Arch. Math. Logic (2003)
Digital Object Identifier (DOI):
10.1007/s00153-003-0188-9

Mathematical Logic

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Birkhoff variety theorem and fuzzy logic

Received: 17 September 2002 / Revised Version: 14 November 2002 /
Published online: ■■ 2003 – © Springer-Verlag 2003

Abstract. An algebra with fuzzy equality is a set with operations on it that is equipped with similarity $\approx$, i.e. a fuzzy equivalence relation, such that each operation f is compatible with $\approx$. Described verbally, compatibility says that each f yields similar results if applied to pairwise similar arguments. On the one hand, algebras with fuzzy equalities are structures for the equational fragment of fuzzy logic. On the other hand, they are the formal counterpart to the intuitive idea of having functions that are not allowed to map similar objects to dissimilar ones. In this paper, we present a generalization of the well-known Birkhoff's variety theorem: a class of algebras with fuzzy equality is the class of all models of a fuzzy set of identities iff it is closed under suitably defined morphisms, substructures, and direct products.

1. Introduction

Functions operating on a set in such a way that close (similar) elements are mapped to close elements have traditionally been the subject of study of calculus and functional analysis, the concept of closeness being almost exclusively formalized using the notion of a metric. On the other hand, the very idea calls for a logical treatment since, formulated verbally, it reads "if arguments of a function are pairwise similar then the results are similar as well". From a logical point of view, the situation is thus described using a logical formula that is traditionally being called the compatibility axiom (or congruence axiom). Therefore, congruence relations which are the relations satisfying the compatibility axioms provide us with a logico-algebraic means for handling the above problem. The appropriateness of such an approach is, however, seriously questionable. Namely, congruences are bivalent relations in that every two elements either are congruent or are not. Contrary to that, closeness (or similarity) is a graded notion – any two elements are close to some degree. With the emergence of fuzzy logic, the ability of logic to treat the problem of functions preserving in a natural way a given similarity on the universe set changed. Namely, instead of 0 and 1 only, fuzzy logic allows one to have a whole scale of truth

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Mathematics Subject Classification (2000): 03B52, 08B05

Key words or phrases: Fuzzy logic – Fuzzy equality – Universal algebra – Variety

degrees and, consequently, one can model graded similarity in fuzzy logic. The above mentioned compatibility axiom, compared to its crisp (two-valued) interpretation, becomes much less trivial in fuzzy logic since its meaning depends on the choice of a conjunction operation (usually a t-norm) and has a numerical (if truth degrees are numbers) significance.

In this paper we study algebras equipped with a fuzzy similarity relation. The paper is a continuation of [1] where we proposed a calculus for obtaining logical consequences of a given fuzzy set of identities generalizing Birkhoff's equational logic. We give an algebraic description of classes of algebras with fuzzy equalities axiomatized by a fuzzy set of identities generalizing thus the well-known Birkhoff's variety theorem [3].

Section 2 presents relevant notions and the main result. Section 3 contains the proof and remarks. In addition to the fact that the paper contributes to equational fragment of fuzzy logic, it is also related to some studies of so-called metric algebras [11, 12] (we comment on this issue in Section 3).

2. The result

First, we briefly review basic notions of fuzzy logic needed in the sequel. More information can be obtained in [6, 7, 9], and in [1].

We pick complete residuated lattices as the structures of truth values. Complete residuated lattices, being introduced in the 1930s in ring theory, were introduced into the context of fuzzy logic by Goguen [4, 5]. Fundamental contribution to formal fuzzy logic using residuated lattices as the structures of truth values is due to Pavelka [10]. Later on, various logical logical calculi were investigated using residuated lattices or particular types of residuated lattices. A thorough information about the role of residuated lattices in fuzzy logic can be obtained from monographs [6, 7, 9].

In the following, $\mathbf{L}$ denotes an arbitrary complete residuated lattice. Recall that a (complete) residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes, \rightarrow$ form an adjoint pair, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$ is valid for any $a, b, c \in L$. All properties of complete residuated lattices used in the sequel are well-known and can be found in any of the above mentioned monographs. Note that particular types of residuated lattices (distinguishable by identities) include Boolean algebras, Heyting algebras, algebras of Girard's linear logic, MV-algebras, Gödel algebras, product algebras, and more generally, BL-algebras (see [7, 8]). An $\mathbf{L}$-set (or fuzzy set with truth degrees in $\mathbf{L}$) in a universe set U is any mapping $A : U \rightarrow L$, $A(u) \in L$ being interpreted as the truth value of "u belongs to A". For $A_1, A_2 : U \rightarrow L$ we put $A_1 \subseteq A_2$ iff $A_1(u) \leq A_2(u)$ for each $u \in U$. If $U = U_1 \times \cdots \times U_n$, A is called an n-ary $\mathbf{L}$-relation between $U_1, \ldots, U_n$. Recall that $\mathbf{L}$-equivalence ($\mathbf{L}$-similarity) on a set U is a binary $\mathbf{L}$-relation E on U satisfying $E(u, u) = 1$ (reflexivity), $E(u, v) = E(v, u)$ (symmetry), and $E(u, v) \otimes E(v, w) \leq E(u, w)$ (transitivity). An $\mathbf{L}$-equivalence on U for which $E(u, v) = 1$ implies $u = v$ will be called an $\mathbf{L}$-equality. A function $f : U^n \rightarrow U$ is said to be compatible w.r.t. a binary $\mathbf{L}$-relation R on U if for any

$u_1, v_1, \ldots, u_n, v_n \in X$ we have

$$R(u_1, v_1) \otimes \cdots \otimes R(u_n, v_n) \leq R(f(u_1, \ldots, u_n), f(v_1, \ldots, v_n)).$$

Note that compatibility says that the corresponding logical formula, i.e. $R^{\mathrm{syn}}(x_1, y_1) \circ \cdots \circ R^{\mathrm{syn}} \ (x_n, y_n)$ ì $R^{\mathrm{syn}}(f^{\mathrm{syn}}(x_1, \ldots, x_n), f^{\mathrm{syn}}(y_1, \ldots, y_n))$ is true (in degree 1) in the structure given by U, R, and f.

By a type we mean a collection F of function symbols, each with its arity. An algebra of type F with $\mathbf{L}$-equality (or simply an $\mathbf{L}$-algebra of type F) is a triple $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, F^{\mathbf{M}} \rangle$ such that $\langle M, F^{\mathbf{M}} \rangle$ is an algebra of type F (i.e. $F^{\mathbf{M}} = \{f^{\mathbf{M}} : M^{\mathrm{ar}(f)} \to M \mid f \in F\}$ where $\mathrm{ar}(f)$ is the arity of f), $\approx^{\mathbf{M}}$ is an $\mathbf{L}$-equality on M, and $f^{\mathbf{M}} \in F^{\mathbf{M}}$ is compatible w.r.t. $\approx^{\mathbf{M}}$. Clearly, if $\mathbf{L}$ is the two-element Boolean algebra, $\mathbf{L}$-algebras are exactly (universal) algebras generalizing thus the ordinary case. The set $T(X)$ of terms over a countable set X of variables (defined in the usual way) may be naturally made an $\mathbf{L}$-algebra $\mathbf{T}(X) = \langle T(X), \approx^{\mathbf{T}(X)}, F^{\mathbf{T}(X)} \rangle$: The support of $\mathbf{T}(X)$ is $T(X)$; functions are defined by

$$f^{\mathbf{T}(X)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$$

for any $t_1, \ldots, t_n \in T(X)$; $\mathbf{L}$-equality $\approx^{\mathbf{T}(X)}$ is defined by

$$(t_1 \approx^{\mathbf{T}(X)} t_2) = \begin{cases} 1 & \text{for } t_1 = t_2, \\ 0 & \text{for } t_1 \neq t_2. \end{cases}$$

Note that $\mathbf{T}(X)$ exists whenever X is nonempty or there is some nullary $f \in F$; in the following we always assume that $\mathbf{T}(X)$ exists. The following is a simple example of an $\mathbf{L}$-algebra [1]:

Example 1. Let U be a set equipped with an $\mathbf{L}$-equality $\approx^U$. Let $M = S(U)$ be the set of all permutations of U (i.e. bijective mappings on U) which are compatible with $\approx^U$. The triple $\mathbf{M} = \langle M, \approx^{\mathbf{M}}, \circ^{\mathbf{M}} \rangle$ where $\pi \approx^{\mathbf{M}} \pi' = \bigwedge_{u \in U} (\pi(u) \approx^U \pi'(u))$ and $\circ^{\mathbf{M}}$ denotes the composition of permutations, is an $\mathbf{L}$-algebra.

Let $\approx$ denote the relation symbol for equality which is interpreted by $\approx^{\mathbf{M}}$. Formulas of the type $p \approx q$ are called identities. Let $\mathbf{M}$ be an $\mathbf{L}$-algebra, $v : X \to M$ be a valuation. The interpretation of terms is defined as usual (we denote $\|p\|_{M,v}$ the element of M assigned to the term p by the interpretation given by $\mathbf{M}$ and v). The degree $\|p \approx q\|_{M,v}$ to which $p \approx q$ is true in $\mathbf{M}$ under v is defined by

$$\|p \approx q\|_{M,v} \quad = \quad \|p\|_{M,v} \approx^{\mathbf{M}} \|q\|_{M,v}.$$

The degree $\|p \approx q\|_M$ to which $p \approx q$ is true in $\mathbf{M}$ is defined by

$$\|p \approx q\|_M \quad = \quad \bigwedge_{v : X \to M} \|p \approx q\|_{M,v},$$

and more generally, if $\mathcal{K}$ is a class of $\mathbf{L}$-algebras of type F, we put

$$\|p \approx q\|_{\mathcal{K}} \quad = \quad \bigwedge_{\mathbf{M} \in \mathcal{K}} \|p \approx q\|_M.$$

Given an $\mathbf{L}$-set Σ of identities and an $\mathbf{L}$-algebra $\mathbf{M}$, we say that $\mathbf{M}$ is a model of Σ if $\Sigma(p \approx q) \leq \|p \approx q\|_M$ for each identity $p \approx q$. For an $\mathbf{L}$-set Σ of identities we denote by $\mathrm{Mod}(\Sigma)$ the class of all models of Σ. A class $\mathcal{K}$ of $\mathbf{L}$-algebras of the same type is called an $\mathbf{L}$-equational (or simply equational) class if $\mathcal{K} = \mathrm{Mod}(\Sigma)$ for some $\mathbf{L}$-set Σ. In what follows we attempt to provide an algebraic characterization of equational classes.

We need to recall further notions for $\mathbf{L}$-algebras (see also [1]): A morphism of an $\mathbf{L}$-algebra $\mathbf{M}_1$ to an $\mathbf{L}$-algebra $\mathbf{M}_2$ is a mapping $h : M_1 \to M_2$ such that $(m \approx^{\mathbf{M}_1} m') \leq (h(m) \approx^{\mathbf{M}_2} h(m'))$ and $h(f^{\mathbf{M}_1}(m_1, \ldots, m_n)) = f^{\mathbf{M}_2}(h(m_1), \ldots, h(m_n))$ for each n-ary $f \in F$. If h is, moreover, a bijection and $(m \approx^{\mathbf{M}_1} m') = (h(m) \approx^{\mathbf{M}_2} h(m'))$, we call h an isomorphism. By a congruence on an $\mathbf{L}$-algebra $\mathbf{M}$ we understand an $\mathbf{L}$-equivalence relation θ on M satisfying $(m_1 \approx^{\mathbf{M}} m_2) \leq \theta(m_1, m_2)$ for $m_1, m_2 \in M$, and $\theta(m_1, m_1') \otimes \cdots \otimes \theta(m_n, m_n') \leq \theta(f^{\mathbf{M}}(m_1, \ldots, m_n), f^{\mathbf{M}}(m_1', \ldots, m_n'))$ for each n-ary $f \in F$ and $m_i, m_i' \in M$. Note that condition $(m_1 \approx^{\mathbf{M}} m_2) \leq \theta(m_1, m_2)$ is equivalent to $\theta(m_1, m_2) \otimes (m_1 \approx^{\mathbf{M}} m_1') \otimes (m_2 \approx^{\mathbf{M}} m_2') \leq \theta(m_1', m_2')$. Indeed, from $(m_1 \approx^{\mathbf{M}} m_2) \leq \theta(m_1, m_2)$ we get $\theta(m_1, m_2) \otimes (m_1 \approx^{\mathbf{M}} m_1') \otimes (m_2 \approx^{\mathbf{M}} m_2') \leq \theta(m_1, m_2) \otimes \theta(m_1, m_1') \otimes \theta(m_2, m_2') \leq \theta(m_1', m_2')$ and, conversely, we get $(m_1 \approx^{\mathbf{M}} m_2) = \theta(m_1, m_1) \otimes (m_1 \approx^{\mathbf{M}} m_2) \otimes (m_2 \approx^{\mathbf{M}} m_2) \leq \theta(m_1, m_2)$. The set of all congruences on $\mathbf{M}$ will be denoted by $\mathrm{Con}(\mathbf{M})$. A factor structure of an $\mathbf{L}$-algebra $\mathbf{M}$ by a congruence θ on $\mathbf{M}$ will be understood to be an $\mathbf{L}$-structure $\mathbf{M}/\theta$ defined as follows: M/θ is $M/{}^1\theta$ (with elements $[m]_{^1\theta} = \{m' \mid \theta(m, m') = 1\}$ denoted also simply by $[m]_\theta$ or even $[m]$); $([m]_\theta \approx^{\mathbf{M}/\theta} [m']_\theta) = \theta(m, m')$; $f^{\mathbf{M}/\theta}([m_1]_\theta, \ldots, [m_2]_\theta) = [f^{\mathbf{M}}(m_1, \ldots, m_n)]_\theta$. One can easily verify that the thus-defined notions are correct, cf. [1].

Given $\mathbf{L}$-algebras $\mathbf{M}$ and $\mathbf{N}$ of type F, we say that $\mathbf{M}$ is a subalgebra of $\mathbf{N}$ if $M \subseteq N$, $f^{\mathbf{M}}$ is a restriction of $f^{\mathbf{N}}$ to M for each $f \in F$, and $\approx^{\mathbf{M}}$ is a restriction of $\approx^N$ to M.

A direct product (over an index set I) of $\mathbf{L}$-algebras $\mathbf{M}_i$ of type F is an $\mathbf{L}$-algebra $\times_{i \in I} \mathbf{M_i}$ of type F with its universe being the direct product of M_i's, the operations on $\times_{i \in I} \mathbf{M}_i$ being defined as usual (componentwise), and $\approx^{\times_i \mathbf{M}_i}$ being defined by

$$(m \approx^{\times_i \mathbf{M}_i} n) = \bigwedge_{i \in I} (m(i) \approx^{\mathbf{M}_i} n(i))$$

for $m, n \in \times_{i \in I} \mathbf{M}_i$.

For a class $\mathcal{K}$ of $\mathbf{L}$-algebras of the same type we define four operators: H, I, S, P. $\mathrm{H}(\mathcal{K})$ is the class of all homomorphic images of $\mathbf{L}$-algebras from $\mathcal{K}$, i.e.

$$\mathrm{H}(\mathcal{K}) = \{h(\mathbf{M}) \mid \mathbf{M} \in \mathcal{K},\ h \text{ a morphism}\};$$

$\mathrm{I}(\mathcal{K})$ is the class of all $\mathbf{L}$-algebras isomorphic to some $\mathbf{M} \in \mathcal{K}$, i.e.

$$\mathrm{I}(\mathcal{K}) = \{\mathbf{N} \mid \mathbf{N} \text{ is isomorphic to some } \mathbf{M} \in \mathcal{K}\};$$

$\mathrm{S}(\mathcal{K})$ is the class of all substructures of $\mathbf{L}$-algebras from $\mathcal{K}$, i.e.

$$\mathrm{S}(\mathcal{K}) = \{\mathbf{N} \mid \mathbf{N} \text{ is a substructure of some } \mathbf{M} \in \mathcal{K}\};$$

P($\mathcal{K}$) is the class of all direct products of **L**-algebras from $\mathcal{K}$, i.e.

$$\mathrm{P}(\mathcal{K}) = \{\times_{i\in I}\mathbf{M}_i \mid I \text{ an index set}, \mathbf{M}_i \in \mathcal{K}\}.$$

Here, a homomorphic image of **M** is an **L**-algebra $\mathbf{N} = h(\mathbf{M})$ for which there is a morphism $h : \mathbf{M} \to \mathbf{N}$ such that h is a surjective mapping. The operators may be composed, i.e. we may have HS($\mathcal{K}$), HHPHS($\mathcal{K}$), etc.

Now, a class $\mathcal{K}$ of **L**-algebras of the same type is called a variety if it is closed under H, S, and P (i.e. if $\mathrm{H}(\mathcal{K}) \subseteq \mathcal{K}$, $\mathrm{S}(\mathcal{K}) \subseteq \mathcal{K}$, and $\mathrm{P}(\mathcal{K}) \subseteq \mathcal{K}$). The following is the main result of this paper.

Theorem 1 (variety theorem). *A class $\mathcal{K}$ of* **L***-algebras of a given type is equational iff it is a variety.*

From the above it is immediate that Theorem 1 generalizes the well-known Birkhoff theorem [3] to fuzzy setting.

3. Proof, remarks

Let $\mathcal{K}$ be a class of **L**-algebras of the same type, let **M** be an **L**-algebra generated by $M' \subseteq M$ (i.e. **M** is the least subalgebra of **M** containing $M' \subseteq M$). If for each $\mathbf{N} \in \mathcal{K}$ and for each mapping $g : M' \to N$ preserving $\approx$ (i.e. such that $(m \approx^{\mathbf{M}} m') \leq (g(m) \approx^{\mathbf{N}} g(m'))$) there exists a morphism $h : \mathbf{M} \to \mathbf{N}$ extending g (i.e. $h(m) = g(m)$ for each $m \in M'$), we say that **M** has a universal mapping property for $\mathcal{K}$ over M'; in this case M' is said to be a set of free generators of **M** over $\mathcal{K}$.

Lemma 2. *If* **M** *has a universal mapping property for $\mathcal{K}$ over M' and $\mathbf{N} \in \mathcal{K}$ then for any $g : M' \to N$ there exists a unique morphism $h : M \to N$ extending g.*

Proof. The proof follows from a standard argument: a morphism is uniquely determined by its restriction to the set of generators. □

Lemma 3. $\mathbf{T}(X)$ *has a universal mapping property for any class $\mathcal{K}$ of* **L***-algebras of a given.*

Proof. For any $\mathbf{M} \in \mathcal{K}$ and any mapping $g : X \to M$, define $h : T(X) \to M$ inductively by $h(x) = g(x)$ for $x \in X$; $h(f(t_1, \ldots, t_n)) = f^{\mathbf{M}}(h(t_1), \ldots, h(t_n))$ for $f \in F$ and $t_i \in T(X)$. Trivially, h is a morphism. □

Let $\mathcal{K}$ be a class of **L**-algebras of the same type, let X be a set of variables. Put

$$\Phi_{\mathcal{K}}(X) = \{\phi \in \mathrm{Con}(\mathbf{T}(X)) \mid \mathbf{T}(X)/\phi \in \mathrm{IS}(\mathcal{K})\},$$

i.e. $\Phi_{\mathcal{K}}$ is the set of all congruences ϕ on $\mathbf{T}(X)$ such that the factor **L**-algebra $\mathbf{T}(X)/\phi$ is isomorphic to some substructure of some $\mathbf{M} \in \mathcal{K}$. Denote furthermore

$$\theta_{\mathcal{K}}(X) = \bigcap \Phi_{\mathcal{K}}(X)$$

the intersection of all congruences from $\Phi_{\mathcal{K}}(X)$. It is immediate to verify that $\theta_{\mathcal{K}}(X)$ is a congruence on $\mathbf{T}(X)$. We may thus form a factor $\mathbf{L}$-algebra $\mathbf{T}(X)/\theta_{\mathcal{K}}(X)$. For $x \in X$, denote by $\overline{x}$ the class $[x]_{\theta_{\mathcal{K}}(X)}$ and put $\overline{X} = \{\overline{x} \mid x \in X\}$. $\overline{X}$ is the set of generators of $\mathbf{T}(X)/\theta_{\mathcal{K}}(X)$. For convenience, we denote $\mathbf{T}(X)/\theta_{\mathcal{K}}(X)$ by $\mathcal{F}_{\mathcal{K}}(\overline{X})$ and call it the $\mathcal{K}$-free $\mathbf{L}$-algebra over $\overline{X}$.

For a morphism $h : \mathbf{M} \to \mathbf{N}$ denote by θ_h an $\mathbf{L}$-relation on M defined by $\theta_h(m_1, m_2) = (h(m_1) \approx^{\mathbf{N}} h(m_2))$. For a congruence θ on $\mathbf{M}$ denote by h_θ a mapping from M to M/θ sending m to $[m]_\theta$. It is routine to verify that θ_h is a congruence on $\mathbf{M}$ and that h_θ is a morphism of $\mathbf{M}$ to $\mathbf{M}/\theta$. Moreover, for congruences θ and ψ on $\mathbf{M}$ such that $\theta \subseteq \psi$ we denote by ψ/θ an $\mathbf{L}$-relation on M/θ defined by $\psi/\theta([m_1]_\theta, [m_2]_\theta) = \psi(m_1, m_2)$. An easy verification shows that ψ/θ is a congruence on $\mathbf{M}/\theta$.

Lemma 4. *$\mathcal{F}_{\mathcal{K}}(\overline{X})$ has a universal mapping property for $\mathcal{K}$ over $\overline{X}$.*

Proof. Let $\mathbf{M} \in \mathcal{K}$ and take a mapping $g : \overline{X} \to M$. Let $n : \mathbf{T}(X) \to \mathcal{F}_{\mathcal{K}}(\overline{X})$ denote the natural morphism (i.e. $n(t) = [t]_{\theta_{\mathcal{K}}(X)}$). Letting n_X denote the restriction of n to X, universal mapping property of $\mathbf{T}(X)$ implies that there is a morphism $k : \mathbf{T}(X) \to \mathbf{M}$ extending $n_X \circ g$. As $\mathbf{T}(X)/\theta_k$ is isomorphic to $k(\mathbf{T}(X))$ which is a substructure of $\mathbf{M}$, definition of $\theta_{\mathcal{K}}(X)$ implies that $\theta_{\mathcal{K}}(X) \subseteq \theta_k$.

We claim that there is a morphism $h : \mathcal{F}_{\mathcal{K}}(\overline{X}) \to \mathbf{M}$ such that $n \circ h = k$. Indeed, let $h = h_{\theta_k/\theta_{\mathcal{K}}(X)} \circ i_1 \circ i_2$ where $i_1 : [[t]_{\theta_{\mathcal{K}}(X)}]_{\theta_k/\theta_{\mathcal{K}}(X)} \mapsto [t]_{\theta_k}$ and $i_2 : [t]_{\theta_k} \mapsto k(t)$ are isomorphisms of $\mathcal{F}_{\mathcal{K}}(\overline{X})/(\theta_k/\theta_{\mathcal{K}}(X))$ to $\mathbf{T}(X)/\theta_k$ and of $\mathbf{T}(X)/\theta_k$ to $k(\mathbf{T}(X))$, respectively. Then $n \circ h(t) = h([t]_{\theta_{\mathcal{K}}(X)}) = i_2(i_1(h_{\theta_k/\theta_{\mathcal{K}}(X)}([t]_{\theta_{\mathcal{K}}(X)}))) = k(t)$.

Now, we have $h(\overline{x}) = h(n(x)) = n \circ h(x) = k(x) = n_X \circ g = g(\overline{x})$ showing that h extends g. □

Lemma 5. *If $\mathbf{T}(X)$ exists then for $\mathcal{K} \neq \emptyset$ we have $\mathcal{F}_{\mathcal{K}}(\overline{X}) \in \mathrm{ISP}(\mathcal{K})$. Thus, particularly, if $\mathcal{K}$ is a variety then $\mathcal{F}_{\mathcal{K}}(\overline{X}) \in \mathcal{K}$.*

Proof. First, we claim that $\mathcal{F}_{\mathcal{K}}(\overline{X})$ is isomorphic to a subalgebra of a direct product of $\mathbf{T}(X)/\phi$ for $\phi \in \Phi_{\mathcal{K}}(X)$, i.e. we claim that $\mathcal{F}_{\mathcal{K}}(\overline{X}) \in \mathrm{ISP}(\{\mathbf{T}(X)/\phi \mid \phi \in \Phi_{\mathcal{K}}\})$. Recall that $\mathcal{F}_{\mathcal{K}}(\overline{X}) = \mathbf{T}(X)/\theta_{\mathcal{K}}(X)$ where $\theta_{\mathcal{K}}(X) = \bigcap \Phi_{\mathcal{K}}(X)$. We verify that the mapping

$$i : \mathcal{F}_{\mathcal{K}}(\overline{X}) \to \times_{\phi \in \Phi_{\mathcal{K}}(X)} \mathbf{T}(X)/\phi$$

sending $[t]_{\theta_{\mathcal{K}}(X)}$ to $\langle \ldots, [t]_\phi, \ldots \rangle$ is an isomorphism of $\mathcal{F}_{\mathcal{K}}(\overline{X})$ to the subalgebra $i(\mathcal{F}_{\mathcal{K}}(\overline{X}))$ of $\times_{\phi \in \Phi_{\mathcal{K}}(X)} \mathbf{T}(X)/\phi$. We claim that i is an injection. Indeed, if $i([t_1]_{\theta_{\mathcal{K}}(X)}) = i([t_2]_{\theta_{\mathcal{K}}(X)})$ then for each $\phi \in \Phi_{\mathcal{K}}(X)$ we have $[t_1]_\phi = [t_2]_\phi$, i.e. $\phi(t_1, t_2) = 1$ and thus also $(\theta_{\mathcal{K}}(X))(t_1, t_2) = 1$, i.e. $[t_1]_{\theta_{\mathcal{K}}(X)} = [t_2]_{\theta_{\mathcal{K}}(X)}$. Furthermore,

$$\begin{aligned}
&(\langle \ldots, [t_1]_\phi, \ldots \rangle \approx^{\times \mathbf{T}(X)/\phi} \langle \ldots, [t_2]_\phi, \ldots \rangle) \\
&= \bigwedge_{\phi \in \Phi_{\mathcal{K}}(X)} \phi(t_1, t_2) = (\bigwedge_{\phi \in \Phi_{\mathcal{K}}(X)} \phi)(t_1, t_2) = \theta_{\mathcal{K}}(X)(t_1, t_2) \\
&= ([t_1]_{\theta_{\mathcal{K}}(X)} \approx^{\mathcal{F}_{\mathcal{K}}(\overline{X})} [t_2]_{\theta_{\mathcal{K}}(X)}).
\end{aligned}$$

Obviously, i is a morphism. Therefore, i is an isomorphism of $\mathcal{F}_{\mathcal{K}}(\overline{X})$ to $i(\mathcal{F}_{\mathcal{K}}(\overline{X}))$.

Now, by definition, for each $\phi \in \Phi_{\mathcal{K}}(X)$, $\mathbf{T}(X)/\phi$ belongs to $\mathrm{IS}(\mathcal{K})$. To sum up, we have $\mathcal{F}_{\mathcal{K}}(\overline{X}) \in \mathrm{ISPIS}(\mathcal{K})$.

We need to show that $\mathrm{ISPIS}(\mathcal{K}) \subseteq \mathrm{ISP}(\mathcal{K})$. This is easy: we have $\mathrm{ISPIS}(\mathcal{K}) = \mathrm{ISPS}(\mathcal{K}) \subseteq \mathrm{ISSP}(\mathcal{K}) = \mathrm{ISP}(\mathcal{K})$. □

Lemma 6. *For a class $\mathcal{K}$ of **L**-algebras and $t_1, t_2 \in T(X)$ we have*

$$\|t_1 \approx t_2\|_{\mathcal{K}} = \bigwedge_{\mathbf{M} \in \mathcal{K}} \bigwedge_{h:\mathbf{T}(X) \to \mathbf{M}} (h(t_1) \approx^{\mathbf{M}} h(t_2)).$$

Proof. By definition we have

$$\|t_1 \approx t_2\|_{\mathcal{K}} = \bigwedge_{\mathbf{M} \in \mathcal{K}} \bigwedge_{v:X \to M} (\|t_1\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|t_2\|_{\mathbf{M},v}).$$

The assertion follows from the fact that there is a bijective correspondence between morphisms $h : \mathbf{T}(X) \to \mathbf{M}$ and valuations $v : X \to M$ (this follows from the universal mapping property of $\mathbf{T}(X)$, Lemma 3) and from the fact that for $h : \mathbf{T}(X) \to \mathbf{M}$ and v being the restriction of h to X we have $h(t_i) = \|t_i\|_{\mathbf{M},v}$. □

For a class $\mathcal{K}$ of **L**-algebras and a set X of variables denote by $\mathrm{Id}_X(\mathcal{K})$ the **L**-set of identities over X of $\mathcal{K}$, i.e. for $t_1, t_2 \in T(X)$ we have

$$\mathrm{Id}_X(\mathcal{K})(t_1, t_2) = \|t_1 \approx t_2\|_{\mathcal{K}}.$$

Lemma 7. *For a class $\mathcal{K}$ of **L**-algebras we have*

$$\mathrm{Id}_X(\mathcal{K}) = \mathrm{Id}_X(\mathrm{I}(\mathcal{K})) = \mathrm{Id}_X(\mathrm{H}(\mathcal{K})) = \mathrm{Id}_X(\mathrm{S}(\mathcal{K})) = \mathrm{Id}_X(\mathrm{P}(\mathcal{K})).$$

Proof. First we show $\mathrm{Id}_X(\mathcal{K}) = \mathrm{Id}_X(\mathrm{I}(\mathcal{K}))$. Since $\mathcal{K} \subseteq \mathrm{I}(\mathcal{K})$ we have $\mathrm{Id}_X(\mathcal{K}) \supseteq \mathrm{Id}_X(\mathrm{I}(\mathcal{K}))$. Conversely, by Lemma 6, we have to show that for each $t_1, t_2 \in T(X)$, each isomorphism $g : \mathbf{M} \to \mathbf{N}$ where $\mathbf{M} \in \mathcal{K}$ and each morphism $h : \mathbf{T}(X) \to \mathbf{N}$ we have

$$(\mathrm{Id}_X(\mathcal{K}))(t_1 \approx t_2) \leq (h(t_1) \approx^{\mathbf{N}} h(t_2)).$$

However, this is true since morphisms $h : \mathbf{T}(X) \to \mathbf{N}$ are in a bijective correspondence with morphisms of $\mathbf{T}(X)$ to $\mathbf{M}$ (h corresponds to $h \circ g^{-1}$), we have $(h(t_1) \approx^{\mathbf{M}} h(t_2)) = (h \circ g^{-1}(t_1) \approx^{\mathbf{M}} h \circ g^{-1}(t_2))$, and one can apply Lemma 6.

Next, since $\mathcal{K} \subseteq \mathcal{O}(\mathcal{K})$ we have $\mathrm{Id}_X(\mathcal{K}) \supseteq \mathrm{Id}_X(\mathcal{O}(\mathcal{K}))$ for $\mathcal{O} = \mathrm{H}$, $\mathcal{O} = \mathrm{S}$, or $\mathcal{O} = \mathrm{P}$ (in fact, we have $\mathcal{K} \subseteq \mathrm{IP}(\mathcal{K})$; however, since $\mathrm{Id}_X(\mathcal{K}) = \mathrm{Id}_X(\mathrm{I}(\mathcal{K}))$, we may afford to neglect this). We thus need to establish the converse inclusions, i.e. to verify $(\mathrm{Id}_X(\mathcal{K}))(t_1 \approx t_2) \leq (\mathrm{Id}_X(\mathcal{O}(\mathcal{K})))(t_1 \approx t_2)$.

For $\mathrm{H}(\mathcal{K})$, we need to show that for each $\mathbf{M} \in \mathcal{K}$, morphism $h : \mathbf{M} \to \mathbf{N}$, $\mathbf{N} = h(\mathbf{M})$, and a valuation $v : X \to N$ we have $(\mathrm{Id}_X(\mathcal{K}))(t_1 \approx t_2) \leq (\|t_1\|_{\mathbf{N},v} \approx^{\mathbf{N}} \|t_2\|_{\mathbf{N},v})$. Due to surjectivity of h we may take $w : X \to M$ such that $h(w(x)) = v(x)$. Then we have

$$\begin{aligned}(\mathrm{Id}_X(\mathcal{K}))(t_1 \approx t_2) &\leq (\|t_1\|_{\mathbf{M},w} \approx^{\mathbf{M}} \|t_2\|_{\mathbf{M},w})\\ &\leq (h(\|t_1\|_{\mathbf{M},w}) \approx^{\mathbf{N}} h(\|t_2\|_{\mathbf{M},w})) = (\|t_1\|_{\mathbf{M},w\circ h} \approx^{\mathbf{N}} \|t_2\|_{\mathbf{M},w\circ h})\\ &= (\|t_1\|_{\mathbf{M},v} \approx^{\mathbf{N}} \|t_2\|_{\mathbf{M},v}).\end{aligned}$$

For $\mathrm{S}(\mathcal{K})$, we need to show that for each substructure $\mathbf{N}$ of some $\mathbf{M} \in \mathcal{K}$ and each valuation $v : X \to N$ we have $(\mathrm{Id}_X(\mathcal{K}))(t_1 \approx t_2) \leq (\|t_1\|_{\mathbf{N},v} \approx^{\mathbf{N}} \|t_2\|_{\mathbf{N},v})$. However, this is true since $(\|t_1\|_{\mathbf{N},v} \approx^{\mathbf{N}} \|t_2\|_{\mathbf{N},v}) = (\|t_1\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|t_2\|_{\mathbf{M},v})$.

For $\mathrm{P}(\mathcal{K})$, we need to show that for each $\mathbf{N} = \times_{i\in I}\mathbf{M}_i$ ($\mathbf{M}_i \in \mathcal{K}$) and each valuation $v : X \to N$ we have $(\mathrm{Id}_X(\mathcal{K}))(t_1 \approx t_2) \leq (\|t_1\|_{\mathbf{N},v} \approx^{\mathbf{N}} \|t_2\|_{\mathbf{N},v})$ which is true iff for each $i \in I$ we have $(\mathrm{Id}_X(\mathcal{K}))(t_1 \approx t_2) \leq (\|t_1\|_{\mathbf{N},v(i)} \approx^{\mathbf{M}_i} \|t_2\|_{\mathbf{N},v(i)})$ which is true. □

Lemma 8. *For a class $\mathcal{K}$ of $\mathbf{L}$-algebras and $t_1, t_2 \in T(X)$ we have*

$$\|t_1 \approx t_2\|_{\mathcal{K}} = \|t_1 \approx t_2\|_{\mathcal{F}_{\mathcal{K}}(\overline{X})} = ([t_1]_{\theta_{\mathcal{K}}} \approx^{\mathcal{F}_{\mathcal{K}}(\overline{X})} [t_2]_{\theta_{\mathcal{K}}}) = \theta_{\mathcal{K}}(t_1, t_2).$$

Proof. For convenience, denote $\mathcal{F}_{\mathcal{K}}(\overline{X})$ by $\mathcal{F}$. We prove the assertion by showing $\|t_1 \approx t_2\|_{\mathcal{K}} \leq \|t_1 \approx t_2\|_{\mathcal{F}} \leq ([t_1]_{\theta_{\mathcal{K}}} \approx^{\mathcal{F}} [t_2]_{\theta_{\mathcal{K}}}) \leq \theta_{\mathcal{K}}(t_1, t_2) \leq \|t_1 \approx t_2\|_{\mathcal{K}}$.

$\|t_1 \approx t_2\|_{\mathcal{K}} \leq \|t_1 \approx t_2\|_{\mathcal{F}}$ follows from $\mathcal{F} \in \mathrm{ISP}(\mathcal{K})$ (Lemma 5) and Lemma 7.

$\|t_1 \approx t_2\|_{\mathcal{F}} \leq ([t_1]_{\theta_{\mathcal{K}}} \approx^{\mathcal{F}} [t_2]_{\theta_{\mathcal{K}}})$ is true since for $v' : X \to \mathcal{F}$ sending x to $[x]_{\theta_{\mathcal{K}}}$ we have

$$\begin{aligned}\|t_1 \approx t_2\|_{\mathcal{F}} &= \bigwedge_{v:X\to\mathcal{F}} (\|t_1\|_{\mathcal{F},v} \approx^{\mathcal{F}} \|t_2\|_{\mathcal{F},v}) \\ &\leq (\|t_1\|_{\mathcal{F},v'} \approx^{\mathcal{F}} \|t_2\|_{\mathcal{F},v'}) = ([t_1]_{\theta_{\mathcal{K}}} \approx^{\mathcal{F}} [t_2]_{\theta_{\mathcal{K}}}).\end{aligned}$$

$([t_1]_{\theta_{\mathcal{K}}} \approx^{\mathcal{F}} [t_2]_{\theta_{\mathcal{K}}}) \leq \theta_{\mathcal{K}}(t_1, t_2)$ holds true by definition since $\mathcal{F} = \mathbf{T}(X)/\theta_{\mathcal{K}}(X)$.

$\theta_{\mathcal{K}}(t_1, t_2) \leq \|t_1 \approx t_2\|_{\mathcal{K}}$ is true iff for each $\mathbf{M} \in \mathcal{K}$ and each valuation $v : X \to M$ we have $\theta_{\mathcal{K}}(t_1, t_2) \leq (\|t_1\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|t_2\|_{\mathbf{M},v})$. However, due to the universal mapping property of $\mathbf{T}(X)$, v can be extended to a morphism $h : \mathbf{T}(X) \to \mathbf{M}$ for which we have $\theta_h(t_1, t_2) = (\|t_1\|_{\mathbf{M},v} \approx^{\mathbf{M}} \|t_2\|_{\mathbf{M},v})$. As $\mathbf{T}(X)/\theta_h \in \mathrm{IS}(\mathcal{K})$, definition of $\theta_{\mathcal{K}}(X)$ gives $\theta_{\mathcal{K}}(X) \subseteq \theta_h$. The required inequality now readily follows. □

Lemma 9. *For an infinite set Y of variables we have* $\mathrm{Id}_X(\mathcal{K}) = \mathrm{Id}_X(\mathcal{F}_{\mathcal{K}}(\overline{Y}))$.

Proof. Pick $Z \supseteq X$ such that $|Z| = |Y|$. Then, obviously, $\mathcal{F}_{\mathcal{K}}(\overline{Y})$ is isomorphic to $\mathcal{F}_{\mathcal{K}}(\overline{Z})$. Furthermore, for any identity $p \approx q$ where $p, q \in T(X)$ we have $p, q \in T(Z)$, whence by Lemma 8, $[\mathrm{Id}_X(\mathcal{K})](p \approx q) = \|p \approx q\|_{\mathcal{K}} = \|p \approx q\|_{\mathcal{F}_{\mathcal{K}}(\overline{Z})} = \|p \approx q\|_{\mathcal{F}_{\mathcal{K}}(\overline{Z})}$. □

Lemma 10. *If $\mathcal{V}$ is a variety of $\mathbf{L}$-algebras and X an infinite set of variables, then* $\mathcal{V} = \mathrm{Mod}(\mathrm{Id}_X(\mathcal{V}))$.

Proof. Denote $\mathcal{V}' = \mathrm{Mod}(\mathrm{Id}_X(\mathcal{V}))$. $\mathcal{V}'$ is a variety. Indeed, this follows from Lemma 7: Let $\mathcal{O}$ denote any of H, S, or P. For any identity $t_1 \approx t_2$ we have $\|t_1 \approx t_2\|_{\mathcal{O}(\mathcal{V}')} = \|t_1 \approx t_2\|_{\mathcal{V}'} \geq (\mathrm{Id}_X(\mathcal{V}))(t_1 \approx t_2)$ which means that for any $\mathbf{M} \in \mathcal{O}(\mathcal{V}')$ we have $\|t_1 \approx t_2\|_{\mathbf{M}} \geq (\mathrm{Id}_X(\mathcal{V}))(t_1 \approx t_2)$ whence $\mathbf{M} \in \mathcal{V}'$. Obviously, $\mathcal{V} \subseteq \mathcal{V}'$. This implies $\mathrm{Id}_X(\mathcal{V}') \subseteq \mathrm{Id}_X(\mathcal{V})$. Conversely, $\mathrm{Id}_X(\mathcal{V}) \subseteq \mathrm{Id}_X(\mathcal{V}')$ is true iff for each $\mathbf{M} \in \mathcal{V}'$ we have $(\mathrm{Id}_X(\mathcal{V}))(t_1 \approx t_2) \leq \|t_1 \approx t_2\|_{\mathbf{M}}$ which is true by definition of $\mathcal{V}'$. We thus have $\mathrm{Id}_X(\mathcal{V}') = \mathrm{Id}_X(\mathcal{V})$.

By Lemma 8, $\mathcal{F}_{\mathcal{V}}(\overline{X}) = \mathcal{F}_{\mathcal{V}'}(\overline{X})$. For each infinite set Y of variables we have by Lemma 9 that $\mathrm{Id}_Y(\mathcal{V}') = \mathrm{Id}_Y(\mathcal{F}_{\mathcal{V}'}(\overline{X})) = \mathrm{Id}_Y(\mathcal{F}_{\mathcal{V}}(\overline{X})) = \mathrm{Id}_Y(\mathcal{V})$. Lemma 8 thus implies $\theta_{\mathcal{V}'}(Y) = \theta_{\mathcal{V}}(Y)$, i.e. $\mathcal{F}_{\mathcal{V}'}(\overline{Y}) = \mathcal{F}_{\mathcal{V}}(\overline{Y})$. Now, for each $\mathbf{M} \in \mathcal{V}'$ we have for some infinite Y that $\mathbf{M} \in \mathrm{H}(\mathcal{F}_{\mathcal{V}'}(\overline{Y}))$ (indeed, it suffices to have $|Y| \geq |M|$ and to consider a morphism induced by some surjection of Y to M) and thus $\mathbf{M} \in \mathrm{H}(\mathcal{F}_{\mathcal{V}}(\overline{Y}))$ whence $\mathbf{M} \in \mathcal{V}$. We thus established $\mathcal{V}' \subseteq \mathcal{V}$ and so $\mathcal{V} = \mathcal{V}'$. □

We can now prove Theorem 1.

Proof of Theorem 1. Let $\mathcal{K}$ be equational, i.e. $\mathcal{K} = \mathrm{Mod}(\Sigma)$ for some $\mathbf{L}$-set Σ of identities. Using Lemma 7, one can check that $\mathcal{K}$ is a variety (the same arguments as in the beginning of proof of Lemma 10).

Conversely, if $\mathcal{V}$ is a variety then Lemma 10 implies that for an infinite set X of variables and $\Sigma = \mathrm{Id}_X(\mathcal{V})$ we have $\mathcal{K} = \mathrm{Mod}(\Sigma)$, i.e. $\mathcal{K}$ is equational. □

Remark 1. Using fuzzy sets of identities, we follows the so-called Pavelka's style [7]. Pavelka showed [10] that the requirement of syntactico-semantical completeness implies severe limitations to the structure of truth values. It is thus worth to note that the variety theorem is valid for any complete residuated lattice taken as the structure of truth values. In the light of [1], however, this is not that much surprising.

Remark 2. The concept of an $\mathbf{L}$-algebra is by no means artificial. First, it is obvious that the compatibility axiom expresses a natural constraint on the operations (mapping similar to similar). If one takes, e.g., $L = [0, 1]$, this constraint has a numerical character. Moreover, this constraint is expressed in a simple fragment of first-order fuzzy logic.

Second, allowing for fuzzy sets of axioms is well-recognized as a useful means of representing vaguely specified knowledge (expert observations, requirements, etc.). For example, one may require that particular products must be highly safe and adds therefore formula $(\forall x)(P(x)\ \mathtt{i}\ S(x))$ with degree 0.8 to the fuzzy set of axioms (P and S denote "product" and "safe", respectively). Note that, technically, this can still be achieved without fuzzifying the metalevel (with a set of axioms), see [7]. Having fuzzy sets of axioms, i.e. identities, is natural even in case of the equational fragment. As an example, consider almost reversible processes: Suppose we have objects (e.g. pools filled with water) and a pair of inverse operations o and i transforming these objects (o and i might be "drain one liter of water" and "pour in one liter of water"). When specifying requirements, it may be worth to take certain technical limitations into account and to require that applying i and o consecutively, one gets almost the same object. Therefore, it might be worth to add the identity $x \approx o(i(x))$ with degree, say, 0.9 to the fuzzy sets of axioms. So, in our example, since draining and pouring can never be achieved with absolute accuracy, it makes sense to require that pouring in one liter and then draining one liter from a given pool yields approximately the same pool. Changing the degree (0.9) in the set of axioms sets the requirement on the accuracy and synchronization of operations i and o.

Remark 3. In [12] (see also [11]), the author investigates the so-called metric algebras which are basically algebras with a metric on the support set. It is interesting to note that the author introduces the notion of an atomic inequality which is an expression of the form "$\rho(p, q) \preceq a$" where ρ denotes a metric, p, q are terms, and $a \in [0, \infty]$. An atomic inequality is said to be ϵ-true in a given metric algebra if $\rho(p, q) \leq a + \epsilon$ in the metric algebra. The analogy to the fuzzy logic approach is obvious. Furthermore, an important notion of [12] is that of an equicontinuity of operations of metric algebras: An operation f is equicontinuous if the implication

$$\rho(x_1, y_1) \preceq 0 \wedge \cdots \wedge \rho(x_n, y_n) \preceq 0 \quad \text{i} \quad \rho(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \preceq 0$$

is satisfied equicontinuously. This means that for each $\epsilon > 0$ there is a $\delta > 0$ such that for any interpretation we have that if each $\rho(x_i, y_i) \preceq 0$ is δ-true then $\rho(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \preceq 0$ is ϵ-true. Now it is clear that the idea behind equicontinuity of operations in metric algebras and compatibility of operations of **L**-algebras is essentially the same. In our opinion, however, the explicit formulation of the requirement in the framework of fuzzy logic (**L**-algebras) is far more natural. A paper investigating the relationships of metric algebras and **L**-algebras is in preparation. Note also that the notion of an **L**-similarity is more general than that of a metric [2].

Acknowledgements. Support by grant of the IGS of the University of Ostrava for 2002 is gratefully acknowledged.

References

[1] Bělohlávek R.: Fuzzy equational logic. Arch. Math. Log. **41**, 83–90 (2002)
[2] Bělohlávek R.: Similarity spaces (submitted)
[3] Birkhoff G.: On the structure of abstract algebras. Proc. Camb. Philos. Soc. **31**, 433–454 (1935)
[4] Goguen J. A.: L-fuzzy sets. J. Math. Anal. Appl. **18**, 145–174 (1967)
[5] Goguen J. A.: The logic of inexact concepts. Synthese **18**, 325–373 (1968–9)
[6] Gottwald S.: A Treatise on Many-Valued Logics. Research Studies Press, Baldock, Hertfordshire, England, 2001
[7] Hájek P.: Metamathematics of Fuzzy Logic. Kluwer, Dordrecht, 1998
[8] Höhle U.: On the fundamentals of fuzzy set theory. J. Math. Anal. Appl. **201**, 786–826 (1996)
[9] Novák V., Perfilieva I., Močkoř J.: Math. Principles of Fuzzy Logic. Kluwer, Boston, 1999
[10] Pavelka J.: On fuzzy logic I, II, III. Zeit. Math. Log. Grungl. Math. **25**, 45–52, 119–134, 447–464 (1979)
[11] Weaver N.: Generalized varieties. Algebra Universalis **30**, 27–52 (1993)
[12] Weaver N.: Quasi-varieties of metric algebras. Algebra Universalis **33**, 1–9 (1995)