A CHARACTERIZATION OF CONGRUENCE CLASSES OF QUASIGROUPS

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Abstract. The property of being a congruence class for a subset of a quasigroup is characterized by the notion of closeness under terms. The presented characterization is of polynomial time complexity.

There are various ways to introduce the concept of a normal subgroup. One of them uses the notion of closedness under certain group terms. A nonempty subset N of a group G is a normal subgroup (and hence a congruence class containing the unit e) iff it is closed under the terms $p_1(y_1, y_2) = y_1 \cdot y_2^{-1}$, $p_2(x_1, y_1) = x_1 \cdot y_1 \cdot x_1^{-1}$ in the following sense. Whenever we substitute arbitrary element $a \in G$ for x_1 and arbitrary elements $b_1, b_2 \in N$ for y_1 and y_2 the result will be in N.

In what follows we give a similar characterization of congruence classes of quasigroups. By a quasigroup (cf. [1]) we mean an algebra Q endowed with three binary operations $\cdot, \backslash$ and $/$ which satisfy the following identities :

$$\begin{aligned} x \cdot (x\backslash y) &= y \qquad (y/x) \cdot x = y \\ x\backslash(x \cdot y) &= y \qquad (y \cdot x)/x = y. \end{aligned}$$

As the operations $/$ and $\backslash$ can be interpreted as the right and left division, the quasigroup can be viewed as a uniquely both-side divisible and cancelable groupoid.

Given a quasigroup term p in variables x_i and y_j, $p = p(x_1, \ldots, x_n, y_1, \ldots, y_m)$, we say that a subset C of a quasigroup Q is $y-$closed under p if for all $a_1, \ldots, a_n \in Q$, $c_1, \ldots, c_m \in C$ we have $p(a_1, \ldots, a_n, c_1, \ldots, c_m) \in C$.

Theorem. *A nonempty subset C of a quasigroup Q is a class of some congruence relation on Q iff it is $y-$closed under the following terms:*

$$\begin{aligned} p_1(x_1, x_2, y_1, y_2, y_3) &= (x_1x_2)/(y_3\backslash[(y_3(y_1\backslash x_1))(y_3(y_2\backslash x_2))]) \\ p_2(x_1, x_2, y_1, y_2, y_3) &= (x_1/x_2)/(y_3\backslash[(y_3(y_1\backslash x_1))/(y_3(y_2\backslash x_2))]) \\ p_3(x_1, x_2, y_1, y_2, y_3) &= (x_1\backslash x_2)/(y_3\backslash[(y_3(y_1\backslash x_1))\backslash(y_3(y_2\backslash x_2))]) \\ p_4(y_1, y_2, y_3) &= y_1/(y_2\backslash y_3) \\ p_5(y_1, y_2, y_3) &= y_1(y_2\backslash y_3) \end{aligned}$$

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Proof. Let C be a class of some congruence relation, say θ, i.e. $C = [c]_\theta$ for $c \in C$. Let a_1, a_2 be in Q, c_1, c_2, c_3 in C. By the substitution property of θ we have

$$\langle p_1(a_1, a_2, c_1, c_2, c_3), p_1(a_1, a_2, c, c, c)\rangle \in \theta \ .$$

Since $p_1(a_1, a_2, c, c, c) = c$ we have $p_1(a_1, a_2, c_1, c_2, c_3) \in [c]_\theta = C$, i.e. C is $y-$closed under p_1. One can similarly observe the $y-$closeness under the other terms.

Conversely, let C be $y-$closed under the abovementioned terms, $c \in C$. We shall show that the relation θ definded by

$$\langle x, y\rangle \in \theta \ \Leftrightarrow y/(c\backslash x) \in C$$

is a congruence relation on Q with a class C. It is well-known that the variety of all quasigroups is congruence-permutable. Following [4], it is enough to prove the reflexivity and the substitution property of θ. For any $a \in Q$ we have $a/(c\backslash a) = c \in C$, proving reflexivity of θ. Suppose $\langle a_1, a_2\rangle \in \theta$, $\langle b_1, b_2\rangle \in \theta$. By definition of θ, $a_2/(c\backslash a_1) \in C$ and $b_2/(c\backslash b_1) \in C$. Since C is $y-$closed under p_1, we have

$$p(a_2, b_2, a_2/(c\backslash a_1), b_2/(c\backslash b_1), c) \in C \ .$$

On the other hand,

$$p(a_2, b_2, a_2/(c\backslash a_1), b_2/(c\backslash b_1), c) = (a_2 b_2)/(c\backslash(a_1 b_1)) \ ,$$

which gives $\langle a_1 b_1, a_2 b_2\rangle \in \theta$, proving the substitution property of the operation $\cdot$. Analogously, terms p_2 and p_3 ensure $\langle a_1/b_1, a_2/b_2\rangle \in \theta$ and $\langle a_1\backslash b_1, a_2\backslash b_2\rangle \in \theta$, i.e. θ is a congruence relation.

It remains to prove $C = [c]_\theta$. From $c' \in C$ it follows that

$$c/(c\backslash c') = p_4(c, c, c') \in C \ ,$$

i.e. $\langle c', c\rangle \in \theta$ which means $C \subseteq [c]_\theta$. If, conversely, $c' \in [c]_\theta$, we have $\langle c, c'\rangle \in \theta$, thus $c'' = c'/(c\backslash c) \in C$. But then $c' = c''(c\backslash c) = p_5(c'', c, c) \in C$, hence $[c]_\theta \subseteq C$. We have proved C is a class of the congruence relation θ. $\square$

Remark 1. A useful characterization of congruence classes of algebras was given by Mal'cev, see [3]: For an algebra (A, F) and a subset $C \subseteq A$, C is a class of some congruence relation iff $\tau(C) \cap C = \emptyset$ or $\tau(C) \subseteq C$ holds for any translation (i.e. unary algebraic function) τ of A. This characterization yields an infinite number of polynomials even in the case of finite algebra with finite similarity type. In the case of quasigroup, our Theorem gives only a finite number of translations to testify in Mal'cev characterization.

Remark 2. Call the group $\mathcal{M}(Q)$ of permutations of a quasigroup Q generated by all $R_a, L_a : Q \to Q$, $R_a(x) = x \cdot a$, $L_a(x) = a \cdot x$, $a \in Q$, the group associated to Q [1,2]. For $e \in Q$, the permutation $p \in \mathcal{M}(Q)$ is called an e-inner permutation if $p(e) = e$. Congruence classes of quasigroups (called normal subsets of quasigroups) are characterized in [2] as follows: If $C \subseteq Q$, $e \in C$, then C is a class of some congruence on Q iff (i) any e-inner permutation $p \in \mathcal{M}(Q)$ maps C into C (i.e. $p(C) \subseteq C$), and (ii) if $(a/e) \cdot b = c$ and any two of the elements a, b, c belong to C

then the third one belongs to C. Note that this result, like that of Mal'cev, yields possibly infinite number of functions (e-inner permutations) to testify, i.e. it does not give a finite list.

Remark 3. From the computational complexity point of view we deal with the following problem: Decide whether for a given finite quasigroup and its subset C, C is a congruence class of Q. The conventional approach, i.e. testing all relevant partitions of Q, leads to an algorithm of exponential time complexity with respect to $card\,Q$. It is easy to see that by Theorem, a polynomial number of steps is sufficient. Hence the problem is in P, the class of problems solvable in polynomial time.

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