CONGRUENCE PROPERTIES IN SINGLE ALGEBRAS

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Abstract

It was proved by H.–P. Gumm [9] that the only congruence condition which can be characterized by a Mal'cev condition also for a single algebra instead of a variety is the arithmeticity. On the other hand, the second author showed in previous papers that for finite algebras of a very limited number of elements also other congruence conditions can be characterized in this manner. We proceed these investigations for congruence distributivity, permutability at 0, arithmeticity at 0 and regularity at 0.

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It is a well-known result that if a variety $\mathcal{V}$ satisfies a given congruence identity then there exists a Mal'cev condition characterizing it. It was proved by A. Pixley [11] that for arithmeticity of congruences, the similar statement is true also for a single algebra. More precisely, a finite algebra $\mathcal{A}$ is arithmetical (i.e. congruence distributive and permutable) if and only if there exists a 3–ary function $p(x, y, z)$, usually called a Pixley function, satisfying $p(x, y, y) = x = p(x, y, x) = p(y, y, x)$ which is compatible with every congruence $\theta \in Con\,\mathcal{A}$. H.-P. Gumm [9] asked whether this result could be extended also for other congruence identities. His answer is negative, i.e. arithmeticity is the only property for which the Mal'cev type characterization can be transfered from varieties to single algebras. Moreover, [9] contains an example of 25 element algebra $\mathcal{A}$ which is congruence permutable but no Mal'cev function compatible with $Con\,\mathcal{A}$ exists.

G. Czédli and the second author tried to search for a number $1 \leq n \leq 25$ for which every n–element algebra is congruence permutable if and only if there exists a Mal'cev function compatible with congruences. It was proven in [4] that the assertion is valid for at least $n \leq 8$.

In what follows we are going to testify some other congruence conditions for single algebras.

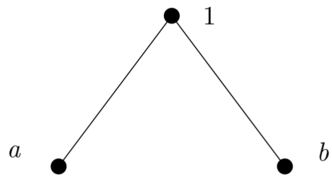

Figure 1: Semilattice S

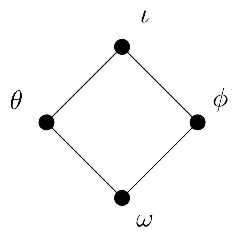

Figure 2: $Con\, S$

1 Congruence distributivity

It was proven by H.-P. Gumm [9] that there exists a 6 element algebra which is congruence distributive but has no Jónsson functions compatible with $Con\,\mathcal{A}$. It is a question if it is true also for algebras of cardinality less than 6.

If $\mathcal{A}$ is an algebra with $card\, A = 2$ then $Con\,\mathcal{A}$ is a two–element chain. Hence, $\mathcal{A}$ is congruence distributive if and only if $\mathcal{A}$ is arithemtical. By [11], there exists a Pixley function compatible with $Con\,\mathcal{A}$. Clearly, a Pixley function yields Jónsson functions (for $n = 3$), hence a two–element algebra is congruence distributive if and only if there exist Jónsson functions compatible with $Con\,\mathcal{A}$, i.e. the functions $p_1, \ldots, p_n$ satisfying

$$\begin{aligned} x &= p_1(x,y,z) \\ z &= p_n(x,y,z) \\ x &= p_i(x,y,x) \qquad \text{for } i = 1,\ldots,n \\ p_i(x,y,y) &= p_{i+1}(x,y,y) \qquad \text{for } i \text{ odd} \\ p_i(x,x,y) &= p_{i+1}(x,x,y) \qquad \text{for } i \text{ even} \end{aligned} \tag{1}$$

Of course, the case $card\, A \leq 2$ is trivial. We will show that it is the only case where the answer is affirmative:

Example 1. There exists a 3–element $\vee$–semilattice S which is congruence distributive but no Jónsson functions compatible with $Con\, S$ exist. Let S be the semilattice depicted in Fig. 1. Then $Con\, S$ is the lattice of Fig. 2, where θ has the classes $\{a, 1\}, \{b\}$, ϕ has the classes $\{a\}, \{b, 1\}$. Thus $Con\, S$ is distributive. However, S has not permutable congruences:

$$\langle a, 1\rangle \in \theta, \langle 1, b\rangle \in \phi \qquad \text{imply} \qquad \langle a, b\rangle \in \theta \cdot \phi$$

but $\langle a, b\rangle \notin \phi \cdot \theta$.

Suppose the existence of compatible 3–ary functions of (1). Since $card\, S = 3$ and p_1 is equal to the unique element, p_n is also equal to the unique element, there is the only one possibility for the other functions p_i for $1 \neq i \neq n$. We infer $n = 3$ enough. Hence, there must be a 3–ary function p_2 which is compatible with θ and ϕ such that

$$\begin{aligned} x &= p_2(x,y,x) \\ x &= p_1(x,y,y) = p_2(x,y,y) \\ z &= p_3(x,x,z) = p_2(x,x,z) \ . \end{aligned}$$

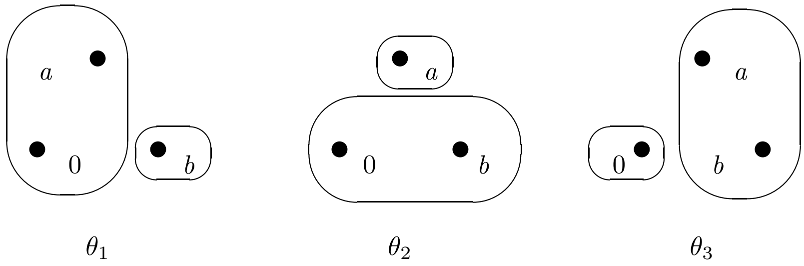

Figure 3: Nontrivial congruences on $\{0, a, b\}$

For $x = a$, $z = 1$ we obtain $a = p_2(a, 1, 1)$. For $x = 1$, $z = b$ we obtain $b = p_2(1, 1, b)$. Hence

$$a = p_2(a, 1, 1)\, \phi\, p_2(a, 1, b)\, \theta\, p_2(1, 1, b) = b,$$

i.e. $\langle a, b\rangle \in \phi \cdot \theta$, a contradiction. □

2 Permutability at 0

Throughout this section we suppose that any algebra under considerations has a constant 0 (i.e. either a nullary basic operation or a nullary term operation). An algebra $\mathcal{A}$ is called *permutable at* 0 (see e.g. [7, 10]) if it satisfies

$$[0]_{\theta\cdot\phi} = [0]_{\phi\cdot\theta}$$

for any $\theta, \phi \in Con\,\mathcal{A}$.

For the following result, see [10] or [2, 7]:

Proposition. *A variety $\mathcal{V}$ is permutable at* 0 *if and only if there exists a binary term $b(x, y)$ satisfying $b(x, x) = 0$, $b(x, 0) = x$.*

We ask whether this result can be extended for single algebras with restricted cardinality. The following lemma is evident.

Lemma 1. *Let A be an algebra with* 0*. If there exists a binary compatible function $b(x, y)$ on $\mathcal{A}$ satisfying $b(x, x) = 0$, $b(x, 0) = x$ then $\mathcal{A}$ is permutable at* 0*.*

For the converse, we have only

Theorem 1. *Let $\mathcal{A}$ be an algebra with* 0 *such that $card\,A \leq 3$. The following are equivalent:*

(1) *$\mathcal{A}$ is permutable at* 0*;*

(2) *there exists a compatible with $Con\,\mathcal{A}$ binary function $b(x, y)$ such that $b(x, x) = 0$, $b(x, 0) = x$.*

Proof: With respect to Lemma 1, it remains to prove only (1) ⇒ (2). If $card\,A \leq 2$ then $Con\,\mathcal{A}$ is a two–element chain. Hence, $\mathcal{A}$ is arithmetical and, by [11], there exists a 3–ary compatible function p such that

$$p(x, y, y) = x = p(x, y, x) = p(y, y, x)\ .$$

Set $b(x, y) = p(x, y, 0)$ and we are done.

Suppose $card\,A = 3$, $A = \{0, a, b\}$. Then there exist exactly 5 equivalences on A, namely the least, denoted by ω, the greatest one denoted by ι and the nontrivial ones shown in Fig. 3.

Hence, $Con\,\mathcal{A} \subseteq \{\omega, \iota, \theta_1, \theta_2, \theta_3\}$.

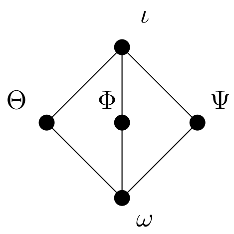

Figure 4: $Con\,\mathcal{A}$ from Example 2

1. If $\theta_1, \theta_3 \in Con\,\mathcal{A}$ then $\mathcal{A}$ is not permutable at 0 since
$$[0]_{\theta_1 \cdot \theta_3} = A \neq \{0, a\} = [0]_{\theta_3 \cdot \theta_1} \ .$$
If $\theta_2, \theta_3 \in Con\,\mathcal{A}$, the proof is analogous.

2. If $Con\,\mathcal{A}$ contains exactly 3 congruences, then $Con\,\mathcal{A} = \{\omega, \iota, \theta_i\}$ for some $i \in \{1, 2, 3\}$. We can put
$$b(x, x) = 0, \qquad b(x, y) = x \quad \text{for } x \neq y$$
for $i = 1$ or $i = 2$ and
$$b(x, 0) = x, \qquad b(x, y) = 0 \quad \text{for } y \neq 0$$
for $i = 3$. It is easy to show that b is compatible with $Con\,\mathcal{A}$.

3. If $Con\,\mathcal{A}$ contains exactly 4 congruences, then, by 1., we have $Con\,\mathcal{A} = \{\omega, \iota, \theta_1, \theta_2\}$. Set $b(x, x) = 0$ and $b(x, y) = x$ for $x \neq y$. Again, $b(x, y)$ is compatible with $Con\,\mathcal{A}$.

4. The remaining case $Con\,\mathcal{A} = \{\omega, \iota\}$ is trivial (we can apply [11] again).

□

Example 2. There exists a 7–element algebra $\mathcal{A}$ such that $Con\,\mathcal{A} \cong M_3$, $\mathcal{A}$ is permutable at 0 but not permutable.

Let $A = \{0, a, b, c, d, p, q\}$, and the non-trivial congruences are given by their partitions

$\Theta \ldots \{a, c\}, \{q, b\}, \{0, d, p\}$

$\Phi \ldots \{a, d\}, \{0, c, q\}, \{p, b\}$

$\Psi \ldots \{0, a, b\}, \{d, q\}, \{c, p\}$.

Then $Con\,\mathcal{A}$ is visualized in Fig. 4. $\mathcal{A}$ is permutable at 0:
$$[0]_{\Theta \cdot \Phi} = [0]_{\Phi \cdot \Theta} = [0]_{\Theta \cdot \Psi} = [0]_{\Psi \cdot \Theta} = [0]_{\Phi \cdot \Psi} = [0]_{\Psi \cdot \Phi} =$$
but
$$\langle a, b \rangle \in \Theta \vee \Phi, \quad \langle a, b \rangle \notin \Theta \cdot \Phi \,,$$
i.e. $\Theta \cdot \Phi \neq \Theta \vee \Phi$ and hence $\mathcal{A}$ is not congruence permutable.

Example 3. There exists a 7–element algebra which is permutable at 0, congruence distributive but not permutable (and hence arithmetical). Let $\mathcal{A} = (A, F)$ where $A = \{0, a, b, c, d, p, q\}$, $F = \{f, g, h, t\}$ – a set of unary operations given by

	f	g	h	t
a	d	c	a	a
b	b	b	p	q
c	0	a	c	c
d	a	0	d	d
p	p	p	b	0
q	q	q	0	b
0	c	d	q	p

Then $Con\,\mathcal{A}$ is the same as the lattice from Fig. 2 where

$\Theta \ldots \{a, c\}, \{p, b\}, \{0, d, q\}$

$\Phi \ldots \{a, d\}, \{0, c, p\}, \{q, b\}$.

Hence, $Con\,\mathcal{A}$ is distributive. Moreover, $[0]_{\Theta\cdot\Phi} = A = [0]_{\Phi\cdot\Theta}$, i.e. $\mathcal{A}$ is permutable at 0. However, $\langle a, b\rangle \notin \Theta\cdot\Phi$, i.e. $\Theta \cdot \Phi \neq \iota = \Theta \vee \Phi$, thus $\mathcal{A}$ is not permutable.

3 Arithmeticity at 0

An algebra $\mathcal{A}$ with 0 is *distributive at* 0 (see [1]) if

$$[0]_{(\Theta\vee\Phi)\wedge\Psi} = [0]_{(\Theta\wedge\Psi)\vee(\Phi\wedge\Psi)}$$

for any $\Theta, \Phi, \Psi \in Con\,\mathcal{A}$. $\mathcal{A}$ is *dually distributive at* 0 if

$$[0]_{(\Theta\wedge\Phi)\vee\Psi} = [0]_{(\Theta\vee\Psi)\wedge(\Phi\vee\Psi)}$$

for any $\Theta, \Phi, \Psi \in Con\,\mathcal{A}$.

As it was pointed out in [6], these conditions are not equivalent either in the case of varieties, see also [1, 2].

An algebra $\mathcal{A}$ is *arithmetical at 0* if $\mathcal{A}$ is both permutable at 0 and distributive at 0. Classes of varieties arithmetical at 0 were characterized by a (strong) Mal'cev condition by J. Duda [7]. Varieties distributive at 0 were characterized by a Mal'cev condition in [1]. For the following lemma, see Lemma 2 in [7]:

Lemma 2. *Let $\mathcal{A}$ be an algebra permutable at* 0*. Then $\mathcal{A}$ is distributive at* 0 *if and only if $\mathcal{A}$ is dually distributive at* 0*.*

As it was shown in [7], varieties arithmetical at 0 can be characterized by the so called Extended Chinese Remainder Theorem at 0 (Extended CRT, for short). We are going to show that, with suitable modification of CRT, it can be done also for single algebras.

An algebra $\mathcal{A}$ satisfies *Weak Chinese Remainder Theorem*, briefly Weak CRT, if for each $a \in A$ and any $\Theta_1, \ldots, \Theta_n, \Phi \in Con\,\mathcal{A}$

$$[0]_{\Theta_1} \cap \ldots \cap [0]_{\Theta_n} \cap [a]_\Phi \neq \emptyset \tag{2}$$

whenever $[0]_{\Theta_i} \cap [a]_\Phi \neq \emptyset$ for $i = 1, \ldots, n$.

Lemma 3. *An algebra $\mathcal{A}$ satisfies Weak CRT if and only if for each $a \in A$ and $\Theta_1, \Theta_2, \Phi \in Con\,\mathcal{A}$,*

$$[0]_{\Theta_1} \cap [a]_\Phi \neq \emptyset \neq [0]_{\Theta_2} \cap [a]_\Phi \Longrightarrow [0]_{\Theta_1} \cap [0]_{\Theta_2} \cap [a]_\Phi \neq \emptyset\ .$$

Proof: By induction on n. For $n = 1$ it is trivial. Suppose the validity of (2) for $n - 1$. Let $a \in A$, $\Theta_1, \ldots, \Theta_n, \Phi \in Con\,\mathcal{A}$ and $[0]_{\Theta_i} \cap [a]_\Phi \neq \emptyset$ for $i = 1, \ldots, n$. Put $\Theta = \Theta_1 \cap \ldots \cap \Theta_{n-1}$. Evidently, $[0]_\Theta = [0]_{\Theta_1} \cap \ldots \cap [0]_{\Theta_{n-1}}$ and, by induction hypothesis, $[0]_\Theta \cap [a]_\Phi \neq \emptyset$. By it and by assumption of Lemma 2 also $[0]_\Theta \cap [0]_{\Theta_n} \cap [a]_\Phi \neq \emptyset$. Since the converse statement is trivial, we are done. □

An algebra $\mathcal{A}$ satisfies *Extended CRT at* 0 (see [7]) if for each $a \in A$ and any $\Theta_1, \ldots, \Theta_n, \Phi \in Con\,\mathcal{A}$ it holds

$$[0]_{\Theta_1} \cap \ldots \cap [0]_{\Theta_n} \cap [a]_\Phi \neq \emptyset \qquad (a)$$

whenever

$$[0]_{\Theta_1} \cap \ldots [0]_{\Theta_{j-1}} \cap [0]_{\Theta_{j+1}} \cap \ldots [0]_{\Theta_n} \cap [a]_\Phi \neq \emptyset \qquad (b)$$

for each $j = 1, \ldots, n$.

If $\mathcal{A}$ satisfies Extended CRT at 0 for some natural n then $\mathcal{A}$ satisfies it also for any $m \geq n$. The converse statement does not hold in general (see [7]). However, we can prove:

Lemma 4. *Let $\mathcal{A}$ be an algebra with 0.*

(1) *If $\mathcal{A}$ satisfies Weak CRT then $\mathcal{A}$ satisfies Extended CRT at 0 for any $n > 0$.*

(2) *If $\mathcal{A}$ satisfies Extended CRT at 0 for $n = 2$ then $\mathcal{A}$ satisfies also Weak CRT.*

Proof: (1) Let $\mathcal{A}$ satisfies (a). Then $[0]_{\Theta_i} \cap [a]\Phi \neq \emptyset$ for any $i = 1, \ldots, n$. Hence, Weak CRT imply (b), i.e. $\mathcal{A}$ satisfies Extended CRT at 0.

(2) It is an immediate observation that for $n = 2$ both of these variants of CRT coincide. By Lemma 3, we conclude the assertion. □

Theorem 2. *Let $\mathcal{A}$ be permutable at 0. The following conditions are equivalent:*

(1) *$\mathcal{A}$ is distributive at 0;*

(2) *$\mathcal{A}$ is dually distributive at 0;*

(3) *$\mathcal{A}$ satisfies Weak CRT.*

Proof: By Lemma 2, (1) $\Leftrightarrow$ (2). Prove (2) $\Rightarrow$ (3): If $\mathcal{A}$ is permutable at 0 then

$$[0]_{\Phi\cdot\Psi} = [0]_{\Phi\vee\Psi}$$

for any $\Phi, \Psi \in Con\,\mathcal{A}$. Suppose $c \in A$ and $\Theta_1, \Theta_2, \Phi \in Con\,\mathcal{A}$. Let

$$[0]_{\Theta_1} \cap [c]_\Phi \neq \emptyset \neq [0]_{\Theta_2} \cap [c]_\Phi \ .$$

Then there exist $a \in [0]_{\Theta_1} \cap [c]_\Phi$ and $b \in [0]_{\Theta_1} \cap [c]_\Phi$, i.e.

$$\langle a, 0\rangle \in \Theta_1, \langle c, a\rangle \in \Phi \text{ and } \langle b, 0\rangle \in \Theta_2, \langle c, b\rangle \in \Phi$$

whence

$$c \in [0]_{\Theta_1\cdot\Phi\cap\Theta_2\cdot\Phi} = [0]_{(\Theta_1\vee\Phi)\wedge(\Theta_2\vee\Phi)} \ .$$

By dual distributivity at 0, we have $c \in [0]_{\Phi\vee(\Theta_1\wedge\Theta_2)}$. Influenced by permutability at 0, it gives $c \in [0]_{\Phi\cdot(\Theta_1\cap\Theta_2)}$. Hence, there exists $d \in A$ with

$$\langle c, d\rangle \in \Phi, \langle d, 0\rangle \in \Theta_1 \cap \Theta_2$$

i.e. $d \in [c]_\Phi$, $d \in [0]_{\Theta_1} \cap [0]_{\Theta_2}$ whence

$$[0]_{\Theta_1} \cap [0]_{\Theta_2} \cap [c]_\Phi \neq \emptyset \ .$$

By Lemma 3, we conclude (3).

(3) $\Rightarrow$ (1): Let $\Theta_1, \Theta_2, \Phi \in Con\,\mathcal{A}$ and let $a \in [0]_{(\Phi\vee\Theta_2)\wedge\Theta_1}$. By permutability at 0, $\langle a, 0\rangle \in (\Phi \cdot \Theta_2) \cap \Theta_1$, i.e. there exists $c \in A$ with $\langle a, c\rangle \in \Phi$, $\langle c, 0\rangle \in \Theta_2$ and $\langle a, 0\rangle \in \Theta_1$. Hence $a \in [0]_{\Theta_1} \cap [c]_\Phi$, $c \in [0]_{\Theta_2} \cap [c]_\Phi$ and, by (3), there exists

$$d \in [0]_{\Theta_1} \cap [0]_{\Theta_2} \cap [c]_\Phi \ .$$

Then $d \in [0]_{\Theta_1} = [a]_{\Theta_1}$, $d \in [c]_\Phi = [a]_\Phi$ whence $\langle a, d\rangle \in \Theta_1 \cap \Phi$. However, $d \in [0]_{\Theta_1}$, $d \in [0]_{\Theta_2}$ give $\langle d, 0\rangle \in \Theta_1 \cap \Theta_2$ thus

$$\langle a, 0\rangle \in (\Theta_1 \cap \Phi) \cdot (\Theta_1 \cap \Theta_2) \subseteq (\Theta_1 \cap \Phi) \vee (\Theta_1 \cap \Theta_2)$$

proving $a \in [0]_{(\Theta_1\wedge\Phi)\vee(\Theta_1\wedge\Theta_2)}$. □

4 Regularity at 0

Another deeply studied congruence property is regularity. An algebra $\mathcal{A}$ whose every two congruences coincide whenever they have a common class is called *regular*. A weaker condition, 0–*regularity*, requires two congruences on an algebra with 0 to be equal whenever their classes containing 0 are equal. Regular resp. 0–regular varieties (i.e. varieties with all members regular resp. 0–regular) were characterized by Csákány [5] and Fichtner [8] respectively. Regularity on single algebras was studied in [3]. For the following result, see [8].

Proposition. *Let $\mathcal{V}$ be a variety with 0. $\mathcal{V}$ is 0–regular if and only if there are binary terms d_i, $i = 1, \ldots, n$, such that*

$$d_1(x,y) = \ldots = d_n(x,y) = 0 \quad \text{iff} \quad x = y \ .$$

Theorem 3. *Let $\mathcal{A}$ be an algebra with 0, $card\, A \leq 3$. Then $\mathcal{A}$ is 0–regular if and only if there are binary functions d_i, $i = 1, \ldots, n$, compatible with $Con\, \mathcal{A}$ such that*

$$d_1(x,y) = \ldots = d_n(x,y) = 0 \quad iff \quad x = y \ . \tag{3}$$

Moreover, for $card\, A \leq 2$ we can put $n = 1$, for $card\, A = 3$ the least possible n is 2. For $card\, A = 4$, there are in general no such functions.

P r o o f: The case $card\, A \leq 2$ is trivial.

Let $\mathcal{A}$ with $A = \{0, a, b\}$ be 0–regular. Put

$$d_i(x,x) = 0, \qquad d_i(x,y) = pr_i(x,y) \quad \text{for } x \neq y,\ i = 1, 2.$$

We have $Con\, \mathcal{A} \subseteq \{\omega, \iota, \theta_{0a}, \theta_{0b}\}$ where , given by their partitions,
$\theta_{0a} \ldots \{0, a\}, \{b\}$
$\theta_{0b} \ldots \{0, b\}, \{a\}$.
Clearly, (3) holds. The compatibility with $Con\, \mathcal{A}$ may be easily verified.

We show that $n = 2$ is the least possible. Suppose that there is a compatible $d(x,y)$ satisfying (3) and let $\mathcal{A}$ be such that $Con\, \mathcal{A} = \{\omega, \iota, \theta_{0a}, \theta_{0b}\}$. It holds

$$d(a,b)\theta_{0a}d(0,b) \quad \text{and} \quad d(a,b)\theta_{0b}d(a,0) \ .$$

We have

$$d(0,a) \neq b \neq d(a,0) \quad \text{and} \quad d(0,b) \neq a \neq d(b,0)$$

(if e.g. $d(a,0) = b$ then $b = d(0,a)\theta_{0a}d(0,0) = 0$, a contradiction). From $d(0,a) \neq b$, $d(0,a) \neq 0$ we conclude $d(0,a) = a$. Similarly, $d(a,0) = a, d(b,0) = b, d(0,b) = b$. But then

$$b = d(0,b)\theta_{0a}d(a,b) \Rightarrow d(a,b) = b$$

and

$$b = d(a,0)\theta_{0b}d(a,b) \Rightarrow d(a,b) = a \ ,$$

a contradiction.

Suppose now $\mathcal{A}$ not be regular and let $d_i(x,y)$, $i = 1, \ldots, n$, exist. Then $\theta = \{0\} \times \{0\} \cup \{a,b\} \times \{a,b\} \in Con\, \mathcal{A}$ which implies

$$d_i(a,b)\theta d_i(a,a) = 0$$

for all $i = 1, \ldots, n$, so $d(a,b) = 0$ for $i = 1, \ldots, n$, a contradiction with (3).

Let $\mathcal{A}$, $A = \{0, a, b, c\}$, be 0–regular such that $\theta_1, \theta_2, \theta_3 \in Con\, \mathcal{A}$ where, by partitions,
$\theta_1 \ldots \{0, a\}, \{b, c\}$
$\theta_2 \ldots \{0, b\}, \{a\}, \{c\}$
$\theta_3 \ldots \{0, c\}, \{a\}, \{b\}$.
One may put e.g. $\mathcal{A} = (A, F)$ where $F = \{f, g\}$ is a set of unary operations given by the following table.

	f	g
0	c	a
a	b	0
b	c	a
c	c	a

By $0 = d_i(b,b)\theta_1 d_i(b,c)$, $i = 1,\dots,n$, we have $d_i(b,c) \in \{0,a\}$. Further, $d_i(0,c) \in \{0,c\}$, $i = 1,\dots,n$, for if e.g. $d_i(0,c) = a$ then $a = d_i(0,c)\theta_3 d_i(0,0) = 0$, a contradiction. For all $i = 1,\dots,n$, it holds $d_i(b,c)\theta_2 d_i(0,c)$, so $d_i(b,c) = 0$ for all $i = 1,\dots,n$, which is impossible by (3). □

We say that an algebra $\mathcal{A}$ has *transferable principal congruences* (briefly TPC) *at* 0 if for every $a,b \in A$ there is $c \in A$ such that $\theta(a,b) = \theta(0,c)$.

Theorem 4. *Let $\mathcal{A}$ with 0 has TPC at 0. Then $\mathcal{A}$ is 0–regular.*

Proof: Let $\theta,\phi \in Con\,\mathcal{A}$ and let $[0]_\theta = [0]_\phi$. Then $\theta \cap \phi \subseteq \theta$ and $[0]_{\theta\cap\phi} = [0]_\theta$. It suffices therefore to consider $\theta,\phi \in Con\,\mathcal{A}$ with $\theta \subseteq \phi$.

Let $\theta \subseteq \phi$, $[0]_\theta = [0]_\phi$. Let $\langle x,y\rangle \in \phi$. By TPC at 0, there is an element $z \in A$ such that $\theta(x,y) = \theta(0,z)$. Clearly $\theta(x,y) \subseteq \phi$, which gives the following series of implications: $\theta(0,z) \subseteq \phi \Rightarrow \langle 0,z\rangle \in \phi \Rightarrow z \in [0]_\phi = [0]_\theta \Rightarrow \langle 0,z\rangle \in \theta \Rightarrow \theta(x,y) = \theta(0,z) \subseteq \theta$, i.e. $\phi \subseteq \theta$. Altogether, $\theta = \phi$ proving $\mathcal{A}$ being 0–regular. □

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