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Concepts and fuzzy sets: Misunderstandings, misconceptions, and oversights

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ABSTRACT

The psychology of concepts has been undergoing significant changes since the early 1970s, when the classical view of concepts was seriously challenged by convincing experimental evidence that conceptual categories never have sharp boundaries. Some researchers recognized already in the early 1970s that fuzzy set theory and fuzzy logic were potentially suitable for modeling of concepts and obtained encouraging results. This positive attitude abruptly changed in the early 1980s, and since that time fuzzy set theory and fuzzy logic have been portrayed as problematic and unsuitable for representing and dealing with concepts. Our aim in this paper is to identify some of the most notorious claims regarding fuzzy set theory and fuzzy logic that have propagated through the literature on psychology of concepts and to show that they are, by and large, false. We trace the origin and propagation of these claims within the literature in this area. It is shown in detail that these claims are consistently erroneous and that they are based on various misunderstandings, misconceptions, and oversights. The ultimate purpose of this paper is to document these various erroneous claims.

1. Introduction

The paper is organized as follows. First, we show in Section 2 that it is generally recognized in the literature on the psychology of concepts that classes of things (referred to as categories), which are mentally represented as concepts, have very rarely (if ever) sharp boundaries, with no borderline cases. We also show that the principal motivation for generalizing classical set theory to fuzzy set theory was to introduce the capability for dealing with categories that lack sharp boundaries in a rigorous, mathematical way. In Section 3, we present some relevant general comments regarding fuzzy set theory and fuzzy logic. In Section 4, which is the main part of the paper, we address in detail the various fallacious claims regarding fuzzy set theory and fuzzy logic, which have been quite commonly made in the literature on the psychology of concepts. We show that these claims are not sustainable and that they are based on various misunderstandings, misconceptions, or oversights. We attempt to trace the origin of these claims and their propagation within the literature on the psychology of concepts and clarify why they are erroneous. Finally, we make a few general conclusions and suggestions in Section 5.

2. The psychology of concepts and fuzzy set theory

2.1. Major views of concepts: an overview

In early views of concepts, philosophers, linguists, and psychologists had generally assumed that most concepts were basically well understood and possessed clear, sharp boundaries. All concepts were seen as possessing necessary and

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sufficient conditions for their application, and we could easily use these conditions to test any object to see if it matched the essence of any particular concept. An object would be considered either completely inside or outside the concept depending on whether the object possessed all the required properties for that particular concept – there could be no middle ground.

This view of concepts has come to be known as the *classical view* of concepts, also known as the definitional view. Here meanings of words and their associated concepts can be completely defined and are contained by clear boundaries. A commitment to some version of the classical view can be seen in much of the Western tradition in philosophy from the ancient Greeks through to the present day. However, the classical view of concepts has also generated many serious criticisms from philosophy, linguistics, and psychology. These criticisms range from our continuing inability to specify sets of defining features for most concepts to the fact that there are numerous cases where it simply is not clear whether an instance belongs to one category or another, no matter how much information we have about the instance (for example, is a rug considered furniture?). However, it was not until the groundbreaking work of Eleanor Rosch in the 1970s [35–37] that psychology came to fully reject the classical view.

Rosch played an important role in dramatically changing the view of concepts. Her work not only provided much of the important empirical evidence that demonstrated the inadequacies of the classical view of concepts, but she also created the groundwork for a movement of an alternative view known as the *prototype view* of concepts. Rosch, influenced by Wittgenstein, Zadeh and Lakoff, whom she cites in her work, established experimental cognitive research programs to demonstrate centrality, family resemblance, and basic-level categorization.

Although there are several versions of the prototype view that have developed since Rosch's first papers, all of them are based on the idea that our concepts are graded. More recently, Rosch has argued that the phenomenon of judging graded structures is universal and that all people for all conceptual categories judge members of the same category to have different degrees of membership [38].

Another related view of concepts is the *exemplar view*, where it is hypothesized that concepts have no summary representation but rather are represented by all of the examples a particular person has encountered of that concept. A new instance is judged by its similarity to past remembered examples. Again, there are several versions of the exemplar view of concepts, but all are compatible with the claim that individual concepts have ill-defined boundaries.

A third major view of conceptual structures is the *knowledge view*, which is also known as the *theory view of concepts*. The proponents of the knowledge view argue that similarity is not powerful enough to account for conceptual coherence. What guides our categorization is knowledge or theories about the world. Again, this view is compatible with ill-defined boundaries of concepts.

There are problems with each of the three major views of concepts and it goes beyond the scope and purpose of this paper to address these problems. Murphy [29] in his comprehensive book on concepts gives an extensive and detailed overview of these views, their problems, and the empirical results that need to be accounted for in any view of concepts. However, in his conclusion, Murphy states on page 488: "If you've been keeping score, there is no clear dominate winner."

In this paper, we attempt neither to support one view over another nor to solve the problem of developing an integrated theory of concepts. Our only aim is to expose errors, mathematical as well as conceptual, regarding fuzzy set theory and fuzzy logic within the literature in the area of the psychology of concepts.

2.2. Fuzzy sets in the psychology of concepts

In the 1970s, various authors suggested using fuzzy sets for representing and dealing with concepts. Recognizing Rosch for her experimental demonstration that concept categories involve borderline cases, Lakoff [23] argues that this is also the case for statements in natural language:

> Logicians have, by and large, engaged in the convenient fiction that sentences of natural languages (at least declarative sentences) are either true or false or, at worst, lack a truth value, or have a third value often interpreted as 'nonsense'. ....Yet students of language, especially psychologists and linguistic philosophers, have long been attuned to the fact that natural language concepts have vague boundaries and fuzzy edges and that, consequently, natural language sentences will very often be neither true, nor false, nor nonsensical, but rather *true to a certain extent and false to a certain extent, true in certain respects and false in other respects* (italics added).

Lakoff further argues that fuzzy set theory as suggested by Zadeh [41] is capable of dealing with degrees of membership and, hence with categories that do not have sharp boundaries:

> Fuzzy concepts have had a bad press among logicians, especially in this century when the formal analysis of axiomatic and semantic systems reached a high degree of sophistication. It has been generally assumed that such concepts were not amenable to serious formal study. I believe that the development of fuzzy set theory ... makes such serious study possible.

The role of fuzzy set theory in dealing formally with vagueness in natural language was clearly recognized by Hersh and Caramazza [16]:

> Recently, there has been considerable interest on the part of linguists in such problems as the role of vagueness in language and the quantification of meaning. Much of this interest has been the result of the development of fuzzy set theory, a *generalization of the traditional theory of sets* (italics added).

Another author who recognized that there is a fruitful connection between fuzzy sets and categories is Gregg C. Oden. He addressed this connection in two papers, both published in 1977. In his first paper [30], he builds on previous psychological studies, which "have shown that many subjective categories are fuzzy sets", by studying how proper rules (or operations) of conjunction and disjunction of statements that are true to some degree can be determined experimentally within each given context. His overall conclusion is that "it is not unreasonable for different rules to be used under various situations". This, of course, is well-known in fuzzy set theory, where classes of conjunctions, disjunctions, and other types of operations on fuzzy propositions are well delimited and have been extensively researched [20].

In his second paper [31], the author addresses the issue of the capability of human beings to make consistent judgments regarding degrees of membership (or degrees of truth). His conclusion, based on experiments he performed, is positive:

> Recent research indicates that class membership may subjectively be a continuous type of relationship. The processing of information about the degree to which items belong to a particular class was investigated in an experiment in which subjects compared two statements describing class membership relationships. The results strongly supported a simple model which describes the judgment process as directly involving subjective degree-of-truthfulness values. The success of the model indicates that the subjects were able to process this kind of fuzzy information in a consistent and systematic manner.

Needless to say, experiments of this kind are considerably more attuned to the spirit of fuzzy set theory than the more traditional experiments. Their advantage is well described by Oden: "The subject is not required to make a choice between two extremes, neither of which may be considered by him to be very appropriate, but rather is explicitly allowed to respond in a continuous manner". These two types of experiments were compared by an experimental study performed by McCloskey and Glucksberg [27]. The study clearly demonstrates that results for experiments that allow subjects to judge category memberships in terms of degrees showed a significantly higher consistency, especially for intermediate-typicality items, than those in which membership degrees were not allowed.

It is possible that the increasing awareness of cognitive psychologists in the 1970s that categories associated with most concepts are inherently imprecise was stimulated, at least to some extent, by the emergence of the idea of fuzzy sets in the 1960s. In fact, the primary motivation by Lotfi Zadeh for proposing fuzzy sets in his seminal paper [41] was to provide a mathematical representation of classes of objects that do not have precisely defined criteria of membership.

The idea of a set whose membership is a matter of degree is of course an extremely radical idea in mathematics. It requires a new way of thinking – thinking in terms of degrees rather than absolutes. When accepted, it affects virtually all areas of mathematics. Since classical sets are special cases of fuzzy sets, mathematics based on fuzzy sets is clearly a wholesale generalization of classical mathematics. This generalization is essential for formalizing and dealing with 'inexact concepts' of our everyday life.

It is worth noting that some cognitive psychologists recognized early that fuzzy sets are more realistic for representing and dealing with concepts than classical sets. Some of them even started to examine fuzzy sets and explore their utility in the psychology of concepts, as is exemplified by a few publications described earlier in this section. This all happened more or less in the 1970s. Unfortunately, this positive attitude toward fuzzy sets drastically changed in the early 1980s, and fuzzy set theory started to be portrayed as unsuitable for representing and dealing with concepts. It was virtually abandoned in the psychology of concepts as a viable alternative to classical set theory, contrary to numerous other areas, where its expressive power was already recognized and utilized in quite profound ways. What happened in the psychology of concepts in the early 1980s? Why had this positive attitude toward fuzzy set theory changed abruptly to a negative attitude?

Although there do not seem to be any hard and fast answers to these questions, we intend to show that this abrupt change in attitudes toward fuzzy set theory was, by and large, a result of a number of negative claims regarding fuzzy set theory that were advanced in respectable publications by some highly influential cognitive psychologists. We also intend to show that, upon closer scrutiny, none of these negative claims are justified and that they basically exemplify several types of fundamental fallacies regarding fuzzy set theory and fuzzy logic.

3. Fuzzy set theory and fuzzy logic: general remarks

One of the requirements of classical set theory is that, for any given object and any given set, the object either is or is not a member of the set. That is, each classical set is required to have a sharp boundary, which precisely delimits which objects are its members. In fuzzy set theory, this requirement is abandoned. This means that fuzzy sets are not required to have sharp boundaries and the membership of each object in a fuzzy set is a matter of degree.

In each particular application of classical or fuzzy set theory, all objects that are relevant to the application are members of a classical set that is called a *universal set*. The application deals with the various subsets of the universal sets, which may be classical sets or fuzzy sets.

Let X denote a universal set relevant to a given application. Then each classical subset of X is conveniently defined by a *characteristic function*

$$A : X \to \{0, 1\}$$

such that for any object x in the universal set X, $A(x) = 1$ means that x is a member of set A, while $A(x) = 0$ means that x is not a member of set A. Clearly, the two numbers, 0 and 1, are used here solely as two symbols (convenient, for example, for computer processing) by which elements of the universal set X are sorted into those that belong to set A and those that do not belong to it. That is, the role of 0 and 1 in characteristic functions is strictly symbolic, not numeric. Any other pair of symbols (e.g., *no* and *yes*) could be used as well.

Since fuzzy sets allow degrees of membership, each fuzzy set relevant to a given application (a fuzzy subset of X) is defined, in general, by a function

$$A : X \to D,$$

where D denotes a set of recognized degrees of membership and $A(x) = d$ means for each x in X and a given value d in D that the element x belongs to the set A with the membership degree d. This function is usually called a *membership function*. Several distinct types of membership functions and, hence, types of fuzzy sets have been recognized and investigated, each characterized by a particular type of set D. This set may be finite or infinite, numeric or non-numeric, etc., but it must be partially ordered and conform to the mathematical structure of a lattice. In the most common type of fuzzy sets, which are usually called *standard fuzzy sets*, D is the closed interval of real numbers between 0 and 1. That is, membership functions of standard fuzzy sets have the form

$$A : X \to [0, 1].$$

Although there is a growing interest in various nonstandard fuzzy sets, we restrict in this paper to standard fuzzy sets since these have been the only fuzzy sets referred to in the literature dealing with concepts.

Contrary to the symbolic role of numbers 0 and 1 in characteristic functions of classical sets, numbers assigned to objects in X by membership functions of standard fuzzy sets have a numerical significance. This significance is preserved when classical sets are viewed (from the standpoint of fuzzy set theory) as special fuzzy sets, usually referred to as *crisp sets*. There are two important consequences of this change from symbolic to numerical membership: (i) fuzzy set theory is a generalization of classical set theory; and (ii) fuzzy sets can be manipulated in various numerical ways, which have no counterparts in classical set theory.

We assume that readers of this journal are familiar with basic ideas of standard fuzzy set theory, in particular with the various classes of operations on fuzzy sets (complements, modifiers, intersections, unions, and averaging operations). Our terminology and notation are adopted, by and large, from [20].

Fuzzy set theory needs to be distinguished from *fuzzy logic*. In addition, it has repeatedly been argued by Zadeh [21, e.g. pp. 776, 784, 797–798, 811] that it is important to distinguish two distinct meanings of the term "fuzzy logic": *fuzzy logic in a broad sense* and *fuzzy logic in a narrow sense*.

In the broad sense, fuzzy logic refers to a system of concepts, principles, and methods for dealing with problems that involve classes with unsharp boundaries. In the narrow sense, fuzzy logic refers to logical calculi for reasoning that involves degrees of truth rather than just truth and falsity. It is concerned with the development of syntactic aspects (based on the notion of proof) and semantic aspects (based on the notion of truth) of these calculi. In order to be acceptable, each of the calculi is expected to be sound (provability implies truth) and complete (truth implies provability). These issues have already been successfully addressed and are well covered in the literature, including several monographs, for example those by Hájek [14], Gottwald [13], and Gerla [10]. Moreover, and most importantly, several interesting phenomena, which are degenerate in classical logic, have been studied in fuzzy logic, such as reasoning from partially true premises, degrees of entailment, reasoning about similarity, truth qualification, etc. Results in this area provide us with solid foundations for dealing with various modes of reasoning that are approximate rather than exact.

A relationship between fuzzy logic and fuzzy set theory is analogous to the relationship of classical logic and classical set theory. Namely, truth functions of logical connectives of fuzzy logic are used in fuzzy set theory to define basic operations with fuzzy sets. For instance, t-norms which are the functions used for defining intersections of fuzzy sets are used as truth functions of fuzzy logic conjunctions in fuzzy logic. As a consequence, a degree to which an object x belongs to the intersection of fuzzy sets A and B can be viewed as the truth degree of a proposition "x belongs to A and x belongs to B". Just as there is no unique fuzzy set theory, there is no unique system of fuzzy logic. Rather, formal properties of systems of fuzzy logic depend on the properties of the intended truth functions of logical connectives (t-norms, negations, etc.). Both syntactic and semantic properties of fuzzy logic are nowadays well understood and further studied.

4. The supposed irrelevance of fuzzy set theory in the psychology of concepts: common but unsustainable claims

In this section, we address the various claims made in the literature on the psychology of concepts that fuzzy sets and fuzzy logic are not useful in representing and dealing with concepts. First, we show in Section 4.1 that these claims originated in a paper by Osherson and Smith [32] and that this paper has had an enormous influence on the whole field. Then, in Section 4.2, we analyze, on a point-by-point basis, claims made over time in the literature on the psychology of concepts that exhibit various misconceptions, flaws in argumentation, misunderstanding, and oversights regarding fuzzy set theory and fuzzy logic.

4.1. Osherson and Smith's [32] paper and its influence

In [32], Osherson and Smith discuss the use of fuzzy set theory in representing and combining concepts within the prototype view of concepts. Although the overall conclusion of the paper is based on several errors and misunderstandings, as is shown in detail in [3], it has been highly influential within the literature on the psychology of concepts. In the following, we demonstrate by several quotations from publications of highly respected scholars how the conclusion presented in Osherson and Smith [32] was uncritically accepted and propagated. Armstrong et al. [1] write (p. 270):

> Osherson and Smith [32], have described this kind of prototype model formally, and distinguished it from the featural interpretations of prototype theory. For both types of models, these authors demonstrate that prototype theory, amalgamated with combinatorial principles from fuzzy-set theory [41], cannot account for our intuitions about conceptual combination.

In a review of work on human concepts and concept formation in the mid 1980s, Medin and Smith [28] write (p. 133):

> More recent studies provide extensive arguments against the use of fuzzy-set theory as an account of complex concepts (see [18,32,33,39]).

Kamp and Partee [19, p. 130] write:

> Osherson and Smith appeal to fuzzy-set theory [41] as the most likely means for extending prototype theory to deal with conceptual combination. O&S then succeed in clearly showing that fuzzy-set theory cannot support a compositional semantics whose input consists of prototype concepts.

Hampton [15, p. 139] writes:

> Although fuzzy logic had some success in accounting for intuitions about the conjunction of unrelated statements [30,31], it soon became clear, following a key article by Osherson and Smith [32], that not only the minimum rule, but in fact any rule that takes as input solely the truth value of the two constituent statements is doomed to failure.

In a book edited by Margolis and Laurence [26], which consists of 652 pages of selected important writings on concepts, starting with Plato and going to modern treatments of concepts, the editors wrote in their introductory chapter (p. 38):

> In a seminal discussion of the Prototype Theory's reliance on fuzzy sets, Daniel Osherson and Edward Smith presented a number of forceful objections to this treatment of compositionality [32].

Let us now mention two papers that argue with some of the conclusions of Osherson and Smith [32]. The first one is a paper by Zadeh [42], which was published in *Cognition*, as a direct response to the paper by Osherson and Smith. This paper addresses in general terms most of the arguments made by Osherson and Smith. However, it is a rather short paper, in which only two pages are devoted to these arguments. Moreover, their refutation is not presented in sufficiently specific terms. The rest of the paper deals with Zadeh's own proposal for a model of prototypes. This, perhaps, is why his arguments were misunderstood or misinterpreted. This can be seen from Osherson and Smith [33], a reply to Zadeh's paper, on which we comment in Section 4.2. The second paper is by Fuhrmann [9]. Both [42,9] have virtually been ignored within the community of cognitive psychology and did not change the attitude towards fuzzy logic. A likely reason for this is that the responses were not specific enough and addressed only particular flaws in [32]. To provide a comprehensive analysis of various misconceptions, flaws in argumentation, misunderstanding, and oversights regarding fuzzy set theory and fuzzy logic in the literature on the psychology of concepts is the aim of the next section.

4.2. Examination of fallacies regarding fuzzy sets and fuzzy logic in the literature on the psychology of concepts

4.2.1. Ill-conceived assumption that fuzzy set theory is a theory of concepts

Several erroneous and misleading claims made in the literature on the psychology of concepts stem from confusing fuzzy set theory with a theory of concepts/categories and concept manipulation. Fuzzy set theory is a mathematical theory. As any mathematical theory, fuzzy set theory has no fixed interpretation. Rather, it is open to many interpretations. That is, fuzzy set theory provides us with a calculus, in a similar manner to what, for example, the theory of differential equations or probability theory do. Mathematical theories can be used to build models for various application areas. Differential equations can be used to build models of physical processes, employed, for example, for weather forecasting. Likewise, probability theory can be used to build models of economic processes, employed, for example, for risk assessment. If a particular model does not perform well in experiments, then it is the model that has to be rejected and reconsidered, not the mathematical theory in terms of which the model is described.

The following quotations show that, surprisingly, the above simple facts are not recognized, by and large, in the literature on the psychology of concepts. That is, fuzzy set theory is often presented as a theory of conceptual combinations. For instance, Laurence and Margolis [25] write in their introductory chapter (p. 38):

> The standard model for composing graded categories was a version of fuzzy set theory—a modification of standard set theory that builds on the notion of graded membership (see esp. [41]).

Fuzzy set theory is not a model for composing categories. As explained above, it is a calculus that can be used for formalizing our intuitions about compositions of graded categories and other issues.

Cohen and Murphy [5], in explaining the concept of a fuzzy set, write (p. 30–31):

> For each object x and concept C in question, Zadeh's system assigns a value in [0,1], called a characteristic value, or a degree of typicality... Zadeh's rules (including recent developments, 1978, 1981) seem to be the only systematic attempt to describe how prototype-based concepts should be combined and evaluated.

Then, when discussing extensional and other models of concepts on p. 33, they make the statement:

> A paradigm case of an extensional model is the fuzzy set model [41].

Later, on p. 35, they make the same statement even more explicitly:

> The standard extensional model of fuzzy concept theory is the original fuzzy set model proposed by Zadeh [41] and formalized by Osherson and Smith [32].

Here, referring to "fuzzy set model" [41] as an extensional model of concepts is confusing. Zadeh did not propose any model of concepts. He proposed a mathematical theory – a theory of fuzzy sets. Neither did he propose rules "how prototype-based concepts should be combined and evaluated". He only proposed basic operations with fuzzy sets, such as union, intersection, etc., as possible generalizations of the corresponding operations of classical set theory.

Osherson and Smith [33], when speaking of conjunctive concepts, introduce a so-called *Simple Functional Hypothesis* (SFH, p. 303–304):

> The gradient theorist may reasonably expect that $c_{A\&B}$ is related to c_A and c_B in the simple functional manner described in the (SFH):
>
> (SFH) There is some function, f, such that for all conjunctive concepts,
>
> $$A\&B : (\forall x \in D) c_{A\&B}(x) = f(c_A(x), c_B(x)).$$
>
> Specification of the function, f, of (SFH) would solve the problem of conjunctive concepts within the gradient framework. Zadeh [41] proposed that the function in (SFH) is *min* (the minimum); Goguen [11] suggested it is the product. Both of the hypotheses are open to counterexamples discussed in Osherson and Smith [32, Section 2.4.1].

Neither Zadeh [41] nor Goguen [11] proposed any function for the Simple Functional Hypothesis about how conjunctive concepts are combined. Zadeh and Goguen proposed *min* and *product* as possible functions for the definition of an intersection of fuzzy sets. It is Osherson and Smith who came up with (SFH), a hypothesis about conceptual combination, and suggested to test whether Zadeh's *min* (or Gogen's *product*) satisfies (SFH).

The following quotation is taken from the abstract of Roth and Mervis [39, p. 509]. It summarizes the main conclusion of the paper:

> It is argued that fuzzy set theory, in principle, cannot specify the relationships among representativeness values in different categories, because the theory does not take the attributes of exemplars directly into account.

By the same logic, one could argue that probability theory and statistics, in principle, cannot specify the relationships among personal income and spending, because the theory does not take the inflation rate into account. Needless to say, such an argument is absurd. It is the economist, not the theory of probability and statistics, who proposes that a model of income and spending needs to take inflation rate into account. In the same spirit, it is the psychologist using fuzzy set theory, and not fuzzy set theory itself, who proposes that a model of representativeness values in categories needs to take the attributes of exemplars directly into account.

Similar misconception is presented in the following two quotations. The first one is from Roth and Mervis [39, p. 511]:

> The first experiment was designed to examine in detail the representativeness of exemplars in categories within the animal taxonomy to determine if fuzzy set theory makes accurate predictions for that domain.

The second one is from Medin and Smith [28, p. 133]:

> Zadeh [41] fuzzy-set theory claims that typicality in the composite is the minimum of the typicalities in the constituents, which means that something cannot be a better example of *pet fish* than it is of *pet* or *fish*.

To examine this misconception is particularly important. First, the misconception confuses fuzzy set theory with a theory of conceptual combination (and perhaps with theories of other topics in the psychology of concepts), which is certainly not correct. Second, and more importantly, the misconception leads to the conclusion that fuzzy set theory is not appropriate for modeling several issues in the psychology of concepts. Let us demonstrate this by the last quotation from [28]. If the quotation is right, fuzzy set theory is not appropriate for modeling typicalities in composite concepts and is to be rejected for this purpose. However, fuzzy set theory does not claim (and, in principle, cannot claim) anything about typicality in a composite concept since it is not a theory of typicality. A correct statement here is the following: A model M in which typicalities are modeled by membership functions of fuzzy sets, and in which typicalities of a composite are modeled by a membership function of intersection of fuzzy sets corresponding to the constituents, claims that typicality in the composite is the minimum of the typicalities in the constituents. It was argued by several psychologists (using for example the concept of *pet fish*) that this claim is wrong. This implies that what has to be rejected is model M, not fuzzy set theory itself. In other words, the problem is in the model, not in the mathematical theory used.

4.2.2. Claims concerning inclusion of fuzzy sets and universal quantification

One of the notions of fuzzy set theory that seems to be grossly misunderstood in the literature on the psychology of concepts is the notion of inclusion in fuzzy sets. Recall that for classical sets A and B we say that A is a subset of B (A is included in B), written $A \subseteq B$, if each object that belongs to A belongs also to B. If A and B are fuzzy sets, then an appropriate notion that generalizes the classical subsethood relation is the notion of a degree of subsethood (degree of inclusion). A degree of inclusion of a fuzzy set A in a fuzzy set B is a truth degree, $S(A,B)$, which can be understood as a degree to which it is true that each object from A belongs to B. One of the first proposals to define $S(A,B)$, which appeared in [11], was to use the formula

$$S(A,B) = \inf_{x \in X}(A(x) \Rightarrow B(x)), \tag{1}$$

where X is the universal set, $\Rightarrow$ is a fuzzy implication (i.e. a binary operation on $[0,1]$ that corresponds to the implication connective), and inf denotes infimum in $[0,1]$ (infimum coincides with minimum if X is finite). This formula for S has a clear meaning. By the basic principles of semantics of predicate fuzzy logic, $S(A,B)$ is the truth degree of the proposition:

For each object x from the universe X: if x belongs to A then x belongs to B.

A natural requirement for S, which is satisfied by the above definition, is that $S(A,B) = 1$ if and only if for each x from X, the degree $B(x)$ (to which x belongs to B) is greater than or equal to the degree $A(x)$ (to which x belongs to A). In this way, S induces a bivalent relation S_{biv} on fuzzy sets in X such that A and B are in relation S_{biv} if and only if $S(A,B) = 1$. That is,

$S_{biv}(A,B) = 1$ if for each x in X we have $A(x) \leqslant B(x)$, and $S_{biv}(A,B) = 0$ otherwise.

Clearly, S_{biv} is a limiting case of S. In fuzzy set terminology, $S_{biv} = 1$ is the 1-cut of S (or the core of S), and $S_{biv}(A,B)$ can be read as "A is fully included in B".

The misunderstanding of inclusion of fuzzy sets is best illustrated by the following example presented in [32], to which several authors refer. To describe this example, we keep Osherson and Smith's original notation. The example concerns so-called "truth conditions of thoughts" by which Osherson and Smith basically mean the issue of determining the truth degree of a proposition of the form

$$\text{All } A\text{'s} \quad \text{are } B\text{'s}. \tag{2}$$

The authors claim that propositions of this form are assigned the truth condition

$$(\forall x \in D)(c_A(x) \leqslant c_B(x)), \tag{3}$$

where D is a universe of discourse and c_A and c_B are membership functions of fuzzy sets representing the terms A and B in (2), respectively. Then, they present the following "counterintuitive result": let D denote the universe of all animals and let A and B denote, respectively, the concepts *grizzly bear* and *inhabitant of North America*. That is, (2) becomes "All grizzly bears are inhabitants of North America". Then, if there is a squirrel (call it Sam) who lives on Mars and if $c_A(Sam) = a > 0$ and $c_B(Sam) < a$, it follows that $c_A\ (Sam) \leqslant c_B\ (Sam)$ is not the case, and thus the existence of a squirrel on Mars makes the truth value of "All grizzly bears are inhabitants of North America" equal to 0 (false). The authors conclude that "...fuzzy set theory does not render prototype theory compatible with the truth conditions of inclusion".

The mistake made here is the failure to recognize that (3) is not the definition of the truth degree of (2). Since "All A's are B's" means "for all x from D: if x belongs to A then x belongs to B", (2) needs to be interpreted as a degree $S(c_A, c_B)$ of inclusion of c_A in c_B given by (1). As an illustration, if the Lukasiewicz implication [14]

$$a \Rightarrow b = \min(1, 1 - a + b)$$

is used, then we get

$$S(c_A, c_B) = \inf_{x \in D} min(1, 1 - c_A(x) + c_B(x)).$$

Therefore, Sam's existence implies that the truth degree of "All grizzly bears are inhabitants of North America" is at most $\min[1, 1 - c_A(Sam) + c_B(Sam)]$. If, for instance, $c_A(Sam) = 0.1$ and $c_B(Sam) = 0.05$, then the truth degree of (2) is at most 0.95 (the existence of other animals that count to a certain degree for grizzly bears and live outside North America can make $S(c_A, c_B)$ still smaller). The meaning of (2), when interpreted correctly, therefore satisfies the basic intuitive requirements of Osherson and Smith.

Note that in his reply to the grizzly bears example by Osherson and Smith [32], Zadeh [42] suggested:

> An alternative way of resolving the difficulty is to fuzzify the concept of inclusion, as is done in the papers by Bandler and Kohout [2], Gottwald [12] and Wilmott [40].

As is shown above, this suggestion removes the difficulty caused by Osherson and Smith's misunderstanding of inclusion of fuzzy sets. Note also that the definition of $S(c_A, c_B)$, which we use, is covered in [11], a classical paper that is included in the reference list of [32]. Nevertheless, they state the following in their 1981 paper (their Eq. (3.2) is our Eq. (3)):

> To replace (3.2) with a "partial falsification" scheme so as to allow for truth values between *truth* and *falsity* requires non-trivial theory construction of a kind apparently not yet undertaken.

In [33], their reply to [42] is:

> Zadeh [42] states that an alternative approach to inclusion is to embrace fuzzy truth (*via* the fuzzification of inclusion itself). We have already expressed our misgivings about this approach [32, Section 3.5]; the papers cited by Zadeh [42] do not allay these anxieties.

There is another interesting issue regarding statement (2) and the grizzly bears example. This issue is briefly addressed by Zadeh [42]:

> ...in a natural language the quantifier *all* is usually not meant in its strict logical sense. Thus, in general, *all* in a proposition in a natural language should be interpreted as a fuzzy proportion which is close to unity.

The point here is that *all* is very often used with the meaning of *almost all* or *many*. This is particularly the case if one feels that the more squirrels on Mars, the smaller the truth degree of (2) should be, which is an issue discussed by Osherson and Smith [32]. In this case, one actually has in mind a proposition of the form "Almost all A's are B's" or "Many A's are B's". Quantifiers like "many", "almost all", etc. are traditionally studied in logic under the name "generalized quantifiers". A simple example to interpret the quantifier "many" is based on the notion of a cardinality of a fuzzy set. Let us illustrate it in the case when the universe of discourse D is finite. For a fuzzy set c_A, its cardinality $|c_A|$ is defined to be $\sum_{x\in D} c_A(x)$. Then the truth degree of "Many A's are B's" can be defined by $|c_A \cap c_B|/|c_A|$, where $\cap$ is the minimum-based intersection of fuzzy sets. It is easy to see that if $c_A(x) \leqslant c_B(x)$ for each x then $|c_A \cap c_B|/|c_A| = 1$. Moreover, $|c_A \cap c_B|/|c_A|$ decreases with growing numbers of x for which $c_A(x) > c_B(x)$ and the bigger the difference $c_A(x) - c_B(x)$, the more it decreases, meeting thus the basic intuitive requirements set for the proposition "Many A's are B's". Zadeh [42] suggested this approach, with references to the literature, as well as its modification by considering only those x in D for which c_A exceeds some threshold k. This proposal was again misunderstood by Osherson and Smith [33]. They formalized Zadeh's proposal by stating that (2) is true if $(\forall x \in D)(c_A(x) > k \Rightarrow c_A(x) \leqslant c_B(x))$. That is, they assigned again only 0 or 1 to (2), and then argued that this did not work either. Note that a good and up-to-date overview of generalized quantifiers can be found in [22]; in the context of fuzzy logic, generalized quantifiers are studied in [14, Chapter 8].

The confusion about fuzzy set inclusion and quantifiers is spread throughout in the literature on the psychology of concepts. For example, Cohen and Murphy [5, p. 42], in a section entitled "Objections to fuzzy models" say (in the quote below, $\mathrm{TYP}_A(x)$ denotes a typicality value):

> The fuzzy set rule for universal quantification is given by this test for "All A's are B's"
>
> $$(\forall x \in D)[Ax \Rightarrow Bx] \quad \text{iff} \quad (\forall x \in D)[\mathrm{TYP}_A(x) \leqslant \mathrm{TYP}_B(x)],$$
>
> where D is some domain.

Here, in addition to erroneously replacing a degree $S(A,B)$ of inclusion of fuzzy sets by $S_{biv}(A,B)$, the authors confuse readers as to what a rule of universal quantification in fuzzy logic says. In fact, the rule says that if F is a formula of predicate fuzzy logic then a truth degree $\|(\forall x)F\|$ of a formula $(\forall x)$ F is defined by $\|(\forall x)F\| = \inf_v \|F\|_v$, where v ranges over all valuations of variable x, and $\|F\|_v$ is a truth degree of F under valuation v.

4.2.3. Misunderstandings of logically empty and logically universal concepts

A widespread claim concerning inadequacies of fuzzy set theory is based on what was termed logically empty and logically universal concepts in [32]. The argument goes as follows. Consider a concept *red apple* and a fuzzy set R such that $R(x)$ is a degree to which x is a red apple. Then, using the standard operations of complement (negation), intersection (conjunction), and union (disjunction), a sentence "x is a red apple and x is not a red apple" has a truth degree $\min(R(x), 1 - R(x))$. For instance, for an apple x such that $R(x) = 0.7$, we get $\min(R(x), 1 - R(x)) = 0.3$. Therefore, the conjunctive concept of *object that is both a red apple and not a red apple* is not empty since it contains x in degree 0.3. A similar argument can be used to show that the disjunctive concept of *object that is a red apple or is not a red apple* is not universal since it contains x in degree 0.7. This argument leads several authors to reject fuzzy set theory. We include some examples. Osherson and Smith [32, p. 45]:

> We conclude that fuzzy set theory does not render prototype theory compatible with strong intuition pertaining to logically empty and logically universal concepts.

Johnson-Laird [17, p. 199]:

> ...the conjunction of the two is greater than zero. This consequence is absurd, because a self-contradiction surely merits a truth value of zero.

Kamp and Partee [19, p. 134]:

> According to the most familiar versions of fuzzy logic the degree to which *a* satisfies the conjunctive concept *apple which is not an apple* is then the minimum of the *c*-values values of the conjuncts, and thus greater than 0. Clearly this is not the right result.

Requiring that the concept *object which is both a red apple and not a red apple* be empty and that the concept *object which is a red apple or is not a red apple* be universal is equivalent to requiring the well-known *law of contradiction* and the *law of excluded middle*, which are two of the basic laws of classical logic. The authors of the above quotations require, in fact, that these laws hold in fuzzy logic and fuzzy set theory in order to correctly represent concepts. To require this, however, needs a justification that they do not provide. The requirement that both these laws be valid has already been disputed in numerous ways; see, e.g., [8, pp. 323, 324; 24, p. 141; 42, p. 588]. For example, Zadeh [42] says:

> The principle of the excluded middle is not accepted as a valid axiom in the theory of fuzzy sets because it does not apply to situations in which one deals with classes which do not have sharply defined boundaries.

Instead of presenting here the various arguments against the requirement, we would like to point out the fact that contrary to classical logic, the truth degree $\min(R(x), 1 - R(x))$ of "x is a red apple and x is not a red apple" provides us with non-trivial information about x. Namely, in classical logic, this truth degree is always 0 and hence, not interesting. In fuzzy logic, however, if we know that $\min(R(x), 1 - R(x))$ is, say, 0.4, we know that x is a borderline case since then either $R(x) = 0.4$ (x is red in degree 0.4) or $1 - R(x) = 0.4$ and thus $R(x) = 0.6$. If, on the other hand, $\min(R(x), 1 - R(x)) = 0$, we know that x is not a borderline case at all. That is, $\min(R(x), 1 - R(x))$ ranges between 0 and 0.5 and the closer it is to 0.5, the more x is a borderline case of *red apple*.

Our next point is the following. If, for some reason, we have to require that the law of contradiction and the law of the excluded middle hold, fuzzy set theory is capable of satisfying this requirement. Although the two laws do not hold under some combinations of operations (such as the standard operations of complement, intersection, and union), they hold under numerous other combinations that fuzzy set theory offers. For instance, if we take the so-called Lukasiewicz operations, which are defined for all $a, b \in [0, 1]$ by the formulas

$$c(a) = 1 - a,$$
$$i(a, b) = \max(0, a + b - 1), \text{ and}$$
$$u(a, b) = \min(1, a + b),$$

then the law of contradiction, as well as the law of excluded middle, are satisfied. It is important to realize that for each chosen combination of operations, some properties of the algebra of classical set theory are inevitably violated. This is a consequence of imprecise boundaries of fuzzy sets. However, different combinations violate different properties, and this is crucial for our argument. The standard operations, for example, violate only the law of contradiction and the law of excluded middle. Some other combinations preserve these laws, but violate some other properties, usually distributivity and idempotence. A procedure is well established [20, Section 3.5] by which classes of operations can be constructed that satisfy the two laws. Hence, it is clear that fuzzy set theory can represent cognitive situations in which the laws of contradiction and/or excluded middle should hold according to experimental evidence. That is, Osherson and Smith [32] are wrong in their conclusions.

4.2.4. Failure to recognize the expressive power of fuzzy set theory and fuzzy logic

Many erroneous and mistaken claims about inadequacies of fuzzy set theory were made because the full expressive power of fuzzy set theory was not recognized. By fuzzy set theory, the authors of these claims often mean just the notion of a fuzzy set and the operations of union, intersection, and complement based on max, min, and $1 - x$, as proposed in Zadeh's seminal paper [41]. As is well-known, fuzzy set theory has considerably more expressive power than its fragment based on max, min, and $1 - x$. The following two examples demonstrate in detail two of the many oversights regarding fuzzy set theory that can be found in the literature on the psychology of concepts.

Our first example is a well-known example from [32]. In Section 2.4.3 (pp. 46–48) of their paper, the authors discuss the concept *financial wealth* and its connection to concepts *liquidity* and *investment*. They consider three persons, *A*, *B*, and *C*, whose assets are given in the following table:

Person	Liquidity	Investment
A	\$105,000	\$5,000
B	\$100,000	\$100,000
C	\$5,000	\$105,000

They describe the concepts *liquidity*, *investment*, and *wealth*, respectively, by fuzzy sets *L*, *I*, and *W* (our notation), and argue correctly that the following inequalities should be satisfied on intuitive grounds:

$$L(A) > L(B)$$
$$I(C) > I(B)$$
$$W(B) > W(A)$$
$$W(B) > W(C)$$

The basic issue now is how $W(x)$ is determined for any $x \in X$ in terms of $L(x)$ and $I(x)$. That is, we want to find a function f such that

$$W(x) = f[L(x), I(x)]$$

is sensible, for any $x \in X$, on intuitive and/or experimental grounds. Osherson and Smith argue that

> if fuzzy-set theory is to represent the conceptual connection among liquidity, investment, and wealth, it would seem that the only option is to employ fuzzy union.

What they mean is one particular fuzzy union, the one expressed by the max operation. This argument is clearly unwarranted, since f can be chosen from an infinite set (a continuum) of aggregation functions. Moreover, it is easy to see that the right function in this example should be an averaging function. If there is no special experimental evidence to use some particular averaging function (possibly weighted), we may as well choose the arithmetic average so that

$$W(x) = \frac{L(x) + I(x)}{2}$$

for all $x \in X$. Clearly, this function satisfies the required inequalities. This example is particularly illuminating. It shows that fuzzy set theory has some capabilities that have no counterparts in classical set theory. Indeed, classical sets cannot be averaged!

Our second example concerns a concept of a stack introduced by Jones [18]. In order to avoid the inadequacy of fuzzy set theory to represent conjunctive concepts claimed by Osherson and Smith [32], Jones introduced the concept of a stack and operations with stacks and argued that:

> It was shown that systematization of the stack view allowed the construction of a form of prototype theory of concepts that has important advantages over one that is based on fuzzy-set theory. Unlike fuzzy-set theory, stack theory represents appropriately both conceptual conjunction and disjunction, and furthermore provides a coherent account of certain phenomena raised by a consideration of binary taxonomy.

Jones's main arguments are the operations with stacks he defines. Namely, Jones demonstrates that the operations meet the expectations about conjunctive concepts presented in [32]. In the following, we show that a stack is, in fact, a fuzzy set defined by a particular formula. Furthermore, we show that Jones's operation with stacks, which he introduced to meet the expectations about conjunctive concepts, does, in fact, coincide with a normalization of product-based intersection of fuzzy sets. Using normalization of an intersection of fuzzy sets to meet Osherson and Smith's [32] intuitive requirements for conjunctive concepts was proposed in [42]. Nevertheless, in a reply to Zadeh [42] and Jones [18], Osherson and Smith [33] say:

> Stack theory ingeniously avoids many of the problems encountered by the earlier hypotheses ...

According to Jones, a stack is given by assigning to any so-called instantiation x a value $C(x)$ from $(0,1]$, which he calls *characteristicness*, by

$$C(x) = I(R(d(x,p))). \tag{4}$$

Here, p denotes a prototype, $d(x,p)$ is a distance of x from p, R is a so-called ranking function, and I is an inverse transformation. The ranking function R maps distances into ranks. Ranks are natural numbers 1, 2,... and it is required that R maps distance to ranks in an order-preserving way, i.e. the larger the distance, the larger the rank assigned to it, and that the smallest distance is assigned rank 1. The inverse transformation I maps the ranks to values from $(0,1]$ in an order-reversing way, i.e. the smaller the rank, the larger the value from $(0,1]$ assigned to it. As an example of I, Jones proposes to take $I(n) = 1/n$ and says "However, this particular transformation appears unduly restrictive in practice since, for example, it excludes the existence of C-values between l/2 and 1". Jones also proposes an operation that should account for conjunctive combination of concepts. For two concepts, A and B, represented by characteristicness functions C_A and C_B, given by

$$C_A(x) = I(R(d(x,pA))) \quad \text{and} \quad C_B(x) = I(R(d(x,pB))),$$

the characteristicness function C corresponding to the conjunctive concept

$$C(x) = I(R(-\mathrm{prod}(C_A(x), C_B(x)))), \tag{5}$$

where prod denotes the usual product. There is a technical difficulty in Jones's model with the ranking function R. Namely, when discussing (4), Jones requires R to be order-preserving and to assign the last rank 1 to the smallest value of a distance function d. The smallest value s of d is a non-negative number (possibly s = 0, but Jones does not say it explicitly). In (5), however, R assigns ranks also to negative values, namely to values of the form $-\mathrm{prod}(C_A(x), C_B(x))$, and is required to do this in an order-preserving way to ensure that "it is high-characteristic instantiations that receive low-value ranks" (p. 283). This implies that for the smallest distance s of d, and for negative values, say

$-0.25 = -\mathrm{prod}(C_A(x1), C_B(x1))$ and $-0.9 = -\mathrm{prod}(C_A(x2), C_B(x2))$ one should have $R(-0.9) < R(-0.25) < R(s)$ which is impossible since $R(s) = 1$ is the smallest rank-value. We consider this only a technical difficulty of Jones's model. In fact, it can be removed for instance by using a ranking function R in (5) which is different from R in (4).

Let us now analyze Jones's proposals. Clearly, (4) is only one particular way of defining a membership function of a fuzzy set C. Namely, (4) defines C as a function assigning a value from $(0,1]$ to any element x (instantiation). In fact, Jones's formula defines what are known as triangular-like fuzzy sets in fuzzy set theory. Namely, a membership degree of an element x in a fuzzy set C is inversely related to the distance of x from a fixed element (reference point, denoted by p in Jones's formula), expressed by a suitably defined distance function. The essence of definition (5) can be described as follows: one takes a product, $\mathrm{prod}(C_A(x), C_B(x))$, of the membership degrees and assigns to it a number from $[0,1]$ obtained by applying an order-reversing function – (minus), an order-preserving function R, and an order-reversing function I. The role of R is to accomplish that value 1 be assigned to x for which the value $\text{prod}(C_A(x), C_B(x))$ is the largest one over all x's (or the supremum of them if the largest one does not exist). Since applying an order-reversing function two times yields an order-preserving function, it is easy to see that the point is actually to map, in an order-preserving way, values of $\text{prod}(C_A(x), C_B(x))$ to values from $[0,1]$ in such a way that the largest value (or supremum) is mapped to 1. But this is exactly what one obtains if C is defined to be a normalization of the intersection of fuzzy sets C_A and C_B, i.e.

$$C = N(C_A \cap C_B),$$

where N is a suitable normalization function which depends on R and I, and $C_A \cap C_B$ is an intersection of fuzzy sets C_A and C_B based on the product as the connective of fuzzy conjunction, i.e. $(C_A \cap C_B)(x) = C_A(x) \cdot C_B(x)$. It is not possible here to present the explicit definition of N because it depends on R and I, and Jones does not present specific definitions of R and I.

Therefore, while Osherson and Smith [33] appeal to the ingenuity of Jones's proposal and, at the same time, argue against Zadeh's proposal of taking normalization of fuzzy sets, Jones's model is in fact a special case of Zadeh's proposal.

4.2.5. Erroneous and misleading claims regarding fuzzy logic in the narrow sense

Some erroneous or misleading claims regarding fuzzy logic in the narrow sense have also been made in the literature on concepts. We illustrate them by two examples. The first example is taken from [32, p. 51–52], where the authors express their general distrust of a calculus that allows propositions with intermediate truth degrees. They claim, without any serious justification:

> The problem is that infinite valued logics violate strong intuitions about truth, validity, and consistency.

The following is the only example by which they illustrate what they mean:

> To illustrate with an influential system, consider Lukasiewicz's infinite valued logic L-aleph. The intuitively valid sentence
>
> If John is happy, and if John is happy only if business is good, then business is good.
>
> is ruled nontautologous in L-aleph.

However, this is not true: if p denotes "John is happy" and q denotes "business is good" then the above sentence corresponds to a propositional formula $\varphi = (p \& (p \Rightarrow q)) \Rightarrow q$. If $v(p)$ and $v(q)$ are truth degrees of p and q, then the truth degree assigned to φ in Lukasiewicz's calculus is

$$\min(1, 1 - \max(0, v(p) + \min(1, 1 - v(p) + v(q)) - 1) + v(q)),$$

which is always 1. This means that φ is a tautology, contrary to what Osherson and Smith claim.

Our second example is taken from another paper by Osherson and Smith [34, p. 200]:

> It is useful to note how committed fuzzy logic is to the non-standard reading of contradictions. Suppose that fuzzy truth-assignment μ over a propositional language has the well-known properties:
>
> $$(a) \quad \mu(\theta \wedge \varphi) = \min[\mu(\theta), \mu(\varphi)]. \qquad (14)$$
> $$(b) \quad \mu(\neg\theta) = 1 - \mu(\theta).$$
>
> Then it is easy to prove the following: suppose that $\mu(\theta \wedge \neg\theta) = \mu(\varphi \wedge \neg\varphi)$, for every pair of propositions θ, φ. Then the range of μ is limited to no more than two-values. In other words, in the presence of (14), contradictions must admit a range of truth values on pain of μ collapsing to a non-fuzzy, two-value truth-assignment. (A similar argument is presented in Elkan [6].)

Fuzzy logic is a wrong target here, because a system satisfying (14)(a) and (b), and $\mu(\theta \wedge \neg\theta) = \mu(\varphi \wedge \neg\varphi)$, is not a system of fuzzy logic. Namely, the assumption that $\mu(\theta \wedge \neg\theta) = \mu(\varphi \wedge \neg\varphi)$ is strange to fuzzy logic. As far as we know, such assumption has never been used in theory or applications of fuzzy logic and fuzzy set theory. Therefore, the authors miss the point and commit the fallacy of *ignoratio elenchi*. To see how unnatural the assumption $\mu(\theta \wedge \neg\theta) = \mu(\varphi \wedge \neg\varphi)$ is, recall the above observation regarding what the truth degree of "x is a red apple and x is not a red apple" tells us about x. Obviously, due to (14)(a) and (b), the value of $\mu(\theta \wedge \neg\theta)$ can range from 0 to 0.5. In fact, $\mu(\theta \wedge \neg\theta) = 0$ if $\mu(\theta) = 0$ or $\mu(\theta) = 1$, i.e. if θ is what is sometimes called a crisp proposition, while $\mu(\theta \wedge \neg\theta) = 0.5$ if $\mu(\theta) = 0.5$, i.e. if θ is what is considered a fuzzy proposition. Therefore, $\mu(\theta \wedge \neg\theta)$ can be seen as measuring the fuzziness of proposition θ. The closer $\mu(\theta \wedge \neg\theta)$ is to 0.5, the more fuzzy θ is. Now, the assumption that $\mu(\theta \wedge \neg\theta) = \mu(\varphi \wedge \neg\varphi)$ for every pair of propositions θ and φ says that the fuzziness of any two propositions is the same. Needless to say, such an assumption is absurd.

Note also what Elkan's result, to which Osherson and Smith [34] refer, actually says. Elkan [6] presents a theorem which allegedly shows fatal inconsistencies of fuzzy logic. A modified version of this result appears in a special issue of *IEEE Expert* [7]. As shown in [4], the assumptions of Elkan's theorems contain conditions which are never used in fuzzy logic and, moreover, are counterintuitive from the point of view of fuzzy logic.

5. Conclusions

The literature on the psychology of concepts has lately been characterized by disputes regarding the "correctness" of the various views of concepts, especially disputes between the prototype and exemplar theories. In this paper, we do not take any stand on this issue. Our aim is to discuss and correct the many misunderstandings, misconceptions, and oversights regarding the role of fuzzy set theory and fuzzy logic in the psychology of concepts that are repeated again and again in the literature of this field, and are accepted, by and large, uncritically. We have identified an appreciable number of them and these are addressed in Section 4. The following is a list summarizing the principal issues involved:

1. Fuzzy set theory is not a theory of concepts, but it can be utilized for representing and dealing with concepts, regardless of the view of concepts which one takes. This fact is not recognized and, as a consequence, fuzzy set theory is confused with theories of concepts. As a result, fuzzy set theory is often criticized as a presumed theory of concepts.
2. It is not recognized that fuzzy set theory is a generalization of classical set theory. Instead, these two theories are viewed as mutually exclusive.
3. Some concepts of fuzzy set theory and fuzzy logic remain overlooked or misunderstood and, as a consequence, unjustified arguments are made to reject this area of mathematics.
4. Some conclusions are either formally incorrect or are based on unjustified assumptions.

We do not expect that this paper alone will change the negative attitudes toward fuzzy set theory and fuzzy logic by most researchers in the psychology of concepts. However, we consider it a necessary first step. To change the attitudes will require that the utility of fuzzy set theory and fuzzy logic for representing and dealing with concepts be properly demonstrated. This will not be possible without extensive research involving experts from both areas. We intend to make efforts to facilitate such cooperation in the future.

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