What is a fuzzy concept lattice?*

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Abstract. The paper is an overview of several approaches to the notion of a concept lattice from the point of view of fuzzy logic. The main aim is to clarify relationships between the various approaches.

Keywords: formal concept analysis, fuzzy logic, fuzzy attribute

1 Introduction

Formal concept analysis (FCA) deals with a particular kind of analysis of data which, in the basic setting, has the form of a table with rows corresponding to objects, columns corresponding to attributes, and table entries containing 1's and 0's depending on whether an object has or does not have an attribute (we assume basic familiarity with FCA and refer to [24] for information). The basic setting is well-suited for attributes which are crisp, i.e. each object of the domain of applicability of the attribute either has (1) or does not have (0) the attribute. Many attributes are fuzzy rather than crisp. That is to say, it is a matter of degree to which an object has a (fuzzy) attribute. For instance, when asking whether a man with a height of 182 cm is tall, one probably gets an answer like "not completely tall but almost tall" or "to a high degree tall", etc. A natural idea, developed in fuzzy logic [12, 28, 31], is to assign to an object a truth degree to which the object has a (fuzzy) attribute. Degrees are taken from an appropriate scale L of truth degrees. A favorite choice of L is the real unit interval $[0, 1]$ or some subset of $[0, 1]$. Then, we can say that a man with a height of 182 cm is tall to a degree, say, 0.8. Doing so, the entries of a table describing objects and attributes become degrees from L instead of values from $\{0, 1\}$ as is the case of the basic setting of FCA.

FCA provides means to process tables with degrees from a scale L. Namely, one can consider a table with degrees from L a many-valued context and use a so-called conceptual scaling [24]. There is, however, another way to process such data tables. The way is based on considering the table entries as truth degrees in fuzzy logic and proceed analogously as we do in the basic setting of FCA, just "replacing classical (bivalent) logic with fuzzy logic". Recently, the second way

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gained a considerable interest. The structures which result this way are called fuzzy concept lattices, fuzzy Galois connections, etc. However, the particular approaches, and hence the resulting structures, differ.

The basic aim of this paper is to compare them and to show some of the basic relationships between them. Due to a limited scope, we omit some technical details (mostly proofs). These are left for a full version of this paper.

Note that we do not study mathematical or any other (like epistemic) relationships of the discussed "fuzzy logic approaches" to the above-mentioned possibility of using conceptual scaling. This, again, is left for a forthcoming paper.

2 Why fuzzy logic in formal concept analysis and a basic standpoint

2.1 Why fuzzy logic in formal concept analysis?

The main reason is the following. FCA is based on working with formulas like "an object belongs to (a set) A iff it has all attributes from (a set) B". In the basic setting (i.e., table entries contain 0's and 1's), this is done in classical logic. When the truth degrees in a data table are taken from scale L, do the same as in the basic setting of FCA, only "replace classical logic by fuzzy logic". Doing so, the results of FCA in a fuzzy setting will have the same verbal interpretation as those in the classical setting. Namely, the underlying formulas are the same in both the fuzzy setting and the classical setting and the difference is only in the interpretation of the formulas (many-valued interpretation in fuzzy setting vs. bivalent interpretation in classical setting).

2.2 A basic standpoint: first-order fuzzy logic

However, "replace classical logic by fuzzy logic" is ambiguous. Namely, there are several ways to perform the "replacement". Nevertheless, there is a way which might be considered direct and simple. Namely, one can use a framework of first-order fuzzy logic (and fuzzy relations) the same way as one uses first-order classical logic (and classical relations) in the basic setting of FCA. Since the standpoint of a first-order logic (and relations) is simple and easy to understand, we take it for a basic standpoint in our paper. It is the one to which we compare the other approaches. In the rest of this section, we recall some basic concepts we need. We refer to [12, 28, 30] for further information.

As a set of truth degrees equipped with suitable operations (truth functions of logical connectives) we use a complete residuated lattice i.e. an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\rightarrow$ satisfy so-called adjointness property: $a \otimes b \leq c$ iff $a \leq b \rightarrow c$; for each $a, b, c \in L$. Elements a of L are called truth degrees (usually, $L \subseteq [0, 1]$). $\otimes$ and

$\to$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication". By $\mathbf{L}^U$ (or L^U) we denote the collection of all fuzzy sets in a universe U, i.e. mappings A of U to L. For $A \in L^U$ and $a \in L$, a set ${}^aA = \{u \in U \mid A(u) \geq a\}$ is called an a-cut of A.

2.3 What else than a first-order fuzzy logic?

Other approaches can be based on working with the truth degrees from L in the data table in any way which, if taking $L = \{0, 1\}$, yields the same what we have in the basic setting of FCA (we will see some of them later). Of course, this requirement follows from a requirement saying that we want to have a generalization of the basic setting of FCA. If the requirement is satisfied, we can consider the results (over a general scale L of truth degrees) as "having the same meaning as in the basic setting of FCA".

3 Overview and comparison of existing approaches to fuzzy concept lattices

A common point of all of the presented approaches is the notion of a fuzzy context, i.e. the input data. Let L be a scale of truth degrees. L might be a support set of some structure $\mathbf{L} = \langle L, \ldots \rangle$ of truth degrees like a complete residuated lattice described above. Then, a fuzzy context (L-context, or $\mathbf{L}$-context) is a triplet $\langle X, Y, I \rangle$ where X and Y are sets of objects and attributes, respectively, and $I : X \times Y \to L$ is a fuzzy relation (L-relation, $\mathbf{L}$-relation) between X and Y. A degree $I(x, y) \in L$ is interpreted as a degree to which object x has attribute y. Note that if one takes $L = \{0, 1\}$ then the notion of a fuzzy context coincides in an obvious way with the notion of an (ordinary) context.

3.1 Basic setting of FCA

To fix notation for further reference, we recall the following [24]. A formal context (formal counterpart to a data table) is a triplet $\langle X, Y, I \rangle$ where X and Y are sets of objects and attributes, respectively, and $I \subseteq X \times Y$ is a binary relation "to have" ($\langle x, y \rangle \in I$ means that object x has attribute y). A formal concept in $\langle X, Y, I \rangle$ is a pair $\langle A, B \rangle$ of a set $A \subseteq X$ of objects and a set $B \subseteq Y$ of attributes such that

$$B \text{ is the collection of all attributes shared by all objects from } A \tag{1}$$

$$A \text{ is the collection of all objects sharing all the attributes from } B. \tag{2}$$

For $A \subseteq X$ and $B \subseteq Y$, put

$$A^{\uparrow} = \{y \mid \text{for each } x \in A :\ \langle x, y \rangle \in I\}, \tag{3}$$

$$B^{\downarrow} = \{x \mid \text{for each } y \in B :\ \langle x, y \rangle \in I\}. \tag{4}$$

This says that, $A^{\uparrow}$ is the set of all attributes from Y shared by all objects from A, and $B^{\downarrow}$ is the set of all objects from X sharing all attributes from B. Therefore, a formal concept is a pair $\langle A, B\rangle$ for which $A^{\uparrow} = B$ and $B^{\downarrow} = A$. For a formal concept $\langle A, B\rangle$, A is called an extent (collection of objects covered by $\langle A, B\rangle$), B is called an intent (collection of attributes covered by $\langle A, B\rangle$). Denote by $\mathcal{B}(X, Y, I)$ the set of all formal concepts in $\langle X, Y, I\rangle$, i.e. $\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B, B^{\downarrow} = A\}$. Introduce a relation $\leq$ on $\mathcal{B}(X, Y, I)$ by $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (or, equivalently $B_2 \subseteq B_1$). The structure $\langle \mathcal{B}(X, Y, I), \leq\rangle$, i.e. set $\mathcal{B}(X, Y, I)$ equipped with $\leq$, is called a concept lattice of $\langle X, Y, I\rangle$. Basic structure of concept lattices is described by the so-called main theorem of concept lattices [24, 39].

3.2 The approach by Burusco and Fuentes-Gonzáles

Burusco and Fuentes-Gonzáles are the authors of the first paper on FCA in a fuzzy setting. Namely, in [21], they proceed as follows. Let $\mathbf{L} = \langle L, \leq, ', \oplus, 0, 1\rangle$ be a structure such that $\langle L, \leq, 0, 1\rangle$ is a complete lattice bounded by 0 and 1, $'$ be a unary operation of complementation, and $\oplus$ be a t-conorm on L (i.e. a binary operation which is associative, commutative, and has 0 as its neutral element). For an L-context $\langle X, Y, I\rangle$, define mappings $^{\uparrow} : L^X \to L^Y$ and $^{\downarrow} : L^Y \to L^X$ by

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x \in X}(A(x)' \oplus I(x, y)), \tag{5}$$

$$B^{\downarrow}(x) = \textstyle\bigwedge_{y \in Y}(B(y)' \oplus I(x, y)), \tag{6}$$

for $A \in L^X$ and $B \in L^Y$. Furthermore, put $\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \in L^X \times L^Y \mid A^{\uparrow} = B, B^{\downarrow} = A\}$ and define a partial order $\leq$ on $\mathcal{B}(X, Y, I)$ by $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (iff $B_2 \subseteq B_1$).

Note first that for $L = \{0, 1\}$, $'$ being a classical negation (i.e. $0' = 1$ and $1' = 0$), and $\oplus$ being a classical disjunction (i.e. $a \oplus b = \max(a, b)$), $\mathcal{B}(X, Y, I)$ equipped with $\leq$ coincides in an obvious way with the ordinary concept lattice.

In [21], the authors show that $\mathcal{B}(X, Y, I)$ equipped with $\leq$ is a complete lattice and show some of the basic properties of $^{\uparrow}$ and $^{\downarrow}$. However, $^{\uparrow}$ and $^{\downarrow}$ do not satisfy some useful properties which hold true in the classical case, like $A \subseteq A^{\uparrow\downarrow}$ a $B \subseteq B^{\downarrow\uparrow}$. The authors extended (generalized) their approach later to include so-called implication operators [22]. Note also that the authors discussed other extensions of FCA in their setting, see e.g. [23]. But since some useful properties are missing in their approach, we stop our visit to this approach here and leave a more detailed discussion to a full paper.

3.3 The approach by Pollandt and Bělohlávek

Pollandt [36] and, independently, Bělohlávek ([3] is the first publication, an overview can be found in [12]), elaborated the following approach which turned out to be a feasible way to develop FCA and related structures in a fuzzy setting. Let $\mathbf{L}$ be a complete residuated lattice (see above). The choice of a residuated lattice structure, and particularly the adjointness property of conjunction and implication turns out to be a crucial property. Let $\langle X, Y, I\rangle$ be an **L**-context, i.e. $I : X \times Y \to L$. For fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ (denoted also $A^{\uparrow_I}$ and $B^{\downarrow_I}$) defined by

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x\in X}(A(x) \to I(x, y)), \tag{7}$$

$$B^{\downarrow}(x) = \textstyle\bigwedge_{y\in Y}(B(y) \to I(x, y)). \tag{8}$$

Using basic rules of predicate fuzzy logic [12, 28], one can easily see that $A^{\uparrow}(y)$ is the truth degree of "y is shared by all objects from A" and $B^{\downarrow}(x)$ is the truth degree of "x has all attributes from B". That is, (7) and (8) properly generalize (3) and (4). Putting

$$\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\},$$

$\mathcal{B}(X, Y, I)$ is the set of all pairs $\langle A, B\rangle$ such that (a) A is the collection of all objects that have all the attributes of (the intent) B and (b) B is the collection of all attributes that are shared by all the objects of (the extent) A. Elements of $\mathcal{B}(X, Y, I)$ are called formal concepts of $\langle X, Y, I\rangle$; $\mathcal{B}(X, Y, I)$ is called a (fuzzy) concept lattice (**L**-concept lattice) of $\langle X, Y, I\rangle$. Note that as in the case of Burusco and Fuentes-Gonzáles, both the extent A and the intent B of a formal concept $\langle A, B\rangle$ are in general fuzzy sets. This corresponds to the fact that in general, concepts apply to objects and attributes to various intermediate degrees, not only 0 and 1. Putting

$$\langle A_1, B_1\rangle \le \langle A_2, B_2\rangle \quad \text{iff} \quad A_1 \subseteq A_2 \quad (\text{iff } B_1 \supseteq B_2) \tag{9}$$

for $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X, Y, I)$, $\le$ models the subconcept-superconcept hierarchy in $\mathcal{B}(X, Y, I)$.

Note that if **L** is a two-element Boolean algebra, i.e. $L = \{0, 1\}$, the notions of an **L**-context, a formal concept, and an **L**-concept lattice coincide with the ordinary notions [24].

For this approach, several issues of FCA have been generalized:

- Main theorem of **L**-concept lattices has two versions. The first one deals with the above defined ordinary partial order on formal concepts (a proof via reduction to the ordinary case can be found in [36] and in [9], a direct proof can be found in [8]). The second one deals with a fuzzy order on formal concepts, see [14]. In particular, $\mathcal{B}(X, Y, I)$ is a complete lattice w.r.t. $\le$.
- Related structures like Galois connections and closure operators are studied in [5, 10, 11].
- A direct algorithm for generating all formal concepts based on Ganter's algorithm [24] is presented in [13].
- Issues which are degenerate in the ordinary case have been studied, see e.g. [7, 6].
- Several further issues like many-valued contexts, relationship to approximate reasoning, attribute implications are studied in [36]. Attribute implications are studied also in [16, 17, 19, 20].

3.4 The aproach by Yahia *et al.* and Krajči

Yahia [40] and, independently, Krajči [32] proposed the following approach, called "one-sided fuzzy approach" by Krajči. Note first that the approaches are not the same but the only difference is that for an L-context $\langle X, Y, I\rangle$, Yahia's definitions yield what Krajči's definitions yield for $\langle Y, X, I^{-1}\rangle$ with $I^{-1} \in L^{Y\times X}$ defined by $I^{-1}(y, x) = I(x, y)$, i.e. the approaches are the same up to the roles of objects and attributes. Note also that both Yahia and Krajči consider $L = [0, 1]$.

For an L-context $\langle X, Y, I\rangle$, define mappings $f : 2^X \to L^Y$ (assigning a *fuzzy set* $f(A) \in L^Y$ of attributes to a *set* $A \subseteq X$ of objects) and $h : L^Y \to 2^X$ (assigning a *set* $h(B) \subseteq$ of objects to a *fuzzy set* $B \in L^Y$ of attributes) by

$$f(A)(y) = \textstyle\bigwedge_{x\in A} I(x, y) \tag{10}$$

and

$$h(B) = \{x \in X \mid \text{for each } y \in Y : B(y) \leq I(x, y)\}. \tag{11}$$

Then, put

$$\mathcal{B}_{f,h}(X, Y, I) = \{\langle A, B\rangle \in 2^X \times L^Y \mid f(A) = B, h(B) = A\}.$$

The set $\mathcal{B}_{f,h}(X, Y, I)$, equipped with a partial order $\leq$ defined by (9) is what Yahia and Krajči call a (one-sided) fuzzy concept lattice (it is, indeed, a complete lattice [40, 32]). Note that extents of concepts from $\mathcal{B}_{f,h}(X, Y, I)$ are crisp sets while the intents are fuzzy sets. Needless to say, one can modify this approach and have fuzzy extents and crisp intents.

The following theorem (partly contained in [32, Observation 2]) shows a basic hint to the relationship of f and h, and $^\uparrow$ and $^\downarrow$ defined by (7) and (8).

Theorem 1. *For $A \subseteq X$ and $B \in L^Y$ and $^\uparrow$ and $^\downarrow$ defined by (7) and (8), we have*

$$f(A) = A^\uparrow, \qquad \text{and} \qquad h(B) = {}^1(B^\downarrow).$$

Proof. Easy, omitted due to lack of space.

Here, $A^\uparrow$ for A means $A'^\uparrow$ where A' is a fuzzy set in X corresponding to A, i.e. $A'(x) = 1$ for $x \in A$ and $A'(x) = 0$ for $x \notin A$. Therefore, f and h can be expressed by $^\uparrow$ and $^\downarrow$. In fact, $\mathcal{B}_{f,h}(X, Y, I)$ is isomorphic to (and, in fact, almost the same as) a substructure of Pollandt's and Bělohlávek's $\mathbf{L}$-concept lattice $\mathcal{B}(X, Y, I)$. The details will be shown in the next section.

3.5 Crisply generated fuzzy concepts

In [15], when dealing with the problem of a possibly large number of formal concepts in Pollandt's and Bělohlávek's $\mathbf{L}$-concept lattice $\mathcal{B}(X, Y, I)$, the authors proposed the following approach. Instead of considering the whole $\mathcal{B}(X, Y, I)$, they consider only its part $\mathcal{B}_c(X, Y, I) \subseteq \mathcal{B}(X, Y, I)$ consisting of what they call crisply generated fuzzy concepts: $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ is called crisply if there is a crisp (i.e., ordinary) set $B_c \subseteq Y$ of attributes such that $A = B_c^{\downarrow}$ (and thus $B = B_c^{\downarrow\uparrow}$). Then, $\mathcal{B}_c(X,Y,I) = \{\langle A,B\rangle \in \mathcal{B}(X,Y,I) \mid$ there is $B_c \subseteq Y : A = B_c^{\downarrow}\}$. Restricting oneself to crisply generated fuzzy concepts, one eliminates "strange" concepts like concepts $\langle A,B\rangle$ such that for any $x \in X$ and $y \in Y$ we have $A(x) = 1/2$ and $B(y) = 1/2$ which might be difficult to interpret.

Note that in the modified approach of Yahia and Krajči, extents are fuzzy sets and intents are crisp sets while in case of crisply generated concepts, both extents and intents are in general fuzzy sets. As the next theorem shows, this is only a negligible difference since there is a natural isomorphism between the lattice of "one-sided fuzzy concepts" and the lattice of crisply generated fuzzy concepts:

Theorem 2. *For an* $\mathbf{L}$*-context* $\langle X,Y,I\rangle$*, the "one-sided fuzzy concept lattice"* $\mathcal{B}_{f,h}(X,Y,I)$ *with fuzzy extents and crisp intents (equipped with the above-defined partial order) is a complete lattice which is isomorphic to* $\mathcal{B}_c(X,Y,I)$ *(equipped with a partial order inherited from* $\mathcal{B}(X,Y,I)$*). Moreover, there exists an isomorphism such that for the corresponding concepts* $\langle A,B\rangle \in \mathcal{B}_{f,h}(X,Y,I)$ *and* $\langle C,D\rangle \in \mathcal{B}_c(X,Y,I)$ *we have* $A = C$ *and* $B = {}^1D$*.*

Proof. Omitted due to lack of space (use definitions and Theorem 1).

For further properties of crisply generated fuzzy concepts we refer to [15].

3.6 The approach of Snášel, Vojtáš *et al.*

Another approach appeared in [38] and in a series of related papers. The basic idea is to consider, for a given L-context $\langle X,Y,I\rangle$, formal contexts $\langle X,Y,{}^aI\rangle$ for $a \in K$ for a suitable subset $K \subseteq L$. That is, one takes a-cuts ${}^aI = \{\langle x,y\rangle \mid I(x,y) \geq a\}$ of the original fuzzy relation I for each $a \in K$. K contains truth degrees which are considered important, relevant, sufficiently covering L, etc. Since ${}^aI$ is an ordinary relation, $\langle X,Y,{}^aI\rangle$ is an ordinary formal context. Therefore, one can apply (ordinary) formal concept analysis to each $\langle X,Y,{}^aI\rangle$. Particularly, one can form concept lattices $\mathcal{B}(X,Y,{}^aI)$. All $\mathcal{B}(X,Y,{}^aI)$'s are then "merged" into a structure $\bigsqcup_{a\in K}\mathcal{B}(X,Y,{}^aI)$ defined by

$$\begin{aligned}&\textstyle\bigsqcup_{a\in K}\mathcal{B}(X,Y,{}^aI) = \{\langle A,B\rangle \mid A \in \mathrm{Ext}({}^aI) \text{ for some } a \in K;\\ &B \text{ is a multiset in } Y \text{ with } B(y) = |\{a \in K;\ A \in \mathrm{Ext}({}^aI),\, y \in A^{\uparrow_{{}^aI}}\}|\}.\end{aligned}$$

$\bigsqcup_{a\in K}\mathcal{B}(X,Y,{}^aI)$ then represents the resulting structure derived from the fuzzy context $\langle X,Y,I\rangle$. Here, $\mathrm{Ext}({}^aI) = \{A \subseteq X \mid \langle A, A^{\uparrow_{{}^aI}}\rangle \in \mathcal{B}(X,Y,{}^aI)\}$ denotes the set of all extents of formal concepts from $\mathcal{B}(X,Y,{}^aI)$. Recall that a multiset may contain multiple occurrences of elements. Therefore, a multiset B in a universe set Y may be understood as a function assigning to each $y \in Y$ a nonnegative integer $B(y)$ (the number of occurrences of y in B). That is, $\bigsqcup_{a\in K}\mathcal{B}(X,Y,{}^aI)$ contains pairs $\langle A,B\rangle$ where A is the extent of some formal concept of ${}^aI$, and B is a multiset of attributes with $B(y)$ being the number of cut-levels $a \in K$ for which A is an extent of $\langle X,Y,{}^aI\rangle$ and y belongs to $A^{\uparrow_{{}^aI}}$.

Although this approach is very different from the other ones, there is still the following relationship:

Theorem 3. *For a linearly ordered complete residuated lattice* $\mathbf{L}$ *of truth values with* $a \otimes b = \min(a, b)$ *and any truth degree* $a \in L$ *we have*

$$\mathrm{Ext}({}^{a}I) = \{{}^{a}A \mid \langle A, B\rangle \in \mathcal{B}_c\,(X, Y, I)\}.$$

Proof. Omitted due to lack of space.

Therefore, extents of concepts from $\bigsqcup_{a\in K} \mathcal{B}\,(X, Y, {}^{a}I)$ are just a-cuts of (crisply generated) concepts from $\mathcal{B}_c\,(X, Y, I)$. A more detailed discussion is left to a full paper.

3.7 Concept lattices and Galois connections with hedges

As a generalization of both fuzzy concept lattcies of Pollandt and Bělohlávek and lattices of crisply generated fuzzy concepts, the authors proposed the following approach in [18]. The approach consists in emplyoing, as parameters, two unary functions on L, called hedges. For a complete residuated lattice $\mathbf{L}$, a (truth-stressing) hedge is a unary function * satisfying (i) $1^* = 1$, (ii) $a^* \leq a$, (iii) $(a \to b)^* \leq a^* \to b^*$, (iv) $a^{**} = a^*$, for all $a, b \in L$. A hedge * is a (truth function of) logical connective "very true" [29]. The largest hedge (by pointwise ordering) is identity, the least hedge is globalization which is defined by $a^* = 1$ for $a = 1$ and $a^* = 0$ for $a < 1$.

Given an $\mathbf{L}$-context $\langle X, Y, I\rangle$, one can proceed as follows. Let $*_X$ and $*_Y$ be hedges (their meaning will become apparent later). For $\mathbf{L}$-sets $A \in \mathbf{L}^X$ ($\mathbf{L}$-set of objects), $B \in \mathbf{L}^Y$ ($\mathbf{L}$-set of attributes) we define $\mathbf{L}$-sets $A^{\uparrow} \in \mathbf{L}^Y$ ($\mathbf{L}$-set of attributes), $B^{\downarrow} \in \mathbf{L}^X$ ($\mathbf{L}$-set of objects) by

$$A^{\uparrow}(y) = \textstyle\bigwedge_{x\in X} \big(A(x)^{*_X} \to I(x, y)\big),$$
$$B^{\downarrow}(x) = \textstyle\bigwedge_{y\in Y} \big(B(y)^{*_Y} \to I(x, y)\big).$$

We put $\mathcal{B}(X^{*_X}, Y^{*_Y}, I) = \{\langle A, B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\uparrow} = B,\, B^{\downarrow} = A\}$. For $\langle A_1, B_1\rangle, \langle A_2, B_2\rangle \in \mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, put $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (or, iff $B_2 \subseteq B_1$; both ways are equivalent). Operators $^{\downarrow}, ^{\uparrow}$ form a so-called Galois connection with hedges [18]. $\langle \mathcal{B}(X^{*_X}, Y^{*_Y}, I), \leq\rangle$ is called a *concept lattice with hedges* induced by $\langle X, Y, I\rangle$ (it is, indeed, a complete lattice [18]). The following is the basic relationship to the above described approaches.

Theorem 4. *For* ^{*_X} *and* ^{*_Y} *being identities,* $\mathcal{B}\,(X^{*_X}, Y^{*_Y}, I)$ *coincides with* $\mathcal{B}\,(X, Y, I)$ *(fuzzy concept lattice of Pollandt and Bělohlávek). For* ^{*_X} *being identity and* ^{*_Y} *being globalization,* $\mathcal{B}\,(X^{*_X}, Y^{*_Y}, I)$ *coincides with* $\mathcal{B}_c\,(X, Y, I)$ *(set of all crisply generated fuzzy concepts).*

Note, however, that there are other choices of ^{*_X} and ^{*_Y} possible. As demonstrated in [18], ^{*_X} and ^{*_Y} can be seen as parameters controlling the size of the resulting $\mathcal{B}\,(X^{*_X}, Y^{*_Y}, I)$.

3.8 Krajči's generalized concept lattice

In [33–35], Krajči studies a so-called generalized concept lattice. In the previous approaches to concept lattices, formal concepts are defined as certain pairs $\langle A, B\rangle$ where for A and B we can have the following possibilities: both A and B are crisp sets, both A and B are fuzzy sets, A is crisp and B is fuzzy, A is fuzzy and B is crisp. Krajči suggests to consider three sets of truth degrees in general, namely, a set L_X (for objects), L_Y (for attributes), and L (for degrees to which objects have attributes, i.e. table entries).

Given sets X and Y of objects and attributes, one can consider a fuzzy context as a triplet $\langle X, Y, I\rangle$ where I is an L-relation between X and Y, i.e. $I \in L^{X\times Y}$. Furthermore, one can consider L_X-sets A of objects and L_Y-sets B of attributes, i.e. $A \in L_X^X$ and $B \in L_Y^Y$. Furthermore, Krajči assumes that L_X and L_Y are complete lattices and L is a partially ordered set. For convenience, we denote all the partial orders (on L_X, L_Y, and L) by $\leq$. Likewise, infima and suprema in both $\langle L_X, \leq\rangle$ and $\langle L_Y, \leq\rangle$ will be denoted by $\bigwedge$ and $\bigvee$.

In order to define arrow-operators, Krajči assumes that there is an operation $\otimes : L_X \times L_Y \to L$ satisfying

$$a_1 \leq a_2 \text{ implies } a_1 \otimes b \leq a_2 \otimes b, \tag{12}$$

$$b_1 \leq b_2 \text{ implies } a \otimes b_1 \leq a \otimes b_2, \tag{13}$$

$$\text{if } a_j \otimes b \leq c \text{ for each } j \in J \text{ then } (\textstyle\bigvee_{j\in J} a_j) \otimes b \leq c, \tag{14}$$

$$\text{if } a \otimes b_j \leq c \text{ for each } j \in J \text{ then } a \otimes (\textstyle\bigvee_{j\in J} b_j) \leq c, \tag{15}$$

for each index set J and all $a, a_j \in L_X$, $b, b_j \in L_Y$, and $c \in L$. That is, we have a three-sorted structure $\langle L_1, L_2, L, \otimes, \leq, \ldots\rangle$ of truth degrees. Call such a structure $\langle L_1, L_2, L, \otimes, \leq, \ldots\rangle$ satisfying (12)–(15) Krajči's structure.

Then, Krajči introduces mappings $^{\nearrow} : L_X^X \to L_Y^Y$ and $^{\swarrow} : L_Y^Y \to L_X^X$ by

$$A^{\nearrow}(y) = \bigvee\{b \in L_Y \mid \text{for each } x \in X : A(x) \otimes b \leq I(x, y)\}, \tag{16}$$

$$B^{\swarrow}(x) = \bigvee\{a \in L_X \mid \text{for each } y \in Y : a \otimes B(y) \leq I(x, y)\}, \tag{17}$$

and defines a formal concept in $\langle X, Y, I\rangle$ as a pair $\langle A, B\rangle \in L_X^X \times L_Y^Y$ satisfying $A^{\nearrow} = B$ and $B^{\swarrow} = A$.

At first sight, it looks like there is no adjointness involved. Also, compared to (7) and (8), (16) and (17) seem a bit complicated. However, the following theorem shows that Krajči's structure $\langle L_1, L_2, L, \otimes, \leq, \ldots\rangle$ can be looked at as a three-sorted structure with two implication connectives satisfying adjointness-like conditions.

Theorem 5 (residuated structure for generalized concept lattices). *Let L_1, L_2 be complete lattices and L be a partially ordered set (all partial orders are denoted by $\leq$).*
(1) For a Krajči's structure $\langle L_1, L_2, L, \otimes, \leq, \ldots\rangle$ define for $c \in L_1$, $d \in L_2$, and $p \in L$,

$$c \to_1 p = \bigvee\{d \in L_2 \mid c \otimes d \leq p\}, \tag{18}$$

$$d \to_2 p = \bigvee\{c \in L_1 \mid c \otimes d \leq p\}. \tag{19}$$

Then we have the following form of adjointness:

$$c \otimes d \leq p \quad \textit{iff} \quad c \leq d \rightarrow_2 p \quad \textit{iff} \quad d \leq c \rightarrow_1 p. \tag{20}$$

(2) If $\langle L_1, L_2, L, \otimes, \rightarrow_1, \rightarrow_2, \ldots \rangle$ satisfies (20), then $\langle L_1, L_2, L, \otimes, \leq, \ldots \rangle$ is a Krajči's structure.

Proof. Omitted due to lack of space.

Then, arrow operators have the form similar to (7) and (8).

Theorem 6 (arrows via residuum for generalized concept lattices). *We have*

$$A^{\nearrow}(y) = \bigwedge_{x \in X}(A(x) \rightarrow_1 I(x, y)), \tag{21}$$

$$B^{\swarrow}(x) = \bigwedge_{y \in Y}(B(y) \rightarrow_2 I(x, y)). \tag{22}$$

Proof. Omitted due to lack of space.

Due to Theorem 5, a structure $\langle L_1, L_2, L, \otimes, \rightarrow_1, \rightarrow_2, \ldots \rangle$ satisfying (20) can be called a three-sorted complete residuated lattice. Theorems 5 and 6 show that Krajči's approach can be seen as another approach (a very general one) in the framework of residuated structures of truth degrees.

A main theorem for a generalized concept lattice is presented in [33]. In particular, each generalized concept lattice is a complete lattice. Note furthermore that in a recent paper [35], Krajči shows that each concept lattice with hedges is isomorphic to some generalized concept lattice. The question of whether each generalized concept lattice is isomorphic to some concept lattice with hedges seems to be an open problem.

4 Concluding remarks

- As mentioned above, we did not discuss the relationship of the approaches presented here with so-called conceptual scaling of fuzzy contexts considered as many-valued contexts. This issue will be discussed in a forthcoming paper.
- Proofs of the results presented here, more detailed comments, further results, and examples will be presented in a full version of this paper. In particular, we did not discuss concept lattices and related structures over structures of truth degrees equipped with non-commutative conjunction. This has been developed in [25, 26]. A discussion of relationships to this approach will also be included in the full paper.

References

1. Arnauld A., Nicole P.: *La logique ou l'art de penser.* 1662. Also in German: *Die Logik oder die Kunst des Denkens.* Darmstadt, 1972.

2. Barbut M., Monjardet B.: *L'ordre et la classification, algèbre et combinatoire, tome II.* Paris, Hachette, 1970.
3. Bělohlávek R.: Lattices generated by binary fuzzy relations (extended abstract). In: Abstracts of the Fourth International Conference on Fuzzy Sets Theory and Its Applications. Liptovsky Jan, Slovakia, 1998, p.11.
4. Bělohlávek R.: Fuzzy concepts and conceptual structures: induced similarities. Joint Conf. Inf. Sci.'98 Proceedings, Vol. I, pp. 179-182, Durham, USA, 1998. [Assoc. Intel. Machinery, ISBN 0-9643456-7-6]
5. Bělohlávek R.: Fuzzy Galois connections. *Math. Log. Quart.* **45**(4)(1999), 497–504.
6. Bělohlávek R.: Logical precision in concept lattices. *J. Logic and Computation* Vol. **12** No. 6(2002), 137–148.
7. Bělohlávek R.: Similarity relations in concept lattices. *J. Logic Comput.* 10(6):823–845, 2000.
8. Bělohlávek R.: Lattices of fixed points of fuzzy Galois connections. *Math. Logic Quarterly* **47**,1(2001), 111–116.
9. Bělohlávek R.: Reduction and a simple proof of characterization of fuzzy concept lattices. *Fundamenta Informaticae* **46**(4)(2001), 277–285.
10. Bělohlávek R.: Fuzzy closure operators. *J. Math. Anal. Appl.* **262**(2001), 473–489.
11. Bělohlávek R.: Fuzzy closure operators II. *Soft Computing* **7**(2002) 1, 53–64.
12. Bělohlávek R.: *Fuzzy Relational Systems: Foundations and Principles.* Kluwer Academic/Plenum Publishers, New York, 2002.
13. Bělohlávek R.: Algorithms for fuzzy concept lattices. Proc. Fourth Int. Conf. on Recent Advances in Soft Computing. Nottingham, United Kingdom, 12-13 December, 2002, pp.67-68 (extended abstract); pp. 200-205 (full paper on the included CD). [ISBN 1-84233-0764]
14. Bělohlávek R.: Concept lattices and order in fuzzy logic. *Ann. Pure Appl. Logic* **128**(2004), 277–298.
15. Bělohlávek R., Sklenář V., Zacpal J.: Crisply generated fuzzy concepts. In: B. Ganter and R. Godin (Eds.): ICFCA 2005, *Lecture Notes in Computer Science* **3403**, pp. 268–283, Springer-Verlag, Berlin/Heidelberg, 2005.
16. Bělohlávek R., Chlupová M., Vychodil V.: Implications from data with fuzzy attributes. AISTA 2004 in Cooperation with the IEEE Computer Society Proceedings, 2004, 5 pages, ISBN 2–9599776–8–8.
17. Bělohlávek R., Vychodil V.: Implications from data with fuzzy attributes vs. scaled binary attributes. In: FUZZ-IEEE 2005, The IEEE International Conference on Fuzzy Systems, May 22–25, 2005, Reno (Nevada, USA), pp. 1050–1055 (proceedings on CD), abstract in printed proceedings, p. 53, ISBN 0–7803–9158–6.
18. Bělohlávek R., Vychodil V.: Reducing the size of fuzzy concept lattices by hedges. In: FUZZ-IEEE 2005, The IEEE International Conference on Fuzzy Systems, May 22–25, 2005, Reno (Nevada, USA), pp. 663–668 (proceedings on CD), abstract in printed proceedings, p. 44, ISBN 0–7803–9158–6.
19. Bělohlávek R., Vychodil V.: Fuzzy attribute logic: attribute implications, their validity, entailment, and non-redundant basis. In: Yingming Liu, Guoqing Chen, Mingsheng Ying (Eds.): *Fuzzy Logic, Soft Computing & Computational Intelligence: Eleventh International Fuzzy Systems Association World Congress* (Vol. I), 2005, pp. 622–627. Tsinghua University Press and Springer, ISBN 7–302–11377–7.
20. Bělohlávek R., Vychodil V.: Fuzzy attribute logic: syntactic entailment and completeness. *Proceedings of the 8th Joint Conference on Information Sciences*, 2005, pp. 78–81, ISBN 0–9707890–3–3.
21. Burusco A., Fuentes-Gonzáles R.: The study of the L-fuzzy concept lattice. *Mathware & Soft Computing*, 3:209–218, 1994.

22. Burusco A., Fuentes-Gonzáles R.: Concept lattice defined from implication operators. *Fuzzy Sets and Systems* (2000).
23. Burusco A., Fuentes-Gonzáles R.: Contexts with multiple weighted values. *Int. J. Uncertainty, Fuzziness and Knowledge-Based Systems* (2001).
24. Ganter B., Wille R.: *Formal Concept Analysis. Mathematical Foundations.* Springer-Verlag, Berlin, 1999.
25. Georgescu G., Popescu A.: Concept lattices and similarity in non-commutative fuzzy logic. *Fundamenta Informaticae* **53**(1)(2002), 23-54.
26. Georgescu G., Popescu A.: Non-commutative Galois connections. *Soft Computing* **7**(2003), 458-467.
27. Goguen J. A.: The logic of inexact concepts. *Synthese* **18**(1968-9), 325–373.
28. Hájek P.: *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.
29. Hájek P.: On very true. *Fuzzy sets and systems* **124**(2001), 329–333.
30. Höhle U.: On the fundamentals of fuzzy set theory. *J. Math. Anal. Appl.* **201**(1996), 786–826.
31. Klir G. J., Yuan B.: *Fuzzy Sets and Fuzzy Logic. Theory and Applications.* Prentice Hall, Upper Saddle River, NJ, 1995.
32. Krajči S.: Cluster based efficient generation of fuzzy concepts. *Neural Network World* **5**(2003), 521–530.
33. Krajči S.: The basic theorem on generalized concept lattice. In Bělohlávek R., Snášel V.: *CLA 2004, Proc. of 2nd Int. Workshop*, Ostrava, 2004, pp. 25–33.
34. Krajči S.: A generalized concept lattice. *Logic J. of IGPL* (to appear).
35. Krajči S.: Every concept lattice with hedges is isomorphic to some generalized concept lattice. Submitted to *CLA 2005, Proc. of 3nd Int. Workshop.*
36. Pollandt S.: *Fuzzy Begriffe.* Springer-Verlag, Berlin/Heidelberg, 1997.
37. Takeuti G., Titani S.: Globalization of intuitionistic set theory. *Annal of Pure and Applied Logic* **33**(1987), 195–211.
38. Ďuráková D., Krajči S., Snášel V., Vojtáš P.: Conceptual structures. Arbeitstagung Allgemeine Algebra, 2002.
39. Wille R.: Restructuring lattice theory: an approach based on hierarchies of concepts. In: Rival I.: *Ordered Sets.* Reidel, Dordrecht, Boston, 1982, 445–470.
40. Yahia S., Jaoua A.: Discovering knowledge from fuzzy concept lattice. In: Kandel A., Last M., Bunke H.: *Data Mining and Computational Intelligence*, pp. 167–190, Physica-Verlag, 2001.