any of the properties MOD–RPC (see Table 9) has seven elements and is depicted in Fig. 2 (middle). The lattice in Fig. 2 (right) is the least lattice which is only modular, complemented, and relatively complemented. Tables 10 and 11

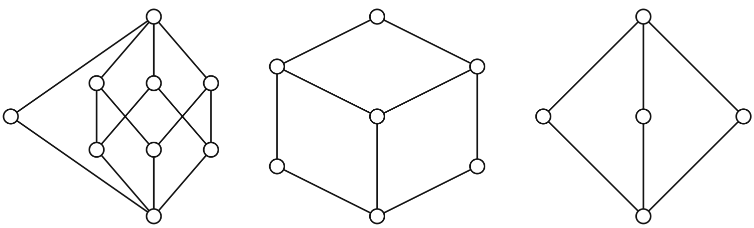

Fig. 2 Least lattices that have a specific group of properties

depict dependencies among properties of the generated residuated lattices. Again, some combinations of properties are rare and some of them appear only in larger structures. For illustration, the least residuated lattice which satisfies only (MOD) and (Π2) has nine elements.

Acknowledgements Supported by research plan MSM 6198959214. Partly supported by grant P202/10/0262 of the Czech Science Foundation. The paper is an extended version of a contribution presented at IFSA 2007 Congress.

Appendix

Table 2 Numbers of non-isomorphic finite lattices and residuated lattices up to 12 elements

	1	2	3	4	5	6	7	8	9	10	11	12
Lattices	1	1	1	2	5	15	53	222	1,078	5,994	37,622	262,776
Residuated lattices	1	1	2	7	26	129	723	4,712	34,698	290,565	2,779,183	30,653,419
Linear res. lattices	1	1	2	6	22	94	451	2,386	13,775	86,417	590,489	4,446,029
Residuated lattice reducts	1	1	1	2	3	7	18	61	239	1,125	6,138	38,165

Table 3 Numbers of 12 -element lattices with given heights and widths

	1	2	3	4	5	6	7	8	9	10
3										1
4					99	395	288	98	17	
5				3,847	14,418	9,536	2,115	176		
6			3,531	37,813	43,394	12,050	952			
7		87	15,501	48,261	23,595	2,507				
8		666	14,735	17,380	3,117					
9		849	4,704	1,792						
10		350	456							
11		45								
12	1									

Table 4 Numbers of 12-element residuated lattices with given heights and widths

	1	2	3	4	5	6	7	8	9
4									1
5				3	127	165	88	48	
6			240	9,383	22,627	9,638	1,335		
7		236	99,088	332,299	161,275	18,546			
8		121,970	1,363,290	1,009,364	142,551				
9		1,732,870	3,563,657	733,266					
10		6,007,716	2,709,365						
11		8,168,242							
12	4,446,029								

Table 5 Numbers of 12-element lattice reducts with given heights and widths

	1	2	3	4	5	6	7	8	9
4									1
5				2	123	159	72	15	
6			92	2,215	3,295	1,139	126		
7		11	2,362	8,498	4,397	518			
8		241	4,549	5,377	973				
9		455	2,183	805					
10		239	280						
11		37							
12	1								

Table 6 Numbers of non-isomorphic lattices with selected properties

	1	2	3	4	5	6	7	8	9	10	11	12
All lattices	1	1	1	2	5	15	53	222	1,078	5,994	37,622	262,776
Modular	1	1	1	2	4	8	16	34	72	157	343	766
Distributive	1	1	1	2	3	5	8	15	26	47	82	151
Complemented	1	1	0	1	2	6	18	71	307	1,594	9,446	63,461
Boolean	1	1	0	1	0	0	0	1	0	0	0	0
Relatively complemented	1	1	0	1	1	1	1	2	2	4	6	13
Pseudo-complemented	1	1	1	2	4	10	29	99	391	1,775	9,214	54,151
relatively pseudo-complemented	1	1	1	2	3	5	8	15	26	47	82	151

Table 7 Numbers of residuated lattices with selected properties

	1	2	3	4	5	6	7	8	9	10	11	12
All res. lattices	1	1	2	7	26	129	723	4,712	34,698	290,565	2,779,183	30,653,419
Modular	1	1	2	7	26	125	660	3,923	25,445	180,113	1,389,782	11,798,582
Distributive	1	1	2	7	26	124	645	3,792	24,268	169,553	1,290,956	10,823,436
(Π1) identity	1	1	1	4	9	46	240	1,610	12,679	118,052	1,280,764	16,074,272
Prelinear	1	1	2	7	23	99	464	2,453	14,087	88,188	601,205	4,516,962
(Π2) identity	1	1	1	3	8	30	143	794	5,090	37,036	306,456	2,897,889
Strict	1	1	1	3	7	27	129	726	4,713	34,705	290,565	2,779,212
(WNM) identity	1	1	2	5	11	30	78	238	771	2,908	12,812	67,467
Divisible	1	1	2	5	10	23	49	111	244	545	1,203	2,697
Involutive	1	1	1	3	3	12	15	70	112	493	980	4,325
Idempotent	1	1	1	2	3	5	8	15	26	47	82	151

Table 8 Numbers of selected algebras (particular residuated lattices)

	1	2	3	4	5	6	7	8	9	10	11	12
All res. lattices	1	1	2	7	26	129	723	4,712	34,698	290,565	2,779,183	30,653,419
MTL-algebras	1	1	2	7	23	99	464	2,453	14,087	88,188	601,205	4,516,962
SMTL-algebras	1	1	1	3	7	24	99	467	2,454	14,094	88,188	601,231
WNM-algebras	1	1	2	5	9	21	40	90	180	378	757	1,584
BL-algebras	1	1	2	5	9	20	38	81	160	326	643	1,314
SBL-algebras	1	1	1	3	5	10	20	41	82	165	326	655
IMTL-algebras	1	1	1	3	3	8	12	35	61	167	333	971
Heyting algebras	1	1	1	2	3	5	8	15	26	47	82	151
G-algebras	1	1	1	2	2	3	3	5	6	8	8	12
NM-algebras	1	1	1	2	1	2	1	4	3	3	2	6
MV-algebras	1	1	1	2	1	2	1	3	2	2	1	4
Π-algebras	1	1	0	1	0	0	0	1	0	0	0	0
ΠMTL-algebras	1	1	0	1	0	0	0	1	0	0	0	0

Table 9 Numbers of lattices sharing selected properties

	MOD	DIS	COM	BOO	REL	PCO	RPC
168,660							
72,930			×				
62,811						×	
1,945			×			×	
580	×					×	
473	×						
338	×	×				×	×
19			×		×		
10	×		×		×		
4	×	×	×	×	×	×	×

MOD modular,
DIS distributive,
COM complemented,
BOO boolean,
REL relatively complemented,
PCO pseudo-complemented,
RPC relatively pseudo-complemented

Table 10 Numbers of residuated lattices sharing selected properties

	MTL-algebra	SMTL-algebra	WNM-algebra	BL-algebra	SBL-algebra	IMTL-algebra	G-algebra	NM-algebra	MV-algebra	Π-algebra	ΠMTL-algebra
28,539,974											
4,511,103	×										
705,260	×	×									
2,954	×		×								
1,556	×					×					
1,258	×	×		×	×						
1,234	×			×							
48	×	×	×	×	×		×				
39	×		×	×							
19	×		×			×		×			
13	×			×		×			×		
4	×		×	×		×		×	×		
4	×	×	×	×	×	×	×	×	×	×	×

Table 11 Numbers of residuated lattices sharing selected properties (detail)

	MOD	DIS	MTL	Π1	Π2	STR	WEA	DIV	INV	IDE
12,275,528				×						
6,404,789										
3,252,773	×	×								
3,025,931	×	×	×							
3,009,686	×	×		×						
1,509,962					×	×				
1,485,172	×	×	×	×						
781,939	×	×			×	×				
705,260	×	×	×		×	×				
653,138	×			×						
317,172	×									
110,710	×				×	×				
101,016					×					
55,732				×			×			
34,162	×	×			×					
15,501							×			
5,761	×	×					×			
2,713	×	×	×				×			
2,271				×					×	
1,720	×	×		×					×	
1,719	×						×			
1,556	×	×	×	×					×	
1,509	×				×					
1,258	×	×	×		×	×		×		
1,234	×	×	×					×		
1,189	×			×			×			
977	×	×		×			×			
757	×	×			×	×		×		
630	×	×						×		
537	×	×			×			×		
419	×			×					×	
241	×	×	×	×			×			
152	×	×			×	×	×	×		×
138	×	×			×		×	×		×
77	×	×					×	×		
48	×	×	×		×	×	×	×		×
39	×	×	×				×	×		
19	×	×	×	×			×		×	
13	×	×	×	×				×	×	
10	×	×		×			×		×	
4	×	×	×	×			×	×	×	
4	×	×	×	×	×	×	×	×	×	×

References

1. Aït-Kaci, H., Boyer, R., Lincoln, P., Nasr, R.: Efficient implementation of lattice operations. ACM Trans. Program. Lang. Syst. **11**(1), 115–146 (1989)
2. Bartušek, T., Navara, M.: Program for generating fuzzy logical operations and its use in mathematical proofs. Kybernetika **38**(3), 235–244 (2002)
3. Birkhoff, G.: Lattice Theory. Publ. AMS, Providence, RI (1967)
4. Belohlavek, R.: Fuzzy Relational Systems: Foundations and Principles. Kluwer Academic/Plenum Publishers, New York (2002)
5. Blount, K., Tsinakis, C.: The structure of residuated lattices. Int. J. Algebra Comput. **13**, 437–461 (2003)
6. Chajda, I., Glazek, K.: A Basic Course on General Algebra. Technical University Press, Zielona Góra (2000)
7. De Baets, B.: The number of linearly ordered abelian integral monoids of size n; the number of t-norms on an n-chain. The On-Line Encyclopedia of Integer Sequences (database entry A030453) (1999)
8. De Baets, B., Mesiar, R.: Triangular norms on product lattices. Fuzzy Sets Syst. **104**, 61–75 (1999)
9. Dilworth, R.P.: Abstract residuation over lattices. Bull. Am. Math. Soc. **44**, 262–268 (1938)
10. Erne, M., Stege, K.: Counting finite posets and topologies. Order **8**, 247–265 (1991)
11. Esteva, F., Godo, L.: Monoidal t-norm based logic: towards a logic for left-continuous t-norms. Fuzzy Sets Syst. **124**, 271–288 (2001)
12. Freese, R.S., Ježek, J., Nation, J.B.: Free Lattices. Math. Surveys and Monographs, vol. 42. AMS (1995)
13. Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics. Studies in Logic and the Foundations of Mathematics, vol. 151. Elsevier (2007)
14. Goguen, J.A.: L-fuzzy sets. J. Math. Anal. Appl. **18**, 145–174 (1967)
15. Goguen, J.A.: The logic of inexact concepts. Synthese **18**, 325–373 (1968–9)
16. Gottwald, S.: A Treatise on Many-Valued Logics. Research Studies Press, Beldock, England (2001)
17. Grätzer, G.A.: General Lattice Theory, 2nd edn. Birkhauser (1998)
18. Hájek, P.: Metamathematics of Fuzzy Logic. Kluwer, Dordrecht (1998)
19. Heitzig, J., Reinhold, J.: The number of unlabeled orders on fourteen elements. Order **17**(4), 333–341 (2000)
20. Heitzig, J., Reinhold, J.: Counting finite lattices. Algebra Univers. **48**(1), 43–53 (2002)
21. Höhle, U.: On the fundamentals of fuzzy set theory. J. Math. Anal. Appl. **201**, 786–826 (1996)
22. Jipsen, P.: Small Residuated Lattices. http://www1.chapman.edu/~jipsen/gap/rl.html
23. Jipsen, P., Tsinakis, C.: A survey of residuated lattices. In: Martinez, J. (ed.) Ordered Algebraic Structures, pp. 19–56. Kluwer Academic Publishers, Dordrecht (2002)
24. Kleitman, D.J., Rothschild, B.L.: Asymptotic enumeration of partial orders on a finite set. Trans. Am. Math. Soc. **205**, 205–220 (1975)
25. Kleitman, D.J., Winston, K.J.: The asymptotic number of lattices. Ann. Discrete Math. **6**, 243–249 (1980)
26. Klement, E.P., Mesiar, R., Pap, E.: Triangular Norms. Kluwer (2000)
27. Klir, G.J., Yuan, B.: Fuzzy Sets and Fuzzy Logic. Theory and Applications. Prentice Hall (1995)
28. Kyuno, S.: An inductive algorithm to construct finite lattices. Math. Comput. **33**(145), 409–421 (1979)
29. Lygeros, N., Zimmermann, P.: Computation of $P(14)$, the Number of Posets with 14 Elements: 1.338.193.159.771. http://www.lygeros.org/Math/poset.html
30. Miller, G.T.: The magical number seven, plus or minus two: some limits on our capacity for processing information. Psychol. Rev. **63**, 81–97 (1956)
31. Pavelka, J.: On fuzzy logic I, II, III. Z. Math. Log. Grundl. Math. **25**, 45–52, 119–134, 447–464 (1979)
32. The On-Line Encyclopedia of Integer Sequences. http://www.research.att.com/~njas/sequences/
33. Ward, M., Dilworth, R.P.: Residuated lattices. Trans. Am. Math. Soc. **45**, 335–354 (1939)