Reducing the size of fuzzy concept lattices by hedges

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***Abstract*— We study concept lattices with hedges. The principal aim is to control, in a parametrical way, the size of a concept lattice. The paper presents theoretical insight, comments, and examples. We show that a concept lattice with hedges is indeed a complete lattice which is isomorphic to an ordinary concept lattice. We describe the isomorphism and its inverse. These mappings serve as translation procedures. As a consequence, we obtain a theorem characterizing the structure of concept lattices with hedges which generalizes the so-called main theorem of concept lattices. Furthermore, the isomorphism and its inverse enable us to compute a concept lattice with hedges using algorithms for ordinary concept lattices. Further insight is provided in case one uses hedges only for attributes. We demonstrate by experiments that the size reduction using hedges as a parameter is smooth.**

I. Problem setting

Tabular data describing objects and their attributes represents a basic form of data. Among the several methods for analysis of object-attribute data, formal concept analysis (FCA) is becoming increasingly popular, see [11], [10]. The main aim in FCA is to extract interesting clusters (called formal concepts) from tabular data. Formal concepts correspond to maximal rectangles in a data table. The number of formal concepts in data can be large. A large collection of formal concepts is not directly comprehensible by a user. Development of methods which help to overcome the problem of large number of extracted formal concepts is thus an important task.

In [4], we proposed a way to reduce the number of formal concepts extracted from data with attributes by using so-called (truth-stressing) hedges. Definitions, basic results, and illustrative examples are given in [4]. The aim of the present paper is twofold. First, we present further theoretical results. The results are related to the structure of extracted formal concepts. Second, we present results of experiments. The results show that hedges, considered as parameters, provide a way to change smoothly the number of clusters extracted from data. Roughly speaking, tuning hedges (parameter) makes the set of extracted clusters smaller by omitting the less important ones.

II. Preliminaries

We use sets of truth degrees equipped with operations (logical connectives) so that is becomes a complete residuated lattice with a truth-stressing hedge. A complete residuated lattice with truth-stressing hedge (shortly, a hedge) [12], [13] is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property:

$$a \otimes b \leq c \quad \text{iff} \quad a \leq b \rightarrow c \tag{1}$$

for each $a, b, c \in L$; hedge * satisfies

$$1^* = 1, \tag{2}$$
$$a^* \leq a, \tag{3}$$
$$(a \rightarrow b)^* \leq a^* \rightarrow b^*, \tag{4}$$
$$a^{**} = a^*, \tag{5}$$

for each $a, b \in L$, $a_i \in L$ ($i \in I$). Elements a of L are called truth degrees. $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". Hedge * is a (truth function of) logical connective "very true", see [12], [13]. Properties (3)–(5) have natural interpretations, e.g. (3) can be read: "if formula φ is very true, then φ is true", (4) can be read: "if it is very true that φ implies ψ and if φ is very true, then ψ is very true", etc. Note that hedges other than truth-stressing ones like "at least a little bit true" have different properties and are not considered in our paper.

A common choice of $\mathbf{L}$ is a structure with $L = [0, 1]$ (unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\rightarrow$. Three most important pairs of adjoint operations on the unit interval are:

$$\text{Łukasiewicz:} \quad \begin{aligned} a \otimes b &= \max(a + b - 1, 0), \\ a \rightarrow b &= \min(1 - a + b, 1), \end{aligned} \tag{6}$$

$$\text{Gödel:} \quad \begin{aligned} a \otimes b &= \min(a, b), \\ a \rightarrow b &= \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise,} \end{cases} \end{aligned} \tag{7}$$

$$\text{Goguen (product):} \quad \begin{aligned} a \otimes b &= a \cdot b, \\ a \rightarrow b &= \begin{cases} 1 & \text{if } a \leq b, \\ \frac{b}{a} & \text{otherwise.} \end{cases} \end{aligned} \tag{8}$$

In applications, we usually need a finite linearly ordered $\mathbf{L}$. For instance, one can put $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0, 1]$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Such an $\mathbf{L}$ is called a finite Łukasiewicz chain. Another possibility is a finite Gödel chain which consists of L and restrictions of Gödel operations on $[0, 1]$ to L.

Two boundary cases of (truth-stressing) hedges are (i) identity, i.e. $a^* = a$ ($a \in L$); (ii) globalization [17]:

$$a^* = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

A special case of the complete residuated lattice with hedge is the two-element Boolean algebra $\langle \{0,1\}, \wedge, \vee, \otimes, \rightarrow, {}^*, 0, 1\rangle$, denoted by $\mathbf{2}$, which is the structure of truth degrees of the classical logic. That is, the operations $\wedge, \vee, \otimes, \rightarrow$ of $\mathbf{2}$ are the truth functions (interpretations) of the corresponding logical connectives of the classical logic and $0^* = 0$, $1^* = 1$. Note that if we prove an assertion for general $\mathbf{L}$, then, in particular, we obtain a "crisp version" of this assertion for $\mathbf{L}$ being $\mathbf{2}$.

Having $\mathbf{L}$, we define usual notions: an $\mathbf{L}$-set (fuzzy set) A in universe U is a mapping $A\colon U \rightarrow L$, $A(u)$ being interpreted as "the degree to which u belongs to A". If $U = \{u_1, \ldots, u_n\}$ then A can be denoted by $A = \{{}^{a_1}/u_1, \ldots, {}^{a_n}/u_n\}$ meaning that $A(u_i)$ equals a_i for each $i = 1, \ldots, n$. Let $\mathbf{L}^U$ denote the collection of all $\mathbf{L}$-sets in U. The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in \mathbf{L}^U$ is an $\mathbf{L}$-set $A \cap B$ in U such that $(A \cap B)(u) = A(u) \wedge B(u)$ for each $u \in U$, etc. $\mathbf{2}$-sets (operations with $\mathbf{2}$-sets) can be identified with the ordinary (crisp) sets (operations with ordinary sets) of the naive set theory. Binary $\mathbf{L}$-relations (binary fuzzy relations) between X and Y can be thought of as $\mathbf{L}$-sets in the universe $X \times Y$.

Given $A, B \in \mathbf{L}^U$, we define a subsethood degree

$$S(A,B) = \textstyle\bigwedge_{u \in U} \big(A(u) \rightarrow B(u)\big), \quad (10)$$

which generalizes the classical subsethood relation $\subseteq$ (note that unlike $\subseteq$, S is a binary $\mathbf{L}$-relation on $\mathbf{L}^U$). Described verbally, $S(A,B)$ represents a degree to which A is a subset of B. In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. As a consequence, $A \subseteq B$ iff $A(u) \leq B(u)$ for each $u \in U$.

In the sequel we will take advantage of one the common methods of representing $\mathbf{L}$-sets (fuzzy sets) by $\mathbf{2}$-sets (ordinary sets) [2]: for $A \in \mathbf{L}^U$ we define $\lfloor A \rfloor \in \mathbf{2}^{U \times L}$ by

$$\lfloor A \rfloor = \{\langle u, a\rangle \in U \times L \mid a \leq A(u)\}. \quad (11)$$

Described verbally, $\lfloor A \rfloor$ can be considered as an area under the membership function $A : U \rightarrow L$. For $B \in \mathbf{2}^{U \times L}$ we define $\lceil B \rceil \in \mathbf{L}^U$ by

$$\lceil B \rceil(u) = \textstyle\bigvee \{a \in L \mid \langle u, a\rangle \in B\} \quad (12)$$

for each $u \in U$. In the following we use well-known properties of residuated lattices and fuzzy structures which can be found in monographs [2], [12].

III. Concept lattices with hedges

A. Definition and remarks

We suppose that we are given a complete residuated lattice $\mathbf{L}$, and two hedges, $*_X$ and $*_Y$ on $\mathbf{L}$. Let X and Y be sets of objects and attributes, respectively, I be a fuzzy relation between X and Y. That is, $I : X \times Y \rightarrow L$ assigns to each $x \in X$ and each $y \in Y$ a truth degree $I(x,y) \in L$ to which object x has attribute y. The triplet $\langle X, Y, I\rangle$ represents a data table with rows and columns corresponding to objects and attributes, and table entries containing degrees $I(x,y)$.

For fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$) defined by

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x \in X} (A(x)^{*_X} \rightarrow I(x,y)) \quad (13)$$

and

$$B^{\downarrow}(x) = \textstyle\bigwedge_{y \in Y} (B(y)^{*_Y} \rightarrow I(x,y)). \quad (14)$$

Using basic rules of predicate fuzzy logic, $A^{\uparrow}(y)$ is the truth degree of "for each $x \in X$: if it is very true that x belongs from A then x has y" where "very true" is interpreted by $*_X$. Similarly for $B^{\downarrow}$ with "very true" interpreted by $*_Y$. That is, $A^{\uparrow}$ is a fuzzy set of attributes common to all objects for which it is very true that they belong to A, and $B^{\downarrow}$ is a fuzzy set of objects sharing all attributes for which it is very true that they belong to B. The set

$$\mathcal{B}(X^{*_X}, Y^{*_Y}, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$$

of all fixpoints of $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ thus contains all pairs $\langle A, B\rangle$ such that A is the collection of all objects that have all the attributes of "very B", and B is the collection of all attributes that are shared by all the objects of "very A". For the sake of brevity, we use also $\mathcal{B}(X^*, Y^*, I)$ instead of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. Also, we omit $*$ if it is the identity and write e.g. only $\mathcal{B}(X, Y^{*_Y}, I)$. Given $*_X$ and $*_Y$ as parameters, elements $\langle A, B\rangle \in \mathcal{B}(X^*, Y^*, I)$ will be called formal concepts of $\langle X, Y, I\rangle$; A and B are called the extent and intent of $\langle A, B\rangle$, respectively; $\mathcal{B}(X^*, Y^*, I)$ will be called a concept lattice of $\langle X, Y, I\rangle$. Both the extent A and the intent B are in general fuzzy sets. This corresponds to the fact that in general, concepts apply to objects and attributes to intermediate degrees, not necessarily 0 and 1.

For $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, put

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ iff } A_1 \subseteq A_2 \text{ (iff } B_2 \subseteq B_1).$$

This defines a subconcept-superconcept hierarchy on $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$.

For convenience, define mappings ${}^{\Uparrow} : L^X \rightarrow L^Y$ and ${}^{\Downarrow} : L^Y \rightarrow L^X$ by $A^{\Uparrow}(y) = \bigwedge_{x \in X} A(x) \rightarrow I(x,y)$ and $B^{\Downarrow}(x) = \bigwedge_{y \in Y} B(y) \rightarrow I(x,y)$. That is, $\langle {}^{\Uparrow}, {}^{\Downarrow}\rangle$ is $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ with both $*_X$ and $*_Y$ being identities (these mappings are studied in [1]). We have $A^{\uparrow} = (A^{*_X})^{\Uparrow}$ and $B^{\downarrow} = (B^{*_Y})^{\Downarrow}$.

Example 1: (1) Let both $*_X$ and $*_Y$ be identities on L. Then $\mathcal{B}(X, Y, I)$, i.e. $\mathcal{B}(X^*, Y^*, I)$, is what is called a (fuzzy) concept lattice, see e.g. [9], [2], [16]. Axiomatic characterization of mappings ${}^{\uparrow}$ and ${}^{\downarrow}$ is given in [1].

(2) Recall from [7] that a crisply generated formal concept of $\langle X, Y, I\rangle$ is a formal concept $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ ($*_X$ and $*_Y$ are identities) which is generated by a crisp (fuzzy) set of attributes, i.e. there is $D \in \{0,1\}^Y$ such that $A = D^{\Downarrow}$ and $B = A^{\Uparrow}$. Crisply generated formal concepts may be thought of as the important ones. The number of crisply generated concepts is considerably smaller than the number of

all formal concepts, see [7]. Now, it can be shown (we omit the proof for lack of space) that if $*_X$ is the identity and $*_Y$ is the globalization on L, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is just the set of all crisply generated concepts.

(3) It can be shown (we omit details) that what is called a fuzzy concept lattice in [18] is in fact a structure isomorphic to $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ with $*_X$ and $*_Y$ being identity and globalization, respectively. If, on the other hand, $*_X$ and $*_Y$ are globalization and identity, respectively, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic to what is called a one-sided fuzzy concept lattice in [14].

(4) An attribute implication [6], [8] is an expression $A \Rightarrow B$ where $A, B \in L^Y$ are fuzzy sets of attributes. The degree $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ to which $A \Rightarrow B$ is true in $\langle X, Y, I\rangle$ is defined by

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = \bigwedge_{x\in X} S(A, I_x)^* \to S(B, I_x).$$

Here, $I_x \in L^Y$ is a fuzzy set of attributes of object x, i.e. $I_x(y) = I(x, y)$, and $*$ is globalization on L. Then, $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ is the truth degree of "each object from X having all attributes from A has also all attributes from B". It can be shown that a set T of attribute implications is a base, i.e. T semantically entails exactly the set of all attribute implications which are fully true (i.e., in degree 1) in $\langle X, Y, I\rangle$, if and only if the set of all models of T (a fuzzy set of attributes in which all implications of T are true) equals the set of all intents of formal concepts from $\mathcal{B}(X^*, Y, I)$, see [6], [8] for details.

B. The structure of concept lattices with hedges

A concept lattice (without hedges, i.e. with both $*_X$ and $*_Y$ being identity) is a complete lattice with infima and suprema corresponding to conceptual specifications and generalizations. Moreover, a characterization of concept lattices up to an isomorphism is known (see [11] for crisp case and [2] for fuzzy setting). The question we are going to answer is: What is the structure of concept lattices with hedges, i.e. the structure of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$? The answer is not obvious. For instance, neither of the composed mappings $^{\uparrow\downarrow}$ and $^{\downarrow\uparrow}$ is a closure operator. Indeed, neither $A \subseteq A^{\uparrow\downarrow}$ nor $B \subseteq B^{\downarrow\uparrow}$ is true in general [4]. In order to answer our question, we proceed as follows: First, we find an ordinary Galois connection $\langle^\curlywedge, ^\curlyvee\rangle$ between sets such that $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic to the the lattice of fixpoints of $\langle^\curlywedge, ^\curlyvee\rangle$. In addition to that, we describe the isomorphism and its inverse. Second, since $\langle^\curlywedge, ^\curlyvee\rangle$ is a Galois connection between sets, the lattice of its fixpoint obeys the so-called main theorem of concept lattices. Applying the isomorphism and its inverse, we get the theorem describing the structure of $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$.

Denote

$$*_X(L) = \{a^{*_X} \mid a \in L\} \text{ and } *_Y(L) = \{a^{*_Y} \mid a \in L\}.$$

Furthermore, for $A \in L^U$, $A' \subseteq U \times L$, and $* : L \to L$, define $A^* \in L^U$ and $A'^* \subseteq U \times L$ by $A^*(u) = (A(u))^*$ and $A'^* = \{\langle x, a^*\rangle \mid \langle x, a\rangle \in A'\}$.

Lemma 1: For $A \subseteq X \times *_X(L)$ we have $A \subseteq \lfloor\lceil A\rceil^{*_X}\rfloor^{*_X}$. If $B = \lfloor B'\rfloor$ for some $B' \in L^Y$ then $\lceil B^{*_Y}\rceil = \lceil B\rceil^{*_Y}$.

Proof: Easy, omitted due to lack of space. ■

Define mappings $^\curlywedge : X \times *_X(L) \to Y \times *_Y(L)$ and $^\curlyvee : Y \times *_Y(L) \to X \times *_X(L)$ by

$$A^\curlywedge = \lfloor\lceil A\rceil^\uparrow\rfloor^{*_Y} \text{ and } B^\curlyvee = \lfloor\lceil B\rceil^\downarrow\rfloor^{*_X}. \quad (15)$$

Lemma 2: The pair $\langle^\curlywedge, ^\curlyvee\rangle$ forms a Galois connection between sets $X \times *_X(L)$ and $Y \times *_Y(L)$.

Proof: Antitony: $A_1 \subseteq A_2$ implies $\lceil A_1\rceil \subseteq \lceil A_2\rceil$ which implies $\lceil A_2\rceil^\uparrow \subseteq \lceil A_1\rceil^\uparrow$ which implies $\lfloor\lceil A_2\rceil^\uparrow\rfloor \subseteq \lfloor\lceil A_1\rceil^\uparrow\rfloor$ which implies $A_2^\curlywedge = \lfloor\lceil A_2\rceil^\uparrow\rfloor^{*_X} \subseteq \lfloor\lceil A_1\rceil^\uparrow\rfloor^{*_X} = A_1^\curlywedge$. Dually, $B_1 \subseteq B_2$ implies $B_2^\curlyvee \subseteq B_1^\curlyvee$.

Extensivity: Using Lemma 1, $A^{\curlywedge\curlyvee} = \lfloor\lceil\lfloor\lceil A\rceil^\uparrow\rfloor^{*_Y}\rceil^\downarrow\rfloor^{*_X} = \lfloor\lceil\lfloor\lceil A\rceil^\uparrow\rfloor\rceil^{*_Y\downarrow}\rfloor^{*_X} = \lfloor\lceil A\rceil^{\uparrow *_Y\downarrow}\rfloor^{*_X} = \lfloor\lceil A\rceil^{\uparrow\downarrow}\rfloor^{*_X} \supseteq \lfloor\lceil A\rceil^{*_X}\rfloor^{*_X} \supseteq A$. Dually, $B \subseteq B^{\curlyvee\curlywedge}$. ■

It is well-known (see e.g. [11]) that each Galois connection $\langle^\curlywedge, ^\curlyvee\rangle$ between sets U and V is induced by some binary relation $I_{\langle^\curlywedge,^\curlyvee\rangle} \subseteq U \times V$. Namely, $I_{\langle^\curlywedge,^\curlyvee\rangle}$ is given by $\langle u, v\rangle \in I_{\langle^\curlywedge,^\curlyvee\rangle}$ iff $v \in \{u\}^\curlywedge$. Then we have $A^\curlywedge = \{v \in V \mid \text{for each } u \in A : \langle u, v\rangle \in I_{\langle^\curlywedge,^\curlyvee\rangle}\}$ for any $A \subseteq U$, and $B^\curlyvee = \{u \in U \mid \text{for each } v \in B : \langle u, v\rangle \in I_{\langle^\curlywedge,^\curlyvee\rangle}\}$ for any $B \subseteq V$. Furthermore, in such a case, the set $\mathcal{B}(U, V, \langle^\curlywedge, ^\curlyvee\rangle) = \{\langle A, B\rangle \in 2^U \times 2^V \mid A^\curlywedge = B, B^\curlyvee = A\}$ of all fixpoints of $\langle^\curlywedge, ^\curlyvee\rangle$ (which is in fact the ordinary concept lattice $\mathcal{B}(U, V, I_{\langle^\curlywedge,^\curlyvee\rangle})$) obeys the so-called main theorem of concept lattices:

Theorem 3 ([11]): (1) $\mathcal{B}\left(U, V, I_{\langle^\curlywedge,^\curlyvee\rangle}\right)$ is under $\leq$, defined by $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$, a complete lattice where the infima and suprema are given by

$$\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\curlyvee\curlywedge}\rangle \ ,$$
$$\bigvee_{j\in J} \langle A_j, B_j\rangle = \langle (\bigcup_{j\in J} A_j)^{\curlywedge\curlyvee}, \bigcap_{j\in J} B_j\rangle \ .$$

(2) Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K, \leq\rangle$ is isomorphic to $\mathcal{B}\left(U, V, I_{\langle^\curlywedge,^\curlyvee\rangle}\right)$ iff there are mappings $\gamma : U \to K$, $\mu : V \to K$ such that

(i) $\gamma(U)$ is $\bigvee$-dense in K, $\mu(V)$ is $\bigwedge$-dense in V;
(ii) $\gamma(u) \leq \mu(v)$ iff $\langle u, v\rangle \in I_{\langle^\curlywedge,^\curlyvee\rangle}$.

Lemma 4: The (ordinary) relation $I^\times = I_{\langle^\curlywedge,^\curlyvee\rangle}$ between $X \times *_X(L)$ and $Y \times *_Y(L)$ corresponding to a Galois connection $\langle^\curlywedge, ^\curlyvee\rangle$ defined by (15) is given by

$$\langle\langle x, a\rangle, \langle y, b\rangle\rangle \in I^\times \text{ iff } a \otimes b \leq I(x, y). \quad (16)$$

Proof: By the above remark, we have $\langle\langle x, a\rangle, \langle y, b\rangle\rangle \in I^\times$ iff $\langle y, b\rangle \in \{\langle x, a\rangle\}^\curlywedge$. By definition of $^\curlywedge$, this is equivalent to $\langle y, b\rangle \in \lfloor\lceil\{\langle x, a\rangle\}\rceil^\uparrow\rfloor^{*_Y}$. Since $\lfloor\lceil\{\langle x, a\rangle\}\rceil^\uparrow\rfloor^{*_Y} = \lfloor\{{}^a/x\}^\uparrow\rfloor^{*_Y}$ and since the largest c such that $\langle y, c\rangle \in \lfloor\{{}^a/x\}^\uparrow\rfloor^{*_Y}$ is $c = (\{{}^a/x\}^\uparrow(y))^{*_Y}$, the last assertion is equivalent to $b \leq (\{{}^a/x\}^\uparrow(y))^{*_Y}$. Since $b = b^{*_Y}$, this is equivalent to $b \leq \{{}^a/x\}^\uparrow(y)$. Now, $\{{}^a/x\}^\uparrow(y) = a^{*_X} \to I(x, y) = a \to I(x, y)$, whence $b \leq \{{}^a/x\}^\uparrow(y)$ is equivalent to $a \otimes b \leq I(x, y)$ by adjointness. ■

In the rest, $I^\times$ always denotes the relation from Lemma 4.

Theorem 5: $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ (concept lattice with hedges) is isomorphic to $\mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^\times)$ (ordinary concept lattice). The isomorphism $h : \mathcal{B}(X^{*_X}, Y^{*_Y}, I) \to \mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^\times)$ and its inverse $g : \mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^\times) \to \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ are given by

$$h(\langle A, B\rangle) = \langle \lfloor A\rfloor^{*_X}, \lfloor B\rfloor^{*_Y}\rangle, \quad (17)$$

$$g(\langle A', B'\rangle) = \langle \lceil A'\rceil^{\uparrow\downarrow}, \lceil B'\rceil^{\downarrow\uparrow}\rangle. \quad (18)$$

Proof: The theorem can be proven by showing that (a) h and g are defined correctly, (b) h is order-preserving, and (c) $g(h(\langle A, B\rangle)) = \langle A, B\rangle$ and $h(g(\langle A', B'\rangle)) = \langle A', B'\rangle$. We give only a sketch.

"(a)": We need to show that for $\langle A, B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ and $\langle A', B'\rangle \in \mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^\times)$ we have $g(A', B') \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ and $h(A', B') \in \mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^\times)$. This can be done using previous propositions, we omit details. "(b)" is evident. "(c)": Can be shown using previous propositions. ∎

The following is our main theorem describing the structure of concept lattices with hedges.

Theorem 6 (main theorem for concept lattices with hedges): (1) $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is under $\leq$ a complete lattice where the infima and suprema are given by

$$\textstyle\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle (\bigcap_{j\in J} A_j)^{\uparrow\downarrow}, (\bigcup_{j\in J} B_j^{*_Y})^{\downarrow\uparrow}\rangle, \quad (19)$$

$$\textstyle\bigvee_{j\in J} \langle A_j, B_j\rangle = \langle (\bigcup_{j\in J} A_j^{*_X})^{\uparrow\downarrow}, (\bigcap_{j\in J} B_j)^{\downarrow\uparrow}\rangle. \quad (20)$$

(2) Moreover, an arbitrary complete lattice $\mathbf{K} = \langle K, \leq\rangle$ is isomorphic to $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ iff there are mappings $\gamma : X \times *_X(L) \to K$, $\mu : Y \times *_Y(L) \to K$ such that

(i) $\gamma(X \times *_X(L))$ is $\bigvee$-dense in K, $\mu(Y \times *_Y(L))$ is $\bigwedge$-dense in V;
(ii) $\gamma(x, a) \leq \mu(y, b)$ iff $a \otimes b \leq I(x, y)$.

Proof: Again, we give only a sketch. Use Theorem 5 and apply Theorem 3 to $\mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^\times)$. Then, using h and g, translate the theorem characterizing $\mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^\times)$ to a theorem characterizing $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$. Doing that, we obtain our theorem with formulas for $\bigwedge_{j\in J} \langle A_j, B_j\rangle$ and $\bigvee_{j\in J} \langle A_j, B_j\rangle$ which are cumbersome. However, they can be simplified to (19) and (20). ∎

C. Case $*_X = \mathrm{id}$

If $*_X = \mathrm{id}$, a hedge is applied only to attributes. In such a case, we denote the concept lattice with hedges by $\mathcal{B}(X, Y^{*_Y}, I)$. This is an important special case. If $*_Y$ is globalization (first boundary possibility), $\mathcal{B}(X, Y^{*_Y}, I)$ is just the lattice of crisply generated concepts [7]. If $*_Y$ is identity (second boundary possibility), $\mathcal{B}(X, Y^{*_Y}, I)$ is the whole fuzzy concept lattice [3]. In general, $*_Y$ (possibly between globalization and identity) controls the meaning of "having all attributes from (the intent) B". Loosely speaking, paying attention to $\langle A, B\rangle \in \mathcal{B}(X, Y^{*_Y}, I)$ means that we do not put any restriction on extents A (any closed fuzzy set of objects of objects is good), but use $*_Y$ to impose a restriction on intents B. This follows intuition. For instance, an intent B which contains each attribute in degree 0.5 might seem not natural. If this is our view, we can take globalization for $*_Y$ and the corresponding concept $\langle A, B\rangle$ disappears (does not belong to $\mathcal{B}(X, Y^{*_Y}, I)$).

We are going to show that if $*_X = \mathrm{id}$, one can answer several important questions. The first theorem shows that concepts from $\mathcal{B}(X, Y^{*_Y}, I)$ are particular concepts from the whole $\mathcal{B}(X, Y, I)$.

Theorem 7: $\mathcal{B}(X, Y^{*_Y}, I) \subseteq \mathcal{B}(X, Y, I)$. Moreover, $\mathcal{B}(X, Y^{*_Y}, I) = \{\langle A, B\rangle \in \mathcal{B}(X, Y, I) \mid A = D^\downarrow$ for some $D \in *_Y(L)^Y\}$.

Proof: "$\subseteq$": If $\langle A, B\rangle \in \mathcal{B}(X, Y^{*_Y}, I)$ then clearly, $D := B^{*_Y} \in *_Y(L)^Y$ and $A = D^\downarrow$. Furthermore, $A^\Uparrow = A^\uparrow = B$ and $A = B^\downarrow = B^{*_Y\Downarrow} \supseteq B^\Downarrow = A^{\Uparrow\Downarrow} \supseteq A$, whence $B^\Downarrow = A$. That is, $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$.

"$\supseteq$": Let $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ such that $A = D^\downarrow$ for some $D \in *_Y(L)^Y$. We need to show $\langle A, B\rangle \in \mathcal{B}(X, Y^{*_Y}, I)$ for which it is sufficient to see $A = B^\downarrow$. We have $A = D^\downarrow = D^{*_Y\Downarrow} = D^{*_Y\Downarrow\Uparrow*_Y\Downarrow} = D^{\Downarrow\Uparrow*_Y\Downarrow} = B^{*_Y\Downarrow} = B^\downarrow$. ∎

In the general case, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ need not be a subset of $\mathcal{B}(X, Y, I)$ as shown by the following example.

Example 2: Take a Łukasiewicz structure on $[0, 1]$, let both $*_X$ and $*_Y$ be globalizations, and consider the following data table

I	y_1	y_2
x_1	1	0.5
x_2	0.7	0.1

One can check that for $A = \{{}^{1}/x_1, {}^{0.7}/x_2\}$ and $B = \{{}^{1}/y_1, {}^{0.5}/y_2\}$, $\langle A, B\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ but $\langle A, B\rangle \notin \mathcal{B}(X, Y, I)$.

$*(L)$ is in fact the set of all fixpoints of $*$, i.e. those $a \in L$ for which $a^* = a$. The next theorem shows that the smaller the set of fixpoints of $*_Y$, the larger the reduction.

Theorem 8: If $*_1(L) \subseteq *_2(L)$ then $\mathcal{B}(X, Y^{*_1}, I) \subseteq \mathcal{B}(X, Y^{*_2}, I)$.

Proof: Follows immediately from Theorem 7. ∎

Theorem 9: If $*_X = \mathrm{id}$, formula (19) simplifies to any of the following forms:

$$\textstyle\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}\rangle, \quad (21)$$

$$\textstyle\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\Downarrow\Uparrow}\rangle. \quad (22)$$

Proof: First, we show that in this case, $(\cap_j A_j)^{\uparrow\downarrow} = \cap_j A_j$: On the one hand, $\cap_j A_j \subseteq (\cap_j A_j)^{\Uparrow\Downarrow} = (\cap_j A_j)^{\uparrow\Downarrow} \subseteq (\cap_j A_j)^{\uparrow\downarrow}$. On the other hand, $(\cap_j A_j)^{\uparrow\downarrow} \subseteq \cap_j A_j$ iff for each $j \in J$ we have $(\cap_j A_j)^{\uparrow\downarrow} \subseteq A_j$ which is true. Indeed, $(\cap_j A_j) \subseteq A_j$ implies $(\cap_j A_j)^{\uparrow\downarrow} \subseteq A_j^{\uparrow\downarrow}$ and $A_j^{\uparrow\downarrow} = A_j$ since A_j is an extent.

Second, since $\langle(\cap_j A_j), (\cup_j B_j^{*_Y})^{\downarrow\uparrow}\rangle = \langle(\cap_j A_j)^{\uparrow\downarrow}, (\cup_j B_j^{*_Y})^{\downarrow\uparrow}\rangle \in \mathcal{B}(X, Y^{*_Y}, I)$, Theorem 7 yields $\langle(\cap_j A_j), (\cup_j B_j^{*_Y})^{\downarrow\uparrow}\rangle \in \mathcal{B}(X, Y, I)$. Now, observe that the intent corresponding to $\cap_j A_j$ in $\mathcal{B}(X, Y, I)$ is $(\cup_j B_j)^{\Downarrow\Uparrow}$ (see e.g. [2], [3]). This yields $(\cup_j B_j^{*_Y})^{\downarrow\uparrow} = (\cup_j B_j)^{\Downarrow\Uparrow}$. Furthermore, $(\cup_j B_j^{*_Y})^{\downarrow\uparrow} \subseteq (\cup_j B_j)^{\downarrow\uparrow} \subseteq (\cup_j B_j)^{\Downarrow\Uparrow} = (\cup_j B_j^{*_Y})^{\downarrow\uparrow}$, whence $(\cup_j B_j^{*_Y})^{\downarrow\uparrow} = (\cup_j B_j)^{\downarrow\uparrow}$. The proof is finished. ■

As a corollary, we get the following assertion.

Theorem 10: If $*_X = \text{id}$, then $\mathcal{B}(X, Y^{*_Y}, I)$ is a $\bigwedge$-sublattice of $\mathcal{B}(X, Y, I)$.

Proof: Follows from Theorem 9 and the fact that the infimum of $\langle A_j, B_j\rangle$'s in $\mathcal{B}(X, Y, I)$ is given by (22), see [2]. ■

The following example shows that $\mathcal{B}(X, Y^{*_Y}, I)$ need not be a $\bigvee$-sublattice of $\mathcal{B}(X, Y, I)$.

Example 3: Take a Łukasiewicz structure on $[0, 1]$ (but this works for Gödel and product as well), let $*_Y$ be globalization, and consider the following data table

I	y_1	y_2	y_3
x_1	0.3	0.5	0.4
x_2	0.2	0.6	0.1

Then both $B_1 = \{\,{}^1/y_1,\ {}^1/y_2\}^{\downarrow\uparrow} = \{\,{}^1/y_1,\ {}^1/y_2,\ {}^{0.9}/y_3\}$ and $B_2 = \{\,{}^1/y_2,\ {}^1/y_3\}^{\downarrow\uparrow} = \{\,{}^{0.9}/y_1,\ {}^1/y_2,\ {}^1/y_3\}$ are intents in $\mathcal{B}(X, Y^{*_Y}, I)$. However, since $B_1 \cap B_2 \neq (B_1 \cap B_2)^{\downarrow\uparrow}$, suprema in $\mathcal{B}(X, Y^{*_Y}, I)$ and $\mathcal{B}(X, Y, I)$ are different.

The following theorem shows that if both $*_X$ and $*_Y$ are globalizations (boundary case, largest restriction), $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is in fact isomorphic to an ordinary concept lattice given by 1-cut ${}^1I = \{\langle x, y\rangle \mid I(x, y) = 1\}$ of I. Note that the data table $\langle X, Y, {}^1I\rangle$ results from $\langle X, Y, I\rangle$ by keeping entries with 1's and deleting (replacing by 0) all other entries.

Theorem 11: If $*_X$ and $*_Y$ are globalizations, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic to (ordinary) concept lattice $\mathcal{B}\left(X, Y, {}^1I\right)$.

Proof: Easy, omitted. ■

D. Algorithms

A study of algorithms for constructing concept lattices with hedges is an important issue which we do not attempt to investigate in this paper. For the sake of completeness, we only mention that according to Theorem 5, one can proceed as follows: Transform the original table $\langle X, Y, I\rangle$ to $\langle X \times *_X(L), Y \times *_Y(L), I^{\times}\rangle$. Using algorithms for ordinary concept lattices, compute $\mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^{\times})$. Using mapping g from Theorem 5, "translate" $\mathcal{B}(X \times *_X(L), Y \times *_Y(L), I^{\times})$ to $\mathcal{B}(X, Y, I)$.

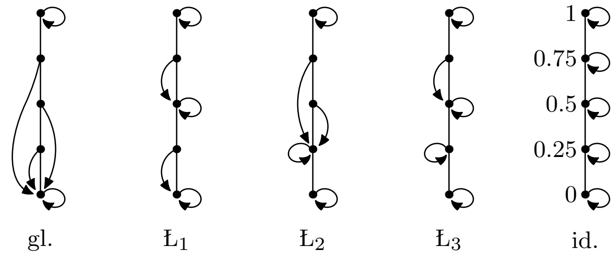

Fig. 1. Truth stressers

IV. Examples

Consider a finite Łukasiewicz chain $\mathbf{L}$ such that $L = \{0, 0.25, 0.5, 0.75, 1\}$, $\otimes$ and $\rightarrow$ given by (6). For $\mathbf{L}$, there are five truth-stressing hedges satisfying (2)–(5). That is, except for globalization and identity, there are three nontrivial hedges which will be denoted by $Ł_1$, $Ł_2$, $Ł_3$, see Fig. 1. Note that globalization has two fixpoints (0 and 1), $Ł_1$ has three fixpoints $(0, 0.5$, and $1)$, $Ł_2$ has also three fixpoints $(0, 0.25$, and $1)$, $Ł_3$ has four fixpoints $(0, 0.25, 0.5$, and $1)$, and finally, identity has five fixpoints. Also note that the number of truth stressing hedges defined on a finite chain depends on the chosen $\otimes$ and $\rightarrow$. For instance, for five-element Gödel chain there are eight truth-stressing hedges satisfying (2)–(5).

Consider a data table $\langle X, Y, I\rangle$ given by Table I. The set X of object consists of objects "Mercury", "Venus", $\ldots$, set Y contains four attributes: size of the planet (small / large), distance from the sun (far / near). Since we have five hedges on $\mathbf{L}$, $*_X$ and $*_Y$ can be defined in 25 possible ways. According to (13) and (14), each couple $*_X$ and $*_Y$ induces a couple of operators $^{\uparrow}$ and $^{\downarrow}$. As a consequence, we obtain 25 concept lattices from the input data table $\langle X, Y, I\rangle$ just by assigning various hedges to $*_X$ and $*_Y$. If both $*_X$ and $*_Y$ are identities then the resulting concept lattice consists of 216 clusters (formal concepts), while if $*_X$ and $*_Y$ are globalizations, the concept lattice consists of 8 clusters. These are two borderline cases. Concept lattices resulting by all the possible choices of $*_X$ and $*_Y$ are depicted in Fig. 2 (rows and columns determine the definition of $*_X$ and $*_Y$, respectively).

Table II contains a summary of average number of clusters in randomly generated data tables according to the density of input data tables: randomly generated tables have 40 objects, 5 attributes, $\mathbf{L}$ is the same structure of truth degrees as in the previous example, and the density of the generated tables varies from 5 % to 80 %. As one can see, the reduction of concept lattices using hedges as a parameter is smooth.

V. Conclusion

The main motivation to study concept lattices with hedges is to control, in a parametrical way, the size of a concept lattice. Concept lattices with hedges generalize several previous approaches to formal concept analysis of data with fuzzy attributes. The paper presents theoretical insight to reducing the size of a fuzzy concept lattice using hedges. In particular, we showed a generalization of the main theorem of concept lattices. According to this, a concept lattice with hedges is

TABLE I

DATA TABLE WITH FUZZY ATTRIBUTES

	size		distance	
	small (s)	large (l)	far (f)	near (n)
Mercury (Me)	1	0	0	1
Venus (Ve)	0.75	0	0	1
Earth (Ea)	0.75	0	0	0.75
Mars (Ma)	1	0	0.5	0.75
Jupiter (Ju)	0	1	0.75	0.5
Saturn (Sa)	0	1	0.75	0.5
Uranus (Ur)	0.25	0.5	1	0.25
Neptune (Ne)	0.25	0.5	1	0
Pluto (Pl)	1	0	1	0

TABLE II

AVERAGE NUMBER OF CLUSTERS

80 %	gl.	Ł$_1$	Ł$_2$	Ł$_3$	id.
gl.	16	31	32	32	32
Ł$_1$	85	120	121	121	180
Ł$_2$	84	107	107	108	108
Ł$_3$	299	337	337	338	501
id.	560	928	637	951	1512

55 %	gl.	Ł$_1$	Ł$_2$	Ł$_3$	id.
gl.	12	27	31	31	31
Ł$_1$	53	89	92	93	150
Ł$_2$	66	95	99	100	100
Ł$_3$	146	186	190	191	410
id.	212	448	271	540	1148

30 %	gl.	Ł$_1$	Ł$_2$	Ł$_3$	id.
gl.	9	17	21	22	22
Ł$_1$	26	48	52	53	78
Ł$_2$	33	54	58	59	59
Ł$_3$	48	72	77	77	181
id.	59	137	91	201	425

5 %	gl.	Ł$_1$	Ł$_2$	Ł$_3$	id.
gl.	4	7	7	8	8
Ł$_1$	8	14	15	15	17
Ł$_2$	9	15	16	16	16
Ł$_3$	10	16	17	17	31
id.	11	21	18	32	52

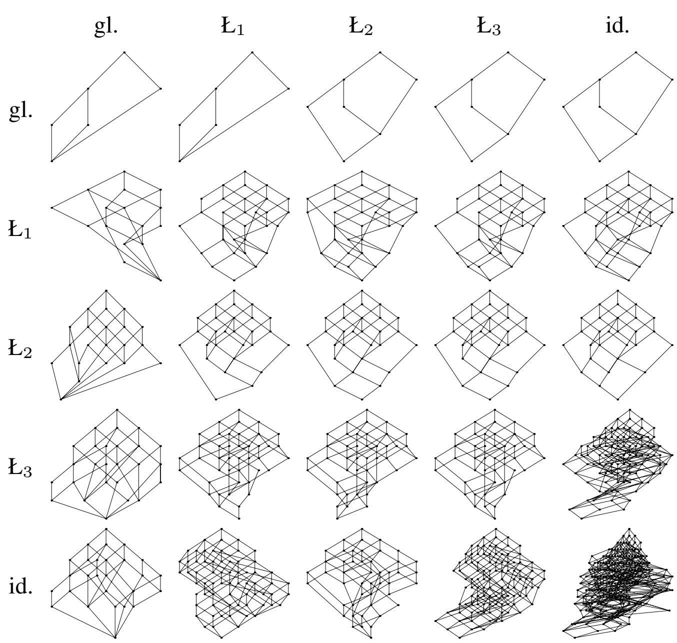

Fig. 2. Concept lattices generated from data in Table I by all combinations of truth stressers $*_X$ and $*_Y$ from Fig. 1.

indeed a complete lattice. Furthermore, it is isomorphic to an ordinary concept lattice, with a well-described isomorphism and its inverse which serve as translation procedures. Among other things, this enables us to compute a concept lattice with hedges using algorithms for ordinary concept lattices. Further insight is provided in case one uses hedges only for attributes. Examples demonstrate that the size reduction using hedges as a parameter is smooth. Future research needs to focus on further theoretical insight (e.g., for case when both hedges are used simultaneously) and on combination of using hedges with other methods for reduction of the size of a fuzzy concept lattice.

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