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# CUTLIKE SEMANTICS FOR FUZZY LOGIC AND ITS APPLICATIONS

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Each fuzzy set can be represented by a nested system of ordinary sets—its $a$-cuts. There is an extensive literature on fuzzy sets devoted to problems of the following kind: is it possible to reduce operations with fuzzy sets to operations with their $a$-cuts? Is it possible to reduce properties of fuzzy relations to properties of their $a$-cuts? More generally, can a fuzzy concept be represented by a collection of corresponding crisp concepts? Klir and Yuan (1995) speak of cutworthiness.

We attempt to provide a general solution to this problem. The way we proceed is thus: a structure for fuzzy predicate logic can be represented by a nested system of crisp structures. The system of crisp structures can be used to define semantics of fuzzy predicate logic in an alternative way by using the nested structure and Boolean connectives only. Answers to the above questions are then obtained by simple application of the obtained general results; we present some examples (extension principle, properties of binary fuzzy relations, fuzzy automata).

*Keywords*: Fuzzy set; Alpha cut; Predicate fuzzy logic; Semantics

## 1. INTRODUCTION

Recall that an $a$-cut of a fuzzy set $A$ in a universe $X$ (Zadeh, 1965) is an ordinary set ${}^aA = \{x \in X | A(x) \geq a\}$. It is a kind of folklore that each fuzzy set is uniquely determined by the collection of all of its $a$-cuts ($a$ from the set $L$ of truth values). Moreover, collections $\{A_a \subseteq X | a \in L\}$ which are systems of $a$-cuts of fuzzy sets are easily characterized (see Definition 1 and Theorem 2). Therefore, fuzzy sets may be looked at as special nested systems of ordinary sets. It has been observed that

- some properties of fuzzy sets are equivalent to corresponding properties of their $a$-cuts (e.g. a fuzzy relation is symmetric iff each of its $a$-cuts is symmetric), some not (it is in general, i.e. under a general $t$-norm, not true that a fuzzy relation is transitive iff all of its $a$-cuts are transitive);

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- some operations with fuzzy sets may be performed "cut by cut" ($a$-cut of intersection of fuzzy sets is the intersection of $a$-cuts of these fuzzy sets), some not (e.g. a complement to an $a$-cut of $A$ is not equal to the $a$-cut of the complement of $A$);
- some "fuzzy concepts" can be represented by (collections of) their crisp counterparts (see e.g. Bělohlávek, 1999; Močkoř, 1999).

Klir and Yuan (1995) speak of *cutworthy properties* in this connection (the notion of cutworthiness goes back to Bandler and Kohout).

The aim of this paper is to show that this situation can be approached in general from the point of view of fuzzy logic (in narrow sense). The idea behind it is to define semantics of fuzzy predicate logic in a cut-like manner: we show that the notion of a truth degree of a formula in a given fuzzy structure can be defined using the notion of a truth degree of a formula in a crisp (bivalent, non-fuzzy) structure and Boolean connectives only. The crisp structures used in this cut-like definition of semantics are $a$-cuts of the fuzzy structure in which a formula is to be evaluated (see later). Section 2 sets the problem; Section 3 presents the result; Section 4 contains some applications.

## 2. PROBLEM SETTING AND MOTIVATION

In this section, we first present in more detail some well-known examples discussed in the literature in connection with cutworthiness. Then we make some observations helping us to identify a "common core" of the examples and formulate basic requirements on the general approach to cutworthiness and the formal framework for discussing the computation with $a$-cuts.

### 2.1 Properties of Fuzzy Relations, *a*-cuts and Cutworthiness

Since the beginning of fuzzy set theory, various generalizations of properties of crisp relations have been proposed for fuzzy relations. For example, a binary fuzzy relation $R$ in a set $M$ is called reflexive if $R(m,m) = 1$ for each $m \in M$; symmetric if $R(m,n) = R(n,m)$ for each $m$, $n \in M$; transitive with respect to a "fuzzy conjunction" $\otimes$ (e.g. a $t$-norm) if $R(m_1,m_2) \otimes R(m_2,m_3) \leq R(m_1,m_3)$ for each $m_1,m_2,m_3 \in M$. These are proper generalizations in that if one considers a crisp fuzzy relation $R$ (i.e. $R(m,n)$ is either 0 or 1 for any $m,n \in M$) then $R$ is reflexive (symmetric, transitive) as an ordinary relation iff $R$ is reflexive (symmetric, transitive) as a fuzzy relation. It can be easily shown (it is in fact a well-known fact) that a fuzzy relation $R$ is reflexive iff each ${}^aR$ of its $a$-cuts (i.e. $a$ ranges over all truth degrees) is reflexive (as an ordinary relation) iff ${}^1R$ (the 1-cut) is reflexive (as an ordinary relation). Likewise, $R$ is symmetric iff each ${}^aR$ is symmetric. However, an analogous result does not hold true for transitivity. The only case where it is true that $R$ is transitive iff each ${}^aR$ is transitive is when $\otimes$ is minimum (or infimum in general).

### 2.2 Operations over Fuzzy Sets, *a*-cuts, and Cutworthiness

An operation $f$ which maps (possibly tuples of) fuzzy sets to fuzzy sets is said to be cutworthy if ${}^af(A,\ldots) = f({}^aA,\ldots)$. That is, each $a$-cut of the result is the image of $a$-cuts of the arguments. As an example, the standard intersection of fuzzy sets ($(A \cap B)(x) = A(x) \wedge B(x)$) is cutworthy while the standard negation ($\bar{A}(x) = \overline{A(x)} = 1 - A(x)$) is not.

### 2.3 Problem Setting

A problem arising from the above discussion can be formulated as follows. Develop a general framework that will enable us to treat the problem of $a$-cuts so that it explains how manipulation over fuzzy sets and fuzzy relations is related to manipulation over their $a$-cuts. In particular, the framework should provide general results that answer the above two particular problems, i.e. (1) how properties of fuzzy relations are related to the corresponding properties of their $a$-cuts; (2) how computation over fuzzy relations is related to computation over their $a$-cuts; and possibly also (3) how "fuzzy notions" are related to the corresponding "ordinary notions" in general. In the following, we attempt to provide such a framework which, however limited, covers a sufficiently wide set of particular examples discussed so far in the literature.

### 2.4 Initial Observations

First, let us mention that all the above-mentioned examples (properties of fuzzy relations, operations with fuzzy sets) can be described using formulas of first-order fuzzy logic. Take, e.g. transitivity of fuzzy relations. A moment's inspection shows that $R$ is transitive iff the formula $(\forall x,y,z)(r_R(x,y) \otimes r_R(y,z) \Rightarrow r_R(x,z)$ has truth degree 1 under an interpretation assigning $R$ to the binary relational symbol $r_R$ (see Section 4 for more details). Furthermore, the degree $(A \cap B)(m)$ to which $m$ belongs to the intersection $A \cap B$ is the truth degree of formula $r_A(x) \otimes r_B(x)$ under an interpretation where $r_A$, $r_B$ are interpreted by fuzzy sets $A$ and $B$, respectively, and $x$ is interpreted by $m$. This suggests first-order fuzzy logic as a suitable candidate for the framework in question.

Second, note that even if the property or operation in question is not cutworthy, there might be a connection to appropriate $a$-cuts. As an example, it is easy to verify that for a fuzzy set $A$ and for the standard negation we have ${}^a\overline{A} = \overline{\bigcup_{1-a<b} {}^bA}$. That is, any $a$-cut of $\bar{A}$ can be obtained from appropriate $b$-cuts using (Boolean) set union, complement, and the structure of truth degrees (ordering and negation). We wish to extend this particular case to a general one.

Inspecting the above remarks from the point of view of first-order fuzzy logic, one can see that the central problem is the following. We have a logical formula $\varphi$ which is evaluated in some fuzzy structure $\mathbf{M}$ (a system of fuzzy relations $r^{\mathbf{M}}, \ldots$, and possibly functions). On the other hand, $\varphi$ (and it subformulas) can be evaluated in crisp structures ${}^a\mathbf{M}$ obtained from $\mathbf{M}$ by the operation of a cut (i.e. the crisp relations $r^{{}^a\mathbf{M}}$ of ${}^a\mathbf{M}$ are $a$-cuts of the corresponding fuzzy relations $r^{\mathbf{M}}$ of $\mathbf{M}$). Now, what is the connection of the truth degree $\|\varphi\|_{\mathbf{M}}$ of $\varphi$ in $\mathbf{M}$ to the truth degrees $\|\psi\|_{{}^a\mathbf{M}}$ of appropriate subformulas $\psi$ of $\varphi$ in appropriate $a$-cuts ${}^a\mathbf{M}$ of $\mathbf{M}$?

## 3. CUTLIKE SEMANTICS FOR PREDICATE FUZZY LOGIC

We need to clarify our conception of predicate fuzzy logic. Predicate fuzzy logic which is used as a formal framework for our treatment is basically the so-called monoidal fuzzy logic introduced by Höhle (1996). We assume that the reader is familiar with basic ideas and formalism of fuzzy sets as explained for example by Klir and Yuan (1995). We start by delineating structures of truth values of our fuzzy logic and then go to syntax and semantics.

We use complete residuated lattices as structures of truth values (see Goguen, 1968–1969; Höhle, 1996; Hájek, 1998) and assume basic familiarity with their properties.

Throughout the paper, $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ denotes a complete residuated lattice. Recall that a (complete) residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice with the least element 0 and the greatest element 1, $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is a commutative and associative binary operation on $L$ satisfying $a \otimes 1 = a$), and for $\otimes$ and $\rightarrow$ we have $a \otimes b \leq c$ if and only if $a \leq b \rightarrow c$ for each $a, b, c \in L$. The set $L$ plays the role of a set of truth degrees, 0 and 1 representing full falsity and full truth, respectively. In the following, $\mathbf{L}$ denotes an arbitrary complete residuated lattice (with $L$ being the universe set of $\mathbf{L}$). All properties of complete residuated lattices used in the sequel are well known and can be found in the above-mentioned literature. Note that particular types of residuated lattices include Boolean algebras, Heyting algebras, algebras of Girard's linear logic, MV-algebras, Gödel algebras, product algebras, and more generally, BL-algebras (see Höhle, 1996; Hájek, 1998). These special residuated lattices are distinguishable by identities corresponding to logical requirements. For example, Boolean algebras are BL-algebras satisfying $a = (a \rightarrow 0) \rightarrow 0$ which is the identity corresponding to the law of double negation (note that $b \rightarrow 0$ is the negation of $b$).

Of particular interest are complete residuated lattices defined on the real unit interval [0,1] or on some subchain of [0,1]. It is well known that $\mathbf{L} = \langle [0, 1], \min, \max, \otimes, \rightarrow, 0, 1\rangle$ is a complete residuated lattice if and only if $\otimes$ is a left-continuous $t$-norm and $\rightarrow$ is defined by $a \rightarrow b = \max\{c | a \otimes c \leq b\}$. Recall that $t$-norm is a binary operation on [0,1] which is associative, commutative, monotone and has 1 as its neutral element, and hence captures the basic properties of conjunction. A $t$-norm is called left-continuous if, as a real function, it is left-continuous in both arguments. Most commonly used are continuous $t$-norms, the basic three of which are Łukasiewicz $t$-norm (given by $a \otimes b = \max(a + b - 1, 0)$ with the corresponding residuum $a \rightarrow b = \min(1 - a + b, 1)$), minimum (also called Gödel) $t$-norm ($a \otimes b = \min(a, b)$, $a \rightarrow b = 1$ if $a \leq b$ and $= b$ else), and product $t$-norm ($a \otimes b = a \cdot b$, $a \rightarrow b = 1$ if $a \leq b$ and $= b/a$ else). It can be shown (see e.g. Hájek, 1998) that each continuous $t$-norm is composed out of the three above-mentioned $t$-norms by a simple construction (ordinal sum). Any finite subchain of [0,1] containing both 0 and 1, equipped with restrictions of the minimum $t$-norm and its residuum is a complete residuated lattice. Furthermore, the same holds true for any equidistant finite chain $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ equipped with restrictions of Łukasiewicz operations. The only residuated lattice on the two-element chain {0,1} (with $0 < 1$) has the classical conjunction operation as $\otimes$ and classical implication operation as $\rightarrow$. That is, the two-element residuated lattice is the two-element Boolean algebra of classical logic.

An $\mathbf{L}$-set (or fuzzy set with truth degrees from a complete residuated lattice $\mathbf{L}$) in a universe $X$ is any mapping $A$ from $X$ to $L$ (in general, we use "$\mathbf{L}$-$\cdots$" instead of "fuzzy ..." to make the structure of truth degrees explicit). Thus, for a residuated lattice $\mathbf{L}$ with $L = [0, 1]$, we get the usual notion of a fuzzy set $A$ in a universe $X$ as a mapping $A$ from $X$ to $[0, 1]$. This is how the notion of an $\mathbf{L}$-set generalizes the usual notion of a fuzzy set. The truth degree $A(x) \in L$ is interpreted as the truth degree to which $x$ belongs to $A$. An $n$-ary $\mathbf{L}$-relation on a universe set $X$ is an $\mathbf{L}$-set in the universe set $X^n$. For example, a binary relation $R$ on $X$ is a mapping $R: X \times X \rightarrow L$. Finally, for an $\mathbf{L}$-set $A$ in $X$ and a truth degree $a \in L$, the $a$-cut ${}^aA$ of $A$ is an ordinary subset of $X$ defined by

$$ {}^aA = \{x \in X | A(x) \geq a\}. $$

That is, ${}^aA$ consists of those elements of $X$ which belong to $A$ in degree at least $a$.

We need basic concepts of fuzzy predicate logic. The language $\mathscr{J}$ of our logic contains: a non-empty set $R$ of relation symbols, each $r \in R$ with it arity $\sigma(r)$; a (possibly empty) set $F$ of function symbols, each $f \in F$ with its arity $\sigma(f)$; object variables (we denote object variables by $x_1, x_2, \ldots, x, y, z, \ldots, \xi, \nu, \ldots$etc.); logical connectives $\owedge$, $\ovee$, $\otimes$, $\Rightarrow$; truth constants

$\mathbb{0}$ and $\mathbb{1}$ (and possibly others); quantifiers $\forall$ and $\exists$ (universal and existential); auxiliary symbols. $R$ may contain a binary relation symbol $\approx$ (equality) which is then handled in a special way (it is interpreted as fuzzy equivalence and is required to be compatible with fuzzy relations and functions that interpret symbols from $R$ and $F$); in this paper, however, we will not pay attention to language with $\approx$. The above language $\mathscr{J}$ is also said to be of type $\langle R, F, \sigma\rangle$. Terms and formulas are defined as usual. Terms: each variable is a term; if $t_i$ are terms and $f \in F$ then $f(\ldots, t_i, \ldots)$ is a term. Formulas: truth constants are formulas; $r(\ldots, t_i, \ldots)$ are formulas ($r \in R$, $t_i$ terms); if $\varphi$ and $\psi$ are formulas then $(\varphi \curlywedge \psi)$, $(\varphi \curlyvee \psi)$, $(\varphi \otimes \psi)$, $(\varphi \Rightarrow \psi)$, $(\forall x)\varphi$, and $(\exists x)\varphi$ are formulas.

Semantics is defined as usual: An $\mathbf{L}$-structure $\mathbf{M} = \langle M, R^{\mathbf{M}}, F^{\mathbf{M}}\rangle$ for $\mathscr{J}$ consists of a non-empty set $M$, a set $R^{\mathbf{M}} = \{r^{\mathbf{M}} \in L^{M^n} | r \in R,\ \sigma(r) = n\}$ of $\mathbf{L}$-relations, and a set $F^{\mathbf{M}} = \{f^{\mathbf{M}} : M^n \to M | f \in F,\ \sigma(r) = n\}$ of functions (such that $\approx^{\mathbf{M}}$ is an $\mathbf{L}$-equivalence relation on $M$ and each $r^{\mathbf{M}} \in R^{\mathbf{M}}$ and $f^{\mathbf{M}} \in F^{\mathbf{M}}$ is compatible w.r.t. $\approx^{\mathbf{M}}$). That is, for each relation symbol $r \in R$, $\mathbf{M}$ contains a corresponding $\mathbf{L}$-relation $r^{\mathbf{M}}$, and for each function symbol $f \in F$, $\mathbf{M}$ contains a corresponding (ordinary) function $f^{\mathbf{M}}$. An $\mathbf{M}$-valuation of object variables is a mapping $v$ assigning an element $v(x) \in M$ to any variable $x$. If $v$ and $v'$ are valuations, and $x$ a variable we write $v =_x v'$ if for each variable $y \neq x$ we have $v(y) = v'(y)$. A value $\|t\|_{\mathbf{M},v}$ of a term $t$ under the $\mathbf{M}$-valuation $v$ is defined by: for a variable $x$, $\|x\|_{\mathbf{M},v} = v(x)$; for $t = f(t_1, \ldots, t_n)$, $\|t\|_{\mathbf{M},v} = f^{\mathbf{M}}(\|t_1\|_{\mathbf{M},v}, \ldots, \|t_n\|_{\mathbf{M},v})$. A truth degree $\|\varphi\|_{\mathbf{M},v}$ of a formula $\varphi$: for atomic formulas, $\|r(t_1, \ldots, t_n)\|_{\mathbf{M},v} = r^M(\|t_1\|_{M,v}, \ldots, \|t_n\|_{M,v})$, $\|\mathbb{0}\|_{\mathbf{M},v} = 0$, $\|\mathbb{1}\|_{\mathbf{M},v} = 1$; $\|\varphi \curlywedge \psi\|_{\mathbf{M},v} = \|\varphi\|_{\mathbf{M},v} \wedge \|\psi\|_{\mathbf{M},v}$, $\|\varphi \curlyvee \psi\|_{\mathbf{M},v} = \|\varphi\|_{\mathbf{M},v} \vee \|\psi\|_{\mathbf{M},v}$, $\|\varphi \otimes \psi\|_{\mathbf{M},v} = \|\varphi\|_{\mathbf{M},v} \otimes \|\psi\|_{\mathbf{M},v}$, $\|\varphi \Rightarrow \psi\|_{\mathbf{M},v} = \|\varphi\|_{\mathbf{M},v} \to \|\psi\|_{\mathbf{M},v}$, $\|(\forall x)\varphi\|_{\mathbf{M},v} = \bigwedge\{\|\varphi\|_{\mathbf{M},v'} | v' =_x v\}$, $\|(\exists x)\varphi\|_{\mathbf{M},v} = \bigvee\{\|\varphi\|_{\mathbf{M},v'} | v' =_x v\}$.

The foregoing definitions can be easily extended for many-sorted case (see e.g. Hájek, 1998).

We now turn to representation by $a$-cuts.

DEFINITION 1 An $L$-indexed system $\mathscr{S} = \{A_a \subseteq X | a \in L\}$ is called $\mathbf{L}$-nested if

(1) $a \leq b$ implies $A_b \subseteq A_a$,
(2) for each $x \in X$ the set $\{a | x \in A_a\}$ has the greatest element.

For an $\mathbf{L}$-set $A$ in $X$, and a system $\mathscr{S} = \{A_a \subseteq X | a \in L\}$ of subsets of $X$ we define a system $\mathscr{S}_A$ of subsets of $X$, and an $\mathbf{L}$-set $A_{\mathscr{S}}$ in $X$ by

$$\mathscr{S}_A = \{{}^aA | a \in L\}, \quad A_{\mathscr{S}}(x) = \bigvee_{x \in A_a} a.$$

Therefore, $\mathscr{S}_A$ is the system of all $a$-cuts. The following theorem shows that there is a bijective correspondence between $\mathbf{L}$-set in $X$ and $L$-nested systems of subsets of $X$.

THEOREM 2 (FOLKLORE) Let $A$ be an $\mathbf{L}$-set in $X$, $\mathscr{S}$ be an $\mathbf{L}$-nested system of subsets of $X$. Then $\mathscr{S}_A$ is $\mathbf{L}$-nested, and we have $A = A_{\mathscr{S}_A}$ and $\mathscr{S} = \mathscr{S}_{A_{\mathscr{S}}}$.

### 3.1 Cut-like Semantics

Theorem 2 says that fuzzy sets in $M$ can be thought of as (special types of) nested systems of (ordinary) sets in $M$. Therefore, each $\mathbf{L}$-structure $\mathbf{M}$ of type $\langle R,F,\sigma\rangle$ can be represented by a system $\mathscr{C}_{\mathbf{M}} = \{{}^a\mathbf{M} | a \in L\}$ of $\mathbf{2}$-structures ${}^a\mathbf{M}$ defined by ${}^aM = M$; $f^{{}^a\mathbf{M}} = f^{\mathbf{M}}$ for $f \in F$; $r^{{}^a\mathbf{M}} = {}^ar^{\mathbf{M}}$ for $r \in R$. Thus, ${}^a\mathbf{M}$ has the same universe and functions as $\mathbf{M}$; relation in ${}^a\mathbf{M}$

corresponding to $r$ is the $a$-cut of the **L**-relation that corresponds to $r$ in **M**. Conversely, having an **L**-indexed system $\mathscr{C} = \{\mathbf{M}_a | a \in L\}$ of **2**-structures with a common universe $M$ (i.e. $M_a = M$ for each $a \in L$) and common functions (i.e. $f^{\mathbf{M}_a} = f^{\mathbf{M}_b}$ for each $a, b \in L$) such that for each $r \in R$, $\{r^{\mathbf{M}_a} | a \in L\}$ is an **L**-nested system (cf. Definition 1), $\mathscr{C}$ induces an **L**-structure $\mathbf{M}_{\mathscr{C}}$ given by $M_{\mathscr{C}} = M$; $f^{\mathbf{M}_{\mathscr{C}}} = f^{\mathbf{M}_a}$ (for any $a \in L$);

$$r^{\mathbf{M}_{\mathscr{C}}}(m_1, \ldots, m_n) = \bigvee \left\{ a | \langle m_1, \ldots, m_n \rangle \in r^{\mathbf{M}_a} \right\}.$$

Definition 3 An **L**-indexed system $\{\mathbf{M}_a | a \in L\}$ of **2**-structures (i.e. ordinary structures) of the same type is called **L**-nested if for each $a, b \in L$ we have $M_a = M_b$ (common universe), $f^{\mathbf{M}_a} = f^{\mathbf{M}_b}$ for each $f \in F$ (common functions), and for each $r \in R$, $\{r^{\mathbf{M}_a} | a \in L\}$ is an **L**-nested system.

Therefore, we have

Lemma 4 The mappings sending an **L**-structure **M** to an **L**-nested system $\mathscr{C}_{\mathbf{M}}$, and an **L**-nested system $\mathscr{C}$ of structures into an **L**-structure $\mathbf{M}_{\mathscr{C}}$ are mutually inverse.

*Proof* Directly by the above considerations and Theorem 2. □

Let us have an **L**-structure **M**. Given a valuation $v{:}X \to M$, a formula $\varphi$ evaluates to $\|\varphi\|_{\mathbf{M},v}$. As $\mathbf{M}_a$ ($a \in L$) is a **2**-structure, one can consider the truth degree $\|\varphi\|_{\mathbf{M}_a, v}$ to which $\varphi$ evaluates under $v$. By definition, $\|\varphi\|_{\mathbf{M}_a, v}$ is either 0 or 1. There is a natural question of whether the truth degree $\|\varphi\|_{\mathbf{M},v}$ of $\varphi$ under $v$ can be in some way composed of truth degrees $\|\varphi\|_{\mathbf{M}_a, v}$ that are assigned to $\varphi$ or its subformulas in $\mathbf{M}_a$ under $v$. One way to answer this question is elaborated in the following. Denote

$${}^a\|\varphi\|_{\mathbf{M},v} = \begin{cases} 1 & \text{if } a \leq \|\varphi\|_{\mathbf{M},v} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, ${}^a\|\varphi\|_{\mathbf{M},v}$ is the $a$-cut of $\|\varphi\|_{\mathbf{M},v}$. Note that $\|\varphi\|_{\mathbf{M},v}$ can be seen as a truth degree assigned to $v$ given **M** and $\varphi$; this way a formula and an **L**-structure induce an **L**-set in the set of all valuations.

*Example 5* Theorem 2 presented in another way serves as a demonstration: consider a language with a unary relation symbol $r$. In an **L**-structure **M**, $r^{\mathbf{M}}$ (**L**-relation corresponding to $r$) is an **L**-set in $M$. Clearly, for a valuation $v$ with $v(x) = m$ and $\varphi = r(x)$, we have $r^{\mathbf{M}}(m) = \|\varphi\|_{\mathbf{M},v}$. Therefore, Theorem 2 then says

$$\|\varphi\|_{\mathbf{M},v} = \bigvee \left\{ a \mid {}^a\|\varphi\|_{\mathbf{M},v} = 1 \right\}$$

which is a way to obtain $\|\varphi\|_{\mathbf{M},v}$ from ${}^a\|\varphi\|_{\mathbf{M},v}$ for the simple case of atomic formula.

In the following we denote the operations in **2** (the two-element Boolean algebra) by $\wedge^{\mathbf{2}}$, $\vee^{\mathbf{2}}$, $\otimes^{\mathbf{2}}$, $\to^{\mathbf{2}}$, $\bigwedge^{\mathbf{2}}$ and $\bigvee^{\mathbf{2}}$ (note that $\otimes^{\mathbf{2}} = \wedge^{\mathbf{2}}$).

Let $\mathscr{C} = \{\mathbf{M}_a \mid a \in L\}$ be an **L**-nested system with a universe $M$. For a term $t$ we put

$$\|t\|_{\mathscr{C},v} =_{df} \|t\|_{\mathbf{M}_a,v}$$

for $a \in L$ (this is correct since each $f \in F$, $f^{\mathbf{M}_a}$ are the same for $a \in L$). Furthermore, for a formula $\varphi$ and $a \in L$ we define $\|\varphi\|^a_{\mathscr{C},v}$ as follows.

(i) for atomic formulas:

$$\|r(t_1,\ldots,t_n)\|^a_{\mathscr{C},v} = r^{\mathbf{M}_a}(\|t_1\|_{\mathscr{C},v},\ldots,\|t_n\|_{\mathscr{C},v}),$$

$$\|\mathbb{b}\|^a_{\mathscr{C},v} = 1 \quad \text{for } a \le b$$

$$0 \quad \text{otherwise}$$

($\mathbb{b}$ a truth constant);

(ii) if $\varphi$ and $\psi$ are formulas then

$$\|\varphi \mathbin{\&\!\!\&} \psi\|^a_{\mathscr{C},v} = \|\varphi\|^a_{\mathscr{C},v} \wedge^{\mathbf{2}} \|\psi\|^a_{\mathscr{C},v},$$

$$\|\varphi \mathbin{\vee\!\!\vee} \psi\|^a_{\mathscr{C},v} = \bigvee^{\mathbf{2}}_{a\le b\vee c} \|\varphi\|^b_{\mathscr{C},v} \wedge^{\mathbf{2}} \|\psi\|^c_{\mathscr{C},v},$$

$$\|\varphi\otimes\psi\|^a_{\mathscr{C},v} = \bigvee^{\mathbf{2}}_{a\le b\otimes c} \|\varphi\|^b_{\mathscr{C},v} \wedge^{\mathbf{2}} \|\psi\|^c_{\mathscr{C},v},$$

$$\|\varphi \Rightarrow \psi\|^a_{\mathscr{C},v} = \bigwedge^{\mathbf{2}}_{b\in L} \|\varphi\|^b_{\mathscr{C},v} \to^{\mathbf{2}} \|\psi\|^{a\otimes b}_{\mathscr{C},v};$$

(iii) if $\varphi$ is a formula and $x$ a variable then

$$\|(\forall x)\varphi\|^a_{\mathscr{C},v} = \bigwedge^{\mathbf{2}}_{v'=_x v} \|\varphi\|^a_{\mathscr{C},v'}$$

$$\|(\exists x)\varphi\|^a_{\mathscr{C},v} = \bigvee^{\mathbf{2}}_{a\le\bigvee_{v'=_x v} b_{v'}} \bigwedge^{\mathbf{2}}_{v'=_x v} \|\varphi\|^{b_{v'}}_{\mathscr{C},v'}.$$

First, we show that $\|\varphi\|^a_{\mathscr{C},v}$ is exactly the $a$-cut of $\|\varphi\|_{\mathbf{M}_{\mathscr{C}},v}$.

LEMMA 6 For an **L**-structure **M** and the corresponding **L**-indexed system $\mathscr{C}_{\mathbf{M}}$ we have

$${}^a\|\varphi\|_{\mathbf{M},v} = \|\varphi\|^a_{\mathscr{C}_{\mathbf{M}},v}.$$

*Proof* We prove the assertion by induction over $\varphi$.

If $\varphi$ is atomic (i.e. either $r(t_1,\ldots,t_n)$ or a truth constant) then the assertion follows directly by definition and from $\|t_i\|_{\mathscr{C}_{\mathbf{M}},v} = \|t_i\|_{\mathbf{M},v}$.

Assume that the assertion is valid for $\varphi$ and $\psi$.

We have ${}^a\|\varphi \mathbin{\&\!\!\&} \psi\|_{\mathbf{M},v} = 1$ iff $a \le \|\varphi \mathbin{\&\!\!\&} \psi\|_{\mathbf{M},v}$ iff $a \le \|\varphi\|_{\mathbf{M},v}$ and $a \le \|\psi\|_{\mathbf{M},v}$ iff ${}^a\|\varphi\|_{\mathbf{M},v} = 1$ and ${}^a\|\psi\|_{\mathbf{M},v} = 1$ iff $\|\varphi\|^a_{\mathscr{C}_{\mathbf{M}},v} = 1$ and $\|\psi\|^a_{\mathscr{C}_{\mathbf{M}},v} = 1$ iff $\|\psi\|^a_{\mathscr{C}_{\mathbf{M}},v} \wedge^{\mathbf{2}} \|\psi\|^a_{\mathscr{C}_{\mathbf{M}},v} = 1$ iff $\|\varphi \mathbin{\&\!\!\&} \psi\|^a_{\mathscr{C}_{\mathbf{M}},v} = 1$, proving that the assertion is valid for $\varphi \mathbin{\&\!\!\&} \psi$.

For $\mathbin{\vee\!\!\vee}$ we have ${}^a\|\varphi \mathbin{\vee\!\!\vee} \psi\|_{\mathbf{M},v} = 1$ iff $a \le \|\varphi\|_{\mathbf{M},v} \vee \|\psi\|_{\mathbf{M},v}$ iff there exist $b$, $c \in L$ with $a \le b \vee c$ and $b \le \|\varphi\|_{\mathbf{M},v}$, $c \le \|\psi\|_{\mathbf{M},v}$ iff there exist $b$, $c \in L$ with $a \le b \vee c$ and ${}^b\|\varphi\|_{\mathbf{M},v} = 1$, ${}^c\|\psi\|_{\mathbf{M},v} = 1$ iff there exist $b, c \in L$ with $a \le b \vee c$ and $\|\varphi\|^b_{\mathscr{C}_{\mathbf{M}},v} = 1$,

$\|\psi\|\mathscr{C}_{\mathbf{M},v} = 1$ iff there exist $b,c \in L$ with $a \leq b \vee c$ and $\|\varphi\|^b_{\mathscr{C}_{\mathbf{M},v}} \wedge^{\mathbf{2}} \|\psi\|\mathscr{C}^c_{\mathbf{M},v} = 1$ iff $\|\varphi \Cup \psi\|^a_{\mathscr{C}_{\mathbf{M},v}} = 1$, showing that the assertion holds true for $\varphi \Cup \psi$.

Consider $\otimes$: ${}^a\|\varphi \otimes \psi\|_{\mathbf{M},v} = 1$ iff $a \leq \|\varphi\|_{\mathbf{M},v} \otimes \|\psi\|_{\mathbf{M},v}$ if there exist $b,c \in L$ such that $a \leq b \otimes c$ and $b \leq \|\varphi\|_{\mathbf{M},v}, c \leq \|\psi\|_{\mathbf{M},v}$ iff there exist $b,\ c \in L$ with $a \leq b \otimes c$ and ${}^b\|\varphi\|_{\mathbf{M},v} = 1$, ${}^c\|\psi\|_{\mathbf{M},v} = 1$ iff there exist $b,\ c \in L$ with $a \leq b \otimes c$ and $\|\varphi\|^b_{\mathscr{C}_{\mathbf{M},v}} = 1$, $\|\psi\|\mathscr{C}^c_{\mathbf{M},v} = 1$ iff there exist $b,\ c \in L$ with $a \leq b \otimes c$ and $\|\varphi\|^b_{\mathscr{C}_{\mathbf{M},v}} \wedge^{\mathbf{2}} \|\psi\|\mathscr{C}^c_{\mathbf{M},v} = 1$ iff $\|\varphi \otimes \psi\|_{\mathscr{C}_{\mathbf{M},v}a} = 1$.

For $\Rightarrow$, we have ${}^a\|\varphi \Rightarrow \psi\|_{\mathbf{M},v} = 1$ iff $a \leq \|\varphi \Rightarrow \psi\|_{\mathbf{M},v}$ iff $a \otimes \|\varphi\|_{\mathbf{M},v} \leq \|\psi\|_{\mathbf{M},v}$. On the other hand, $\bigwedge^{\mathbf{2}}_{b\in L} \|\varphi\|^b_{\mathscr{C}_{\mathbf{M},v}} \to^{\mathbf{2}} \|\psi\|^{a\otimes b}_{\mathscr{C}_{\mathbf{M},v}} = 1$ iff for each $b \in L$, $\|\varphi\|^b_{\mathscr{C}_{\mathbf{M},v}} \to^{\mathbf{2}} \|\psi\|^{a\otimes b}_{\mathscr{C}_{\mathbf{M},v}} = 1$. Therefore, we have to show that $a \otimes \|\varphi\|_{\mathbf{M},v} \leq \|\psi\|_{\mathbf{M},v}$ is true iff for each $b \in L$, $\|\varphi\|^b_{\mathscr{C}_{\mathbf{M},v}} \to {}^{\mathbf{2}}\|\psi\|^{a\otimes b}_{\mathscr{C}_{\mathbf{M},v}} = 1$. Thus let $a \otimes \|\varphi\|_{\mathbf{M},v} \leq \|\psi\|_{\mathbf{M},v}$ and take any $b \in L$. If $\|\varphi\|^b_{\mathscr{C}_{\mathbf{M},v}} = 1$ then, by assumption $b \leq \|\varphi\|_{\mathbf{M},v}$ and so $a \otimes b \leq a \otimes \|\varphi\|_{\mathbf{M},v} \leq \|\psi\|_{\mathbf{M},v}$, whence ${}^{a\otimes b}\|\psi\|_{\mathbf{M},v} = 1$ thus $\|\psi\|^{a\otimes b}_{\mathscr{C}_{\mathbf{M},v}} = 1$ by assumption. We thus have $\|\varphi\|^b_{\mathscr{C}_{\mathbf{M},v}} \to^{\mathbf{2}} \|\psi\|^{a\otimes b}_{\mathscr{C}_{\mathbf{M},v}} = 1$. Conversely, suppose that for each $b \in L$ we have $\|\varphi\|^b_{\mathscr{C}_{\mathbf{M},v}} \to^{\mathbf{2}} \|\psi\|^{a\otimes b}_{\mathscr{C}_{\mathbf{M},v}} = 1$. Then for $b = \|\varphi\|_{\mathbf{M},v}$ we particularly have $\|\varphi\|^{\|\varphi\|_{\mathbf{M},v}}_{\mathscr{C}_{\mathbf{M},v}} \to^{\mathbf{2}} \|\psi\|^{a\otimes\|\varphi\|_{\mathbf{M},v}}_{\mathscr{C}_{\mathbf{M},v}} = 1$. Now, since $\|\varphi\|_{\mathbf{M},v} \leq \|\varphi\|_{\mathbf{M},v}$, assumption yields $\|\psi\|^{\|\varphi\|_{\mathbf{M},v}}_{\mathscr{C}_{\mathbf{M},v}} = 1$, and thus $\|\psi\|^{a\otimes\|\varphi\|_{\mathbf{M},v}}_{\mathscr{C}_{\mathbf{M},v}} = 1$. Invoking the assumption again we get $a \otimes \|\varphi\|_{\mathbf{M},v} \leq \|\psi\|_{\mathbf{M},v}$ proving that the assertion is true for $\varphi \Rightarrow \psi$.

For the general quantifier we have ${}^a\|(\forall x)\varphi\|_{\mathbf{M},v} = 1$ if $a \leq \|(\forall x)\varphi\|_{\mathbf{M},v}$ if for each valuation $v' =_x v$ we have $a \leq \|\varphi\|_{\mathbf{M},v'}$ iff for each valuation $v' =_x v$ we have $\|\varphi\|^a_{\mathscr{C}_{\mathbf{M},v'}} = 1$ iff $\bigwedge^{\mathbf{2}}_{v'=_x v} \|\varphi\|^a_{\mathscr{C}_{\mathbf{M},v'}} = 1$ iff $\|(\forall x)\varphi^a_{\mathscr{C}_{\mathbf{M},v}} = 1$.

Existential quantifier: ${}^a\|(\exists x)\varphi\|_{\mathbf{M},v} = 1$ iff $a \leq \bigvee_{v'=_x v} \|\varphi\|_{\mathbf{M},v'}$ iff there exist $b_{v'} \in L$ over all valuations $v' =_x v$ with $a \leq \bigvee_{v'=_x v} b_{v'}$ such that $b_{v'} \leq \|\varphi\|_{\mathbf{M},v'}$ iff there exist $b_{v'} \in L$ $(v' =_x v)$ with $a \leq \bigvee_{v'=_x v} b_{v'}$ such that $\|\varphi\|^{b_{v'}}_{\mathscr{C}_{\mathbf{M},v'}} = 1$ iff $\bigvee^{\mathbf{2}}_{b_{v'}\in L,\, v'=_x v, a\leq \bigvee_{v'=_x v} b_{v'}} \bigwedge^{\mathbf{2}}_{v'=_x v} \|\varphi\|^{b_{v'}}_{\mathscr{C}_{\mathbf{M},v'}} = 1$ iff $\|(\exists x)\varphi\|^a_{\mathscr{C}_{\mathbf{M},v}} = 1$.

The proof is complete. □

If **L** is linearly ordered the situation for disjunction and the existential quantifier simplifies.

LEMMA 7 For a linearly ordered **L** we have

$$\|\varphi \Cup \psi\|^a_{\mathscr{C}v} = \|\varphi\|^a_{C,v} \vee^{\mathbf{2}} \|\psi\|^a_{\mathscr{C},v};$$

if $M$ is, moreover, finite then

$$\|(\exists x)\varphi\|^a_{\mathscr{C},v} = \bigvee^{\mathbf{2}}_{v'=_x v} \|\varphi\|^a_{\mathscr{C},v'}.$$

*Proof* Note that for $a \leq b$ we have $\|\varphi\|^b_{\mathscr{C},v} \leq \|\varphi\|^a_{\mathscr{C},v}$ (by Lemma 6 and by ${}^b\|\varphi\|_{\mathbf{M},v} \leq {}^a\|\varphi\|_{\mathbf{M},v}$ for $a \leq b$). Due to linearity of **L** we have $\|\varphi \Cup \psi\|^a_{\mathscr{C},v} = \bigvee^{\mathbf{2}}_{a\leq b\vee c} \|\varphi\|^b_{\mathscr{C},v} \wedge^{\mathbf{2}} \|\psi\|^c_{\mathscr{C},v} = \bigvee^{\mathbf{2}}_{a\leq \max(b,c)} \|\varphi\|^b_{\mathscr{C},v} \wedge^2 \|\psi\|^c_{\mathscr{C},v} = \bigvee^{\mathbf{2}}_{a\leq b \text{ or } a\leq c} \|\varphi\|^b_{\mathscr{C},v} \wedge^{\mathbf{2}} \|\psi\|^c_{\mathscr{C},v} = \|\varphi\|^a_{\mathscr{C},v} \vee^2 \|\psi\|^a_{\mathscr{C},v}$. Furthermore, if the support $M$ of **M** is finite then $\|(\exists x)\varphi\|^a_{\mathscr{C},v} = \bigvee^{\mathbf{2}}_{a\leq \max_{v'=_x v} b_{v'}} \bigwedge^{\mathbf{2}}_{v'=_x v} \|\varphi\|^{b_{v'}}_{\mathscr{C},v'} = \bigvee^{\mathbf{2}}_{v'=_x v} \|\varphi\|^a_{\mathscr{C},v'}$. □

Now, for an **L**-nested system $\mathscr{C}$ of structures, a valuation $v$, and a formula $\varphi$ we define the degree $\|\varphi\|_{\mathscr{C},v}$ to which $\varphi$ is true in $\mathscr{C}$ by

$$\|\varphi\|_{\mathscr{C},v} =_{\text{df}} \bigvee \left\{ a \mid \|\varphi\|^a_{\mathscr{C},v} = 1 \right\}.$$

Then we have

THEOREM 8 For an **L**-structure **M** and the corresponding **L**-indexed system $\mathscr{C}_{\mathbf{M}}$ we have

$$\|\varphi\|_{\mathbf{M},v} = \|\varphi\|_{\mathscr{C}_{\mathbf{M}},v}.$$

*Proof* The assertion follows directly by Lemma 6 using the obvious fact $||\varphi||_{\mathbf{M},v} = \bigvee\{a \mid {}^{a}||\varphi||_{\mathbf{M},v} = 1\}$. □

*Remark 9* Note that Theorem 8 provides an alternative way to the definition of semantics. Instead of interpreting a formula $\varphi$ directly in a fuzzy structure **M**, it is, in principle, possible to consider subformulas of $\varphi$, interpret the subformulas in an **L**-nested system of crisp structures (the crisp structures $\mathbf{M}_a$ to which **M** can be decomposed by $a$-cuts), and then to use the (classical) truth values of the subformulas of $\varphi$ to obtain the truth degree of $\varphi$ in **M**.

We will see applications of this result in the next section.

## 4. APPLICATIONS

### 4.1 Cutworthy Properties and Representation of Fuzzy Notions by Crisp Notions

Some concepts and properties in fuzzy setting are in a cut-like manner related to their bivalent counterparts. For instance, a fuzzy set $A$ is the intersection of fuzzy sets $B$ and $C$ iff each $a$-cut of $A$ is the intersection of the $a$-cut of $B$ and the $a$-cut of $C$, i.e. ${}^{a}(B \cap C) = {}^{a}B \cap {}^{a}C$. Indeed, $x \in {}^{a}B \cap C$ iff $a \leq (B \cap C)(x)$ iff $a \leq B(x) \wedge C(x)$ iff $a \leq B(x)$ and $a \leq C(x)$ iff $x \in {}^{a}B$ and $x \in {}^{a}C$ iff $x \in {}^{a}B \cap {}^{a}C$.

Klir and Yuan (1995) write (p. 23): "Any property generalized from classical set theory into the domain of fuzzy set theory that is preserved in all $\alpha$-cuts for $\alpha \in (0, 1]$ in the classical sense is called a *cutworthy property*;...". Later on (p. 36): "... the standard fuzzy intersection and fuzzy union are both cutworthy...". We shall see that although Klir and Yuan speak of cutworthy properties and cutworthy operations, the problem has a common core.

The aim of this section is to show how results of Section 3 can shed light on "cutworthiness". The point is whether the fact that a given property applies to a fuzzy relation or a collection of fuzzy relations (in general, to a given fuzzy structure) can be "translated" into saying that the property applies to (suitable) $a$-cuts of the fuzzy relation or a collection of fuzzy relations. We shall show that as far as properties expressible by logical formulas are concerned, cutworthiness can be approached from a unifying point of view: what plays a role is a logical formula $\varphi$ and the question of whether ${}^{a}||\varphi||_{\mathbf{M},v}$ can be expressed by $||\psi||_{{}^{b}\mathbf{M},v}$ (for some $b$'s and subformulas $\psi$ of $\varphi$); moreover, we show that Section 3 provides a complete solution to the problem. We take several examples which have been discussed in literature in connection with cutworthiness and discuss them in detail; the main emphasis is on showing how Lemma 6, Lemma 7 and Theorem 8 apply. Although some conclusions we will obtain can be inferred directly and more easily, we still prefer to use Section 3 since it provides a general method.

We start by a definition of cutworthiness (of a formula); the definition will be illustrated in subsequent examples.

Definition 10 A formula $\varphi$ is *cutworthy* for **L** if for each $a \in L$ there exists $C_a \subseteq L$ such that

$${}^{a}||\varphi||_{\mathbf{M},v} = 1 \quad \text{iff for each } b \in C_a : ||\varphi||_{{}^{b}\mathbf{M},v} = 1.$$

That is, $\varphi$ is cutworthy if for each $a \in L$, testing whether truth degree of $\varphi$ in **M** is at least $a$ is equivalent to testing whether $\varphi$ is true in $b$-cuts of **M** for all $b$ from some $C_a \subseteq L$.

### 4.2 Cutworthiness and Operations with Fuzzy Sets

First, consider intersection. Cutworthiness of intersection was discussed directly (not using Section 3) above. Let $A$ and $B$ be $\mathbf{L}$-sets in a universe $X$; consider a language $\mathscr{J}$ with unary relation symbols $r_A$ and $r_B$, and an $\mathbf{L}$-structure $\mathbf{M}$ for $\mathscr{J}$ such that $r_A^{\mathbf{M}} = A$ and $r_B^{\mathbf{M}} = B$. Then the intersection $A \cap B$ "is defined" by formula $r_A(\xi) \wedge r_B(\xi)$: for a valuation $v$ such that $v(\xi) = x$ we have $(A \cap B)(x) = \|r(\xi) \wedge s(\xi)\|_{\mathbf{M},v}$. Now, applying Lemma 6 we get ${}^a(A \cap B)(v(\xi)) = {}^a\|r(\xi) \wedge s(\xi)\|_{\mathbf{M},v} = \|r(\xi) \wedge s(\xi)\|^a_{\mathscr{C}_{M,v}} = \|r(\xi)\|^a_{\mathscr{C}_{M,v}} \wedge^{\mathbf{2}} \|s(\xi)\|^a_{\mathscr{C}_{M,v}} = {}^aA(v(\xi)) \wedge^{\mathbf{2}} {}^aB(v(\xi))$, i.e. ${}^a(A \cap B) = {}^aA \cap {}^aB$ showing that the formula $r(\xi) \wedge s(\xi)$ which represents intersection is cutworthy (for $C_a = \{a\}$). Note that cutworthiness of intersection as understood in Klir and Yuan (1995) means exactly cutworthiness of $r(\xi) \wedge s(\xi)$.

For union, the situation is in general different. Namely, if the structure $\mathbf{L}$ of truth values is not linearly ordered, we do not have ${}^a(A \cup B) = {}^aA \cup {}^aB$. Indeed, let $a, b \in L$ be non-comparable and let $A$ and $B$ be $\mathbf{L}$-sets in $X$ such that for some $x \in X$ we have $A(x) = a$ and $B(x) = b$. Then we have $(A \cup B)(x) = a \vee b$ and thus $x \in^{a \vee b} (A \cup B)$; on the other hand, $x \notin^{a \vee b} A$ and $x \notin^{a \vee b} B$. However, if $\mathbf{L}$ is linearly ordered, union of fuzzy sets is cutworthy, i.e. ${}^aA \cup B = {}^aA \cup {}^aB$. This can be proved as above for intersection using Lemma 7.

Consider now negation. Again, we do not have ${}^a\neg A = \neg^a A$ (where $\neg{}^aA$ is the set-theoretical complement of ${}^aA$); the reader can easily find a counter-example. However, Lemma 6 yields how ${}^a\neg A$ can be expressed in terms of $b$-cuts of $A$. Consider an $\mathbf{L}$-structure $\mathbf{M}$ for a language $\mathscr{J}$ containing a unary relation symbol $r_A$ and suppose $r_A^{\mathbf{M}} = A$. Then for a valuation $v$ such that $v(\xi) = x$ we have by Lemma 6 ${}^a\neg A(x) = {}^a\|\neg r_A(\xi)\|_{\mathbf{M},v} = \|\neg r_A(\xi)\|^a_{\mathscr{C}_{\mathbf{M},v}} = \|r_A(\xi) \Rightarrow \mathbb{0}\|^a_{\mathscr{C}_{\mathbf{M},v}} = \bigwedge^{\mathbf{2}}_{b \in L} \|r_A(\xi)\|^b_{\mathscr{C}_{\mathbf{M},v}} \Rightarrow^{\mathbf{2}} \|\mathbb{0}\|^{a \otimes b}_{\mathscr{C}_{\mathbf{M},v}} = \bigwedge^{\mathbf{2}}_{b \in L} {}^bA(x) \Rightarrow^{\mathbf{2}} {}^{a \otimes b}0(x)$ (recall that ${}^{a \otimes b}0(x) = 1$ iff $a \otimes b = 0$ iff $b \leq \neg a$). We thus get that $x \in {}^a\neg A$ iff for each $b \in L$ we have that if $x \in {}^bA$ then $b \leq \neg a$. This can be simplified: it is easy to see that it is sufficient to test the latter condition only for $b = A(x)$; this yields $x \in {}^a\neg A$ iff $A(x) \leq \neg a$.

We thus have

*Observation 11* Intersection of $\mathbf{L}$-sets is cutworthy (for $C_a = \{a\}$); union of $\mathbf{L}$-sets is cutworthy (for $C_a = \{a\}$) iff $\mathbf{L}$ is linearly ordered; negation is not cutworthy.

### 4.3 Cutworthiness and Reflexivity, Symmetry and Transitivity

Let $R$ be a binary $\mathbf{L}$-relation on a set $X$; let $\mathbf{M}$ be an $\mathbf{L}$-structure for a language $\mathscr{J}$ containing a binary relation symbol $r$ such that $R = r^{\mathbf{M}}$. Recall that reflexivity, symmetry and transitivity of $R$ are defined using formulas $(\forall \xi) r(\xi, \xi)$, $(\forall \xi, \nu)(r(\xi, \nu) \Rightarrow r(\nu, \xi))$, and $(\forall \xi, \nu, \varsigma)((r(\xi, \nu) \otimes r(\nu, \varsigma) \Rightarrow r(\xi, \varsigma))$, abbreviated for now by ref, sym and tra, respectively. $R$ is called $a$-reflexive if $a \leq \|\mathrm{ref}\|_{\mathbf{M}}$; $a$-symmetric if $a \leq \|\mathrm{sym}\|_{\mathbf{M}}$; $a$-transitive if $a \leq \|\mathrm{sym}\|_{\mathbf{M}}$. For $a = 1$ we omit the prefix $a$- and speak of reflexivity instead of 1-reflexivity etc.

Consider first reflexivity. Using Lemma 6 we have ${}^a\|\mathrm{ref}\|_{\mathbf{M},v} = \|\mathrm{ref}\|^a_{\mathscr{C}_{\mathbf{M},v}} = \bigwedge^{\mathbf{2}}_{v' =_x v} \|r(\xi, \xi)\|^a_{\mathscr{C}_{\mathbf{M},v}} = \bigwedge^{\mathbf{2}}_{x \in X} {}^a r^{\mathbf{M}}(x, x)$ (note that $v$ can be taken arbitrarily since $\|\mathrm{ref}\|_{\mathbf{M},v} = \|\mathrm{ref}\|_{\mathbf{M}}$). As $R$ is $a$-reflexive iff ${}^a\|\mathrm{ref}\|_{\mathbf{M},v} = 1$, observing that $\bigwedge^{\mathbf{2}}_{x \in X} {}^a r^{\mathbf{M}}(x, x) = 1$ is equivalent to $a$-reflexivity of ${}^a r^{\mathbf{M}}$, we conclude that $R$ is $a$-reflexive iff ${}^aR$ is reflexive (as a bivalent relation). Particularly, $R$ is reflexive iff ${}^1R$ is a reflexive relation. Note also that since ${}^aR \subseteq {}^{\mathrm{a}}R$, $R$ is reflexive iff each ${}^aR$ ($a \in L$) is reflexive.

Symmetry: Lemma 6 gives ${}^a\|\mathrm{sym}\|_{\mathbf{M},v} = \|\mathrm{sym}\|^a_{\mathscr{C}_{\mathbf{M},v}} = \bigwedge^{\mathbf{2}}_{v' =_{\xi,\nu} v} \|r(\xi, \nu) \Rightarrow r(\nu, \xi)\|^a_{\mathscr{C}_{\mathbf{M},v}} = \bigwedge^{\mathbf{2}}_{v' =_{\xi,\nu} v} \bigwedge^{\mathbf{2}}_{b \in L} \|r(\xi, \nu)\|^b_{\mathscr{C}_{\mathbf{M},v}} \rightarrow^{\mathbf{2}} \|r(\nu, \xi)\|^{a \otimes b}_{\mathscr{C}_{\mathbf{M},v}} = \bigwedge^{\mathbf{2}}_{x,y \in X} \bigwedge^{\mathbf{2}}_{b \in L} {}^b r^{\mathbf{M}}(x, y) \rightarrow^{\mathbf{2}} {}^{a \otimes b} r^{\mathbf{M}}(y, x)$. $R$ is $a$-symmetric iff ${}^a\|\mathrm{sym}\|_{\mathbf{M},v} = 1$. Therefore, $R$ is $a$-symmetric iff for every $x,y \in X$ and for each $b \in L$ we have that whenever $\langle x, y \rangle$ belongs to $b$-cut of $R$ then $\langle y, x \rangle$ belongs to $(a \otimes b)$-cut of $R$. That is, we have a condition equivalent to $a$-symmetry of $R$ which uses only

$c$-cuts of $R$. Particularly, for symmetry of $R$ ($a = 1$) we get that $R$ is symmetric iff for each $b \in L$, if $\langle x, y\rangle$ belongs to $b$-cut of $R$ then $\langle y, x\rangle$ belongs to $(1\otimes b)$-cut, i.e. to $b$-cut of $R$. That is, $R$ is symmetric iff each $a$-cut of $R$ is symmetric.

Transitivity: By Lemma 6, ${}^{a}\|\text{tra}\|_{\mathbf{M},v} = \|\text{tra}\|^{a}_{\mathscr{C}_{\mathbf{M},v}} = \bigwedge^{\mathbf{2}}_{v'=_{\xi,\nu,\varsigma}v}\|(r(\xi,\nu) \otimes r(\nu,\varsigma)) \Rightarrow r(\xi,\varsigma)\|^{a}\mathscr{C}_{\mathbf{M},v} = \bigwedge^{\mathbf{2}}_{v'=_{\xi,\nu,\varsigma}v}\bigwedge^{\mathbf{2}}_{b\in L} \left(\bigvee^{\mathbf{2}}_{b\le c\otimes d}\|r(\xi,\nu)\|^{c}_{\mathscr{C}_{\mathbf{M},v}} \wedge^{\mathbf{2}} \|r(\nu,\varsigma)\|^{d}_{\mathscr{C}_{\mathbf{M},v}}\right) \rightarrow^{\mathbf{2}} \|r(\xi,\varsigma)\|^{a\otimes b}_{\mathscr{C}_{\mathbf{M},v}} = \bigwedge^{\mathbf{2}}_{x,y,z}\bigwedge^{\mathbf{2}}_{b\in L}(\bigvee^{\mathbf{2}}_{b\le c\otimes d}{}^{c}r^{\mathbf{M}}(x,y) \wedge^{\mathbf{2}} {}^{d}r^{\mathbf{M}}(y,z)) \rightarrow^{\mathbf{2}} {}^{a\otimes b}r^{\mathbf{M}}(x,z)$. As $a$-transitivity of $R$ is equivalent to ${}^{a}\|\text{tra}\|_{\mathbf{M},v} = 1$, we get that $R$ is $a$-transitive iff for each $x,y,z \in X$ and each $b \in L$ we have that if $\langle x, y\rangle \in {}^{c}R$ and $\langle y,z\rangle \in {}^{d}R$ for some $c\otimes d \ge b$ then $\langle x, z\rangle \in {}^{a\otimes b}R$. This is how $a$-transitivity of $R$ is expressed using $b$-cuts of $R$ and bivalent logical operations. Particularly, consider transitivity and $\mathbf{L}$ satisfying $x\otimes y = x \wedge y$ (i.e. $\mathbf{L}$ is a Heyting algebra). Then from $\|r(\xi,\nu)\otimes r(\nu,\varsigma)\|^{b}_{\mathscr{C}_{\mathbf{M},v}} = \|r(\xi,\nu)\|^{b}_{\mathscr{C}_{\mathbf{M},v}}\otimes^{\mathbf{2}}\|r(\nu,\varsigma)\|^{b}_{\mathscr{C}_{\mathbf{M},v}}$ we get that $R$ is transitive iff for every $x,y,z \in X$ we have that for each $b \in L$, if $\langle x,y\rangle \in {}^{b}R$ and $\langle y,z\rangle \in {}^{b}R$ then $\langle x,z\rangle \in {}^{b}R$; that is, $R$ is transitive iff each $b$-cut of $R$ is transitive.

To sum up, we have

*Observation 12* $a$-reflexivity is cutworthy (for $C_a = \{a\}$); symmetry is cutworthy (for $C_a = L$); transitivity is cutworthy (for $C_a = \{a\}$) whenever $\mathbf{L}$ is a Heyting algebra.

### 4.4 Fuzzy Galois Connections as Systems of Galois Connections

Fuzzy Galois connections are the basic structures behind so-called fuzzy concept lattices (loosely speaking, hierarchical structures of formal concepts hidden in data). Our next example shows how fuzzy Galois connections can be viewed as systems of classical Galois connections. The result we are going to prove using results from Section 3 was obtained directly in Bělohlávek (1999).

An $\mathbf{L}$-Galois connection between sets $X$ and $Y$ is a pair $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ of mappings ${}^{\uparrow}: L^{X} \rightarrow L^{Y}$ and ${}^{\downarrow}: L^{Y} \rightarrow L^{X}$ satisfying $S(A_1,A_2) \le S(A_2^{\uparrow},A_1^{\uparrow})$, $S(B_1,B_2) \le S(B_2^{\uparrow},B_1^{\uparrow})$, $A \subseteq A^{\uparrow\downarrow}$, $B \subseteq B^{\downarrow\uparrow}$ for any $A,A_1,A_2 \in L^{X}$, $B,B_1,B_2 \in L^{Y}$ (note that $S(A_1,A_2) = \bigwedge_{x\in X}(A_1(x) \rightarrow A_2(x))$ is the degree to which $A_1$ is a subset of $A_2$). For a binary $\mathbf{L}$-relation $I$ between $X$ and $Y$, the pair $\langle {}^{\uparrow_I}, {}^{\downarrow_I}\rangle$ of mappings ${}^{\uparrow_I}: L^{X} \rightarrow L^{Y}$ and ${}^{\downarrow_I}: L^{Y} \rightarrow L^{X}$ defined by

$$A^{\uparrow_I}(y) = \bigwedge_{x\in X} A(x) \rightarrow I(x,y)$$

and

$$B^{\downarrow_I}(x) = \bigwedge_{y\in Y} B(y) \rightarrow I(x,y)$$

is an $\mathbf{L}$-Galois connection; conversely, each $\mathbf{L}$-Galois connection between $X$ and $Y$ is induced by some $I$ in the above manner, see Bělohlávek (1999). Note that $\mathbf{2}$-Galois connections ($\mathbf{2}$ denotes the two-element Boolean algebra) are exactly the well-known Galois connections.

A system $\{\langle {}^{\uparrow_a}, {}^{\downarrow_a}\rangle | a \in L\}$ of $\mathbf{2}$-Galois connections is called $\mathbf{L}$-nested if (1) for each $a,b \in L$, $a \le b, A \in 2^{X}, B \in 2^{Y}$, it holds $A^{\uparrow_a} \supseteq A^{\uparrow_b}$, $B^{\downarrow_a} \supseteq B^{\downarrow_b}$, (2) the set $\{a \in L |\ y \in \{x\}^{\uparrow_a}\}$ contains a greatest element, and (3) $A^{\uparrow_0} = Y$, $B^{\downarrow_0} = X$, for any $A \in 2^{X}, B \in 2^{Y}$.The following theorem showing a way to consider $\mathbf{L}$-Galois connections as systems of Galois connections was obtained in Bělohlávek (1999):

THEOREM 13 For an $\mathbf{L}$-Galois connection $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ between $X$ and $Y$ denote $\mathscr{C}\langle {}^{\uparrow}, {}^{\downarrow}\rangle = \{\langle {}^{\uparrow_a}, {}^{\downarrow_a}\rangle | a \in L\}$ where ${}^{\uparrow_a}: 2^{X} \rightarrow 2^{Y}$ and ${}^{\downarrow_a}: 2^{Y} \rightarrow 2^{X}$ are defined by $A^{\uparrow_a} = {}^{a}(A^{\uparrow})$ and $B^{\downarrow_a} = {}^{a}(B^{\downarrow})$

for $A \in 2^X$, $B \in 2^Y$. For an **L**-nested system $\mathscr{C} = \{\langle {}^{\uparrow_a}, {}^{\downarrow_a} \rangle | a \in L\}$ of **2**-Galois connections between $X$ and $Y$ denote $\langle {}^{\uparrow_\mathscr{C}}, {}^{\downarrow_\mathscr{C}} \rangle$ the pair of mappings ${}^{\uparrow_\mathscr{C}}: L^X \to L^Y$ and ${}^{\downarrow_\mathscr{C}}: L^Y \to L^X$ defined by

$$A^{\uparrow_\mathscr{C}}(y) = \bigvee \left\{ a | y \in \bigcap_{b \in L} ({}^bA)^{\uparrow_{a\otimes b}} \right\}, \quad B^{\downarrow_\mathscr{C}}(x) = \bigvee \left\{ a | x \in \bigcap_{b \in L} ({}^bB)^{\downarrow_{a\otimes b}} \right\}$$

for $A \in L^X$, $B \in L^Y$. Then it holds

(1) $\mathscr{C}_{\langle {}^{\uparrow_\mathscr{C}}, {}^{\downarrow_\mathscr{C}} \rangle}$ is a nested system of **L**-Galois connections between $X$ and $Y$,

(2) $\langle {}^{\uparrow_\mathscr{C}}, {}^{\downarrow_\mathscr{C}} \rangle$ is an **L**-Galois connection between $X$ and $Y$,

(3) $\mathscr{C} = \mathscr{C}\langle {}^{\uparrow_\mathscr{C}}, {}^{\downarrow_\mathscr{C}} \rangle$ and $\langle {}^{\uparrow}, {}^{\downarrow} \rangle = \langle {}^{\uparrow_{\mathscr{C}_{\langle \uparrow, \downarrow \rangle}}}, {}^{\downarrow_{\mathscr{C}_{\langle \uparrow, \downarrow \rangle}}} \rangle$.

The crucial point in proving the foregoing theorem is the following lemma, particularly (2) of the lemma. We will see that (2) is an easy consequence of results from Section 3.

LEMMA 14 Let $I \in L^{X \times Y}$ be an **L**-relation, $\langle {}^{\uparrow}, {}^{\downarrow} \rangle$ be the **L**-Galois connection induced by $I$, and for $a \in L$ let $\langle {}^{\uparrow_a}, {}^{\downarrow_a} \rangle$ be the **2**-Galois connection induced by the **2**-relation ${}^aI$. Then (1) for every **2**-sets $A \in 2^X$, $B \in 2^Y$, $a \in L$, we have

$$^a(A^{\uparrow}) = A^{\uparrow_a}, \quad ^a(B^{\downarrow}) = B^{\downarrow_a}, \tag{1}$$

and (2) for every **L**-sets $A \in L^X$, $B \in L^Y$, $a \in L$, we have

$$^a(A^{\uparrow}) = \bigcap_{b \in L} ({}^bA)^{\uparrow_{a\otimes b}}, \quad ^a(B^{\downarrow}) = \bigcap_{b \in L} ({}^bB)^{\downarrow_{a\otimes b}}. \tag{2}$$

*Proof* For (1) see Bělohlávek (1999). Prove (2): this is an easy consequence of Lemma 6. Consider a two-sorted language $\mathscr{J}_{\mathrm{Con}}$ with sorts $\mathscr{X}$ and $\mathscr{Y}$ that contains unary relation symbols $r_A$ (of sort $\mathscr{X}$) and $r_B$ (of sort $\mathscr{Y}$), and a binary relation symbol $r_I$ (with arguments of sorts $\mathscr{X}$ and $\mathscr{Y}$). For a formula $\varphi_{\uparrow} = (\forall \xi)(r_A(\xi) \Rightarrow r_I(\xi, \nu))$, the **L**-structure **M** for $\mathscr{J}_{\mathrm{Con}}$ that corresponds to $\langle X, Y, I \rangle$ and $A$ and $B$ (i.e. $X$ and $Y$ are universes of sorts $\mathscr{X}$ and $\mathscr{Y}$, $r_A$, $r_B$, $r_I$ are interpreted by $A$, $B$, $I$), and a valuation $v$ such that $v(\nu) = y$ we have

$$A^{\uparrow}(y) = \|\varphi_{\uparrow}\|_{\mathbf{M},v}.$$

Lemma 6 thus yields

$$
\begin{aligned}
({}^aA^{\uparrow})(y) = {}^a\|\varphi_{\uparrow}\|_{\mathbf{M},v} &= \bigwedge_{v'=_\xi v}^{\mathbf{2}} \bigwedge_{b \in L}^{\mathbf{2}} \|r_A\|^b_{\mathscr{C}_{\mathbf{M},v'}} \to^{\mathbf{2}} \|r_I\|^{a\otimes b}_{\mathscr{C}_{M,v'}} \\
&= \bigwedge_{b \in L}^{\mathbf{2}} \bigwedge_{x \in X}^{\mathbf{2}} ({}^bA)(x) \to^{\mathbf{2}} ({}^{a\otimes b}I)(x, y) \\
&= \bigcap_{b \in L}^{\mathbf{2}} ({}^bA)^{\uparrow a\otimes b}.
\end{aligned}
$$

For $B$ one can proceed analogously. □

## 4.5 Fuzzy Automata as Nested Systems of (non-deterministic) Automata

In this section, we assume that the structure **L** of truth values is linearly ordered. The notion of a fuzzy automaton generalizes that of a non-deterministic automaton (see e.g. Klir and Yuan, 1995). An **L**-*automaton* $\mathscr{M}$ over a finite alphabet $\Sigma$ is given by a finite set $Q$ of states, an **L**-set $Q_I$ in $Q$ (for $q \in Q$, $Q_I(q)$ is the degree to which $q$ is an initial state); an **L**-set $Q_F$ in $Q$ (for $q \in Q$, $Q_F(q)$ is the degree to which $q$ is a final state); an **L**-relation $\delta$ between $Q, \Sigma$ and $Q$ (for $q,q' \in Q$ and $s \in \Sigma$, $\delta(q, s, q')$ is the degree to which the **L**-automaton can transfer from $q$ to $q'$ if the actual input symbol is $s$). Then, for an input word $s_1 \ldots s_n$ we define the degree $(\mathscr{L}(\mathscr{M}))$ $(s_1 \ldots s_n)$ to which $\mathscr{M}$ accepts $s_1 \ldots s_n$ by

$$(\mathscr{L}(\mathscr{M}))(s_1 \ldots s_n) = \bigvee_{q_1,\ldots,q_{n+1} \in Q} Q_I(q_1) \wedge \delta(q_1, s_1, q_2) \wedge \cdots \wedge \delta(q_n, s_n, q_{n+1}) \wedge Q_F(q_{n+1}).$$

The thus-defined **L**-set $\mathscr{L}(\mathscr{M})$ is called the **L**-language recognized by $\mathscr{M}$. We can easily see that for $\mathbf{L} = \mathbf{2}$ we get the above notion of a non-deterministic automaton and the recognized language.

An **L**-automaton can be viewed as an **L**-relational system: Consider an $S$-sorted language $\mathscr{J}_{\mathrm{Aut}}$ where $S = \{\mathscr{Q}, \mathscr{S}\}$, $R = \{r_\delta, r_{Q_I}, r_{Q_F}\}$, $F = \emptyset$. Then an **L**-automaton $\mathscr{M}$ can be viewed as an **L**-structure **M** for $\mathscr{J}_{\mathrm{Aut}}$ where $r_\delta^{\mathbf{M}} = \delta$, $r_{Q_I}^{\mathbf{M}} = Q_I$, and $r_{Q_F}^{\mathbf{M}} = Q_F$. Using the rules for evaluating truth degrees of formulas we easily see that the degree $(\mathscr{L}(\mathscr{M}))(s_1 \ldots s_n)$ to which $\mathscr{M}$ accepts $s_1 \ldots s_n$ equals the truth degree of a formula ($\xi_i$ are variables of sort $\mathscr{Q}$, $\nu_i$ are variables of sort $\mathscr{S}$)

$$(\exists \xi_1, \ldots \xi_{n+1})[r_{Q_I}(\xi_1) \mathbin{\&\!\!\&} r_\delta(\xi_1, \nu_1, \xi_2) \mathbin{\&\!\!\&} \cdots \mathbin{\&\!\!\&} r_\delta(\xi_n, \nu_n, \xi_{n+1}) \mathbin{\&\!\!\&} r_{Q_F}(\xi_1)]$$

for a valuation $v$ such that $v(\nu_i) = s_i$. Denoting this formula accept$_n$ we thus have

$$(\mathscr{L}(\mathscr{M}))(s_1 \ldots s_n) = \|\mathrm{accept}_n\|_{\mathbf{M},v}.$$

For an $S$-sorted **L**-structure **M** which represents an **L**-automaton $\mathscr{M}$, ${}^a\mathbf{M}$ is a bivalent $S$-sorted structure which represents in a natural way a crisp non-deterministic automaton ${}^a\mathscr{M}$: the alphabet and the set of states of ${}^a\mathscr{M}$ is $\Sigma$ and $Q$, respectively; the transition relation is ${}^a\delta$, the set of initial states is ${}^aQ_I$, the set of final states is ${}^aQ_F$. The collection of all ${}^a\mathbf{M}$ ($a \in L$) is **L**-nested; we can thus call **L**-nested the collection of all ${}^a\mathscr{M}$ ($a \in L$).

Using Lemma 6 and 7 we get that for $a \in L$ we have

$$\|\mathrm{accept}_n\|^a_{\mathscr{C}_{\mathbf{M},v}} = \bigvee^{2}_{v' = \xi_1,\ldots,\xi_{n+1}^{\ v}} \|r_{QI}(\xi_1)\|^a_{\mathscr{C}_{\mathbf{M},v'}} \wedge^{\mathbf{2}} \|r_\delta(\xi_1, \nu_1, \xi_2)\|^a_{\mathscr{C}_{\mathbf{M},v'}} \wedge^{\mathbf{2}} \ldots$$

$$\wedge^{\mathbf{2}} \|r_\delta(\xi_1, \nu_n, \xi_{n+1})\|^a_{\mathscr{C}_{\mathbf{M},v'}} \wedge^{\mathbf{2}} \|r_{Q_F}(\xi_{n+1})\|^a_{\mathscr{C}_{\mathbf{M},v'}}.$$

Recalling $\|r_{QI}(\xi_1)\|^a_{\mathscr{C}_{\mathbf{M},v'}} = {}^ar^{\mathbf{M}}_{QI}(v'(\xi_1))$, $\|r_\delta(\xi_i, \nu_i, \xi_{i+1})\|^a_{\mathscr{C}_{\mathbf{M},v'}} = {}^ar^{\boldsymbol{M}}_\delta(v'(\xi_i), v'(\nu_i), v'(\xi_{i+1}))$, and $\|r_{QF}(\xi_{n+1})\|^a_{\mathscr{C}_{\mathbf{M},v'}} = {}^ar^{\mathbf{M}}_{QF}(v'(\xi_{n+1}))$, we obtain

$$\|\mathrm{accept}_n\|^a_{\mathscr{C}_{\mathbf{M},v}} = \|\mathrm{accept}_n\|^a_{\mathbf{M},v}.$$

This means that for a word $s_1 \ldots s_n$ ($s_i \in \Sigma$), the degree to which $s_1 \ldots s_n$ is accepted by an **L**-automaton $\mathscr{M}$ is at least $a$ if and only if $s_1 \ldots s_n$ is accepted by the non-deterministic automaton ${}^a\mathscr{M}$. Taking into account Theorem 8, we can summarize the observed results:

THEOREM 15 The mapping sending an **L**-automaton $\mathscr{M}$ to a system $\{{}^a\mathscr{M} | a \in L\}$ is a bijective correspondence between **L**-automata over an alphabet $\Sigma$ and **L**-nested systems of non-deterministic automata over $\Sigma$. Moreover, we have

$$^{a}\mathscr{L}(\mathscr{M}) = \mathscr{L}(^{a}\mathscr{M})$$

and thus

$$(\mathscr{L}(\mathscr{M}))(s_1 \dots s_n) = \bigvee \{a | s_1 \dots s_n \in \mathscr{L}(^{a}\mathscr{M})\}$$

for each word $s_1 \dots s_n$ of symbols from $\Sigma$.

*Remark 16* Note that Theorem 15 is the main result obtained (in terms of category theory) in Močkoř (1999).

### 4.6 Cutworthiness and Extension Principle: Computing $\bar{g}$ "cut-by-cut"

Note that extension principle (see e.g. Klir and Yuan, 1995) enables one to obtain a function $\bar{g}: L^X \to L^Y$ from a function $g: X \to Y$: for a fuzzy set $A \in L^X$, $\bar{g}(A)$ is a fuzzy set in $Y$ defined by

$$(\bar{g}(A))(y) = \bigvee \{A(x) | g(x) = y\}.$$

In bivalent case (only 0 and 1 as truth values), extension principle yields the following: for a function $g: X \to Y$, $\bar{g}$ is a function assigning subsets of $Y$ to subsets of $X$ by

$$\bar{g}(A) = \{g(x) | x \in A\}.$$

For now, we will denote the function obtained from $g$ by the extension principle in the bivalent case by $g^*$ (and not by $\bar{g}$). A natural question arises as to whether it is possible to reduce the general case (**L** also contains other truth values than 0 and 1) to the bivalent one; particularly, whether $\bar{g}(A)$ can be computed cut by cut, i.e. whether we have

$$^{a}\bar{g}(A) = g^{*}(^{a}A) \tag{3}$$

for each $a \in L$. We shall see that if **L** is linearly ordered and if $X$ and $Y$ are finite then this is indeed the case (note that this assumption can be still weakened).

To this end, observe that $\bar{g}$ is in fact "defined' by logical formula. Indeed, let $S = \{\mathscr{X}, \mathscr{Y}\}$ and consider an $S$-sorted language $\mathscr{J}_{\mathrm{EP}}$ with a unary relation symbol $r_A$, binary relation symbols $\approx_{\mathscr{X}}$ and $\approx_{\mathscr{Y}}$ (equalities for respective sorts), and a unary function symbol $f$ such that for the sorts we have $\sigma(r_A) = \mathscr{X}$, $\sigma(\approx_{\mathscr{X}}) = \mathscr{X}\mathscr{X}$, $\sigma(\approx_{\mathscr{Y}}) = \mathscr{Y}\mathscr{Y}$, $\sigma(f) = \mathscr{X}\mathscr{Y}$. An **L**-structure **M** for $\mathscr{J}_{\mathrm{EP}}$ thus consists of a set $M_{\mathscr{X}}$, a set $M_{\mathscr{Y}}$, equivalence relations $\approx_{\mathscr{X}}^{\mathbf{M}}$ on $M_{\mathscr{X}}$ and $\approx_{\mathscr{Y}}^{\mathbf{M}}$ on $M_{\mathscr{Y}}$, and a function $f^{\mathbf{M}}: M_{\mathscr{X}} \to M_{\mathscr{Y}}$ which is compatible with $\approx_{\mathscr{X}}^{\mathbf{M}}$ and $\approx_{\mathscr{Y}}^{\mathbf{M}}$. Let EP($f$) be the formula

$$(\exists \xi)(r_A(\xi) \wedge (f(\xi) \approx_{\mathscr{Y}} \nu))$$

where $\xi$ and $\nu$ are variables of sort $\mathscr{X}$ and $\mathscr{Y}$, respectively. Now, EP($f$) induces a mapping assigning to an **L**-set $A$ in $X$, a function $g: X \to Y$, and an **L**-equivalence $E$ on $Y$ the **L**-set $\mathrm{EP}(f)^{(\mathbf{L},\mathbf{M})}$ in $Y$ which is given by

$$(\mathrm{EP}(f)^{(\mathbf{L},\mathbf{M})})(y) = \|\mathrm{EP}(f)\|_{\mathbf{M},v}$$

where $y \in Y$, **M** is an **L**-structure for $\mathscr{J}_{\mathrm{EP}}$ such that $M = \{M_{\mathscr{X}}, M_{\mathscr{Y}}\}$, $M_{\mathscr{X}} = X$, $M_{\mathscr{Y}} = Y$, $f^{\mathbf{M}} = g$, $\approx_{\mathscr{Y}}^{\mathbf{M}} = E$, and $r_A^{\mathbf{M}} = E = A$, and $v$ is a valuation such that $v(\nu) = y$. If $E$ is the identity on $Y$ ($E(y_1, y_2) = 1$ for $y_1 = y_2$; $E(y_1, y_2) = 0$ otherwise) then we get

$$(\mathrm{EP}(f)^{(\mathbf{L},\mathbf{M})})(y) = \bigvee \{A(x) | g(x) = y\} = (\bar{g}(A))(y), \tag{4}$$

i.e. EP($f$)$^{(\mathrm{L,M})}$ is exactly $\bar{g}(A)$–thus, $\bar{g}$ is in the above sense defined by formula EP($f$). Now, Lemma 6 yields

$$
\begin{aligned}
{}^{a}\bar{g}(A) &= {}^{a}\|(\exists\xi)(r_A(\xi) \mathbin{\&} (f(\xi) \approx \mathscr{Y}\nu))\|_{\mathbf{M},v} \\
&= \|(\exists\xi)(r_A(\xi) \mathbin{\&} (f(\xi) \approx \mathscr{Y}\nu))\|^{a}_{\mathscr{C}_{\mathbf{M},v}} \\
&= \bigvee^{2}_{v'=_{\xi}v} \|r_A(\xi)\|^{a}_{\mathscr{C}_{\mathbf{M},v'}} \wedge^{\mathbf{2}} \|f(\xi) \approx_{\mathscr{Y}} \nu)\|^{a}_{\mathscr{C}_{\mathbf{M},v'}} \\
&= \bigvee^{2}_{v'=_{\xi}v} r_A^{{}^{a}\boldsymbol{M}}(v'(\xi)) \wedge^{2} (g(v'(\xi)) \approx^{{}^{a}\mathbf{M}}_{\mathscr{Y}} v'(\nu)) \\
&= \|(\exists\xi)(r_A(\xi) \mathbin{\&} (f(\xi) \approx \mathscr{Y}\nu))\|_{{}^{a}\boldsymbol{M},v} = g^{*}({}^{a}A)
\end{aligned}
$$

verifying (3).


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