# BOOLEAN PART OF BL-ALGEBRAS

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ABSTRACT. The set of elements of a Heyting algebra (the algebraic counterpart of intuitionistic logic) which are closed under double negation forms a Boolean algebra. We present similar results for BL-algebras, the algebraic couterpart of the logic of continuous t-norms.

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## 1. BL-algebras

Each continuous t-norm $\otimes$ (i.e. an isotone associative commutative operation on $[0,1]$ with 1 as the neutral element) is "composed" of three basic ones (for details see [8]): Łukasiewicz ($a \otimes b = \max(0, a+b-1)$), minimum (also called Gödel t-norm; $a \otimes b = \min(a,b)$), and product ($a \otimes b = ab$).

The interest in many-valued calculi with conjunction defined by a t-norm (and implication by the corresponding residuum $\rightarrow$ where $a \rightarrow b = \max\{c \mid a \otimes c \leq b\}$) has a long tradition (see [7], [4], and [5] for Łukasiewicz, Gödel, and product logics, respectively, and [6] for completeness, further results, and historical information). Recently, there has been a strong interest in t-norm based logics in the context of investigations in fuzzy logic, i.e. "logic of graded truth". The three above mentioned logics have a common generalization—they are axiomatic extensions of so-called basic logic. Basic logic is a syntactico-semantically complete calculus; semantics is defined in the usual manner using so-called BL-algebras ("BL" stands for "basic logic") that play the role of structures of truth values [6]. A BL-algebra is a residuated lattice [2, 6] (i.e. an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded lattice, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $x \otimes y \leq z$ iff $x \leq y \rightarrow z$ (adjointness condition)) satisfying prelinearity ($(x \rightarrow y) \vee (y \rightarrow x) = 1$) and divisibility ($x \wedge y = x \otimes (x \rightarrow y)$; equivalently: for every $x \leq y$ there is $z$ such that $x = y \otimes z$).

The class $\mathcal{BL}$ of all BL-algebras is a variety of algebras (i.e. an equationally defined class). For a continuous t-norm $\otimes$, the algebra $[0,1]_{\otimes} = \langle [0,1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ ($\rightarrow$ is the residuum corresponding to $\otimes$) is a BL-algebra, so-called t-norm algebra corresponding to $\otimes$. $\mathcal{BL}$ is the variety generated by all t-norm algebras corresponding to continuous t-norms (i.e. $\mathcal{BL}$ is the smallest variety containing $\{[0,1]_{\otimes} \mid \otimes$ is a continuous t-norm $\}$), see [1]. Another example of a BL-algebra is the Lindenbaum algebra of propositional basic logic (i.e. the algebra of provably equivalent formulas), see [6]. There are three special BL-algebras corresponding to the basic t-norms (we abbreviate $x \rightarrow 0$ by $\neg x$; all of the following statements are reformulation of results from [6]): MV-algebras, i.e. BL-algebras satisfying $\neg\neg x = x$ (the variety $\mathcal{MV}$ of MV-algebras is generated by the Łukasiewicz t-norm algebra; there are other definitions [6]), G-algebras, i.e. BL-algebras satisfying $x \otimes x = x$ (the variety $\mathcal{G}$ of G-algebras is generated by the t-norm algebra that corresponds to Gödel t-norm; G-algebras are Heyting algebras satisfying prelinearity), and Π-algebras, i.e. BL-algebras satisfying $x \wedge \neg x = 0$ and $\neg\neg z \leq ((x \otimes z \rightarrow y \otimes z) \rightarrow (x \rightarrow y))$ (the variety $\mathcal{P}$ of Π-algebras is generated by the t-norm algebra that corresponds to the product t-norm). Along this line, a Boolean algebra is a BL-algebra $\mathbf{L}$ which is both an MV-algebra and a G-algebra. Note that the correspondence to the usual definition (i.e. a Boolean algebra as a complemented distributive lattice) is the following one: if **L** is a BL-algebra which is both an MV-algebra and a G-algebra then putting $x' = x \to 0$, $\langle L, \wedge, \vee, ', 0, 1\rangle$ is a complemented distributive lattice; conversely, if $\langle L, \wedge, \vee, ', 0, 1\rangle$ is a complemented distributive lattice then putting $x \to y = x' \vee y$, $\mathbf{L} = \langle L, \wedge, \vee, \wedge, \to, 0, 1\rangle$ is a BL-algebra which is both an MV-algebra and a G-algebra.

## 2. Boolean parts

For a BL-algebra **L**, denote

$$D(\mathbf{L}) = \{a \in L \mid a = \neg\neg a\},$$

the set of all elements satisfying the law of double negation, and

$$H(\mathbf{L}) = \{a \in L \mid a = a \otimes a\},$$

the set of all elements idempotent w.r.t. conjunction.

A well-known result, essentially due to Glivenko [3], says that if **L** is a Heyting algebra then $D(\mathbf{L})$ is a Boolean algebra where the meet is inherited from **L** and the supremum of $a$ and $b$ in $D(\mathbf{L})$ is $\neg\neg(a \vee b)$.

**Lemma 1** *If* **L** *is a BL-algebra then* $H(\mathbf{L})$ *is the largest subalgebra of* **L** *that is a G-algebra.*

*Proof.* First, $0, 1 \in H(\mathbf{L})$. Now, observe that if $a \in H(\mathbf{L})$ then $a \otimes b = a \wedge b$ for any $b \in L$. Indeed, $a \wedge b = a \otimes (a \to b) = a \otimes a \otimes (a \to b) = a \otimes (a \wedge b) \leq a \otimes b$; $a \otimes b \leq a \wedge b$ follows from the isotony of $\otimes$. We prove that $H(\mathbf{L})$ is a subalgebra. Take any $a, b \in H(\mathbf{L})$. Since $\otimes$ is distributive over $\wedge$ [6, proof of Lemma 2.3.10], we have $(a \wedge b) \otimes (a \wedge b) = (a \otimes a) \wedge (a \otimes b) \wedge (b \otimes b) = a \wedge b$, i.e. $H(\mathbf{L})$ is closed under $\wedge$. Furthermore, $(a \vee b) \otimes (a \vee b) = (a \otimes a) \vee (a \otimes b) \vee (b \otimes b) = a \vee (a \wedge b) \vee b = a \vee b$, i.e. $H(\mathbf{L})$ is closed under $\vee$. Finally, $(a \otimes b) \otimes (a \otimes b) = (a \otimes a) \otimes (b \otimes b) = a \otimes b$, proving closedness under $\otimes$. We prove that $H(\mathbf{L})$ is closed under $\to$: Each BL-algebra is a subdirect product of linearly ordered BL-algebras [6, Lemma 2.3.16]. We may therefore safely assume that $\mathbf{L}$ is linearly ordered. If $a \leq b$ then $a \to b = 1 \in H(\mathbf{L})$. Let $a > b$. We show that $a \to b = b$. Since $b \leq a \to b$ is always true, it suffices to show that $b < a \to b$ is impossible. Let then $b < a \to b$. Since $a \in H(\mathbf{L})$, we have $a \wedge (a \to b) = a \otimes (a \to b) \leq b$. By linearity of $\mathbf{L}$, $a \wedge (a \to b) = \min(a, a \to b) > b$, a contradiction.

If $H' \supseteq H(\mathbf{L})$ is another subalgebra of $\mathbf{L}$ that is a G-algebra then for any $a \in H'$, $a \otimes a = a$, i.e. $a \in H(\mathbf{L})$, thus $H' = H(\mathbf{L})$. This proves that $H(\mathbf{L})$ is the largest subalgebra that is a G-algebra. □

**Lemma 2** *If $\mathbf{L}$ is a BL-algebra then $D(\mathbf{L})$ is the largest subalgebra of $\mathbf{L}$ that is an MV-algebra.*

*Proof.* First, we show that $D(\mathbf{L})$ is a subalgebra of $\mathbf{L}$. Since $\neg x = \neg\neg\neg x$ is valid in $\mathbf{L}$, $D(\mathbf{L}) = \{\neg a \mid a \in L\}$. Clearly, $0, 1 \in D(\mathbf{L})$. Since $(a \to 0) \wedge (b \to 0) = (a \vee b) \to 0$ (easy to prove by adjointness), $D(\mathbf{L})$ is closed w.r.t. $\wedge$. To see that $D(\mathbf{L})$ is closed w.r.t. $\vee$, we verify $(a \to 0) \vee (b \to 0) = (a \wedge b) \to 0$: The "$\leq$" part follows by antitony of negation. Conversely, $(a \wedge b) \to 0 = ((a \wedge b) \to 0) \otimes ((a \to b) \vee (b \to a)) = ((a \to b) \otimes ((a \wedge b) \to 0)) \vee ((b \to a) \otimes ((a \wedge b) \to 0)) \leq (a \to 0) \vee (b \to 0)$. $x \otimes (x \to y) \leq y$ yields $\neg a \to \neg b = \neg(\neg a \otimes b)$ (indeed, applying adjointness to $b \otimes (\neg a \otimes (\neg a \to \neg b)) \leq 0$ and to $(\neg a \otimes b) \otimes ((\neg a \otimes b) \to 0) \leq 0$ gives the "$\leq$" and "$\geq$" inequalities). Now, introduce a binary operation $\odot$ on $D(\mathbf{L})$ by $a \odot b = \neg\neg(a \otimes b)$. We show that $\langle D(\mathbf{L}), \odot, 1\rangle$ is a commutative monoid: Clearly, $a \odot b \in D(\mathbf{L})$. Furthermore, $\odot$ is obviously commutative and since $\neg\neg(\neg a \otimes 1) = \neg a$, 1 is its neutral element. To verify associativity, we reason as follows: $\neg\neg(\neg\neg(a \otimes b) \otimes c) \leq \neg\neg(a \otimes \neg\neg(b \otimes c))$ iff $\neg(a \otimes \neg\neg(b \otimes c)) \leq \neg(\neg\neg(a \otimes b) \otimes c)$ iff $\neg\neg(a \otimes b) \otimes c \otimes \neg(a \otimes \neg\neg(b \otimes c)) \leq 0$ iff $c \otimes \neg(a \otimes \neg\neg(b \otimes c)) \leq \neg\neg\neg(a \otimes b) = \neg(a \otimes b)$ iff $a \otimes b \otimes c \otimes \neg(a \otimes \neg\neg(b \otimes c)) \leq 0$ which follows from $b \otimes c \leq \neg\neg(b \otimes c)$. We proved $(a \odot b) \odot c \leq a \odot (b \odot c)$, the converse inequality is symmetric. Therefore, $\langle D(\mathbf{L}), \odot, 1\rangle$ is a commutative monoid.

Furthermore, as $\neg a \to \neg b = \neg(\neg a \otimes b)$, $D(\mathbf{L})$ is closed under $\to$. We now verify that $\odot$ and $\to$ satisfy adjointness: Since $a \otimes b \leq \neg\neg(a \otimes b)$, $a \odot b \leq c$ implies $a \leq b \to c$ by adjointness of $\otimes$ and $\to$. If $a \leq b \to c$ then $a \otimes b \leq c$, and so $a \odot b = \neg\neg(a \otimes b) \leq \neg\neg c = c$. Now, we have $a \otimes b \leq a \odot b$ iff $a \leq b \to (a \odot b)$ iff $a \odot b \leq a \odot b$, i.e. $a \otimes b \leq a \odot b$. In a similar way one obtains $a \odot b \leq a \otimes b$, thus $a \odot b = a \otimes b$ for any $a, b \in D(\mathbf{L})$. Therefore, $D(\mathbf{L})$ is a subalgebra of $\mathbf{L}$. Obviously, $D(\mathbf{L})$ satisfies $x = \neg\neg x$ and so $D(\mathbf{L})$ is an MV-algebra. It is the largest MV-algebra contained in $\mathbf{L}$ as a subalgebra since otherwise there is an $a \in L - D(\mathbf{L})$ such that $a = \neg\neg a$, a contradiction to the definition of $D(\mathbf{L})$. □

*Remark.* Note that in a different way, the fact that $D(\mathbf{L})$ is an MV-algebra is obtained in [9].

**Theorem 3** (1) *If* $\mathbf{L}$ *is an MV-algebra then* $D(\mathbf{L}) = L$ *and* $H(\mathbf{L})$ *is the largest subalgebra of* $\mathbf{L}$ *that is a Boolean algebra.*

(2) *If* $\mathbf{L}$ *is a G-algebra then* $H(\mathbf{L}) = L$ *and* $D(\mathbf{L})$ *is the largest subalgebra of* $\mathbf{L}$ *that is a Boolean algebra.*

(3) *If* $\mathbf{L}$ *is a* $\Pi$*-algebra then* $D(\mathbf{L}) = H(\mathbf{L})$ *is the largest subalgebra of* $\mathbf{L}$ *that is a Boolean algebra.*

*Proof.* (1): If $\mathbf{L}$ is an MV-algebra then obviously $D(\mathbf{L}) = L$. The second part follows directly from Lemma 1.

(2): Analogously, $\mathbf{L}$ is a G-algebra yields $H(\mathbf{L}) = L$ and the assertion follows from Lemma 2.

(3): As mentioned above, each BL-algebra $\mathbf{L}$ is a subdirect product of linearly ordered BL-algebras [6, Lemma 2.3.16]. Moreover, as it follows from the proof, the linearly ordered factors satisfy all identities of $\mathbf{L}$. Therefore, every $\Pi$-algebra is a subdirect product of linearly ordered $\Pi$-algebras. Let

$\mathbf{L}_i$ be the linearly ordered factors of $\mathbf{L}$. We identify each $a \in L$ with the corresponding element $(\ldots, a_i, \ldots)$ of the direct product of $\mathbf{L}_i$'s.

Let $\mathbf{L}$ be a $\Pi$-algebra. First, we show that $a = (\ldots, a_i, \ldots) \in H(\mathbf{L})$ iff $a_i = 0$ or $a_i = 1$ for all $i$. The right-to-left part is evident. Conversely, let $a \in H(\mathbf{L})$ and $0 < a_i$. Since $\mathbf{L}_i$ is linearly ordered, $\neg a_i = 0$ (see [6, Lemma 4.1.7]), thus $\neg\neg a_i = 1$. Therefore, putting $x = 1$, $y = a_i$, and $z = a_i$, $\neg\neg z \leq ((x \otimes z) \to (y \otimes z)) \to (x \to y)$ yields $1 \leq (a_i \to a_i) \to (1 \to a_i)$, thus $a_i = 1$. Therefore, for each $i$, either $a_i = 0$ or $a_i = 1$.

Second, we verify that $a = (\ldots, a_i, \ldots) \in D(\mathbf{L})$ iff $a_i = 0$ or $a_i = 1$ for all $i$. Again, the right-to-left part is evident. Conversely, since $\mathbf{L}_i$ is linearly ordered and $a_i \wedge \neg a_i = 0$, $0 < a_i$ implies $\neg a_i = 0$. It folows that $0 < a_i$ and $a_i \in D(\mathbf{L}_i)$ imply $a_i = \neg\neg a_i = 1$. Therefore, $H(\mathbf{L}) = D(\mathbf{L})$, and the claim directly follows by Lemma 1 and Lemma 2. □

*Remark.* (1) Note that (1) of Theorem 3 can also be proved by the subdirect representation method: $a = (\ldots, a_i, \ldots) \in H(\mathbf{L})$ implies $a_i \in H(\mathbf{L}_i)$, i.e. $a_i \otimes a_i = a_i$. We claim that $a_i = 0$ or $a_i = 1$. By contradiction, let $0 < a_i < 1$. Since $\mathbf{L}_i$ is linearly ordered, $0 < a_i \otimes a_i$ yields $\neg a_i < a_i$ ($a_i \leq \neg a_i$ gives $a_i \otimes \neg a_i = 0$). As $x \vee y = (x \to y) \to y$ and $x \to \neg y = \neg(x \otimes y)$, we conclude $a = a \vee \neg a = (a \to \neg a) \to \neg a = \neg(a \otimes a) \to \neg a = \neg a \to \neg a = 1$, a contradiction to $a < 1$. The rest is clear. In a similar way, one can prove (2) of Theorem 3.

(2) A direct consequence of (2) of Theorem 3 is that if a Heyting algebra $\mathbf{L}$ satisfies $(x \to y) \vee (y \to x) = 1$ then the join in the Boolean algebra $D(\mathbf{L})$ coincides with the join in $\mathbf{L}$.

We therefore have the following theorem.

**Corollary 4** *If $\mathbf{L}$ is a BL-algebra then $D(\mathbf{L}) \cap H(\mathbf{L})$ is the largest subalgebra of $\mathbf{L}$ which is a Boolean algebra.*

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