Algorithms for fuzzy concept lattices

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Abstract. Let I be a fuzzy relation between a set X of objects and a set Y of attributes, $I(x, y)$ being interpreted the degree to which object x has attribute y. A fuzzy concept lattice induced by $\langle X, Y, I\rangle$ is a set $\mathcal{B}(X, Y, I)$ of all so-called formal fuzzy concepts hidden in $\langle X, Y, I\rangle$ together with the subconcept-superconcept hierarchy. A formal fuzzy concept in $\langle X, Y, I\rangle$ is a pair $\langle A, B\rangle$ of a fuzzy set A of objects from X and a fuzzy set B of attributes from Y satisfying $A^{\uparrow} = B$ and $B^{\downarrow} = A$. Here, $A^{\uparrow}$ is the fuzzy set of all attributes common to all objects from A and $B^{\downarrow}$ is the fuzzy set of all objects sharing all attributes from B. Generating all fuzzy concepts from $\mathcal{B}(X, Y, I)$ naively using directly the definition leads to an algorithm with exponential time complexity: one has to check all pairs $\langle A, B\rangle$ and there are $|L|^{|X|}$ choices for A and $|L|^{|Y|}$ choices for B (L is the set of truth values, e.g. a suitable subset of $[0, 1]$). In the ordinary (crisp, non-fuzzy) case, Ganter's "next extent" algorithm provides a better way.

In this paper we show two approaches to deal with the problem of generating all concepts of a fuzzy concept lattice. First, using previously published results, generating a fuzzy concept lattice can be reduced to the problem of generating an (ordinary) concept lattice and thus, Ganter's algorithm can be applied. Second, we present a generalization of Ganter's algorithm for fuzzy setting. Presented is a discussion and an outline of further research.

Key words: context, fuzzy context, concept lattice, fuzzy concept lattice

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1 Introduction and problem setting

Formal concept analysis [11] (FCA) provides us with mathematical methods for analysis of data which is in the form of a table describing a relationship between objects and attributes. In the following, we are interested in concept lattices which are structures extracted from the data by FCA. A concept lattice is a partially ordered set (in fact, it is a complete lattice) of so-called formal concepts, a formal concept consists of a collection of objects which fall under the concept and a collection of attributes which fall under the concept. The partial order is given by the subconcept-superconcept relation (conceptual hierarchy where the higher the concept the more general it is). Each formal concept represents a granule (a cluster) of objects and attributes. From this point of view, the concept lattice is thus a partially ordered set of clusters. However, unlike most of clustering methods, clusters produced by FCA have a clearly defined meaning: they are concepts in the traditional sense. The state of art of concept lattices and formal concept analysis is well covered in [11]; the applications are described in dozens of articles in edited monographs, conference proceedings, and journals.

Our aim in this paper is to address the problem of generating all fuzzy concepts from $\mathcal{B}(X, Y, I)$. In the ordinary case, Ganter's algorithm is used to solve this problem [11]. Ganter's algorithm is based on lexicographically ordering all formal concepts and on the observation enabling us to generate a lexicographic successor to a given extent (a set of objects). We show that using [4], the corresponding problem of generating all concepts of a fuzzy concept lattice can be reduced to the problem of generating all concepts from a corresponding ordinary (crisp) concept lattice and thus Ganter's algorithm can be applied. Furthermore, we present a generalization of Ganter's algorithm for fuzzy setting. The advantage of having a generalization of the Ganter's algorithm for fuzzy setting is that, one does not lose the direct control over the truth degrees involved which can be utilized in modifying the algorithm.

2 Preliminaries

We recall basic facts of concept lattices and formal concept analysis (for details see the list of references, particularly [11, 10, 18]), fuzzy sets [6, 14], and fuzzy concept lattices [6].

A fuzzy set A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$ where L is some partially ordered set of truth degrees containing at least 0 (full falsity) and 1 (full truth). In the following, we assume that L is equipped with a structure of a complete residuated lattice, i.e. L is a support of $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ [6, 13, 12]. Usually, L is the unit interval $[0, 1]$ or some of its subsets. The most studied and applied set of truth values is the real interval $[0, 1]$; with $a \wedge b = \min(a, b)$, $a \vee b = \max(a, b)$, and with three important pairs of adjoint operations: the Łukasiewicz one ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), Gödel one ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and product one ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ else). Another important set of truth values is the set $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_{\min(n-k+l,n)}$. Another important set of truth values is the set $\{a_0 = 0, a_1, \ldots, a_n = 1\}$ ($a_0 < \cdots < a_n$) with $\otimes$ given by $a_k \otimes a_l = a_{\min(k,l)}$ and the corresponding $\rightarrow$ given by $a_k \rightarrow a_l = a_n$ for $a_k \leq a_l$ and $a_k \rightarrow a_l = a_l$ otherwise (this is a restriction of Gödel structure on $[0, 1]$).

The basic notions of concept lattices can be generalized to fuzzy setting as follows: Suppose some complete residuate lattice $\mathbf{L}$ is given. An $\mathbf{L}$-context (*fuzzy context*) is a triple $\langle X, Y, I\rangle$ where I is a binary fuzzy relation between X and Y; $I(x, y)$ is interpreted as the truth degree to which object x has attribute y. One introduces two operators, $^{\uparrow} : L^X \rightarrow L^Y$ and $^{\downarrow} : L^Y \rightarrow L^X$ by

$$A^{\uparrow}(m) = \bigwedge_{g \in G} (A(g) \rightarrow I(g, m)) \tag{1}$$

and

$$B^{\downarrow}(g) = \bigwedge_{m \in M} (B(m) \to I(g,m)) \tag{2}$$

for any $A \in L^X$ and $B \in L^Y$. Using basic rules of fuzzy logic, one can easily see that $A^{\uparrow}(y)$ is the truth degree of the fact "y is shared by all objects from A" and similarly for $B^{\downarrow}(x)$. Therefore, $\mathcal{B}(X,Y,I) = \{\langle A,B\rangle \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}$ (called a fuzzy concept lattice with its elements $\langle A,B\rangle$ called fuzzy concepts) is the set of all pairs $\langle A,B\rangle$ such that (a) each object of A has all the properties of (the intent) B and (b) each property of B is shared by all the objects of (the extent) A. The conceptual hierarchy in $\mathcal{B}(X,Y,I)$ is modeled by the relation $\leq$ defined on $\mathcal{B}(G,M,I)$ by

$$\langle A_1,B_1\rangle \leq \langle A_1,B_1\rangle \text{ iff } A_1 \subseteq A_2 (\text{iff } B_1 \supseteq B_2) \tag{3}$$

where $A_1 \subseteq A_2$ means that $A_1(x) \leq A_2(x)$ for each $x \in X$. The following theorem generalizes the so-called main theorem of conceptual data analysis [3, 4, 5]:

Theorem 1 *The set $\mathcal{B}(G,M,I)$ is under $\leq$ a complete lattice where the suprema and infima are given by*

$$\bigwedge_{j\in J} \langle A_j,B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}\rangle\ , \tag{4}$$

$$\bigvee_{j\in J} \langle A_j,B_j\rangle = \langle (\bigcup_{j\in J} A_j)^{\uparrow\downarrow}, \bigcap_{j\in J} B_j\rangle\ . \tag{5}$$

Moreover, an arbitrary complete lattice $\mathbf{V} = \langle V,\wedge,\vee\rangle$ is isomorphic to some $\mathcal{B}(G,M,I)$ iff there are mappings $\gamma : G \times L \to V$, $\mu : M \times L \to V$ such that $\gamma(G,L)$ is $\bigwedge$-dense in V, $\mu(M,L)$ is $\bigvee$-dense in V; $\alpha \otimes \beta \leq I(g,m)$ iff $\gamma(g,\alpha) \leq \mu(m,\beta)$.

3 Generating all concepts of a fuzzy concept lattice

Reduction: fuzzy concept lattices as concept lattices

In [4] it is shown how fuzzy Galois connections between X and Y can be represented by special Galois connections between $X \times L$ and $Y \times L$. One of the consequences of this result is the fact that each fuzzy concept lattice can be viewed as a crisp concept lattice, see [4] and also [16]. This relationship is described by the following theorem:

Theorem 2 *Let $\langle X,Y,I\rangle$ be an* **L***-context. Then $\mathcal{B}(X,Y,I)$ (a fuzzy concept lattice given by $\langle X,Y,I\rangle$) is isomorphic to the (ordinary) concept lattice $\mathcal{B}(X \times L, Y \times L, I^{\times})$ where $I^{\times}$ is defined as follows: $\langle\langle x,a\rangle,\langle y,b\rangle\rangle \in I^{\times}$ iff*

$a \otimes b \leq I(x, y)$. *Moreover, the isomorphism is given by the mapping sending each fuzzy concept* $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ *to* $\langle \lfloor A \rfloor, \lfloor B \rfloor \rangle$ *where for an* **L**-*set* C *in* U, $\lfloor C \rfloor$ *is an ordinary set in* $U \times L$ *defined by* $\lfloor C \rfloor = \{\langle u, a\rangle \mid a \leq C(u)\}$.

Therefore, Theorem 2 provides a direct way to reduce the problem of generating all fuzzy concepts from $\mathcal{B}(X, Y, I)$: First, transform a fuzzy context $\langle X, Y, I\rangle$ into an ordinary context $\langle X \times L, Y \times L, I^{\times}\rangle$. Second, apply ordinary Ganter's algorithm to $\langle X \times L, Y \times L, I^{\times}\rangle$. Third, transform $\mathcal{B}(X \times L, Y \times L, I^{\times})$ back to $\mathcal{B}(X, Y, I)$.

Direct way: Ganter's algorithm for fuzzy setting

Recall that Ganter's algorithm in ordinary setting is based on lexicographic ordering of the extents (intents) and on computing immediate successors in this ordering. In fuzzy setting, we may proceed as follows.

Our formulation of the algorithm is in terms of intents. Suppose $Y = \{1, 2, \ldots, n\}$; $L = \{0 = a_1 < a_2 < \cdots < a_k = 1\}$ (the assumption of linearly ordered L may be skipped). Put

$$(i, j) \leq (r, s) \quad \text{iff} \quad i < r \quad \text{or} \quad i = r, a_j \geq a_s.$$

In the following, we do not distinguish between $Y \times L$ and $\{1, \ldots, n\} \times \{1, \ldots, k\}$. The following lemma is obvious.

Lemma 3 $\leq$ *is a total order on* $Y \times L$.

For $B \in L^Y$, $(i, j) \in Y \times L$, put

$$B \oplus (i, j) := ((B \cap \{1, 2, \ldots, i-1\}) \cup \{\, ^{a_j}/i\})^{\downarrow\uparrow}.$$

Here, $B \cap \{1, 2, \ldots, i-1\}$ is the intersection of a fuzzy set B and a crisp set $\{1, 2, \ldots, n\}$, i.e. $(B \cap \{1, 2, \ldots, i-1\})(y) = B(y)$ for $y < i$ and $(B \cap \{1, 2, \ldots, i-1\})(y) = 0$ otherwise. Furthermore, for $B, C \in L^Y$, put

$$B <_{(i,j)} C \quad \text{iff} \quad B \cap \{1, \ldots, i-1\} = C \cap \{1, \ldots, i-1\},\ B(i) < C(i) = a_j.$$

Finally,

$$B < C \quad \text{iff} \quad B <_{(i,j)} C \quad \text{for some } (i, j).$$

The following lemma is obvious.

Lemma 4 $<$ *is a strict total order on* L^Y.

Note that the strict order $<$ on L^Y is the usual lexicographic order. Given a fuzzy context $\langle X, Y, I\rangle$, an intent is a fuzzy set B in Y such that $\langle A, B\rangle \in \mathcal{B}(X, Y, I)$ for some A. We now have the following theorem.

Theorem 5 (next intent) *The least intent B^+ which is greater (w.r.t. $<$) than a given $B \in L^Y$ is given by*

$$B^+ = B \oplus (i, j)$$

where (i, j) is the greatest one with $B <_{(i,j)} B \oplus (i, j)$.

Proof. Let B^+ the successor of B w.r.t. to the lexicographic order $<$. We have $B < B^+$ and thus $B <_{(i,j)}$ for some i and j such that $a_j = B^+(i)$. One can easily verify that $B <_{(i,j)} B \oplus (i, j)$. Furthermore, $B \oplus (i, j) \leq B^+$ which gives $B^+ = B \oplus (i, j)$. It remains to show that (i, j) is the greatest one satisfying $B <_{(i,j)} B \oplus (i, j)$. Suppose $B <_{(k,l)} B \oplus (k, l)$ for $(k, l) > (i, j)$. If $k > i$ then we can easily see that $B \oplus (k, l) <_{(i,j)} B \oplus (i, j)$ which is a contradiction to $B \oplus (i, j) = B^+ < B \oplus (k, l)$. Therefore we have $k = i$. Clearly, if $a_l > a_j$ then $B^+ = B \oplus (i, j) < B \oplus (k, l)$ showing that (i, j) is the greatest one with $B <_{(i,j)} B \oplus (i, j)$. □

Using the above Theorem 5, we can justify the following algorithm.

INPUT: $\langle X, Y, I \rangle$
OUTPUT: $\{B \mid \langle A, B \rangle \in \mathcal{B}(X, Y, I)\}$

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B := ∅↓↑
store(B)
while B ≠ Y do
  B := B+
  store(B) /* possibly compute and store extent B↓ */
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Note that Ganter's algorithm [11, p. 67] is a particular case of Algorithm 1 for $L = \{0, 1\}$.

4 Discussion, further research

Generating all fuzzy concepts of a fuzzy concept lattice can be done in two ways. First, it is possible to view the fuzzy concept lattice $\mathcal{B}(X, Y, I)$ as a suitable ordinary concept lattice $\mathcal{B}(X \times L, Y \times L, I^{\times})$ (as shown in Theorem 2). This makes it possible to apply Ganter's algorithm for ordinary concept lattices. Second, it is possible to apply the modification of Ganter's algorithm as presented above. The apparent advantage of the second approach is that one can keep control over the truth degrees involved in fuzzy concepts in a natural way (using the first approach, the meaning of the truth degrees gets lost due to the reduction to ordinary case). This enables one to modify the algorithm which is the subject of our further research.

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