Concept Equations

RADIM BĚLOHLÁVEK, *Department of Computer Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic.*
E-mail: radim.belohlavek@upol.cz
and Inst. Res. Appl. Fuzzy Modeling, University of Ostrava, Czech Republic.

Dedicated to George J. Klir

Abstract

Studied in the paper are systems of equations which naturally arise in the formalization of the Port-Royal theory of concepts. The unknown quantity is a relation between objects and attributes. We study the case where the relation is fuzzy with truth values in a complete residuated lattice, covering therefore the special cases of complete Boolean algebras, Heyting algebras, MV-algebras etc. We answer the question of solvability, structure of solutions, and show how solvability of non-solvable systems may be attained by so-called decrease of logical precision.

Keywords: Port-Royal logic, formal concept, relational equation, fuzzy logic.

1 Problem setting

Following Port-Royal logic [1], a concept consists of its extent A and its intent B. Extent is a collection of objects of a given set G of objects, intent is the collection of attributes of a given set M of attributes. By definition, a collection A of objects from G and a collection B of attributes from M forms a concept if each object of A has all the attributes of B and each attribute of B is shared by all the objects of A. Identify a concept determined by the extent A and the intent B with $\langle A, B\rangle$, and denote by $\mathcal{B}(G, M, I)$ the collection of all concepts determined in the above way by a set G of objects, a set M of attributes, and a relation I (interpreted as 'to have') between G and M. Rewriting the verbally expressed conditions into logical formulas yields $\langle A, B\rangle \in \mathcal{B}(G, M, I)$ iff

$$(\forall g)(g \in A \Rightarrow \langle g, m\rangle \in I) \quad \text{iff} \quad m \in B \tag{1.1}$$

and

$$(\forall m)(m \in B \Rightarrow \langle g, m\rangle \in I) \quad \text{iff} \quad g \in A. \tag{1.2}$$

Denote the collection of all $m \in M$ such that $(\forall g)(g \in A \Rightarrow \langle g, m\rangle \in I)$ by $A^{\uparrow_I}$, and the collection of all $g \in G$ such that $(\forall m)(m \in B \Rightarrow \langle g, m\rangle \in I)$ by $B^{\downarrow_I}$. Then, given G and M, the relation I determines a collection of concepts $\langle A, B\rangle$. Consider the inverse problem:

Given $T = \{\langle A^p, B^p\rangle \mid p \in P\}$ where A^p and B^p are collections of elements of G and M, respectively, is there a relation I between G and M such that (1.1) and (1.2) hold for any $\langle A, B\rangle \in T$? Putting in another way, we ask for conceptual consistency of T in the framework of Port-Royal logic: is there a relation I such that all couples of T may be interpreted as concepts determined by I? The above observations show that the question of conceptual consistency is equivalent to the question of solvability of a system $\{A^{\uparrow_I} = B, B^{\downarrow_I} = A \mid \langle A, B\rangle \in T\}$ of relational equations. For the above reasons, we call such equations concept equations.

In our paper we study the problem of solvability of systems of concept equations in the framework of logic with truth values in a complete residuated lattice, covering thus Boolean logic, intuitionistic logic, Łukasiewicz logic and other logical systems as special cases. In Section 2 we summarize the necessary background, Section 3 contains the results.

2 Preliminaries

Formal theory of formal concepts and hierarchical conceptual structures (so-called concept lattices) with emphasis on data analysis was started in [15] and since then substantially developed [8, 9]. The relation I, extents A, and intents B are classical sets, i.e. the truth value of 'an object has an attribute' is either 0 or 1 etc. Since this assumption of bivalence is unrealistic (typically, an object g has more or less an attribute m; for instance: g is a particular mineral and m is 'to be hard'), a generalization for the case that the set L of truth values forms a scale was proposed, see [2, 3, 7, 14]. Under this general approach, neither the truth value of 'an object belongs to the extent of a concept' is forced to be 0 or 1 (for instance, if $L = [0, 1]$, g is 'granite' and A is the extent of 'hard mineral' then the truth value of 'g belongs to A', i.e. 'granite is a hard mineral', may be 0.9), similarly for attributes and intents.

In the rest of this section we recall the necessary notions (for details we refer to [2, 3, 7]). A general structure of truth values is that of a complete residuated lattice, i.e. an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ where $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, $\langle L, \otimes, 1\rangle$ is a commutative monoid, and $\otimes, \rightarrow$ are binary operations which form an adjoint pair, i.e. $x \otimes y \leq z$ iff $x \leq y \rightarrow z$. Residuated lattices play important roles in many-valued logical calculi: for a syntactico-semantically complete first-order logic with truth values in complete residuated lattices see [13], for several other calculi with truth values in special residuated lattices see [12]. Well-known examples of residuated lattices are Boolean algebras (algebraic counterpart of classical logic), Heyting algebras (intuitionistic logic), MV-algebras (Łukasiewicz logic). $\otimes$ and $\rightarrow$ model conjunction and implication, respectively, the existential and universal quantifiers are modelled using suprema and infima, respectively. The most studied and applied set of truth values is the real interval $[0, 1]$ equipped with natural ordering, a continuous t-norm $\otimes$ (i.e. a continuous operation making $\langle [0, 1], \otimes, 1\rangle$ a partially ordered commutative monoid), and the corresponding residuum given by $a \rightarrow b = \max\{c \mid a \otimes c \leq b\}$. Each continuous t-norm may be composed of three basic ones (for details see [12]): Łukasiewicz ($a \otimes b = \max(a + b - 1, 0)$, $a \rightarrow b = \min(1 - a + b, 1)$), minimum ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and product ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ else). In what follows $\mathbf{L}$ will denote a complete residuated lattice.

Recall that an $\mathbf{L}$-set (or fuzzy set with truth values in $\mathbf{L}$) in a universe set X is any mapping $A : X \rightarrow L$ (i.e. $A \in L^X$). The value $A(x)$ is understood as the truth value (degree of truth) of 'x belongs to A'. Similarly, a binary $\mathbf{L}$-relation between G and M is a mapping $R : G \times M \rightarrow L$. For $A_1, A_2 \in L^X$ we put $A_1 \subseteq A_2$ iff $A_1(x) \leq A_2(x)$ holds for all $x \in X$.

Let I be a binary $\mathbf{L}$-relation between the sets G and M. For an $\mathbf{L}$-set A in G define an $\mathbf{L}$-set $A^{\uparrow_I}$ in M by

$$A^{\uparrow_I}(m) = \bigwedge_{g \in G} A(g) \rightarrow I(g, m).$$

Similarly, for $B \in L^M$ put

$$B^{\downarrow_I}(m) = \bigwedge_{m \in M} B(m) \rightarrow I(g, m).$$

For brevity we write only $^{\uparrow}$ and $^{\downarrow}$ if I is apparent from context. Clearly, $A^{\uparrow}(m)$ is the truth value of (1.1), i.e. the truth value of 'each object of A has the attribute m', and analogously for $B^{\downarrow}(g)$. The graded truth approach yields therefore the following definitions which comply with Port-Royal:

given G, M and $I \in L^{G\times M}$, an $\mathbf{L}$-concept is a pair $\langle A, B\rangle \in L^G \times L^M$ such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$. The set of all $\mathbf{L}$-concepts will be denoted by $\mathcal{B}\,(G, M, I)$. A natural hierarchy $\leq$ on $\mathcal{B}\,(G, M, I)$ given by $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (or, equivalently, $B_2 \subseteq B_1$) is the subject of the following theorem [3].

THEOREM 2.1
The set $\mathcal{B}\,(G, M, I)$ is under $\leq$ a complete lattice where the suprema and infima are given by

$$\bigwedge_{j\in J} \langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}\rangle\,, \tag{2.1}$$

$$\bigvee_{j\in J} \langle A_j, B_j\rangle = \langle (\bigcup_{j\in J} A_j)^{\uparrow\downarrow}, \bigcap_{j\in J} B_j\rangle\,. \tag{2.2}$$

Moreover, an arbitrary complete lattice $\mathbf{V} = \langle V, \wedge, \vee\rangle$ is isomorphic to some $\mathcal{B}\,(G, M, I)$ iff there are mappings $\gamma : G \times L \to V$, $\mu : M \times L \to V$ such that $\gamma(G, L)$ is $\bigwedge$-dense in V, $\mu(M, L)$ is $\bigvee$-dense in V; $\alpha \otimes \beta \leq I(g, m)$ iff $\gamma(g, \alpha) \leq \mu(m, \beta)$.

Note that a subset U of an ordered set V is called supremally (infimally) dense if each element of V is the supremum (infimum) of some subset of U.

3 Results

Suppose we are given two sets, G and M, and a set

$$T = \{\langle A^p, B^p\rangle \mid p \in P, A^p \in L^G, B^p \in L^M\}, \tag{3.1}$$

and denote

$$\mathrm{Sol}(T) = \{I \in L^{G\times M} \mid (\forall p \in P)(A^{p\uparrow_I} = B^p,\ B^{p\downarrow_I} = A^p)\},$$

the set of all solutions to the system

$$A^{p\uparrow_I} = B^p, \quad B^{p\downarrow_I} = A^p, \quad p \in P. \tag{3.2}$$

We say that T is *(conceptually) consistent* if $\mathrm{Sol}(T)$ is non-empty. If T is consistent and, moreover, $T = \mathcal{B}\,(G, M, I)$ for some $I \in \mathrm{Sol}(T)$ then T will be called *(conceptually) complete*.

A direct generalization to the above introduced notion 'to be a solution of T' is to consider not only whether an $\mathbf{L}$-relation I is a (exact) solution or not but also the truth degree to which I can be considered as a (approximate) solution. A natural way to measure similarity (or $\mathbf{L}$-equality) of $\mathbf{L}$-sets $A_1, A_2 \in L^X$ is to put $E(A_1, A_2) = \bigwedge_{x\in X} A_1(x) \leftrightarrow A_2(x)$ (here $a \leftrightarrow b = (a \to b) \wedge (b \to a)$). Obviously, $E(A_1, A_2)$ is the truth value of 'for each x: x belongs to A_1 iff x belongs to A_2'. If $L = \{0, 1\}$ (the case of two-valued Boolean logic), $E(A_1, A_2) = 1$ iff A_1 and A_2 are equal sets. It is easy to prove [4] that E is an $\mathbf{L}$-similarity relation on L^X in that $E(A, A) = 1$ (reflexivity), $E(A_1, A_2) = E(A_2, A_1)$ (symmetry), and $E(A_1, A_2) \otimes E(A_2, A_3) \leq E(A_1, A_3)$ (transitivity). Following this, we define the truth value $\mathrm{Sol}(T, I)$ of the fact that I is a solution of T as the truth value of 'for every $\langle A, B\rangle \in T$: $A^{\uparrow}$ is B, and $B^{\downarrow}$ is A', formally

$$\mathrm{Sol}(T, I) = \bigwedge_{\langle A,B\rangle\in T} E(A^{\uparrow_I}, B) \wedge E(B^{\downarrow_I}, A).$$

Clearly, $\mathrm{Sol}(T, I) = 1$ iff $I \in \mathrm{Sol}(T)$ which justifies why both of the predicates are denoted by Sol.

Consider first the structure of solutions of (3.2). For instance, it is well known that the set of all solutions of homogenous linear equations forms a linear space. In our case we have the following.

THEOREM 3.1
Let T be given as in (3.1). For any $I_j \in L^{G\times M}$ $(j \in J)$ we have $\bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \leq \mathrm{Sol}(T, \bigwedge_{j\in J} I_j)$.

PROOF. We have to show that $\bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \leq E(A^{\uparrow_{\wedge I_j}}, B) \wedge E(B^{\downarrow_{\wedge I_j}}, A)$ is true for any $\langle A, B\rangle \in T$. Due to symmetry we prove only $\bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \leq E(A^{\uparrow_{\wedge I_j}}, B)$ which holds iff for each $m \in M$ we have $\bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \leq A^{\uparrow_{\wedge I_j}}(m) \leftrightarrow B(m)$. First, we prove

$$\bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \leq A^{\uparrow_{\wedge I_j}}(m) \to B(m). \tag{3.3}$$

By adjointness, (3.3) is equivalent to $A^{\uparrow_{\wedge I_j}}(m) \otimes \bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \leq B(m)$. Now, in the complete lattice $L^{G\times M}$, the meet $\bigwedge_{j\in J} I_j$ is given by $(\bigwedge_{j\in J} I_j)(g, m) = \bigwedge_{j\in J}(I_j(g, m))$. Since $a \to \bigwedge_{j\in J} b_j = \bigwedge_{j\in J}(a \to b_j)$ holds in $\mathbf{L}$, we have

$$\begin{aligned} & A^{\uparrow_{\wedge I_j}}(m) = \bigwedge_{g\in G}(A(g) \to \bigwedge_{j\in J} I_j(g, m)) \\ = & \bigwedge_{g\in G}\bigwedge_{j\in J}(A(g) \to I_j(g, m)) \\ = & \bigwedge_{j\in J}\bigwedge_{g\in G}(A(g) \to I_j(g, m)) = \bigwedge_{j\in J} A^{\uparrow_{I_j}}(m). \end{aligned}$$

for any $p \in P$. Take any $k \in J$. We have

$$\begin{aligned} & A^{\uparrow_{\wedge I_j}}(m) \otimes \bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \\ = & \bigwedge_{j\in J} A^{\uparrow_{I_j}}(m) \otimes \bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \\ \leq & A^{\uparrow_{I_k}}(m) \otimes (A^{\uparrow_{I_k}}(m) \to B(m)) \leq B(m), \end{aligned}$$

proving (3.3). Second, we prove

$$\bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \leq B(m) \to A^{\uparrow_{\wedge I_j}}(m),$$

which is equivalent to $B(m) \otimes \bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \leq A^{\uparrow_{\wedge I_j}}(m)$. As $A^{\uparrow_{\wedge I_j}}(m) = \bigwedge_{j\in J} A^{\uparrow_{I_j}}(m)$ we need to show $B(m) \otimes \bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \leq A^{\uparrow_{I_j}}(m)$ for any $j \in J$. We have $B(m) \otimes \bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \leq B(m) \otimes (B(m) \to A^{\uparrow_{I_j}}(m)) \leq A^{\uparrow_{I_j}}(m)$, completing the proof. ∎

COROLLARY 3.2
$\mathrm{Sol}(T)$ is a complete meet-semilattice.

PROOF. Let $I_j \in L^{G\times M}$ $(j \in J)$ be solutions to (3.2). Then, by Theorem 3.1, $1 = \bigwedge_{j\in J} \mathrm{Sol}(T, I_j) \leq \mathrm{Sol}(T, \bigwedge_{j\in J} I_j)$, i.e. $\bigwedge_{i\in J} I_j \in \mathrm{Sol}(T)$. ∎

REMARK 3.3
In general, $\mathrm{Sol}(T)$ has no greatest element. Indeed, let $\mathbf{L}$ be the two-element Boolean algebra (each Boolean algebra is a residuated lattice where $\otimes = \wedge$ and $a \to b = a' \vee b$), $G = \{g_1, g_2\}$,

$M = \{m_1, m_2\}$, $T = \{\langle A, B\rangle\}$ where $A(g_1) = 1$, $A(g_2) = 0$, $B(m_1) = 1$, $B(m_2) = 1$. Then $\mathrm{Sol}(T) = \{I_1, I_2, I_3\}$, where $I_1 = \{\langle\langle g_1, m_1\rangle, 1\rangle, \langle\langle g_1, m_2\rangle, 1\rangle, \langle\langle g_2, m_1\rangle, 0\rangle, \langle\langle g_2, m_2\rangle, 0\rangle\}$, $I_2 = \{\langle\langle g_1, m_1\rangle, 1\rangle, \langle\langle g_1, m_2\rangle, 1\rangle, \langle\langle g_2, m_1\rangle, 1\rangle, \langle\langle g_2, m_2\rangle, 0\rangle\}$, $I_3 = \{\langle\langle g_1, m_1\rangle, 1\rangle, \langle\langle g_1, m_2\rangle, 1\rangle, \langle\langle g_2, m_1\rangle, 0\rangle, \langle\langle g_2, m_2\rangle, 1\rangle\}$. I_1 is the least solution, both I_2 and I_3 are maximal, there is no greatest solution.

THEOREM 3.4
Let T be complete, I be the solution such that $T = \mathcal{B}(G, M, I)$. Let the weights I^T be given by

$$I^T(g, m) = \bigvee_{\langle A, B\rangle \in T} A(g) \otimes B(m). \tag{3.4}$$

Then it holds that $I^T = I$.

PROOF. First, we show $I^T \subseteq I$. Let $g \in G$, $m \in M$ be arbitrary elements. For each $\langle A, B\rangle \in T$ we have $A(g) = \bigwedge\{B(m') \to I(g, m') \mid m' \in M\}$. From the well-known lattice properties it follows $A(g) \leq B(m) \to I(g, m)$ from which we get by the adjunction property $A(g) \otimes B(m) \leq I(g, m)$. Since g and m are chosen arbitrarily we conclude

$$I^T(g, m) \quad = \quad \bigvee_{\langle A, B\rangle \in T} \big(A(g) \otimes B(m)\big) \leq I(g, m),$$

i.e. $I^T \subseteq I$ holds.

Conversely, consider the concept $\langle\{\,{}^1/g\}^{\uparrow\downarrow}, \{\,{}^1/g\}^{\uparrow}\rangle \in \mathcal{B}(G, M, I)$. Here, $\{\,{}^1/g\}$ is a fuzzy set in G defined by $\{\,{}^1/g\}(x) = 1$ for $x = g$ and $\{\,{}^1/g\}(x) = 0$ for $x \neq g$. We have

$$\begin{aligned} \{\,{}^1/g\}^{\uparrow}(m) \quad &= \quad \bigwedge\{\{\,{}^1/g\}(g') \to I(g', m) \mid g' \in G\} \\ &= \quad 1 \to I(g, m) = I(g, m) \end{aligned}$$

and $\{\,{}^1/g\}^{\uparrow\downarrow}(g) = 1$. Thus for the concept $\langle A, B\rangle = \langle\{\,{}^1/g\}^{\uparrow\downarrow}, \{\,{}^1/g\}^{\uparrow}\rangle \in T$ we have $A(g) \otimes B(m) = 1 \otimes I(g, m) = I(g, m)$ which gives

$$I^T(g, m) = \bigvee_{\langle A, B\rangle \in T} \big(A(g) \otimes B(m)\big) \geq I(g, m),$$

i.e. $I^T \supseteq I$. We have $I^T = I$ completing the proof. ∎

LEMMA 3.5
If T is consistent then $I^T \subseteq I$ for any $I \in \mathrm{Sol}(T)$.

PROOF. Suppose T is consistent, i.e. $T \subseteq \mathcal{B}(G, M, I)$ for some I. Since, by Theorem 3.4,

$$\begin{aligned} I^T(g, m) &= \bigvee_{\langle A, B\rangle \in T} A(g) \otimes B(m) \\ &\leq \bigvee_{\langle A, B\rangle \in \mathcal{B}(G, M, I)} A(g) \otimes B(m) = I^{\mathcal{B}(G, M, I)}(g, m) = I(g, m) \end{aligned}$$

we have $I^T \subseteq I$. ∎

THEOREM 3.6
Let T be consistent and I^T be given by (3.4). Then $I^T \in \mathrm{Sol}(T)$.

PROOF. By Lemma 3.5, $I^T \subseteq I$. Consider any $\langle A, B\rangle \in T$. We have to prove $\langle A, B\rangle \in \mathcal{B}\left(G, M, I^T\right)$, i.e. $A^{\uparrow_{I^T}} = B$ and $B^{\downarrow_{I^T}} = A$. Due to symmetry we prove only $A^{\uparrow_{I^T}} = B$ by checking both inequalities. Take any $m \in M$. By $I^T \subseteq I$,

$$\begin{aligned} A^{\uparrow_{I^T}}(m) &= \bigwedge_{g\in G} \left(A(g) \to I^T(g,m)\right) \\ &\leq \bigwedge_{g\in G} \left(A(g) \to I(g,m)\right) = A^{\uparrow_I}(m) = B(m). \end{aligned}$$

On the other hand,

$$B(m) \leq A^{\uparrow_{I^T}}(m) = \bigwedge_{g_i\in G} \left(A(g_i) \to I^T(g,m)\right)$$

holds iff for each $g \in G$ it holds

$$B(m) \leq A(g) \to I^T(g,m) = A(g) \to \bigvee_{\langle A',B'\rangle\in T} A'(g) \otimes B'(m).$$

By adjointness, the last inequality is equivalent to

$$A(g) \otimes B(m) \leq \bigvee_{\langle A',B'\rangle\in T} A'(g) \otimes B'(m),$$

which holds since $\langle A, B\rangle \in T$. ∎

Lemma 3.5 and Theorem 3.6 yield the following corollary.

COROLLARY 3.7
T is consistent iff I^T given by (3.4) is the solution of (3.2). In that case, I^T is the least solution.

REMARK 3.8
Corollary 3.7 gives an algorithm for testing conceptual consistency of T: (1) compute I^T; (2) for each $\langle A, B\rangle \in T$ verify $A^{\uparrow_{I^T}} = B$ and $B^{\downarrow_{I^T}} = A$. It is obvious that the algorithm is of a polynomial time complexity (polynomial in $|T|$, $|G|$, and $|M|$). The brute force search (i.e. testing all $I \in L^{G\times M}$) which leads to exponential time complexity may be overcome due to the fact that $I^T \in \mathrm{Sol}(T)$ iff $\mathrm{Sol}(T)$ is non-empty.

The following theorem says that if I_1 is a solution to T and I_1 and I_2 are similar then I_2 is a solution to T as well.

THEOREM 3.9
For any T, I_1 and I_2 we have $\mathrm{Sol}(T, I_1) \otimes E(I_1, I_2) \leq \mathrm{Sol}(T, I_2)$.

PROOF. We have to show that $\mathrm{Sol}(T, I_1) \otimes E(I_1, I_2) \leq E(A^{\uparrow_{I_2}}, B) \wedge E(B^{\downarrow_{I_2}}, A)$ is true for any $\langle A, B\rangle \in T$. Due to the symmetry of both cases we prove only $\mathrm{Sol}(T, I_1) \otimes E(I_1, I_2) \leq E(A^{\uparrow_{I_2}}, B)$, i.e. we show that $\mathrm{Sol}(T, I_1) \otimes E(I_1, I_2) \leq A^{\uparrow_{I_2}}(m) \leftrightarrow B(m)$. First, we prove $\mathrm{Sol}(T, I_1) \otimes E(I_1, I_2) \leq A^{\uparrow_{I_2}}(m) \to B(m)$, which is equivalent to $A^{\uparrow_{I_2}}(m) \otimes E(I_1, I_2) \otimes \mathrm{Sol}(T, I_1) \leq B(m)$. The last inequality is true: since $E(I_1, I_2) \leq E(A^{\uparrow_{I_1}}, A^{\uparrow_{I_2}})$ [4] we have

$$\begin{aligned} & A^{\uparrow_{I_2}}(m) \otimes E(I_1, I_2) \otimes \mathrm{Sol}(T, I_1) \\ \leq\; & A^{\uparrow_{I_2}}(m) \otimes E(A^{\uparrow_{I_1}}, A^{\uparrow_{I_2}}) \otimes E(A^{\uparrow_{I_1}}, B) \\ \leq\; & A^{\uparrow_{I_2}}(m) \otimes (A^{\uparrow_{I_2}}(m) \to A^{\uparrow_{I_1}}(m)) \otimes (A^{\uparrow_{I_1}}(m) \to B(m)) \leq B(m). \end{aligned}$$

Conversely, $\mathrm{Sol}(T, I_1) \otimes E(I_1, I_2) \leq B(m) \rightarrow A^{\uparrow_{I_2}}(m)$ is equivalent to $B(m) \otimes E(I_1, I_2) \otimes \mathrm{Sol}(T, I_1) \leq A^{\uparrow_{I_2}}(m)$ which is true since

$$\begin{aligned} & B(m) \otimes E(I_1, I_2) \otimes \mathrm{Sol}(T, I_1) \\ \leq\ & B(m) \otimes E(A^{\uparrow_{I_1}}, A^{\uparrow_{I_2}}) \otimes E(A^{\uparrow_{I_1}}, B) \\ \leq\ & B(m) \otimes (B(m) \rightarrow A^{\uparrow_{I_1}}(m)) \otimes (A^{\uparrow_{I_1}}(m) \rightarrow A^{\uparrow_{I_2}}(m)) \leq A^{\uparrow_{I_2}}(m). \end{aligned}$$

∎

COROLLARY 3.10
For any T, I_1 and I_2 we have $E(I_1, I_2) \leq \mathrm{Sol}(T, I_1) \leftrightarrow \mathrm{Sol}(T, I_2)$.

PROOF. Directly from Theorem 3.9 by adjointness and by $E(I_1, I_2) = E(I_2, I_1)$. ∎

Adding a linear combination of equations of a given system of linear equations does not affect the solvability of the system. In our case, solvability of T does not change if we add to T new pairs $\langle A, B\rangle$ as follows (note that if T is solvable, the pairs being added are infima and suprema (in $\mathcal{B}\left(X, Y, I^T\right)$) of pairs from T).

THEOREM 3.11
Let T be given as in (3.1). Then T is consistent iff

$$T \cup \{\langle \bigcap_{p \in P'} A^p, (\bigcap_{p \in P'} A^p)^{\uparrow_{I^T}}\rangle \mid P' \subseteq P\} \cup \{\langle (\bigcap_{p \in P'} B^p)^{\downarrow_{I^T}}, \bigcap_{p \in P'} B^p\rangle \mid P' \subseteq P\}$$

is consistent.

PROOF. If T is consistent then $T \subseteq \mathcal{B}\left(G, M, I^T\right)$ by Corollary 3.7. By Theorem 1, $\langle \bigcap_{p \in P'} A^p, (\bigcap_{p \in P'} A^p)^{\uparrow_{I^T}}\rangle = \bigwedge_{p \in P'} \langle A^p, B^p\rangle \in \mathcal{B}\left(G, M, I^T\right)$ and $\langle (\bigcap_{p \in P'} B^p)^{\downarrow_{I^T}}, \bigcap_{p \in P'} B^p\rangle = \bigvee_{p \in P'} \langle A^p, B^p\rangle \in \mathcal{B}\left(G, M, I^T\right)$, i.e. $T \cup \ldots \cup \ldots$ is consistent. The converse implication is trivial. ∎

Consider now the following problem. We are given some T as in (3.1). It may happen that $\mathrm{Sol}(T)$ is empty but there is a near solution I (for instance, $L = [0, 1]$ and $\mathrm{Sol}(T, I) = 0.95$). In that case one might consider the difference negligible. What is taking place here is a phenomenon which was termed logical precision in [6]: We are given a structure $\mathbf{L}_1$ of truth values using which our knowledge is formulated. For several reasons it might not be desirable to discern all the truth values of L_1 (e.g. for computational reasons, or, as in our case, close truth values may appear essentially the same to us). In that case, a kind of rounding off is desirable. In order that the rounding off be systematic, it should be compatible with operations in $\mathbf{L}_1$. Formally, we look for another structure $\mathbf{L}_2$ of truth values such that a suitable onto morphism $h : L_1 \rightarrow L_2$ exists. In our case we require that h is a complete onto morphism, i.e. that h is an onto mapping which preserves the operations of $\mathbf{L}_1$ as well as arbitrary infima and suprema. Morphism h plays the role of rounding off, $a \in L_1$ is rounded to $h(a) \in L_2$. The change-over from $\mathbf{L}_1$ to $\mathbf{L}_2$ then represents a decrease of logical precision: all values $a \in L_1$ rounded to the same element are identified, i.e. considered to be the same. Since the decrease of logical precision means obviously loss of information, one is interested in what is the most economical (i.e. smallest) decrease of logical precision such that under the decreased precision our needs are satisfied. Returning to our problem, we ask for the most economical change of logical precision such that the system of concept equations has a solution.

Before we formulate the result, recall the necessary notions: let h be a complete onto morphism of $\mathbf{L}_1$ onto $\mathbf{L}_2$. That is, h is a surjective mapping of L_1 onto L_2 preserving all residuated lattice operations, including arbitrary infima and suprema. For an $\mathbf{L}_1$-set $A \in L_1^X$ we define an $\mathbf{L}_2$-set $h(A) \in L_2^X$ by $(h(A))(x) = h(A(x))$. The following assertions concerning complete morphisms and congruences can be verified analogously as their well-known counterparts concerning not necessarily complete morphisms and congruences. If $h : \mathbf{L}_1 \to \mathbf{L}_2$ is a complete onto morphism (i.e. $h(\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} h(a_i)$ and dually) then putting $\langle a, b\rangle \in \theta_h$ iff $h(a) = h(b)$ defines a complete congruence relation on $\mathbf{L}_1$ (i.e. a congruence such that $\langle a_i, b_i\rangle \in \theta_h$ ($i \in I$) implies $\langle \bigwedge_{i\in I} a_i, \bigwedge_{i\in I} b_i\rangle \in \theta_h$ and $\langle \bigvee_{i\in I} a_i, \bigvee_{i\in I} b_i\rangle \in \theta_h$). If $\mathbf{L}$ is a complete residuated lattice and θ is a complete congruence relation on $\mathbf{L}$ then the factor algebra $\mathbf{L}/\theta$ is a complete residuated lattice; moreover, $h_\theta(a) = [a]_\theta$ ($[a]_\theta$ is the class of θ containing a) defines a complete onto morphism of $\mathbf{L}$ onto $\mathbf{L}/\theta$. As usual, for $a, b \in L$ we denote by $\theta(a, b)$ the least complete congruence on $\mathbf{L}$ containing $\langle a, b\rangle$.

Coming back to our problem, we look for the smallest decrease of logical precison (represented by some morphism h) such that for a given I we have $h(I) \in \mathrm{Sol}(h(T))$ where $h(T) = \{\langle h(A), h(B)\rangle \mid \langle A, B\rangle \in T\}$. Recall that $\theta(\mathrm{Sol}(T, I), 1)$ is the least congruence on $\mathbf{L}$ containing $\langle \mathrm{Sol}(T, I), 1\rangle$.

THEOREM 3.12
Let T be given as in (3.1), let $I \in L^{G\times M}$. Then putting $\theta = \theta(\mathrm{Sol}(T, I), 1)$, we have $h_\theta(I) \in \mathrm{Sol}(h_\theta(T))$. Furthermore, if $h : \mathbf{L} \to \mathbf{L}'$ is a complete onto morphism such that $h(I) \in \mathrm{Sol}(h(T))$ then there is a complete onto morphism $g : \mathbf{L}/\theta \to \mathbf{L}'$ such that $h_\theta \circ g = h$.

PROOF. Since

$$
\begin{aligned}
& h_\theta(\mathrm{Sol}(T, I)) = h_\theta(\bigwedge_{\langle A,B\rangle\in T} E(A^{\uparrow_I}, B) \wedge E(B^{\downarrow_I}, A)) \\
= \quad & \bigwedge_{\langle A,B\rangle\in T} \bigwedge_{m\in M} (h_\theta(A^{\uparrow_I}(m)) \leftrightarrow h_\theta(B(m))) \wedge \bigwedge_{g\in G} (h_\theta(A(g)) \leftrightarrow h_\theta(B^{\downarrow_I}(g))) = \\
= \quad & \bigwedge_{\langle h_\theta(A),h_\theta(B)\rangle\in h_\theta(T)} \bigwedge_{m\in M} (h_\theta(A)^{\uparrow_{h_\theta(I)}}(m) \leftrightarrow h_\theta(B)(m)) \\
& \wedge \bigwedge_{g\in G} (h_\theta(A)(g) \leftrightarrow h_\theta(B)^{\downarrow_{h_\theta(I)}}(g)) = \mathrm{Sol}(h_\theta(T), h_\theta(I)),
\end{aligned}
$$

$\langle \mathrm{Sol}(T, I), 1\rangle \in \theta$ implies $\mathrm{Sol}(h_\theta(T), h_\theta(I)) = 1$, i.e. $h_\theta(I) \in \mathrm{Sol}(h_\theta(T))$.

If $h : \mathbf{L} \to \mathbf{L}'$ is a complete onto morphism such that $h(I) \in \mathrm{Sol}(h(T))$ then $h(\mathrm{Sol}(T, I)) = 1$, i.e. $\langle \mathrm{Sol}(T, I), 1\rangle \in \theta_h$. As θ is the least congruence containing $\langle \mathrm{Sol}(T, I), 1\rangle$, we conclude $\theta \subseteq \theta_h$. The conclusion then follows by well-known homomorphism theorems from universal algebra. ∎

REMARK 3.13
Theorem 3.12 says that the decrease $h_{\theta(\mathrm{Sol}(T,I),1)}$ of logical precision makes $h_{\theta(\mathrm{Sol}(T,I),1)}(I)$ a solution of the system of concept equations given by $h_{\theta(\mathrm{Sol}(T,I),1)}(T)$. Furthermore, any other decrease h of logical precision which makes $h(I)$ a solution of $h(T)$ may be obtained from $h_{\theta(\mathrm{Sol}(T,I),1)}$ by a decrease of logical precision. Therefore, $h_{\theta(\mathrm{Sol}(T,I),1)}$ is the most economical one.

COROLLARY 3.14
For any T, $\theta = \bigcap_{I\in L^{G\times M}} \theta(\mathrm{Sol}(T, I), 1)$ is the least complete congruence on $\mathbf{L}$ such that $h_\theta(T)$ has a solution.

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